Ramification Points on the Eigencurve and the Two Variable
Symmetric Square $p$-adic $L$-Function

by

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Abstract

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Let $C_{p,N}$ denote the $p$-adic eigencurve of tame level $N$ constructed by Coleman, Mazur, and Buzzard. This curve has a natural weight projection map, $\pi : C_{p,N} \to W_N$, from the eigencurve to the level $N$ weight space, $W_N$. In this thesis, we use methods of Hida which he developed for ordinary modular forms and $p$-adic families of these forms and the theory of overconvergent modular symbols developed by Stevens to construct a locally analytic symmetric square $p$-adic $L$-function, $L(X)$, whose domain is $C_{p,N}$. We show that the zeros of $L(X)$ are precisely the ramification points of $C_{p,N}$ for the weight projection map $\pi$. Also, by following a $p$-adic version of Rankin's method as developed by Hida, we construct a two variable symmetric square $p$-adic $L$-function on the eigencurve $C_{p,N}$ whose diagonal restricts to give $L(X)$. 

Professor Robert Coleman
Dissertation Committee Chair

1
This dissertation is dedicated to my parents,  
Suk Hong Kim and Hyen Yang Kim
Contents

1 Introduction .................................................................................................................. 1
  1.1 History .................................................................................................................. 1
  1.2 Eigencurve Notation .............................................................................................. 3
  1.3 Main Results .......................................................................................................... 5

2 Background Material ................................................................................................. 9
  2.1 Modular Forms ........................................................................................................ 9
    2.1.1 Overconvergent modular forms ....................................................................... 9
    2.1.2 Families of modular forms .............................................................................. 11
  2.2 Abstract Hecke Operators ..................................................................................... 13
  2.3 The Eigencurve ...................................................................................................... 15
    2.3.1 The spectral curve .......................................................................................... 15
    2.3.2 Construction of the eigencurve ...................................................................... 16
  2.4 Cohomology ........................................................................................................... 18
  2.5 Classical Hecke Operators ..................................................................................... 18
  2.6 Modular Symbols .................................................................................................... 20
    2.6.1 Definitions ....................................................................................................... 21
    2.6.2 Actions ............................................................................................................ 22
    2.6.3 Special Values ................................................................................................ 23
    2.6.4 Measures ......................................................................................................... 25
    2.6.5 $p$-adic $L$-function ...................................................................................... 28
  2.7 Overconvergent Modular Symbols ......................................................................... 29
    2.7.1 Overconvergent modules .............................................................................. 30
    2.7.2 Specialization .................................................................................................. 31
    2.7.3 Modular symbols on the eigencurve ............................................................. 32
  2.8 Two Variable $p$-adic $L$-function ......................................................................... 33

3 Ramification Points on the Eigencurve ...................................................................... 35
  3.1 Introduction ............................................................................................................ 35
  3.2 Local Pieces of the Eigencurve .............................................................................. 37
    3.2.1 The eigencurve .............................................................................................. 37
3.2.2 Weight space ................................................. 37
3.2.3 The spectral curve ........................................... 38
3.2.4 An admissible cover ......................................... 38
3.2.5 Local pieces of the eigencurve .............................. 38
3.2.6 Slope $\leq \alpha$ components ............................... 39
3.3 Congruence Modules ............................................ 40
  3.3.1 Components ................................................. 40
  3.3.2 Big congruence module .................................... 40
  3.3.3 Specializing to a weight .................................. 41
  3.3.4 Classical congruence modules .............................. 41
3.4 Ramification Points: Part I ................................... 42
3.5 Cohomology Pairings ............................................ 43
  3.5.1 Modules ..................................................... 43
  3.5.2 Cohomology ................................................ 44
  3.5.3 Cup Product on Specialized Cohomology ................. 45
  3.5.4 Cup Product on Big Cohomology ........................... 46
3.6 Symmetric Square $p$-adic $L$-function ......................... 47
3.7 Gorenstein Error ................................................ 49
  3.7.1 Cohomology Groups ....................................... 50
  3.7.2 Perfection of the Cup Product ............................ 52
  3.7.3 Error from the Parabolic Map ............................. 54
  3.7.4 Error from the Theta Map ................................ 57
  3.7.5 Independent of weight .................................... 58
3.8 Ramification Points: Part II ................................... 59
3.9 Special Values .................................................. 60
  3.9.1 Symmetric square $L$-function ............................ 60
  3.9.2 Eichler-Shimura isomorphism .............................. 61
  3.9.3 Interpolation of special values ........................... 62

4 Two Variable Symmetric Square $p$-adic $L$-function ............ 63
  4.1 Introduction .................................................. 63
  4.2 Modular Forms of Half Integral Weight ..................... 65
    4.2.1 Classical Modular Forms of Half Integral Weight ...... 65
    4.2.2 $p$-adic Modular Forms of Half Integral Weight ...... 65
    4.2.3 Properties of Modular Forms of Integral Weight ...... 66
    4.2.4 Properties of Modular Forms of Half Integral Weight 67
  4.3 Theta Measure ................................................ 68
  4.4 Eisenstein Measure of Half Integral Weight ................ 70
  4.5 Shimura Differential Operators .............................. 72
  4.6 Two variable symmetric square $p$-adic $L$-function ....... 73
  4.7 Construction of the Convoluted Measure .................... 77
  4.8 The $p$-adic Holomorphic Projector .......................... 79
4.9 Linear Form and Periods ...................................... 80
4.10 Proof of the Main Theorem .................................. 82

Bibliography .......................................................... 85
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Chapter 1

Introduction

1.1 History

In the 1980's a large amount of work was done to understand congruences between cuspidal eigenforms modulo various primes $p$. A large portion of this work was done by Hida [9] and strengthened by Ribet [15]. To give a brief summary of their results, let $f_k$ be a cuspidal eigenform of weight $k \geq 2$ and level $N$. Hida constructed a congruence module $C(f_k)$ from the Hecke ring attached to the space of weight $k$ level $N$ cusp forms. The results stated that primes $p$ dividing the order of $C(f_k)$ are precisely the primes for which there exists another cuspidal eigenform $g$ of the same weight and level such that $f \equiv g$ modulo $p$. It was also shown that the order of $C(f_k)$ could be extracted from the special value $L(k, f_k)$ of the symmetric square complex $L$-function attached to $f_k$ at $k$.

For a fixed prime $p$, further work of Hida [10] led to the notion of a $p$-adic Hida family of ordinary modular forms. An ordinary modular eigenform is an eigenform whose $U_p$-eigenvalue is a $p$-adic unit. More generally, if the $p$-adic valuation of the $U_p$-eigenvalue of a modular eigenform is the rational number $\alpha$, one calls the eigenform of slope $\alpha$. Thus ordinary eigenforms are of slope 0. A Hida family of ordinary forms, in a simple situation, can be thought of as a formal $q$-expansion with coefficients in the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ with the property that when one specializes the formal $q$-expansion at the $p$-adic character $\psi(x)x^k$ defined on $\mathbb{Z}_p^\times$ with $\psi$ of finite order and...
$k \geq 2$, one obtains the $q$-expansion of an ordinary modular eigenform of weight $k$ and character $\psi \omega^{-k}$ where $\omega$ denotes the Teichmüller character. The existence of a $p$-adic family of modular forms implies congruences between modular forms of various weights modulo powers of the prime $p$.

Hida further constructed a congruence module over the ring $\Lambda$ for each family with the property that it $p$-adically interpolates the congruence modules of each form in the family. When the annihilator ideal in $\Lambda$ of such a congruence module is principle, a generator of this ideal $p$-adically interpolates, up to a $p$-adic unit, the algebraic part of the special values $L(k, f_k)$ as $f_k$ varies over the Hida family.

Existing notions of $p$-adic modular forms developed by Katz [12] and Serre [16] and the work of Hida led people to believe that modular forms that are not necessarily ordinary could be put into $p$-adic families. This was shown by Coleman [3]. The work of Coleman organizes finite slope modular eigenforms into $p$-adic families. The $q$-expansions that arise when specializing the family at a particular weight are not always classical modular forms, but they do lie in the space of $p$-adic modular forms. Not all finite slope $p$-adic modular forms arise among these specializations, but rather those belonging to a subspace of the space of $p$-adic modular forms called overconvergent $p$-adic modular forms. Overconvergent $p$-adic modular forms were defined prior to the work of Coleman as, roughly speaking, certain $p$-adic modular forms whose domain of definition can be expanded. Coleman’s work gives great significance to this subspace of $p$-adic modular forms.

Coleman’s work led to the construction by Coleman and Mazur [4] of a geometric object called the eigencurve which parametrizes the families of finite slope overconvergent modular forms. This eigencurve is a rigid analytic curve whose construction uses notions from rigid analytic geometry. Their construction was actually done for primes $p \geq 3$ and tame level $N = 1$. In other words, their eigencurve parameterized modular eigenforms whose level is a power of $p \geq 3$. Verification that the construction works for all primes $p$ and all tame levels $N$ was completed by Buzzard [1].
1.2 Eigencurve Notation

Let $p$ be a prime and $N$ be a positive integer relatively prime to $p$. Let $C_{N,p}$ denote the level $N$ $p$-adic eigencurve of Coleman, Mazur, and Buzzard. $C_{N,p}$ is a rigid analytic curve with a natural embedding into $X_p \times A^1$ where $X_p$ is the rigid analytic space attached to a certain universal deformation ring of certain Galois pseudo-representations and $A^1$ is the affine line. The following is one of the main results about the eigencurve:

**Theorem 1.2.1 (Coleman, Mazur, Buzzard)** There is a natural one-to-one correspondence

$$c \leftrightarrow f_c$$

(1.1)

between the $C_p$-valued points on the eigencurve $C_{N,p}$ and finite slope overconvergent modular eigenforms of tame level $N$. The one-to-one correspondence is characterized by the conditions that the projection of $c$ to $X_p$ is the pseudo-representation attached to $f_c$ and that the projection of $c$ to $A^1$ is the reciprocal of the $U_p$-eigenvalue of $f_c$.

We note that the eigencurve was constructed by Coleman and Mazur for the $N = 1$ case and extended for general $N$ by Buzzard. To clarify the situation, Buzzard’s eigencurve is really a generalization of what Coleman and Mazur call the reduced eigencurve which is notated $C_{1,p}^{\text{red}}$ in their paper and defined to be the nilreduction of their $C_{1,p}$. Thus $C_{N,p}$ really is the eigencurve of Buzzard which is the reduced eigencurve of Coleman and Mazur for $N = 1$.

Let $W_N$ denote the level $N$ weight space. This is the rigid analytic space attached to the Iwasawa algebra $\Lambda_N = \varprojlim Z_p[[Z/Np^\infty Z]^\times]$. There is a natural projection map

$$\pi : C_{N,p} \rightarrow W_N$$

(1.2)

giving the weight character of the form attached to the point of the eigencurve. One of the motivating goals of this thesis is to understand the ramification points of this projection map $\pi$.

For the constructions in this thesis, we consider a certain kind of affinoid subspace of the reduced eigencurve which we will refer to as a “flat affinoid subspace.”
Definition 1.2.2 Let $\mathcal{Y} \subset \mathcal{W}_N$ be an affinoid subspace of weight space defined over a $p$-adic field $K$. A "flat affinoid subspace of $C_{N,p}$ over $\mathcal{Y}$" is an affinoid subspace $\mathcal{D} \subset C_{N,p}$ contained in $\pi^{-1}(\mathcal{Y})$ such that the restriction of $\pi$ to $\mathcal{D}$ is finite and flat. If $\mathcal{D}$ contains precisely the $K$-valued points corresponding to forms of slope $\leq \nu$ in $\pi^{-1}(\mathcal{Y})$, we call $\mathcal{Y}$ a "slope $\leq \nu$ flat affinoid subspace of $C_{N,p}$ over $\mathcal{Y}$" and sometimes write $\mathcal{D}_\nu$ for $\mathcal{D}$.

Proposition 1.2.3 For each slope $\nu$, there is a collection of affinoid subspaces of weight space satisfying the following:

(a) Each affinoid subspace in the collection has a slope $\leq \nu$ flat affinoid subspace.

(b) The union of these slope $\leq \nu$ flat affinoid subspaces consists of exactly the $K$-valued points of the reduced eigencurve that correspond to overconvergent forms of slope $\nu$.

We also consider certain irreducible components of a slope $\leq \nu$ flat affinoid subspace $\mathcal{D}_\nu$ over $\mathcal{Y}$. Such subspaces of the eigencurve give rise to families of overconvergent modular forms.

Definition 1.2.4 A "rank 1 family of slope $\nu$" over an affinoid subspace of weight space $\mathcal{Y} \subset \mathcal{W}_N$ is an irreducible affinoid subspace $\mathcal{U} \subset \mathcal{D}_\nu \subset C_{N,p}^{\text{red}}$ whose image under the weight projection map $\pi$ is $\mathcal{Y}$, and the restricted map $\pi|_U : U \to \mathcal{Y}$ is of rank 1 and all $K$-valued points correspond to forms of slope $\nu$.

The situation of this definition is a special situation that we look at to simplify the exposition. One can base extend from $\mathcal{Y}$ to an affinoid extension of $\mathcal{Y}$, to be in a situation analogous to the situation described in the definition but with higher rank.

Let $\mathcal{U} \subset C_{N,p}$ be a rank 1 family of slope $\nu$. Suppose $\mathcal{U}$ is defined over $K$, a finite extension of $\mathbb{Q}_p$. Let $A$ be the affinoid $K$-algebra associated to $\mathcal{U}$. Let $A^0 \subset A$ be the ring of elements of norm $\leq 1$. $\mathcal{U}$ determines a formal $q$-expansion

$$F_{\mathcal{U}}(s) = \sum_{n=1}^{\infty} a_n(s)q^n \in A^0[[q]] \quad (1.3)$$

The coefficients of the $q$-expansion here are viewed as functions of $s$ where $s$ varies in a compact subspace of $\mathbb{Z}_p$ determined by $\mathcal{Y}$. More specifically, given $s$, the elements of $A^0$ are evaluated on the point $\pi^{-1}(\psi(\cdot)^s)$ where $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{Q}_p^\times$ is a character determined by $\mathcal{Y}$ and $\cdot$ denotes the projection map of $\lim_{\leftarrow} (\mathbb{Z}/Np^r\mathbb{Z})^\times \cong (\mathbb{Z}/Np\mathbb{Z})^\times \times \cdots$.
(1 + p\mathbb{Z}_p) \text{ onto } 1 + p\mathbb{Z}_p. \text{ We note that we may have to restrict to a component of } \mathcal{Y} \text{ to determine } \psi.

1.3 Main Results

The main results of this thesis include a relation between ramification points of the map } \pi : C_{N,p} \rightarrow W_N \text{ and a one variable symmetric square } p\text{-adic } L\text{-function on the eigencurve, and when } 2|N, \text{ the construction of a two variable symmetric square } p\text{-adic } L\text{-function on the eigencurve whose restriction to "the diagonal" specializes to the one variable symmetric square } p\text{-adic } L\text{-function. Before stating these results, we first define some periods that will be needed when talking about the } p\text{-adic } L\text{-functions.}

Let } \mathcal{U} \subset C_{N,p} \text{ be a rank 1 family of slope } \nu. \text{ Fix a point } c \in \mathcal{U} \text{ corresponding to a classical normalized eigenform } f_k \text{ of weight } k \text{ and character } \psi : (\mathbb{Z}/Np\mathbb{Z})^\times \rightarrow K(f_k)^\times, \text{ where } K(f_k) \text{ is the field generated by the Fourier coefficients of } f_k. \text{ One can associate to this form a modular symbol } \Phi_{f_k} \in \text{Sym}_1^0(Np')(\text{Sym}^{k-2}(\mathbb{C})). \text{ One can cut the space of modular symbols into } + \text{ and } - \text{ eigenspaces of a certain involution, thus cutting up the modular symbol associated to } f_k \text{ also. So we have}

\begin{equation}
\Phi_{f_k}^\pm \in \text{Sym}_1^0(Np')(\text{Sym}^{k-2}(\mathbb{C}))^\pm.
\end{equation}

Recall that } K \text{ is the } p\text{-adic field that } \mathcal{U} \text{ is defined over. It is also the } p\text{-adic completion of } K(f_k) \text{ in this situation. Now we look at the Hecke eigenspace associated to } f_k \text{ in } \text{Sym}_1^0(Np')(\text{Sym}^{k-2}(K))^\pm; \text{ it is one-dimensional over } K. \text{ We choose a generator}

\begin{equation}
\Psi_{f_k}^\pm \in \text{Sym}_1^0(Np')(\text{Sym}^{k-2}(K))^\pm
\end{equation}

and choose periods } \Omega_{f_k}^\pm \in \mathbb{C}^\times \text{ defined by}

\begin{equation}
\Psi_{f_k}^\pm = \Phi_{f_k}^\pm / \Omega_{f_k}^\pm
\end{equation}

Let us suppose now that } f_k \text{ is a classical cuspidal normalized newform of weight } k \text{ and level } Np \text{ where } (N, p) = 1 \text{ and the character } \psi \text{ is of conductor } Np. \text{ We write the } q\text{-expansion of } f \text{ by}

\begin{equation}
f = \sum_{n=1}^{\infty} a_n q^n
\end{equation}
For each prime $l$, let $\alpha_l$ and $\beta_l$ denote the roots of $1 - \alpha_lX + \psi(l)l^{k-1}X^2$. The complex symmetric square $L$-function is defined by the following Euler product:

$$L(s, f_k) = \prod_l \left[ (1 - \overline{\psi}(l)\alpha_l^2 l^{-s})(1 - \overline{\psi}(l)\alpha_l\beta_l l^{-s})(1 - \overline{\psi}(l)\beta_l^2 l^{-s}) \right]^{-1}$$

(1.8)

**Theorem 1.3.1** Let $\mathcal{U}$ be a rank 1 family of slope $\nu$. There exists a rigid analytic function

$$L_\mathcal{U}(s) : \mathcal{U} \to \mathbb{C}_p$$

(1.9)

satisfying the following interpolation property: If $c' \in \mathcal{U}$ corresponds to a classical form $f_{k'}$ of weight $k'$ and character $\psi$,

$$L_\mathcal{U}(c') = \frac{L(k', f_{k'})}{\pi i (-2\pi i)^k(k' - 1)\Omega_{k'}^+ \Omega_{k'}^-}$$

(1.10)

where the periods $\Omega_{k'}^\pm$ are uniquely determined by the choice of the periods $\Omega_{k'}^\pm$.

The $p$-adic $L$-functions on the local affinoids $\mathcal{U}$ together give a locally analytic $p$-adic $L$-function on the eigencurve.

**Theorem 1.3.2** The locally defined $p$-adic $L$-functions $L_\mathcal{U}(s) : \mathcal{U} \to \mathbb{C}_p$ together to give a locally analytic one variable symmetric square $p$-adic $L$-function

$$L(s) : \mathcal{C}_{N,p} \to \mathbb{C}_p$$

(1.11)

It appears that there may be discrepancies in the choice of period at singular points of the reduced eigencurve, but the following theorem clears up these discrepancies and connects the $p$-adic $L$-function to the geometry of the eigencurve.

**Theorem 1.3.3** The ramification points in $\mathcal{C}_{N,p}$ of the map

$$\pi : \mathcal{C}_{N,p} \to \mathcal{W}_N$$

(1.12)

are precisely the zeros of the one variable symmetric square $p$-adic $L$-function on the eigencurve

$$L(s) : \mathcal{C}_{N,p} \to \mathbb{C}_p$$

(1.13)
In the case where $2|N$ and $p \geq 3$, one in fact has a two variable symmetric square $p$-adic $L$-function on the eigencurve that restricts to give the one variable $p$-adic $L$-function $L(s)$ described above. First we need to define the twisted complex symmetric square $L$-function. It is given by

$$L(s, f, \chi) = \prod_l [(1 - \bar{\psi}(l)\chi(l)\alpha_l^2 l^{-s})(1 - \psi(l)\chi(l)\alpha_l^2 l^{-s})(1 - \bar{\psi}(l)\chi(l)\beta_l^2 l^{-s})]^{-1}$$

(1.14)

where $\chi$ is an arbitrary Dirichlet character.

**Theorem 1.3.4** Suppose $2|N$ and $p \geq 3$. Let $L$ be a positive integer relatively prime to $p$. Let $U \subset C_{N,p}$ be a family of slope $\nu$. Let us denote $Z_L = \varprojlim \mathbb{Z}/Lp^r \mathbb{Z}$. There exists a unique rigid analytic function

$$L_U(w, z) : W_L \times U \to \mathbb{C}_p$$

(1.15)

satisfying the following interpolation property: If $w_{\chi, n} \in W_L$ is the character on $Z_L^\times$ given by $z \mapsto \chi(z)z^n$ for a finite order character $\chi : Z_L^\times \to \mathbb{Q}_p^\times$, $\epsilon \in U$ corresponds to a classical form $f_{k'}$ of weight $k'$ and character $\psi$ on $(\mathbb{Z}/Np\mathbb{Z})^\times$ (so we have $k' > \nu + 1$), and $1 \leq n \leq k' - 1$ then

$$L(w_{\chi, n}, \epsilon) = C(w_{\chi, n}, \epsilon) a_p(k')^{-2} (1 - \chi(p)a_p(k')^{-2}p^{n-1}) \frac{L(n, f_{k'}, \psi)}{(2\pi i)^{n-2} \Omega_{k'}^{\pm} \Omega_{k}^{\pm}}$$

(1.16)

where

$$C(w_{\chi, n}, \epsilon) = -(n-1)! (Lp)^{n-1} G(\chi) N^{-k'/2} W_N(f_{k'})^{-1} G(\psi_p)$$

(1.17)

and where $a_p(k')$ is the $U_p$-eigenvalue of $f_{k'}$, $G$ denotes taking the Gauss sum of a character, $W_N$ denotes taking the root number of a form, and $\psi_p$ denotes the restriction of $\psi$ to $(\mathbb{Z}/p\mathbb{Z})^\times$. Also, the periods $\Omega_{k'}^{\pm}$ are uniquely determined by the choice of the periods $\Omega_{f_k}^{\pm}$ of the particular form $f_k$ given by the fixed point $\epsilon \in U$.

**Theorem 1.3.5** The locally defined two variable $p$-adic $L$-functions $L_U(w, z) : W_L \times U \to \mathbb{C}_p$ together to give a locally analytic two variable symmetric square $p$-adic $L$-function

$$L(w, z) : W_L \times C_{N,p} \to \mathbb{C}_p$$

(1.18)
Now we take $L = N$. We let $\Delta$ denote the diagonal map

$$\Delta : C_{N,p} \to W_N \times C_{N,p} \quad (1.19)$$

where $C_{N,p}$ maps to $W_N$ via $\pi$, the weight projection map, and $C_{N,p}$ maps to $C_{N,p}$ via the identity map. The following theorem relates this two variable $p$-adic $L$-function to the one variable $p$-adic $L$-function.

**Theorem 1.3.6** $L(w, s)$ restricted to the diagonal gives $L(s)$ with the corresponding periods. In other words, $L(s)$ is the map

$$L(s) : C_{N,p} \xrightarrow{\Delta} W_N \times C_{N,p} \xrightarrow{L(s,w)} C_p \quad (1.20)$$
Chapter 2

Background Material

2.1 Modular Forms

Let $p$ be a prime and $N$ a positive integer relatively prime to $p$. In this section we briefly define all the spaces of modular forms, $p$-adic modular forms, and families of these forms that arise in the construction of the eigencurve $C_{p,N}$. We also state a theorem that relates these notions to Katz's notion of a $p$-adic modular form. We refer the reader to [4] for the details.

2.1.1 Overconvergent modular forms

In this subsection, we define the classical spaces of modular forms and overconvergent modular forms of integral weight using the moduli theoretic approach. Let $S$ be a scheme. Let $\mu_{Np^m}/S$ denote the finite flat group scheme over $S$ given by the kernel of multiplication by $Np^m$ in the multiplicative group over $S$. Let $Y_1(Np^m)$ denote the uncompactified modular curve classifying isomorphism classes of pairs $(E, \alpha)$ where $E$ is an elliptic curve over $S$ and $\alpha : \mu_{Np^m} \hookrightarrow E$ is an injection over $S$. We suppose throughout that $Np^m \geq 5$. In this case, the moduli problem just described is rigid and representable, and thus there is a universal family $(u : E_1(Np^m) \to Y_1(Np^m), \alpha)$. Let $\omega$ denote the invertible line bundle on $Y_1(Np^m)/Q_p$ given by

$$\omega := u_* \Omega^1_{E_1(Np^m)/Y_1(Np^m)}.$$ (2.1)
This line bundle $\omega$ extends to a line bundle on the compactification $X_1(Np^m)/\mathbb{Q}_p$ of $Y_1(Np^m)/\mathbb{Q}_p$ which we denote by the same letter $\omega$.

Let $Z_1(Np^m)$ denote the inverse image under reduction to $\text{Spec}(\mathbb{F}_p)$ of the complement of the supersingular points on the irreducible component containing the cusp $\infty$ in the model of $X_1(Np^m)$ over $\text{Spec}(\mathbb{Z}_p)$. $Z_1(Np^m)$ is an affinoid subdomain of the rigid analytic space $X_1(Np^m)$ over $\mathbb{Q}_p$. When defining spaces of $p$-adic modular forms, one typically looks at the moduli space of elliptic curves with a level $p^\infty$-structure; such a structure is commonly known as a trivialization. To make a precise definition, a trivialization is an isomorphism $\hat{\mathcal{G}}_m \cong \hat{\mathcal{E}}$ between formal completions at the origin. We look at $Z_1(Np^m)$ because the elliptic curves that it parameterizes have a canonical level $p^\infty$-structure. In fact, $Z_1(Np^m)$ is a realization of the moduli space of trivialized elliptic curves. The canonical trivializations of the elliptic curves allow us to view points of $Z_1(Np^m)$ as systems of compatible points in the tower of modular curves of level $Np^m$ as $m$ increases.

We now define a system of neighborhoods $Z_1(Np^m)(v)$ of $Z_1(Np^m)$ for $v \in \mathcal{I}_m := \{v \in \mathbb{Q} | 0 \leq v < p^{2-m}/(p+1)\}$. Roughly speaking, points in $Z_1(Np^m)(v)$ are elliptic curves whose level $p^m$-structure can be extended canonically but not as far as to a trivialization unless $v = 0$. To make the definition, let $A$ denote the level $1$ characteristic $p$ Hasse invariant. $Z_1(Np^m)(v)$ is defined to be the affinoid subdomain of $X_1(Np^m)$ consisting of points over $\mathbb{C}_p$ corresponding to pairs $(\mathcal{E}, \alpha)$ over $S = \text{Spec}(\mathcal{O}_{\mathbb{C}_p})$ such that $v(A(\overline{\mathcal{E}}, \overline{\eta})) \leq v$ where $\eta$ is a generator of $\Omega_{\mathcal{E}/S}$, $(\overline{\mathcal{E}}, \overline{\eta})$ is the reduction of $(\mathcal{E}, \eta)$ modulo $p$, and $\alpha$ is the level $Np^m$-structure.

We now define the space of classical modular forms. Let $\omega^k$ denote the $k$-th tensor power of $\omega$. Let $K$ be a complete subfield of $\mathbb{C}_p$. The classical spaces of modular forms can be defined by

$$M_k(\Gamma_1(Np^m), K) := \omega^k(X_1(Np^m)/K)$$

for $k \geq 0$. This space coincides with the more traditional definition of taking complex functions on the upper half plane that satisfy the usual transformation properties, looking at the subspace of such functions defined over $\mathcal{I}$, and tensoring that space with $K$.

We also let $S_k(\Gamma_1(Np^m), K)$ denote the space of cusp forms in $M_k(\Gamma_1(Np^m), K)$. 

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Also, given a character $\psi : (\mathbb{Z}/Np^m)^{\times} \to O_k^{\times}$, we let $M_k(\Gamma_0(Np^m), \psi, K)$ and $S_k(\Gamma_0(Np^m), \psi, K)$ respectively denote the subspaces of $M_k(\Gamma_1(Np^m), K)$ and $S_k(\Gamma_1(Np^m), K)$ where the diamond operators, which will be defined in a later section, act by the character $\psi$.

We define the space of $v$-overconvergent modular forms of weight $k$ for any integer $k$ by

$$M_k(\Gamma_1(Np^m), v, K) := \omega(k) (Z_1(Np^m)(v)/K)$$

(2.3)

By overconvergent modular form we mean an element of one of these spaces for $v > 0$. We refer to an element of $M_k(\Gamma_1(Np^m), v, K)$ for $v = 0$ as a convergent modular form. We remark that there is a notion of “cuspidal-overconvergent,” but this notion is not exactly analogous to the notion of a classical cusp form because “cuspidal-overconvergent,” as defined, implies that the form is 0 only at the infinite cusp. Thus, one may have and indeed there are cuspidal overconvergent forms that are classical but not cuspidal in the classical sense.

2.1.2 Families of modular forms

We now prepare to define families of modular forms parameterized by a rigid analytic space. Let $Z_{p,N}^{\times} = \lim_m (\mathbb{Z}/Np^m \mathbb{Z})^{\times}$ and $\Lambda_N = \lim_m Z_p[(\mathbb{Z}/Np^m \mathbb{Z})^{\times}]$. In the literature, $\Lambda_N$ is commonly referred to as the Iwasawa algebra. There is a canonical factorization $Z_{p,N}^{\times} = (\mathbb{Z}/Np\mathbb{Z})^{\times} \times (1 + p\mathbb{Z}_p)$. This gives a corresponding factorization of the Iwasawa algebra

$$\Lambda_N \cong Z_p[(\mathbb{Z}/Np\mathbb{Z})^{\times}] \otimes Z_p[(1 + p\mathbb{Z}_p)].$$

(2.4)

There is an isomorphism between $Z_p[(1+p\mathbb{Z}_p)]$ and $Z_p[[t]]$ given by mapping $1+p \mapsto 1+t$, and we identify the two spaces via this isomorphism.

Let $\mathcal{W}_N$ denote the rigid analytic space over $\mathbb{Q}_p$ associated to the formal $Spf(\mathbb{Z}_p)$-scheme, $Spf(\Lambda_N)$. We refer to $\mathcal{W}_N$ as weight space of tame level $N$. Let $\mathcal{B} \subset \mathcal{W}_N$ denote the connected component of the identity, and let $\mathcal{D}_N = \mathcal{W}_N/\mathcal{B}$ denote the quotient group. We have the decomposition $\mathcal{W}_N \cong \mathcal{D}_N \times \mathcal{B}$.

For $a \in Z_{p,N}^{\times}$, let $\langle\langle a \rangle\rangle$ denote the projection of $a$ to the $1 + p\mathbb{Z}_p$ component. For $\kappa \in \mathcal{W}_N$, let $\langle \kappa \rangle \in \mathcal{B}$ denote the character $a \mapsto \kappa(\langle\langle a \rangle\rangle)$. For $s \in \mathbb{C}_p$ with $v(s) >$
\(-1 + 1/(p - 1)\), let \(\eta_s: \Lambda_N \to \mathbb{C}_p\) denote the continuous character \(a \mapsto \langle a \rangle^s\). If \(\kappa\) is of the form \(\kappa = \chi \eta_s\) for a finite order character \(\chi\), we also denote \(\kappa\) by \(\kappa = (\chi, s)\).

Let \(\mathcal{V}\) be a rigid analytic space. In practice, \(\mathcal{V}\) will be a subspace of \(\mathcal{W}_N\). An overconvergent family of rigid analytic functions on \(Z_1(Np^m)\) parameterized by \(\mathcal{V}\) is defined to be a function \(F\) on \(Z_1(Np^m) \times \mathcal{V}\) with the property that there is an admissible open cover of \(\mathcal{V}\) by affinoids \(V_j\) such that there are positive numbers \(v_j \in I_m\) where the restriction of \(F\) to \(V_j\) extends to a rigid analytic function on \(Z_1(Np^m)(v_j) \times V_j\). Following the notation of [4], we denote the ring of such functions as \(A^1(Z_1(Np^m)_{\mathcal{V}}/\mathcal{V})\).

We now recall the definition of the Eisenstein family of modular forms. Let \(\zeta^*(\kappa)\) denote the \(p\)-adic \(\zeta\)-function defined by the formula \(\zeta^*(\kappa) = L_p(\chi, 1 - s)\) for \(\kappa = (\chi, s)\) where \(L_p(\chi, s)\) denotes the Kubota-Leopoldt \(p\)-adic \(L\)-function. For \(n \geq 1\), let \(\sigma_\kappa(n) = \sum_{d|n, (d, p) = 1} \kappa(d)d^{-1}\). For \(\kappa \neq 1\) and \(\zeta^*(\kappa) \neq 0\), let

\[
E_\kappa(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \sigma_\kappa(n)q^n.
\]

This defines a formal \(q\)-expansion whose coefficients are rigid analytic functions on the subspace of \(\mathcal{W}_N\) where \(\zeta^*(\kappa) \neq 0\). We note that this subspace contains \(\mathcal{B}\). We let \(\mathcal{E}(q)\) denote the restriction of this Eisenstein family to \(\mathcal{B} \subset \mathcal{W}_N\).

We say that \(F(q) = \sum_{n=0}^{\infty} a_nq^n\) with \(a_n \in K\) is the \(q\)-expansion of a convergent (resp. overconvergent) modular form of tame level \(N\) and weight character \(\kappa \in \mathcal{W}\) over \(K\) if \(F(q)/E_\kappa(q)\) is the \(q\)-expansion of a rigid (resp. overconvergent) function on \(Z_1(Np)\) in \(X_1(Np)\) of character \(\kappa/(\kappa)\) under the action of \((\mathbb{Z}/Np\mathbb{Z})^\times\). We denote the space of such \(q\)-expansions that converge on \(Z_1(Np)(v)\) as \(M_1^1(N, v)\).

Let \(\mathcal{U}\) be an admissible open subspace of \(\mathcal{B}\). Let \(A(\mathcal{U})\) denote the affinoid algebra over \(K\) attached to \(\mathcal{U}\), and let \(A^0(\mathcal{U})\) denote the subring of \(A(\mathcal{U})\) of elements with norm bounded by \(1\). We say that \(F(q) = \sum_{n=0}^{\infty} a_nq^n\) with \(a_n \in A(\mathcal{U})\) is the \(q\)-expansion of a family of convergent (resp. overconvergent) modular forms over \(\mathcal{U}\) of tame level \(N\) if \(F(q)/E(q)\) is the \(q\)-expansion of a rigid (resp. overconvergent) function on \(Z_1(Np)(v) \times \mathcal{U}\). This family has type \(\delta \in \mathcal{D}_N\) if this function has character \(\delta\) under the action of \((\mathbb{Z}/Np\mathbb{Z})^\times\). We denote the \(A(\mathcal{U})\)-module of such forms converging on \(Z_1(Np)(v) \times \mathcal{U}\) as \(M_1^1(N, v)\).
We also define
\[ M_κ^I(N) = \bigcup_{v \in H_1, v > 0} M_κ^I(N, v) \] (2.6)
\[ M_κ^U(N) = \bigcup_{v \in H_1, v > 0} M_κ^U(N, v) \] (2.7)

These are the spaces of overconvergent modular forms of weight character κ and tame level N and families of overconvergent modular forms over U and tame level N respectively.

These definitions of families of modular forms using the Eisenstein family to traverse between weights is consistent with and thus extend earlier definitions of families of modular forms formulated by Katz. We now formulate Katz's definition. Let B be a \( \Lambda_N \)-adic ring. By this we mean that B is a \( \Lambda_n \)-algebra that is complete with respect to the \( \Lambda_N \)-adic topology. By a Katz p-adic modular function over B of level N we mean a function F which assigns to any isomorphism class \((E, \alpha_\infty, \alpha_N)\) of trivialized elliptic curve with level N-structure \(\alpha_N\) over a \(\Lambda_N\)-adic B-algebra D an element \(F(E, \alpha_\infty, \alpha_N) \in D\) and whose formation commutes with base change. The following theorem asserts the consistency of the definitions.

**Theorem 2.1.1** Let F be a convergent family of modular forms over an affinoid space \(U \subset W_N\) with q-expansion coefficients in \(A^0(U)\). Then the q-expansion of F is the q-expansion of a Katz modular function \(\tilde{F}\) over \(A^0(U)\).

This theorem shows that families of modular forms are "good" generalizations of modular forms in the sense that they are modular forms over a \(\Lambda\)-adic base in the sense of Katz.

### 2.2 Abstract Hecke Operators

We define \(\mathcal{H}\) to be the commutative polynomial ring over the topological ring \(\Lambda_N\) in infinitely many variables labeled \(T_l\) for prime numbers \(l\) not dividing \(NP\) and \(U_l\) for prime numbers \(l\) dividing \(NP\). \(\mathcal{H}\) can be viewed as a topological \(\Lambda_N\)-algebra given by its weak topology. Letting \(T_l = T_l\) for prime numbers \(l\) not dividing \(NP\) and \(T_l = U_l\) for prime numbers \(l\) dividing \(NP\).
for prime numbers $l$ dividing $Np$, we can define elements $\mathcal{T}_n \in \mathcal{H}$ for positive integers $n$ recursively by the equality of formal series

$$\sum_{n=1}^{\infty} \mathcal{T}_n \cdot n^s = \prod_l (1 - \mathcal{T}_l \cdot l^{-s} + [l] \cdot l^{-2s})^{-1}$$

(2.8)

where the product is taken over all primes $l$, and where $[l]$ is defined to be 0 if $l$ divides $Np$ and $[l]$ is defined to be the image of $l \in \mathbb{Z}_{p,N}^\times \subset \Lambda_N$ for $l$ not dividing $Np$.

Let $\Psi : \mathcal{H} \to \mathbb{C}_p$ be a continuous ring homomorphism. By the weight character of $\Psi$, we mean the homomorphism $w_\Psi : \Lambda_N \to \mathbb{C}_p$ induced by restricting $\Psi$ to $\Lambda_N$. We refer to

$$\sum_{n=1}^{\infty} \Psi(\mathcal{T}_n) \cdot q^n \in \mathbb{C}_p[[q]]$$

(2.9)

as the Fourier expansion of the character $\Psi$.

**Definition 2.2.1** Given a continuous homomorphism $w : \Lambda_N \to \mathbb{C}_p$ and a power series

$$f(q) = \sum_{n=1}^{\infty} a_n \cdot q^n \in \mathbb{C}_p[[q]],$$

(2.10)

we say that $f(q)$ is a normalized Hecke eigenvector of weight $w$ if there is a character $\Psi : \mathcal{H} \to \mathbb{C}_p$ of weight character $w$ whose Fourier expansion if $f(q)$.

More explicitly, we can define the action of $\mathcal{T}_l$ on $f(q)$ by the following rule: For primes $l$,

$$f|_w \mathcal{T}_l = \sum_{n=1}^{\infty} b_n \cdot q^n \in \mathbb{C}_p[[q]]$$

(2.11)

where $b_n = a_{nl} + l^{-1}w([l]) \cdot a_{n/l}$ where $a_{n/l} = 0$ if $l$ does not divide $n$ and $w([l]) = 0$ if $l|Np$. With this, one can equivalently define $f(q)$ to be a normalized Hecke eigenvector of weight $w$ if $a_1 = 1$ and $f(q)$ is an eigenvector for the weight $w$ action of $\mathcal{T}_l$ with eigenvalue $a_l$ for each prime $l$.

We note that all the spaces of modular forms that were defined in the previous section have compatible actions of Hecke operators $\mathcal{T}_n$ via their $q$-expansions that respect the rigid analytic structure of the spaces involved in the $p$-adic setting. We also note that in the $p$-adic setting the action of $[a]$ for $a \in \mathbb{Z}_{p,N}$ incorporates both the weight and nebentypus character. If we let $(b)$ for $b \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ denote the classical diamond
operator on a space of modular forms with integral weight \( k \) and level \( Np^m \), we have that \( [a] = a^\ell \cdot (\bar{a}) \) where \( \bar{a} \) denotes the image of \( a \) in \( (\mathbb{Z}/Np^m\mathbb{Z})^\times \).

2.3 The Eigencurve

In this section we give a brief summary of the construction of \( C_{p,N} \), the \( p \)-adic eigencurve of tame level \( N \) as constructed by Buzzard in [1]. \( C_{p,N} \) is a rigid analytic curve that was constructed for \( p \geq 3 \) and \( N = 1 \) by Coleman and Mazur and extended to all \( p \) and \( N \) by Buzzard. One of the main theorems about \( C_{p,N} \) is the following:

**Theorem 2.3.1** (Coleman, Mazur [4], Buzzard [1]) There is a natural bijection between the \( \mathbb{C}_p \)-valued points of the eigencurve \( C_{p,N} \) and finite slope overconvergent \( p \)-adic modular eigenforms of tame level \( N \).

We remark that this theorem as stated is slightly imprecise. This will be clarified in Proposition 2.3.4. We refer the reader to [4] and [1] for the complete details about the properties and construction of the eigencurve. We also remark that Buzzard’s construction is a generalization of the construction of the curve denoted \( D \) in [4] which was shown to be isomorphic to what Coleman and Mazur called the reduced eigencurve in the tame level \( N = 1 \) and \( p \geq 3 \) case. To clarify confusion that might arise from this remark, we also remark that the components of Buzzard’s eigencurves arising from eigencurves of strictly lower tame level are not reduced.

An important object attached to \( C_{p,N} \) is its associated weight space of tame level \( N \), \( \mathcal{W}_N \), which is defined in the first section. There is a natural map \( \pi : C_{p,N} \to \mathcal{W}_N \) called the weight projection map that associates to a point on \( C_{p,N} \) the weight of the overconvergent modular form associated to that point.

2.3.1 The spectral curve

Let \( \mathcal{M}_k^\Lambda(N,\nu) \) denote the space of \( \nu \)-overconvergent modular forms of weight character \( \kappa \in \mathcal{W}_N \) and tame level \( N \) defined in the first section. This space is a Banach module on which the \( U_p \) Hecke operator acts completely continuously. Let \( P(\kappa, T) \) denote the Fredholm determinant of the completely continuous system of operators \( U_p \) acting
on the system of Banach modules $M_i^k(N, 1/i)$ for $i$ large enough. The following theorem asserts that these Fredholm determinants vary well as the weight $\kappa$ varies:

**Theorem 2.3.2** (Coleman, Mazur [4], Buzzard [1]) There exists an entire power series $P(T) \in \Lambda_N\{\{T\}\}$ uniquely determined by the property that for each weight $\kappa \in \mathcal{W}_N$, the image of $P(T) \in \Lambda_N\{\{T\}\} \subset \Lambda_N[[T]]$ under the map $\Lambda_N[[T]] \rightarrow \mathbb{C}_p[[T]]$ induced by $\kappa : \Lambda_N \rightarrow \mathbb{C}_p$ is equal to the Fredholm determinant $P(\kappa, T)$.

This Fredholm series $P(T)$ can be viewed as a rigid analytic function on $\mathcal{W}_N \times \mathbb{A}^1$. Its zero locus $Z \subset \mathcal{W}_N \times \mathbb{A}^1$ is a rigid analytic curve which is called the spectral curve for the $U_p$ operator. This space also has a natural projection to weight space which we denote by $\hat{\pi} : Z \rightarrow \mathcal{W}_N$.

**Definition 2.3.3** Let $C$ denote the set of affinoid subspaces of $V \subset Z$ with the following property: there is an affinoid subspace $Y \subset \mathcal{W}_N$ with the property that $V \subset Z_Y := \hat{\pi}^{-1}(Y)$, the induced map $\hat{\pi} : V \rightarrow Y$ is finite flat and surjective, and $V$ is disconnected from its complement in $Z_Y$.

The significance of $C$ is that it is an admissible cover of the spectral curve $Z$. This follows from the theorem of Buzzard which generalizes a theorem of Coleman, Theorem 3.2.4, which we state at the end of this section. The affinoid subsets in the collection $C$ give rise to the affinoid spaces that are glued together to form the eigencurve $C_{p, N}$. We explain roughly how this is done in the paragraphs that follow.

**2.3.2 Construction of the eigencurve**

Let $V \in C$. Let $Y = \hat{\pi}(V)$. Let $A(Y)$ be the affinoid ring attached to $Y$. $V$ corresponds to a factorization of $P(T)$ over $A(Y)$. Let $P(T) = Q_Y(T)H(T)$ be the corresponding factorization. Here $Q_Y(T)$ is a polynomial with constant term 1 and whose leading coefficient is a unit, and $H(T)$ is entire. Let $d$ denote the degree of $Q_Y(T)$, and let $Q_Y^*(T) = T^dQ_Y(T^{-1})$. Then we have $V = \text{Max}(A(Y)(T)/Q_Y^*(T))$. The factorization also corresponds to a direct sum decomposition $M_N^k(\nu) = N(V, \nu) \oplus F(V, \nu)$ for each $\nu$. $N(V, \nu)$ is in fact independent of $\nu$ for small enough $\nu > 0$, and thus we refer to the coinciding spaces as $N(V)$. Let $\mathcal{I}(V)$ denote the image of the Hecke algebra in the
endomorphism ring of $N(V)$ over $A(Y)$. $\mathcal{I}(V)$ is in fact an affinoid ring over $A(Y)$. Let $A^0(Y)$ and $\mathcal{I}^0(V)$ denote the subrings of the corresponding rings of elements with norm $\leq 1$. We have the natural map $A^0(Y) \to \mathcal{I}^0(V)$. Let $D(V)$ denote that affinoid variety attached to $\mathcal{I}(V)$. The eigencurve $C_{p,N}$ is constructed by gluing together the affinoid spaces $D(V)$ for $V \in \mathcal{C}$. The main ingredient for doing this is the fact that $\mathcal{C}$ is an admissible cover of $Z$.

One important fact that we mention here is the following:

**Proposition 2.3.4** (Coleman, Mazur [4]) There is a pairing between $\mathcal{I}(V)$ and $N(V)$ given by $\langle \tau, f \rangle = a_1(\tau \cdot f)$ for $\tau \in \mathcal{I}(V)$ and $f \in N(V)$ which is perfect as long as constants do not arise in $N(V)$.

This proposition is the main fact that leads to the correspondence stated in Theorem 2.3.1 stated earlier.

Now we will give a more detailed description of the local components to have more control when working with slopes. Let $W \subset W_N$ be an affinoid subspace of weight space. Let $D(W,\alpha) \subset C_{p,N}$ be the affinoid subspace consisting of points $x \in C_{p,N}$ whose weight $\pi(x) \in W$ and the slope of the form corresponding to $x$ is $\leq \alpha$. The following is a theorem of Buzzard which generalizes a theorem of Coleman.

**Theorem 2.3.5** (Buzzard [1], Coleman [3]) There exists an admissible covering of $W$ by affinoids $\{U_i\}$ and affinoids $\{D_i\}$ that cover $D(W,\alpha)$ satisfying the following properties:
(a) $\pi(D_i) = U_i$
(b) $D(W,\alpha)_{U_i} \subset D_i$
(c) $D_i = D(U_i, t)$ for some $\alpha^* \in \mathbb{Q}$ independent of $i$ and such that no $s$ satisfying $\alpha < s \leq \alpha^*$ is the valuation of an element of $K$
(d) each $D_i$ is of the form $D(V_i)$ for some $V_i \in \mathcal{C}$

This is the theorem that verifies that $C$ is an admissible cover of the spectral curve $Z$ and thus implies that the local pieces of the eigencurve really do glue together to give a rigid analytic curve. For the constructions in this paper, we work with local pieces of the $D_i = D(V_i)$ of the theorem. This is what causes the constructions in this
paper to be locally analytic as opposed to rigid analytic. We define the local pieces of the eigencurve that we work with in this paper as follows:

**Definition 2.3.6** Let $V$ be an affinoid subspace of one of the $V_i$ described in Theorem 3.2.4 such that $Y := \pi(V) \subset U_i$ is a closed disk. So then we have that $D(V) = D(Y, \alpha^*)$. We call $D(V)$ a local flat piece of the eigencurve of slope $\leq \alpha$.

### 2.4 Cohomology

Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$. In practice, $\Gamma$ will be either $\Gamma_0(Np^r)$ or $\Gamma_1(Np^r)$. Let $\mathbb{H}$ denote the complex upper half plane. Let $\mathbb{H}^*$ denote $\mathbb{H}$ union the rational cusps. Let $R$ be a commutative ring with identity. Let $A$ be a right $R[\Sigma]$-module.

In this thesis, we will make use of the group cohomology groups $H^i(\Gamma, A)$ and the parabolic group cohomology groups denoted $H^i_p(\Gamma, A)$. Let $\tilde{A}$ denote the locally constant sheaf on $\Gamma \backslash \mathbb{H}$ associated to $A$. We will also make use of the sheaf cohomology groups $H^i(\Gamma \backslash \mathbb{H}, \tilde{A})$ and the compactly supported sheaf cohomology groups $H^i_c(\Gamma \backslash \mathbb{H}, \tilde{A})$. We will define these spaces in a later section and state precisely how these different cohomology groups are related.

All of these cohomology groups are right $R[\Sigma]$-modules. The action of $\Sigma$ gives rise to the action of the usual Hecke operators, diamond operators, and involutions via the double coset formulation of the Hecke operators given in the next section. We remark that the significance of these cohomology groups lies in that they are intimately related to spaces of modular forms and have nice presentations as modular symbols which will be defined in a later section.

### 2.5 Classical Hecke Operators

On classical spaces of modular forms and cohomology groups related to these spaces, the action of the Hecke operators can be given explicitly via double coset algebras. We explain how this works in this section.

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup. Let $\Sigma = GL_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$. Let $D(\Gamma, \Sigma)$ denote the double coset algebra associated to the pair $(\Gamma, \Sigma)$. This is the space of

18
$\mathbb{Z}$-valued functions on $\Sigma$ that are bi-invariant with respect to $\Gamma$. If $\Sigma_1$ is a subsemigroup of $\Sigma$ containing $\Gamma$, then $D(\Gamma, \Sigma_1)$ is naturally a subalgebra of $D(\Gamma, \Sigma)$. We let $\Sigma_1^+$ denote the elements of $\Sigma_1$ with positive determinant.

We let $T(g)$ denote the characteristic function of the double coset $\Gamma g \Gamma$. These elements generate the double coset algebra. There is an anti-involution on $\Sigma$ given by $g \mapsto g^* = det(g)g^{-1}$. When $\ast$ preserves $\Gamma$, $\ast$ induces an anti-involution on $D(\Gamma, \Sigma)$ given by $T(g)^* = T(g^*)$.

Fix a positive integer $N$ and a prime $p$ that does not divide $N$. Let

$$\Sigma_1(p^r) = \left\{ g \in \Sigma \middle| g = \begin{bmatrix} 1 & \ast \\ 0 & \ast \end{bmatrix} \mod p^r \right\}.$$  \hspace{1cm} (2.12)

We look at the double coset algebra $D(\Gamma_1(Np^r), \Sigma_1(p^r))$. Let $\mathbb{Z}'$ denote the multiplicative set of integers which are prime to $p$. For each $a \in \mathbb{Z}'$ chose $\gamma_a \in \Gamma_1(N) \cap \Gamma_0(p^r)$ whose lower right hand entry is congruent to $a$ modulo $p^r$. Let $\langle a \rangle_p = T\left( \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \cdot \gamma_a \right)$ in $D(\Gamma_1(Np^r), \Sigma_1(p^r))$. The map $a \mapsto \langle a \rangle_p$ is multiplicative and thus extends to a map defined on $\mathbb{Z}'^*$. For each integer a prime to $N$ chose and element $\beta_a \in \Gamma_0(N)$ whose lower right entry is congruent to a modulo $N$. Define $\langle a \rangle_N = T(\beta_a)$. Let $T = T\left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$. We then define $\langle a \rangle = \langle a \rangle_N \cdot \langle a \rangle_p$ for integers $a$ prime to $Np$, the map $a \mapsto \langle a \rangle$ extends uniquely to a map from $\mathbb{Z}'^*_{p,N}$ to $D(\Gamma_1(Np^r), \Sigma_1(p^r))$ for each $r$. The map $\mathbb{Z}' \to D(\Gamma_1(Np^r), \Sigma_1(p^r))$ given by $a \mapsto \langle a \rangle_p$ extends to a map $\mathbb{Z}$-algebra map $\mathbb{Z}[\mathbb{Z}'] \to D(\Gamma_1(Np^r), \Sigma_1(p^r))$. We define

$$D_p(Np^r) = D(\Gamma_1(Np^r), \Sigma_1(p^r)) \otimes_{\mathbb{Z}[\mathbb{Z}']} \mathbb{Z}_p[[\mathbb{Z}'^*]].$$  \hspace{1cm} (2.13)

Via the $a \mapsto \langle a \rangle$ map, $D_p(Np^r)$ is a $\Lambda_N$-algebra. For $s \geq r$, there are natural $\Lambda_N$-algebra maps $D_p(Np^s) \to D_p(Np^r)$. Because the pair $(\Gamma_1(Np^r), \Sigma_1(p^r))$ is weakly compatible with $(\Gamma_1(N), \Sigma)$, there is a natural $\Lambda_N$-algebra map $D_p(N) \to D_p(Np^r)$ induced by restriction of functions on $\Sigma$ to $\Sigma_1(p^r)$. We also let $D_p^+(Np^r)$ denote the elements of $D_p(Np^r)$ supported on $\Sigma^+(p^r)$.

We now define some standard elements of $D_p(N)$ which determine elements of $D_p(Np^r)$ for each $r$ which we in turn refer to by the same name. For each positive
integer $n$, let $T_n = T\left(\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}\right)$. We also define $W_N = T\left(\begin{bmatrix} 0 & 1 \\ -N & 0 \end{bmatrix}\right)$ and we can extend the $[\cdot]_p$ map to $\mathbb{Z}_p$ by defining $[p]_p = T\left(\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}\right)$.

Given an element $g \in \Sigma$, we can define a right action of $T(g)$ on a right $\Sigma$-module as follows. We can write the following decomposition of a double coset into a union of cosets:

$$\Gamma g \Gamma = \cup_i \Gamma g_i$$

(2.14)

For $m$ an element of a $\Gamma$ invariant $\Sigma$-module, we define $m[T(g)] = \sum_i m[g_i]$.

For a classical modular form of weight $k$, if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Sigma^+$, we define

$$(f[g])(z) = \text{det}(g)^{k-1}(cz + d)^{-k}f(gz)$$

(2.15)

The following theorem states that these Hecke operators defined via double coset are the same as the abstract Hecke operators defined on formal $q$-expansions.

**Theorem 2.5.1** For $r \geq 1$, the action of the subalgebra of $D_p(Np^r)$ generated by the $T_n$'s for positive integers $n$ and the $(\alpha)$'s for $a \in \mathbb{Z}_{p,N}^\times$ on $M_k(\Gamma_1(Np^r), K)$ is the same as the action induced by $\mathcal{H}$.

### 2.6 Modular Symbols

Modular symbols are a presentation of the cohomology groups defined above. They prove to be an incredibly useful tool in studying modular forms. This is given by the Eichler-Shimura theorem which asserts that the space of classical cusp forms is isomorphic to a certain eigenspace of the cohomology group of the modular curve. Modular symbols are also constructed as certain integrals of modular forms. This makes it easy to relate properties of modular forms to the $L$-functions attached to them. Further, Mazur and others have shown how to construct $p$-adic $L$-functions from modular symbols. We give an exposition of the construction below.
2.6.1 Definitions

Let $R$ be a commutative ring, and let $A$ be a right $R[\Sigma]$-module where $\Sigma = \Sigma_1(p^r)$. Let $\Gamma = \Gamma_1(Np^r)$ be the usual congruence subgroup. Let $D = \text{Div}(\mathbb{P}^1(\mathbb{Q}))$ denote the group of divisors supported on the rational cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. Let $D_0 \subset D$ be the subgroup of divisors of degree zero. We let $\Sigma$ act on $D$ by fractional linear transformation.

**Definition 2.6.1** An additive homomorphism $\Phi : D_0 \to A$ is called a modular symbol if $\Phi(\gamma D)|_{\gamma} = \Phi(D)$ for all $D \in D_0$ and $\gamma \in \Gamma$. Let $\text{Sym}_\Gamma(A)$ denote the space of all modular symbols. An additive homomorphism $\Phi : D \to A$ is called a boundary symbol if $\Phi(\gamma D)|_{\gamma} = \Phi(D)$ for all $D \in D$ and $\gamma \in \Gamma$. Let $\text{Bound}_\Gamma(A)$ denote the space of all boundary symbols. We let $\Sigma$ act on these symbols by $\Phi|_g : D \to \Phi(gD)|_g$.

The following theorem shows how modular symbols are related to cohomology groups.

**Theorem 2.6.2** Let $t(\Gamma)$ denote the least common multiple of the orders of the torsion elements of $\Gamma$. Suppose $t(\Gamma)$ is invertible in $R$. Then there is a canonical isomorphism $\text{Sym}_\Gamma(A) \cong H^1_c(\Gamma \backslash \mathbb{H}, A)$. Moreover, there is a canonical exact sequence:

$$0 \to H^0(\Gamma, A) \to \text{Bound}_\Gamma(A) \to \text{Sym}_\Gamma(A) \to H^1_{\text{par}}(\Gamma, A) \to 0 \quad (2.16)$$

**Proof.** See [7].

In particular we care about when $A = \text{Sym}^r(R^2)$, the $R$-module of homogeneous polynomials of degree $r$ in variables $X$ and $Y$ with coefficients in $R$. $\Sigma$ acts on $\text{Sym}^r(R^2)$ by

$$(F|g)(X,Y) = F((X,Y)g^*)$$

where $*$ is the map $g \mapsto g^* = \text{det}(g)g^{-1}$. This module is relevant because

**Theorem 2.6.3** (Eichler-Shimura)

$$S_k(\Gamma, \mathbb{C}) \cong H^1_{\text{par}}(\Gamma, \text{Sym}^{k-2}((\mathbb{C}^2))[\pm] \quad (2.18)$$

where $[\pm]$ denotes the $+1$ eigenspace under the action of $i = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. 

21
Proof. See [7]

Definition 2.6.4 The standard weight $k$ modular symbol associated to a cusp form $f$ is the modular symbol $\Phi_f \in \text{Symb}(\text{Sym}^{k-2}(\mathbb{C}^2))$ given by

$$\Phi_f((c_2) - (c_1)) = 2\pi i \int_{c_1}^{c_2} f(z)(zX + Y)^{k-2} dz$$  \hspace{1cm} (2.19)

We remark that the map $f \mapsto \Phi_f$ is compatible with the maps given in the last two theorems. This is because this map commutes with the action of $\Sigma^+$. In other words, we have $\Phi_{fg} = \Phi_f \mid g$. We verify this fact in the next section.

2.6.2 Actions

Let $L_n(R)$ denote the $R$-module of homogeneous polynomials of degree $n$ in variables $X$ and $Y$ with coefficients in $R$. This space is spanned as an $R$-module by elements of the form $(zX + Y)^n$ where $z \in R$. Let $\hat{L}_n(R)$ denote the $R$-module of polynomials of degree $\leq n$ in the variable $z$. This space is spanned as an $R$-module by elements of the form $(zX + Y)^n$ where $(X, Y) \in R^2$. Let $P(X, Y) = (z_0X + Y)^n \in L_n(R)$ and $Q(z) = (zX_0 + Y_0)^n \in \hat{L}_n(R)$. A pairing on $L_n(R) \otimes_R \hat{L}_n(R)$ is given by linearly extending the following formula

$$\langle P(X, Y), Q(z) \rangle = (z_0X_0 + Y_0)^n = \sum_{j=0}^{n} \binom{n}{j} z_0^{n-j} X_0^{n-j} Y_0^j$$  \hspace{1cm} (2.20)

We define the action of $\Sigma$ on $L_n(R)$ with

$$(P|g)(X, Y) = P((X, Y)g^*)$$  \hspace{1cm} (2.21)

with $*$ being the map $g \mapsto g^* = \text{det}(g)g^{-1}$, and the action of $\Sigma$ on $\hat{L}_n(R)$ with

$$(Q|g)(z) = Q(g(z)(cz + d)^n \text{det}(g)^{-n}$$  \hspace{1cm} (2.22)

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. One verifies that

$$\langle (P|g)(X, Y), (Q|g)(z) \rangle = \langle P(X, Y), Q(z) \rangle$$  \hspace{1cm} (2.23)
Definition 2.6.5 For \( Q(z) \) a polynomial in \( z \) over \( R \) of degree \( \leq k - 2 \), we define
\[
\Phi_f(Q(z))(\{c_2\} - \{c_1\}) = 2\pi i \int_{c_1}^{c_2} f(z)Q(z) \, dz
\]  
(2.24)

Let \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Sigma \). One verifies that \( d(gz) = \frac{\det(g)}{(cz+d)} \cdot dz \). By applying a change of variables \( z \mapsto g^*(z) \), we have
\[
\int_{c_1}^{c_2} (f|g)(z)Q(z) \, dz = \int_{g(c_1)}^{g(c_2)} (f|g)(g^*z)Q(g^*z) \, d(g^*z)
\]
(2.25)
\[
= \int_{g(c_1)}^{g(c_2)} f(z)(Q|g^*)(z) \, dz
\]
(2.26)

This combined with
\[
\langle P, Q|g^* \rangle = \langle P|g, Q|g^* \cdot g \rangle \Rightarrow \langle P|g, Q \rangle
\]
(2.27)
gives the formula \( \Phi_f|g = \Phi_f|g \) of the last section.

2.6.3 Special Values

Let \( \Phi \in Symb(A) \). We define the special value of the \( L \)-function of \( \Phi \) to be the element \( L(\Phi) = \Phi(\{0\} - \{\infty\}) \in A \). Let \( \chi : \mathbb{Z} \rightarrow R \) be a primitive Dirichlet character of conductor \( m \). We define the twist operator \( R_\chi \) to be given by
\[
\Phi|R_\chi = \sum_{a=0}^{m-1} \chi(a)\Phi \begin{bmatrix} 1 & a \\ 0 & m \end{bmatrix}
\]
(2.28)

We define the special value of the \( L \)-function of \( \Phi \) twisted to be \( \chi \) is \( L(\Phi, \chi) = L(\Phi|R_\chi) \).

When considering \( A = \text{Sym}^n(R^2) \), we define the special values \( L(\Phi, \chi, s_0) \in R \) for integers \( 0 \leq s_0 \leq k - 2 \) to be the elements of \( R \) defined by
\[
L(\Phi, \chi) = \sum_{s_0=0}^{n-1} \binom{n}{s_0} \chi(-1)^{n-s_0} L(\Phi, \chi, s_0) X^{s_0} Y^{n-s_0}.
\]
(2.29)

Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be the Fourier expansion of \( f \in S_k(\Gamma, \overline{Q}) \). Let \( \chi : \mathbb{Z} \rightarrow \mathbb{C} \) be a Dirichlet character with conductor \( m \). The complex \( L \)-function of \( f \) twisted by \( \chi \) is
\[
L_\chi(f, \chi, s) = \sum_{n=1}^{\infty} \chi(n)a_n n^{-s}
\]
(2.30)
which converges for $\text{Re}(s) > \frac{k-1}{2}$. This extends to an entire function in $s$. We are interested in its values at integers $s = s_0$ in the critical strip $1 \leq s_0 \leq k - 1$. When $\chi$ is the primitive trivial character, we drop the $\chi$ from the notation.

We define the Gauss Sum by the following formula

$$
\tau(n, \chi) = \sum_{a=0}^{m-1} \chi(a)e^{2\pi ina/m}.
$$

(2.31)

We also let $\tau(\chi) = \tau(1, \chi)$. The following formula holds for all $n$:

$$
\tau(n, \chi) = \overline{\chi(n)} \tau(\chi).
$$

(2.32)

**Theorem 2.6.6** For every primitive Dirichlet character $\chi$ of conductor $m$ and integer $s_0$ with $0 \leq s_0 \leq k - 2$, we have

$$
L(\Phi_f, \chi, s_0) = m^{s_0}s_0!\tau(\chi) \frac{L_{\infty}(f, \overline{\chi}, s_0 + 1)}{(-2\pi i)^{s_0}}
$$

(2.33)

**Proof.** A key ingredient of the proof is the fact that the Mellin transform of $f$ is an integral representation of the complex $L$-function attached to $f$.

$$
L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} f(it)t^{s-1}dt
$$

(2.34)

We use this to relate the special values of $L(f, s)$ to the modular symbol attached to $f$.

For $0 \leq n \leq k - 2$ and using the change of variables $z = it$, we have

$$
i^n \frac{n!}{(2\pi)^n} L(f, n + 1) = i^n2\pi \int_0^{\infty} f(it)t^{n+1}dt
$$

(2.35)

$$
= i^n2\pi(-i)^{n+1} \int_0^{\infty} f(z)z^ndz
$$

(2.36)

$$
= -2\pi i \int_0^{\infty} f(z)z^ndz
$$

(2.37)

$$
= \Phi_f(z^n)(\{0\} - \{\infty\})
$$

(2.38)

We relate twists of $f$ by a character to $f$ using the Gauss Sum.

$$
f_{\chi}(z) = \sum_n \overline{\chi(n)}a_ne^{2\pi iz}
$$

(2.39)

$$
= \sum_n \frac{\tau(n, \chi)}{\tau(\chi)}a_ne^{2\pi iz}
$$

(2.40)

$$
= \frac{1}{\tau(\chi)} \sum_{a=0}^{m-1} \chi(a)f(z + \frac{a}{m})
$$

(2.41)
Now we relate the modular symbol of a twist of $f$ to the twisted modular symbol of $f$.

\[
m^n \Phi_{f,X}(z^n)((\{ \frac{r}{m} \} - \{ \infty \})) = \Phi_{f,X}((mz)^n)((\{ \frac{r}{m} \} - \{ \infty \}))
\]

\[
= \frac{1}{\tau(\chi)} \sum_{a=0}^{m-1} \chi(a) \Phi_{l} \begin{bmatrix} 1/a & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} (z^n)((\{ r \} - \{ \infty \}))
\]

\[
= \frac{1}{\tau(\chi)} \sum_{a=0}^{m-1} \chi(a) \Phi_{l} \begin{bmatrix} 1/a & m \\ 0 & 1 \end{bmatrix} (z^n)((\{ r \} - \{ \infty \}))
\]

\[
= \frac{1}{\tau(\chi)} \sum_{a=0}^{m-1} \chi(a) \Phi_{l} \begin{bmatrix} 1/a & m \\ 0 & m \end{bmatrix} (z^n)((\{ r \} - \{ \infty \}))
\]

Putting together the above formulas, one deduces the theorem. \qed

\section{2.6.4 Measures}

In this section we study measures on a $p$-adic space $Y$. Specifically, we will be looking at the case where $Y$ is $\mathbb{Z}_p^\times$. The results in this section generalize easily to $Y = \mathbb{Z}_{J,p}^\times$ for positive integers $J$ relatively prime to $p$. Let $R$ be a commutative ring with identity with a $p$-adic norm. We will be looking at the case where $R = \mathbb{C}_p$. We let $R^0$ denote the subring of elements with norm bounded by 1. So in our specific case we have $R^0 = \mathcal{O}_{\mathbb{C}_p}$. Let $\mathcal{C}^h(Y, R)$ denote the $R$-module of locally polynomial functions on $Y$ of degree $< h$. So $\mathcal{C}^1(Y, R)$ is the space of locally constant functions. We let $\mathcal{C}^{\text{loc.an.}}(Y, R)$ denote the $R$-module of locally analytic functions on $Y$. Note that

\[
\mathcal{C}^1(Y, R) \subset \mathcal{C}^h(Y, R) \subset \mathcal{C}^{\text{loc.an.}}(Y, R).
\]

For $a \in \mathbb{Z}_p^\times$, let $D(a, \nu) = a + p^\nu \mathbb{Z}_p \subset \mathbb{Z}_p^\times$.

\textbf{Definition 2.6.7} Let $M$ be an $R$-module. A $R$-module homomorphism $\mu : \mathcal{C}^h(Y, R) \to$
$M$ is called a weakly $h$-admissible measure if for each $j$ satisfying $0 \leq j < h$,

$$
\left| \int_{D(a,\nu)} (x-a)^j \, d\mu \right|_p = O(p^{-\nu(j-h)}).
$$

(2.48)

**Theorem 2.6.8** A weakly $h$-admissible measure $\mu : C^h(Y, R) \to M$ uniquely extends to an $R$-module homomorphism $\mu : C^{\text{loc.an.}}(Y, R) \to M$.

**Proof.** By the definition of weakly $h$-admissible, for $0 \leq j < h$ we have

$$
\int_{D(a,\nu)} (x-a)^j \, d\mu \in p^{\nu(j-h)} \cdot M'.
$$

(2.49)

where $M'$ is some $R^0$-module of $M$. Let $F(x) = \sum_n c_n (x-a)^n$ be an analytic function on $D(a,\nu)$. Define the fractional ideal

$$
I(F, a, \nu) = p^{-h \nu} \sum_{n \geq h} c_n p^{n \nu} \cdot R^0
$$

(2.50)

We show the following lemma:

**Lemma 2.6.9** If $D(a', \nu') \subset D(a, \nu)$, then $I(F, a', \nu') \subset I(F, a, \nu)$.

**Proof.** This is clear if $\nu' > \nu$. Assume $\nu' = \nu$.

$$
c_j' = \sum_{n \geq j} \binom{n}{j} (a' - a)^{n-j}
$$

(2.51)

$$
p^{j \nu} c_j' \in \sum_{n \geq j} c_n p^{n \nu} R^0 \subset p^{h \nu} I(F, a, \nu)
$$

(2.52)

\[\square\]

We now show the existence of $\mu : C^{\text{loc.an.}}(Y, R) \to M$. Let $a$ and $a'$ be such that $a' \equiv a \mod p^\nu$. Then $D(a', \nu) = D(a, \nu)$. We write

$$
F(x) = \sum_n c_n (x-a)^n = \sum_n c_n (x-a')^n
$$

(2.53)

We define the following truncations of $F$.

$$
F_a(x) = \sum_{n<h} c_n (x-a)^n
$$

(2.54)
and
\[ F'_a(x) = \sum_{n<h} c'_n (x - a')^n \]  \hspace{1cm} (2.55)

Then we have
\[ F_a(x) - F'_a(x) = \sum_{n<h} b_n (x - a)^n \]  \hspace{1cm} (2.56)

We have the following lemma:

**Lemma 2.6.10**
\[ p^\nu b_n \in p^{h\nu} I(f, a, \nu) \]  \hspace{1cm} (2.57)

**Proof.**
\[ b_n = \sum_{j \geq h} c'_{j} \binom{j}{n} (a' - a)^{j-n} \]  \hspace{1cm} (2.58)

\[ a - a' \in p^\nu \mathbb{Z}_p \] so
\[ p^\nu b_n \in \sum_{j \geq h} c'_{j} p^j \cdot R^0 \subset p^{h\nu} I(f, a, \nu) \]  \hspace{1cm} (2.59)

\[ \square \]

Thus with the lemma and the weakly $h$-admissible condition, we have
\[ \int_{D(a, \nu)} F_a - F'_a \in \left( \frac{p^h}{\alpha} \right)^\nu I(f, a, \nu) \cdot M' \]  \hspace{1cm} (2.60)

Let $U = \cup D(a_i, \nu)$.
\[ \sum_i \int_{D(a_i, \nu)} F_{a_i} = \int_U F \]  \hspace{1cm} (2.61)

is determined modulo $(\frac{p^{k-1}}{\alpha})^\nu I(f, a, \nu) \cdot M'$. These Riemann sums converge and we have the existence of the measure $\mu : C^{loc.an.}(Y, R) \rightarrow M$.

From the above work, the well-definedness of $\mu$ implies that for all $n \geq 0$ we have
\[ \int_{D(a, \nu)} x^n d\mu \in p^{\nu(n-h)} \cdot M. \]  \hspace{1cm} (2.62)

This together with first lemma imply the uniqueness. \[ \square \]
2.6.5 \( p \)-adic \( L \)-function

Let \( f \in S_k(\Gamma_0(Np), \psi, \overline{Q}) \) be a Hecke eigenform of weight \( k \geq 2 \) where \( \psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{Q}^\times \) is a Dirichlet character. Associated to \( f \), we have the modular symbol \( \Psi_f^\pm \in \text{Sym}_r(Sym^2(C^2)) \). We can choose periods \( \Omega_f^\pm \in C^\times \) so that

\[
\Psi_f^\pm = \Phi_f^\pm / \Omega_f^\pm \in \text{Sym}_r(Sym^2(C^2)_K)
\]  

(2.63)

where \( \mathcal{O}_K \) is the ring of integers of \( K \), a finite extension of \( \mathbb{Q}_p \).

Our goal is to construct a \( p \)-adic \( L \)-function attached to \( f \). To do this we define a measure by

\[
\mu_f^\pm(P(z))(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha'} \Psi_f^\pm(P(p^nz + a))(-\frac{a}{p^n}) \in C_p
\]  

(2.64)

where \( P(z) \) is a polynomial of degree \( \leq n \) and \( \alpha \) is the \( U_p \)-eigenvalue of \( f \).

We have the following proposition that shows that \( \mu_f^\pm \) gives a well defined \( C_p \)-module homomorphism from \( C^h(\mathbb{Z}_p^\times, C_p) \to C_p \).

**Proposition 2.6.11 (Distribution Property)**

\[
\mu_f^\pm(P(z))(a + p^n\mathbb{Z}_p) = \sum_{b \equiv a \mod p^n, b \equiv b + p^{n+1} \mod p^{n+1}} \mu_f^\pm(P(z))(b + p^{n+1}\mathbb{Z}_p)
\]  

(2.65)

**Proof.** \( f \) is an eigenform for the \( U_p \) operator so we have

\[
\alpha f = f|U_p = \sum_{b=0}^{p-1} f\begin{bmatrix} 1 & b \\ 0 & p \end{bmatrix}
\]  

(2.66)

This gives the following formula for modular symbols.

\[
\alpha \Psi_f^\pm(P(p^nz + a)(-\frac{a}{p^n}) \in C_p
\]  

(2.67)

\[
= \sum_{b=0}^{p-1} \Psi_f^\pm(P(p^nz + (a - bp^n)(-\frac{a - bp^n}{p^{n+1}}) \in C_p
\]  

(2.68)

This gives the proposition. \(\square\)
Proposition 2.6.12 $\mu_f$ is an h-admissible measure and thus gives a well define map $\mu_f : C^{\text{loc.an.}}(\mathcal{Z}_p^X, \mathbb{C}_p) \to \mathbb{C}_p$.

Proof. By a straightforward calculation, $\mu_f$ satisfies convergence conditions. \hfill \Box

For a fixed measure $\mu : C^{\text{loc.an.}}(\mathcal{Z}_p^X, \mathbb{C}_p) \to \mathbb{C}_p$ and continuous character $\sigma : \mathcal{Z}_p^X \to \mathbb{C}_p^X$, we define the $p$-adic $L$-function attached to the measure $\mu$ by

$$L_p(\mu, \sigma) = \int_{\mathcal{Z}_p^X} \sigma(t) \, d\mu(t) \in \mathbb{C}_p. \quad (2.69)$$

We define the $p$-adic $L$-function attached to the cuspidal eigenform $f$ by

$$L_p(f, \chi, s) = L_p(\mu_f^{\text{sgn}(\chi)}, \chi(\cdot)^s) \quad (2.70)$$

where $\chi : \mathcal{Z}_p^X \to \mathbb{C}_p$ is a continuous character.

Theorem 2.6.13 Let $f \in S_k(\Gamma_0(Np), \psi, \overline{Q})$ be a eigenform, and let $\alpha$ denote the $U_p$-eigenvalue of $f$.

(a) (Analyticity) The $p$-adic $L$-function $L_p(f, \chi, s)$ for a fixed character $\chi : \mathcal{Z}_p^X \to \mathbb{C}_p$ is analytic in the variable $s$.

(b) (Interpolation) For integral $s_0$ with $0 \leq s_0 \leq k - 2$, $L_p(f, \chi, s)$ interpolates the algebraic parts of the special values of the complex $L$-function attached to $f$ as follows:

$$L_p(f, \chi, s_0) = \frac{1}{\alpha^s}(1 - \frac{\chi_{\text{w}^{-s_0}}(p)p^{s_0}}{\alpha}) \cdot p^{s_0} \cdot \Omega(f, \chi, s_0 + 1) \cdot \frac{L_{\infty}(f, \chi, s_0 + 1)}{(\pi i)^{s_0} \cdot \Omega_f^{\text{sgn}(\chi)}}. \quad (2.71)$$

(c) (Functional Equation) For integral $s_0$ with $0 \leq s_0 \leq k - 2$, we have

$$L_p(f, \chi, s_0) = (-\overline{\chi}(-N)(-N)^{-s_0} \cdot L_p(f|_{W_N}, \overline{\chi}, k - 2 - s_0). \quad (2.72)$$

Proof. See [13]. \hfill \Box

2.7 Overconvergent Modular Symbols

In order to construct two variable $p$-adic $L$-functions, one needs to be able to put modular symbols into $p$-adic families. This is done via Stevens' theory of overconvergent modular symbols. We give an exposition of the theory below. For a fuller exposition see [19].

29
2.7.1 Overconvergent modules

For \( r \in \mathbb{Q} \) and \( r \geq 1 \) let \( B[r] = \{ x \in \mathbb{C}_p | |x| \leq r \} \). Let \( Y \) be an affinoid subspace of weight space \( W_N \) defined over \( K \). The affinoid functions on \( B[r] \) have a norm given by

\[
||f||_r = \sup_{x \in B[r]} |f(x)|_p
\]  

(2.73)

Let \( A[r] \) be the affinoid functions on \( Y \times B[r] \) that are defined over \( K \). The norm for the completed tensor product of two Banach spaces gives the norm on \( A[r] \). Let

\[
A^\dagger = \lim_{r \to 1} A[r]
\]  

(2.74)

Explicitly we have

\[
A^\dagger = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(Y)[[z]] \mid \{|a_n| r^n\} \to 0 \quad \forall \quad r > 1 \right\}
\]  

(2.75)

where \( A(Y) \) are the affinoid functions on \( Y \). The inductive limit topology determines a topology on \( A^\dagger \).

\( A^\dagger \) has an action of \( \Sigma^* = \Sigma \cup \{e\} \) given by following: let \( F : Y \times B^\dagger \to \mathbb{C}_p \) and

\[
\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Sigma^*, \text{ define}
\]

\[
(\gamma \cdot F)(\kappa, z) = \kappa (cz + a) \cdot F\left(\kappa, \frac{dz + b}{cz + a}\right)
\]  

(2.76)

\( A^\dagger \) has an action of \( A(Y) \) given by the following: for \( \lambda \in A(Y) \), let

\[
(\lambda \cdot F)(\kappa, z) = \lambda(\kappa) F(\kappa, z)
\]  

(2.77)

Thus, \( A^\dagger \) has an left action of \( A(Y)[\Sigma^*] \).

Let \( D^\dagger \) be the set of \( A(Y) \)-linear maps from \( A^\dagger \) to \( A(Y) \). The action of \( \Sigma^* \) on \( \mu \in D^\dagger \) is given by

\[
(\mu|\gamma)(F) = \mu(\gamma \cdot F)
\]  

(2.78)

\( D^\dagger \) has a right action of \( A(Y)[\Sigma^*] \). If we let \( D[r] \) be the set of \( A(Y) \)-linear maps from \( A[r] \) to \( A(Y) \), we can equivalently define \( D^\dagger \) to be the projective limit of the \( D[r] \)'s for \( r > 1 \). Each \( D[r] \) has a norm given by

\[
||\mu||_r = \sup_{f \in A[r], f \neq 0} \frac{||\mu(f)||}{||f||_r}
\]  

(2.79)
These norms and the projective limit topology determines a topology on $D^\dagger$.

We note that polynomials over $A(Y)$ are dense in $\mathcal{A}^\dagger$. Thus, any element of $D^\dagger$ is determined by its values on the set $\{x^n\}_{n \in \mathbb{N}}$. We define an injective map called the moment map $\mathcal{M} : D^\dagger \rightarrow \prod_{j=0}^{\infty} A(Y)$ which sends $\mu$ to the sequence $(c_j)_{j \in \mathbb{N}}$ where $c_j = \mu(x^j)$.

**Proposition 2.7.1** The image of $\mathcal{M}$ is precisely

$$\{(c_j) \in \prod_{j=0}^{\infty} A(Y) \text{ where } |c_j| \text{ is } o(r^j) \text{ for each } r > 1\}.$$  \hspace{1cm} (2.80)

**Proof.** See [19]. \hfill \square

$\mathcal{A}^\dagger$ and $D^\dagger$ are both modules over $A(Y)$. We can specialize these modules given an element of $Y$. When the character determined by the element of $Y$ is of the form $z \mapsto \varphi(z)z^k$ for some integer $k$ and character $\varphi : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow K^\times$ and $Y$ is small enough to determine $\varphi$, we let $\mathcal{A}^\dagger_k$ and $D^\dagger_k$ denote the corresponding specializations.

### 2.7.2 Specialization

Let $L_k(K)$ be the space of homogeneous polynomials in the variables $W$ and $Z$ of degree $k$ with coefficients in $K$. We define the weight $k$ specialization map $\rho_k$ for $D^\dagger$ to be the composition of maps

$$\rho_k : D^\dagger \rightarrow D^\dagger_k \rightarrow L_k(K)$$  \hspace{1cm} (2.81)

where the first map is the map induced by tensoring with the weight $k$ level $Np$ character in the weight space $Y$, and the second map is given by

$$\mu \mapsto \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \mu(x^{k-j}) W^{k-j} Z^j.$$  \hspace{1cm} (2.82)

These specialization maps induce maps on the corresponding spaces of modular symbols

$$\rho_k : \text{Sym}_k(D^\dagger) \rightarrow \text{Sym}_k(D^\dagger_k) \rightarrow \text{Sym}_k(L_k(K))$$  \hspace{1cm} (2.83)

To understand the relationship between these spaces better, we want to restrict our attention to certain finite slope subspaces. To do this, we need then following proposition:
Proposition 2.7.2 The $U_p$ Hecke operator acts as a completely continuous operator on $\text{Sym}_Y(D^\dagger)$ and $\text{Sym}_Y(D^\dagger_k)$.

Proof. See [19]. □

This gives rise to a characteristic series of the $U_p$ operator which we can factor. For the weight $k$ spaces, we have the following theorem of Stevens:

Theorem 2.7.3 (Stevens) The induced map on the finite slope subspaces

$$\text{Sym}_Y(D^\dagger)_{\leq \alpha} \to \text{Sym}_Y(L_k(K))_{\leq \alpha}$$

is an isomorphism of $\Sigma_p$-modules for $\alpha \leq k + 1$.

Proof. See [19]. □

Theorem 2.7.4 (Stevens) The following map is surjective:

$$\text{Sym}_Y(D^\dagger) \to \text{Sym}_Y(D^\dagger_k)$$

(2.85)

Proof. See [19]. □

2.7.3 Modular symbols on the eigencurve

Let $D(V) \subset C_{p,N}$ be a local flat piece of the eigencurve arising from a $V \in C$ such that $D(V) = D(Y,t)$. Since $Y$ is a closed disk, $A(Y)$ is a PID. This implies that $\mathcal{T}(V)$ is in fact free of rank $d$ over $A(Y)$. To emphasize the dependence of $D^\dagger$ on $Y$, we denote $D^\dagger_Y = D^\dagger$. We refer to $\text{Sym}_Y(D^\dagger_Y)$ as the space of overconvergent modular symbols over $Y$.

Theorem 2.7.5 There is a rank 2 locally free sheaf $\mathcal{E}$ defined on the nonsingular locus of $C_{p,N}$ characterized by the property that for each classical $x \in C_{p,N}$ of level $N\text{p}$ defined over $K$ with $\alpha(x) \leq k(x) + 1$, the fiber

$$\mathcal{E}_x = \text{Sym}_Y(L_{k(x)}(K))_{[\lambda_x]}$$

(2.86)

where $\lambda_x$ is the Hecke map determined by $x$, $k(x)$ is the weight of $x$, $\alpha(x)$ is the slope of $x$, and the $[\lambda_x]$ denotes the $\lambda_x$-eigenspace.

32
Proof. We assume all components of $D(V)$ are primitive for tame level $N$. The space of modular symbols of slope $\leq t$ when we specialize to a weight $k$ is free of rank $2d$ at the classical weights of large enough weight. Since $A(Y)$ is a PID, The space of overconvergent symbols over $Y$ of slope $\leq t$ is free of rank $2d$. Because of the primitive assumption, the characteristic polynomial of a Hecke operator determines the operator. For the large enough classical weights, the characteristic polynomial of a Hecke operator on the overconvergent modular symbols of weight $k$ is the square of the characteristic polynomial of the Hecke operator acting on the space of classical modular forms of slope $\leq t$ by Eichler-Shimura theorem. Thus $\text{Sym}_{n}(D_{1}^{+})$ is a $\mathbb{T}(V)$-module. This also gives the locally free of rank 2 on the nonsingular locus. □

2.8 Two Variable $p$-adic $L$-function

For a family of overconvergent modular forms given by $\lambda : \mathbb{T}(V) \otimes \mathcal{L} \to \mathcal{K}$, one obtains a measure by taking the modular symbol $\Phi_{\lambda}$ attached to the family and evaluating $\mu_{\lambda} = \Phi_{\lambda}(\{0\} - \{\infty\}) \in D_{1}^{+}$. This measure can be extended to evaluate locally analytic functions with coefficients in $A(Y)$ using the same techniques as in the one variable case by verifying that $\mu_{\lambda}$ is an $h$-admissible measure for each $h \geq \alpha$.

$\mu_{\lambda}$ defines a two variable $p$-adic $L$-function.

$$L_{p}(\Phi_{\lambda}, \kappa, \sigma) = \left( \int \sigma(t) d\mu_{\lambda}(t) \right)_{\kappa} \in \mathbb{C}_{p} \quad (2.87)$$

Let $\varphi : (\mathbb{Z}/Np\mathbb{Z})^{\times} \to \mathbb{K}^{\times}$ denote the character determined by $Y$, and let $\psi$ be a finite order $p$-adic character. We define

$$L_{p}(\Phi_{\lambda}, \varphi, k, \psi, s) = L_{p}(\Phi_{\lambda}, \varphi(z)t^{k}, \psi(t)^{s}) \quad (2.88)$$

**Theorem 2.8.1** (a) (Analyticity) $L_{p}(\Phi_{\lambda}, \varphi, k, \psi, s)$ is analytic in the variables $k$ and $s$.
(b) (Interpolation) For classical $k_{0}$, $L_{p}(\Phi_{\lambda}, \varphi, k_{0}, \psi, s) = L_{p}(f_{k_{0}}, \psi, s)$ where $f_{k_{0}}$ is the specialization of the family $\lambda$ at weight $k_{0}$.
(c) (Functional Equation)

$$L_{p}(\Phi, \varphi, k, \psi, s) = -\psi^{-1}(\mathbb{N})(\mathbb{N})^{-1-s} \cdot L_{p}(\Phi|_{W_{N}}, \varphi, k, \varphi\psi^{-1}, k - s + 1). \quad (2.89)$$

33

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**Proof.** Analyticity is clear from the definitions. Interpolation is a straightforward computation. Likewise the functional equation. □

We keep in mind that the one variable $p$-adic $L$-function is determined up to an element of $K^\times$ depending on the choice of generator. In the two variable situation, $\Phi_A$ determines generators for each $k$ that arises as a weight in $Y$. This determines a compatible set of periods for each weight $k$. 
Chapter 3

Ramification Points on the Eigencurve

3.1 Introduction

In this chapter we construct a locally analytic p-adic L-function, $L(X)$, defined on $C_{p,N}$, the p-adic eigencurve of tame level $N$. This p-adic L-function, $L(X)$, when evaluated at a $C_p$-valued point on the eigencurve $C_{p,N}$ that corresponds to a classical cusp form $f$ of integral weight $k$, gives the algebraic part of the special value of the complex symmetric square $L$-function, $L(f,k)$. In addition, we show that the zeros of $L(X)$ are precisely the ramification points of $C_{p,N}$ for the weight projection map $\pi : C_{p,N} \to \mathcal{W}_N$, where $\mathcal{W}_N$ denotes the level $N$ weight space. In the next chapter, we will show that this p-adic L-function has a natural two variable extension interpolating all the expected special values of each complex symmetric square $L$-function attached to each classical cuspidal point of the eigencurve when $2\mid N$.

The constructions we use are extensions of methods of Hida applied to local pieces of the eigencurve $C_{p,N}$ that have properties analogous to those of the ordinary situation studied by Hida. The local pieces on which we work are affinoid subsets $D \subset C_{p,N}$ whose image in weight space is a closed disk $Y$ over which it is flat. Another required property of the $D$ that we consider is that they parametrize precisely the overconvergent modular eigenforms whose weight character is an element of $Y$ and whose slope is $\leq$
some fixed \( t \in \mathbb{Q} \).

For such a local piece \( D \) over its weight space \( Y \), we mimic Hida's construction and construct a congruence module for each family arising in \( D \). These congruence modules interpolate certain congruence modules attached to the individual forms in the family. We then interpret the size of the congruence modules for the individual forms as a measure of distance between the different families arising in \( D \) at each particular classical weight. This allows us to access the ramification points through the congruence modules for the families.

In order to relate the special values of the symmetric square \( L \)-functions to the sizes of the congruence modules, we again follow Hida. Hida studied the parabolic cohomology of a modular curve with coefficients in symmetric power modules. Given a modular form \( f \) and using the Eichler-Shimura isomorphism, one can find two elements of the cohomology group that generate two different eigenspaces with the same Hecke eigenvalues as the form \( f \). One also has a pairing on the cohomology groups that arises from the cup product. Hida shows that the pairing applied to the two cohomology classes attached to \( f \) is related to the special value \( L(k, f_k) \) through formulas involving the Petersson inner product. In the ordinary situation, Hida also shows that this pairing applied to the two cohomology classes gives the size of a congruence module attached to \( f \).

In this paper, we use Stevens' theory of overconvergent modular symbols to interpolate the cohomology classes described above over a \( p \)-adic family of overconvergent modular forms. This allows us to proceed by developing a \( p \)-adic family version of Hida's methods. To construct the \( p \)-adic \( L \)-function \( \mathcal{L}(X) \) of our main result, we construct a pairing on the space of overconvergent modular symbols over the weight space \( Y \) and evaluate this pairing on the overconvergent modular symbols that interpolate the cohomology classes.

We are able to show that \( \mathcal{L}(X) \) interpolates the desired special values by directly following the methods of Hida. Relating \( \mathcal{L}(X) \) to the congruence modules does not go through as smoothly because the Hecke rings involved might not be Gorenstein. In the ordinary situation, the Hecke rings are in fact Gorenstein. Following Hida, we relate \( \mathcal{L}(X) \) to the congruence modules, but an error which we refer to as the "Gorenstein
error" arises in the formula. This error can be controlled by understanding how far the pairing on the cohomology is from being perfect. We bound this error as a function of the slope of the family independent of the weight. With this, we show that no zeros can arise or disappear due to the Gorenstein error. This allows us to show the correspondence between zeros of $L(X)$ and the ramification points in our main result.

We state some notation that will carry through this chapter. Let $p$ be a prime, and $N$ be a positive integer relative prime to $p$. Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ denote the ring of integers of $K$. Let $\alpha \in \mathbb{Q}$ be a rational number that arises as the valuation of an element of $\mathcal{O}_K$.

## 3.2 Local Pieces of the Eigencurve

In this section, we recall the eigencurve notation defined in the previous chapter.

### 3.2.1 The eigencurve

Recall that $C_{p,N}$ denotes the $p$-adic eigencurve of tame level $N$ as constructed by Buzzard in [1]. This is a rigid analytic curve that was constructed for $p \geq 3$ and $N = 1$ by Coleman and Mazur and extended to all $p$ and $N$ by Buzzard. One of the main theorems about $C_{p,N}$ is the following:

**Theorem 3.2.1** (Coleman, Mazur [4], Buzzard [1]) There is a natural bijection between the $C_p$-valued points of the eigencurve $C_{p,N}$ and finite slope overconvergent $p$-adic modular eigenforms of tame level $N$.

### 3.2.2 Weight space

An important object attached to $C_{p,N}$ is its associated weight space of tame level $N$ denoted $W_N$. This is the rigid analytic space attached to the ring $\Lambda_N$. There is a natural map $\pi : C_{p,N} \to W_N$ called the weight projection map.
3.2.3 The spectral curve

Recall that $M^1_k(N, \nu)$ denotes the space of $\nu$-overconvergent modular forms of weight character $\kappa \in \mathcal{W}_N$ and tame level $N$. This space is a Banach module on which the $U_p$ Hecke operator acts completely continuously. Let $P(\kappa, T)$ denote the Fredholm determinant of the completely continuous system of operators $U_p$ acting on the system of Banach modules $M^1_k(N, 1/i)$ for $i$ large enough. The following theorem asserts that these Fredholm determinants vary well as the weight varies:

**Theorem 3.2.2** (Coleman, Mazur [4], Buzzard [1]) There exists an entire power series $P(T) \in \Lambda_N\{\{T\}\}$ uniquely determined by the property that for each weight $\kappa \in \mathcal{W}_N$, the image of $P(T) \in \Lambda_N\{\{T\}\} \subset \Lambda_N[[T]]$ under the map $\Lambda_N[[T]] \to \mathbb{C}_p[[T]]$ induced by $\kappa : \Lambda_N \to \mathbb{C}_p$ is equal to the Fredholm determinant $P(\kappa, T)$.

This Fredholm series $P(T)$ can be viewed as a rigid analytic function on $\mathcal{W}_N \times \mathbb{A}^1$. Its zero locus $Z \subset \mathcal{W}_N \times \mathbb{A}^1$ is a rigid analytic curve which is called the spectral curve for the $U_p$ operator. This space also has a natural projection to weight space which we denote by $\hat{\pi} : Z \to \mathcal{W}_N$.

3.2.4 An admissible cover

**Definition 3.2.3** Let $C$ denote the set of affinoid subspaces of $V \subset Z$ with the following property: there is an affinoid subspace $Y \subset \mathcal{W}_N$ with the property that $V \subset Z_Y := \hat{\pi}^{-1}(Y)$, the induced map $\hat{\pi} : V \to Y$ is finite flat and surjective, and $V$ is disconnected from its complement in $Z_Y$.

The significance of $C$ is that it is an admissible cover of the spectral curve $Z$. The affinoid subsets in the collection $C$ give rise to the affinoid spaces that are glued together to form the eigencurve $C_{p, N}$. We summarize how this is done in the paragraphs that follow.

3.2.5 Local pieces of the eigencurve

Let $V \in C$. Let $Y = \hat{\pi}(V)$. Let $A(Y)$ be the affinoid ring attached to $Y$. $V$ corresponds to a factorization of $P(T)$ over $A(Y)$. Let $P(T) = Q_V(T)H(T)$ be the
corresponding factorization. Here $Q_V(T)$ is a polynomial with constant term 1 and whose leading coefficient is a unit, and $H(T)$ is entire. Let $d$ denote the degree of $Q_V(T)$, and let $Q_V^d(T) = T^d Q_V(T^{-1})$. Then we have $V = \text{Max}(A(Y)(T)/Q_V^d(T))$. The factorization also corresponds to a direct sum decomposition $M^d_{\chi}(\nu) = N(V, \nu) \oplus F(V, \nu)$ for each $\nu$. $N(V, \nu)$ is in fact independent of $\nu$ for small enough $\nu > 0$, and thus we refer to the coinciding spaces as $N(V)$. Let $\mathcal{T}(V)$ denote the image of the Hecke algebra in the endomorphism ring of $N(V)$ over $A(Y)$. $\mathcal{T}(V)$ is in fact an affinoid ring over $A(Y)$. Let $A^0(Y)$ and $\mathcal{T}^0(V)$ denote the subrings of the corresponding rings of elements with norm $\leq 1$. We have the natural map $A^0(Y) \rightarrow \mathcal{T}^0(V)$. Let $D(V)$ denote that affinoid variety attached to $\mathcal{T}(V)$. The eigencurve $C_{p,N}$ is constructed by gluing together the affinoid spaces $D(V)$ for $V \in \mathcal{C}$. The main ingredient for doing this is the fact that $\mathcal{C}$ is an admissible cover of $Z$.

3.2.6 Slope $\leq \alpha$ components

Now we will give a more detailed description of the local components to have more control when working with slopes. Let $W \subset W_N$ be an affinoid subspace of weight space. Let $D(W, \alpha) \subset C_{p,N}$ be the affinoid subspace consisting of points $x \in C_{p,N}$ whose weight $\pi(x) \in W$ and the slope of the form corresponding to $x$ is $\leq \alpha$. The following is a theorem of Buzzard which generalizes a theorem of Coleman.

**Theorem 3.2.4 (Buzzard [1], Coleman [3])** There exists an admissible covering of $W$ by affinoids $\{U_i\}$ and affinoids $\{D_i\}$ that cover $D(W, \alpha)$ satisfying the following properties:

(a) $\pi(D_i) = U_i$

(b) $D(W, \alpha)_{U_i} \subset D_i$

(c) $D_i = D(U_i, \alpha^*)$ for some $\alpha^* \in \mathbb{Q}$ independent of $i$ and such that no $s$ satisfying $\alpha < s \leq \alpha^*$ is the valuation of an element of $K$

(d) each $D_i$ is of the form $D(V_i)$ for some $V_i \in \mathcal{C}$

For the constructions in this chapter, we work with local pieces of the $D_i = D(V_i)$ of the theorem. This is what causes the constructions in this paper to be locally analytic as opposed to rigid analytic. We define the local pieces of the eigencurve that we work with in this paper as follows:

39
Definition 3.2.5 Let \( V \) be an affinoid subspace of one of the \( V_i \) described in Theorem 3.2.4 such that \( Y := \tilde{\pi}(V) \subset U_i \) is a closed disk. So then we have that \( D(V) = D(Y, \alpha^*) \). We call \( D(V) \) a local flat piece of the eigencurve of slope \( \leq \alpha \).

### 3.3 Congruence Modules

#### 3.3.1 Components

Let \( D(V) \subset C_{p,N} \) be a local flat piece of the eigencurve arising from a \( V \in \mathcal{C} \) such that \( D(V) = D(Y, \alpha^*) \). Since \( Y \) is a closed disk, \( A(Y) \) is a PID. This implies that \( \mathfrak{T}(V) \) is in fact free of finite rank over \( A(Y) \); let \( d \) denote this rank. Then we also have that \( \mathfrak{T}^0(V) \) is free of rank \( d \) over \( A^0(Y) \).

Let \( \mathcal{L} \) denote the fraction field of \( A^0(Y) \). Then \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{L} \) is a finite dimensional Artinian algebra over \( \mathcal{L} \) of dimension \( d \). Let \( \mathcal{K} \) be a finite extension of \( \mathcal{L} \) and \( \lambda : \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{L} \to \mathcal{K} \) a surjective map. For the exposition of this chapter we suppose we are in the situation where \( \lambda \) cuts out a direct summand of \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{L} \) isomorphic to \( \mathcal{K} \). So we have \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{L} \cong \mathcal{K} \oplus \mathcal{B}' \) where \( \mathcal{B}' \) is an Artinian algebra. This assumption restricts our constructions to primitive components, or in otherwords, components that do not arise as components of eigencurves of strictly lower tame level. Let \( A_\mathcal{K} \) denote the integral closure of \( A^0(Y) \) in \( \mathcal{K} \). Let \( \mathfrak{T}_{A_\mathcal{K}} = \mathfrak{T}^0(V) \otimes_{A^0(Y)} A_\mathcal{K} \).

#### 3.3.2 Big congruence module

Consider \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{K} \). This has a direct sum decomposition \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{K} \cong \mathcal{K} \oplus \mathcal{B} \) where \( \mathcal{B} \) is an Artinian algebra. Let \( \mathfrak{h}(\mathcal{K}) \) and \( \mathfrak{h}(\mathcal{B}) \) be the images of \( \mathfrak{T}_{A_\mathcal{K}} \) in \( \mathcal{K} \) and \( \mathcal{B} \) respectively. We define the congruence module for \( \mathcal{K} \) by

\[
\mathcal{C}(\mathcal{K}) = (\mathfrak{h}(\mathcal{K}) \oplus \mathfrak{h}(\mathcal{B}))/\mathfrak{T}_{A_\mathcal{K}} \tag{3.1}
\]

Let \( \mathcal{I}_\mathcal{K} \subset A_{\mathcal{K}} \) denote the annihilator ideal of \( \mathcal{C}(\mathcal{K}) \).

**Proposition 3.3.1** \( \mathcal{C}(\mathcal{K}) \) is a finite torsion \( A_\mathcal{K} \)-module.

**Proof.** \( \mathfrak{T}_{A_\mathcal{K}} \) and \( \mathfrak{h}(\mathcal{K}) \oplus \mathfrak{h}(\mathcal{B}) \) are both lattices of rank \( d \) in \( \mathfrak{T}^0(V) \otimes_{A^0(Y)} \mathcal{L} \) with \( \mathfrak{T}_{A_\mathcal{K}} \subset \mathfrak{h}(\mathcal{K}) \oplus \mathfrak{h}(\mathcal{B}) \). This implies that \( \mathcal{C}(\mathcal{K}) \) is finite and torsion. \( \square \)
3.3.3 Specializing to a weight

Let $P$ be a height 1 prime ideal of $A_K$. $P$ determines a prime ideal of $A(Y)$. Let $\chi$ be the weight character corresponding to this prime ideal. Suppose $\chi$ has values in $K^\times$. Let us also denote $P$ by $P_\chi$. Recall that $\lambda : \mathcal{T}(V) \otimes_{A(Y)} \mathcal{L} \to \mathcal{K}$ is the map cutting out a direct summand component isomorphic to $\mathcal{K}$. We also have the restricted map $\lambda : \mathcal{T}(V) \otimes_{A(Y)} A_K \to A_K$. If we tensor $\mathcal{T}(V) \otimes_{A(Y)} A_K$ with $A_K/P_\chi A_K$, we get a Hecke ring defined over $O_K$ which we denote by $\mathcal{T}(V)_\chi$. This Hecke ring acts on the space of weight $\chi$ and slope $\leq \alpha$ overconvergent modular forms which we denote by $N(V)_\chi$. The map $\lambda$ induces a map on the specialized Hecke rings that determines a component of $\mathcal{T}(V)_\chi = \mathcal{T}(V)_\chi \otimes K$. We can write a direct sum decomposition $\mathcal{T}(V)_\chi \cong \mathcal{K}_\chi \oplus B_\chi$. One can define a congruence module for this component as follows:

$$C(P_\chi) = (h(\mathcal{K}_\chi) \oplus h(B_\chi))/\mathcal{T}(V)_\chi \quad (3.2)$$

where $h(\mathcal{K}_\chi)$ and $h(B_\chi)$ denote the image of $\mathcal{T}(V)_\chi$ in $\mathcal{K}_\chi$ and $B_\chi$ respectively.

**Proposition 3.3.2** We have the following isomorphism of Hecke modules:

$$C(\mathcal{K}) \otimes_{A_K} A_K/P_\chi A_K \cong C(P_\chi) \quad (3.3)$$

**Proof.** We have the exact sequence:

$$0 \to \mathcal{T}_A \to h(\mathcal{K}) \oplus h(B) \to C(\mathcal{K}) \to 0 \quad (3.4)$$

Tensoring with $A_K/P_\chi A_K$ is right exact. This gives the proposition. \qed

3.3.4 Classical congruence modules

We now consider classical modular forms whose weight character arises as a point in the weight space $Y$. By possibly shrinking the weight space $Y$, we can assume that all classical forms with weight in $Y$ and level $Np$ have character $\varphi : (\mathbb{Z}/Np\mathbb{Z})^\times \to K^\times$. The following is a useful theorem first proved by Coleman and proved using a different method by Kassaei:

41
Theorem 3.3.3 (Coleman [2], Kassaei [11]) Let \( f \) be an overconvergent \( U_p \)-eigenform of weight \( k \) with eigenvalue \( a_p \). If the \( p \)-adic valuation of \( a_p \) is less than \( k - 1 \), then \( f \) is classical.

We consider \( S_k(\Gamma_0(Np), \psi, K) \) be the space of classical cusp forms on \( \Gamma_0(Np) \) over \( K \) with character \( \psi \). Let \( S_k(\Gamma_0(Np), \psi, \mathcal{O}_K) \) be the corresponding space over the ring of integers of \( K \). Let \( \mathcal{H}(\Gamma_0(Np), \psi, K) \) and \( \mathcal{H}(\Gamma_0(Np), \psi, \mathcal{O}_K) \) be the corresponding spaces of Hecke operators. We let a superscript \( \leq \alpha \) denote the subspace generated by forms with slope \( \leq \alpha \) or the Hecke algebra acting on such a subspace. Let \( f_k \in S_k(\Gamma_0(Np), \psi, \mathcal{O}_K)^{\leq \alpha} \) be a cuspidal eigenform that is the specialization of the family \( \lambda \) at the \( p \)-adic character \( z \mapsto \psi(z)(z)^k \). This eigenform corresponds to a summand \( F \) of \( \mathcal{H}(\Gamma_0(Np), \psi, K)^{\leq \alpha} \cong F \oplus B \). Let \( \mathcal{H}(F) \) and \( \mathcal{H}(B) \) denote the projections of \( \mathcal{H}(\Gamma_0(Np), \psi, \mathcal{O}_K)^{\leq \alpha} \) onto \( F \) and \( B \) respectively. We define the classical congruence module as follows:

\[
C_k(f_k) = (\mathcal{H}(F) \oplus \mathcal{H}(B))/\mathcal{H}(\Gamma_0(Np), \psi, \mathcal{O}_K)^{\leq \alpha}
\] (3.5)

Proposition 3.3.4 For characters \( \chi \in Y \) of the form \( \chi(z) = \psi(z)(z)^k \) for an integer \( k \geq \alpha + 1 \), we have \( C(P_\chi) \cong C_k(f_k) \) where \( f_k \) is the classical normalized eigenform determined by the prime ideal \( P_\chi \).

Proof. For such weights, overconvergent forms of slope \( \alpha \) or less are classical by theorem 3.3.3. Thus the space of overconvergent forms of weight \( \chi \) and slope \( \leq \alpha \) is the space of classical forms of weight \( \chi \) and slope \( \leq \alpha \). Thus they have isomorphic congruence modules. \( \square \)

3.4 Ramification Points: Part I

We now relate the congruence modules that we constructed to the geometry of the eigencurve \( C_{p,N} \).

Definition 3.4.1 Let \( y \in Y \). If the fiber of \( \pi : D(V) \to Y \) above \( y \) is not reduced, we say that \( \pi \) ramifies at the weight \( y \). If \( x \in \pi^{-1}(y) \) is not reduced in the fiber, we say \( x \) is a ramification point.
Let $S \subset D(V)$ be the set of ramification points. The map $\lambda : T^0(V) \otimes A_0(V) \mathcal{L} \to \mathcal{K}$ restricts to a map $\lambda : T^0(V) \otimes A_0(V) A_{K} \to A_{K}$. This induces a map $\tilde{\lambda} : Max(A_{K} \otimes K) \to D(V)$.

**Theorem 3.4.2** Let $Q \subset Max(A_{K} \otimes K)$ be the common zero locus of the annihilator ideal $I_{K}$ of the big congruence module $C(\mathcal{K})$. The image $\tilde{\lambda}(Q) \subset D(V)$ is precisely $S \cap im(\tilde{\lambda})$.

**Proof.** Let $x \in Q$. Then the specialization of $C(\mathcal{K})$ to $x$ has non-torsion elements. The specialization is

$$C(\mathcal{K})_x = ((\mathfrak{h}(\mathcal{K}) \oplus \mathfrak{h}(\mathcal{B}))/I_{A_{K}})_x$$

$$\cong (\mathfrak{h}(\mathcal{K}) \oplus \mathfrak{h}(\mathcal{B}))/I_{A_{K}})_x$$

$$\cong \mathfrak{h}(\mathcal{K})_x/\mathfrak{h}(\mathcal{K})_x \cap (I_{A_{K}})_x$$

where the first isomorphism follows from proposition 3.3.2. So for $C(\mathcal{K})_x$ to have non-torsion elements, we must have $\mathfrak{h}(\mathcal{K})_x \cap (I_{A_{K}})_x = \emptyset$. This implies that the image of $(I_{A_{K}})_x$ in $(\mathfrak{I}_{A_{K}} \otimes \mathcal{L})_x$ is not of full rank. Thus, $(I_{A_{K}})_x$ must have nilpotent elements. This implies that $\mathfrak{I}(V)$ has nilpotents at the image of $x$. The argument reverses for the opposite implication. □

### 3.5 Cohomology Pairings

#### 3.5.1 Modules

In this section we let $R$ be a commutative ring with identity. In practice, $R$ will be either $K$ or $O_{K}$. We let $\Sigma = GL_2(Q) \cap M_2(Z)$, and let $\Sigma^* = \Sigma \cup \{e\}$ where $e = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

We define some contravariant $R[\Sigma^*]$-modules. Let $L_n(R)$ denote $Sym^n(R^2)$, the $R$-module of homogeneous polynomials of degree $n$ in variables $X$ and $Y$. The action of $\Sigma^*$ on $L_n(R)$ is given by

$$(F|g)(X,Y) = F((X,Y)g^*)$$

for $F \in L_n(R)$ and $g \in \Sigma^*$ where $\ast$ is the map $g \mapsto g^* = det(g)g^{-1}$. For a character $\chi : (Z/Np^*Z)^* \to R^*$, we define $L_n(\chi,R)$ to be the contravariant $R[\Sigma^*]$-module with
the same underlying space as $L_n(R)$ but with a different action. We denote the action of $\Sigma^*$ on $L_n(\chi, R)$ by $F|_xg$ for $F \in L_n(\chi, R)$ and $g \in \Sigma^*$ and define it by $F|_xg = \chi(d)F|g$ where $d$ is the lower right hand entry of $g$. We drop the subscript $\chi$ notation on the action when it clear from context.

We let $A$ denote a contravariant $R[\Sigma^*]$-module. We will be looking at situations where $A$ is either $L_n(R)$ or $L_n(\chi, R)$. We let $\Gamma = \Gamma_1(Np)$ if $A = L_n(R)$ and let $\Gamma = \Gamma_0(Np^\infty)$ if $A = L_n(\chi, R)$ with $\chi : (\mathbb{Z}/Np^\infty\mathbb{Z})^\times \to R^\times$. We will also consider when $A$ is a space of overconvergent modular symbols $\mathcal{D}^1_Y$ over an affinoid disk $Y$ in weight space or $\mathcal{D}^1_Y \otimes_{A(Y)} \mathcal{L}$ where $\mathcal{L}$ is the fraction field of $A(Y)$. In this situation, we let $\Gamma = \Gamma_1(Np)$.

### 3.5.2 Cohomology

We will look at three different cohomology groups. Let $H^1_1(A) = H^1(\Gamma, A)$. We note that this space is canonically isomorphic to $H^1(\Gamma \backslash \mathbb{H}, \tilde{A})$ where $\mathbb{H}$ is the complex upper half plane and $\tilde{A}$ is the locally constant sheaf associated to $A$. Let $H^1_c(A)$ be the compactly supported cohomology group on the noncompactified modular curve $\Gamma \backslash \mathbb{H}$ with coefficients in $\tilde{A}$. We note that there is a canonical isomorphism

$$H^1_c(A) \cong Symb_\Gamma(A)$$  \hspace{1cm} (3.10)

Let $H^1_p(A) = H^1_p(\Gamma, A)$ denote the parabolic group cohomology group defined by the following exact sequence:

$$0 \to H^1_p(\Gamma, A) \to H^1(\Gamma, A) \to \prod_{s \in S} H^1(\Gamma_s, A) \to 0$$  \hspace{1cm} (3.11)

where $S$ is the set of equivalence classes of cusps, and $\Gamma_s$ is the stabilizer of a cusp $s$ in $\Gamma$. We note that $H^1_p(A)$ is canonically isomorphic to the image of $H^1_c(A)$ in $H^1(A)$.

When the module $A$ has no $\Gamma$-invariants, it is known that the following sequence is exact:

$$0 \to Bound_\Gamma(A) \to Symb_\Gamma(A) \to H^1_p(\Gamma, A) \to 0$$  \hspace{1cm} (3.12)

**Theorem 3.5.1** When $A = L_n(K)$ or $\mathcal{D}^1_Y \otimes_{A(Y)} \mathcal{L}$ where $\mathcal{L}$ is the fraction field of $A(Y)$, the above sequence splits.

**Proof.** See [7].
3.5.3 Cup Product on Specialized Cohomology

There is a natural pairing \( \langle \cdot, \cdot \rangle : A \times \hat{A} \to R \) where \( \hat{A} \) denotes the \( R \)-dual of \( A \). Via cup product, this induces a pairing on cohomology

\[
\langle \cdot, \cdot \rangle : H^1(A) \otimes_R H^1_*(\hat{A}) \to R
\]

(3.13)

We let a superscript \( fr \) in the notation for a cohomology group denote the maximal free quotient. With the natural map \( H^1_p,fr(A) \to H^1,fr(A) \) and the injective map \( H^1_0,fr(A) \to H^1,fr(A) \) induced by the splitting given in Theorem 3.5.1, one has an induced pairing

\[
\langle \cdot, \cdot \rangle : H^1_p,fr(A) \otimes_R H^1_p,fr(\hat{A}) \to R
\]

(3.14)

The symmetric power modules have the \( R[\Sigma^*] \)-equivariant map

\[
\theta_n : L_n(R) \to \hat{L}_n(R)
\]

(3.15)

given explicitly as follows: The spaces \( L_n(R) \) and \( \hat{L}_n(R) \) can be viewed as \((n + 1)\)-fold products of \( R \) where the duality is given by the usual dot product. Let \( \{e_i\} \) denote the standard basis for the product view of \( L_n(R) \) and likewise let \( \{f_j\} \) denote the standard basis for \( \hat{L}_n(R) \). Then \( \theta_n \) is given by

\[
\theta_n(e_i) = (-1)^{n-i} \binom{n}{i} f_{n-i}.
\]

(3.16)

One can likewise define a \( \theta_n \) map for the modules \( L_n(\chi, R) \). These \( \theta_n \) maps induces maps on the cohomology groups (here the subscript * denotes the various cohomology groups)

\[
\theta_n : H^1,fr(L_n(R)) \to H^1,fr(\hat{L}_n(R))
\]

(3.17)

\[
\theta_n : H^1,fr(L_n(\chi, R)) \to H^1,fr(\hat{L}_n(\chi, R)).
\]

(3.18)

This allows one to define pairings

\[
\langle \cdot, \cdot \rangle_n : H^1_p,fr(L_n(R)) \otimes_{\mathcal{O}_K} H^1_p,fr(L_n(R)) \to R
\]

(3.19)

\[
\langle \cdot, \cdot \rangle_{\chi,n} : H^1_p,fr(L_n(\chi, R)) \otimes_{\mathcal{O}_K} H^1_p,fr(L_n(\chi, R)) \to R.
\]

(3.20)

The action of the matrix \( W_{np} = \begin{bmatrix} 0 & 1 \\ -np & 0 \end{bmatrix} \) gives maps

\[
\cdot |_{W_{np}} : H^1_p(L_n(R)) \to H^1_p(L_n(R)).
\]

(3.21)
\[ \cdot |_{W_{N_p}} : H^1_p(L_n(\chi, R)) \to H^1_p(L_n(\overline{\chi}, R)). \]  

(3.22)

We define the pairing

\[ \langle \cdot, \cdot \rangle_n : H^1_{p,\text{fr}}(L_n(R)) \otimes R H^1_{p,\text{fr}}(L_n(R)) \to R \]  

(3.23)

by \( \langle x, y \rangle_n = [x, y|_{W_{N_p}}]_n \), and define

\[ \langle \cdot, \cdot \rangle_{X,n} : H^1_{p,\text{fr}}(L_n(\chi, R)) \otimes R H^1_{p,\text{fr}}(L_n(\overline{\chi}, R)) \to R \]  

(3.24)

by \( \langle x, y \rangle_{X,n} = [x, y|_{W_{N_p}}]_{X,n} \)

### 3.5.4 Cup Product on Big Cohomology

Let \( D^\perp_{Y,\leq \alpha} \) be the subspace of \( D^\perp_Y \) spanned by the values of modular symbols of slope \( \leq \alpha^* \). We note that this is in fact the space of measures that are \( h \)-admissible for all \( h \geq \alpha^* \).

**Proposition 3.5.2** There exists a \( \Gamma_1(Np) \)-invariant pairing

\[ \langle \cdot, \cdot \rangle_Y : D^\perp_{Y,\leq \alpha} \otimes_{A(Y)} D^\perp_{Y,\leq \alpha} \to A(Y) \]  

(3.25)

such that the following diagram commutes for all integers \( k \geq \alpha + 1 \) with \( (\psi, k) \in Y \):

\[ \begin{array}{ccc}
\langle \cdot, \cdot \rangle_Y : D^\perp_{Y,\leq \alpha} \otimes_{A(Y)} D^\perp_{Y,\leq \alpha} & \to & A(Y) \\
\downarrow & & \downarrow \\
\langle \cdot, \cdot \rangle_n : L_n(K) \otimes L_n(K) & \to & K
\end{array} \]  

(3.26)

**Proof.** We write the elements of \( D^\perp_Y \) as its sequence of moments. To evaluate

\[ ((\alpha_0, \alpha_1, \alpha_2, \ldots), (\beta_0, \beta_1, \beta_2, \ldots))_Y, \]  

(3.27)

we evaluate the usual dot product

\[ (\alpha_0, \alpha_1, \alpha_2, \ldots) \cdot (\beta_0, \beta_1, \beta_2, \ldots) = \sum_{j=0}^{\infty} \alpha_j \beta_j \]  

(3.28)

where

\[ \alpha_i(k) = \lim_{r \to \infty} \langle Np \rangle^k \binom{k}{i} \cdot c_{i+p^r} \]  

(3.29)
where \( (\tau^t_i) \) is given by the polynomial \( \frac{k(k-1) \cdots (k-i+1)}{t^i} \). The limit converges by the analyticity given by Theorem 2.8.1 (a). It is straightforward to verify that for classical weights the big pairing restricts to the specialized pairing. One sees that the dot product in the definition of the big pairing is well-defined because the binomial coefficients in the definition of the \( \alpha \)'s give enough \( p \)'s to make the dot product sum converge. 

\[ \boxed{\text{Theorem 3.5.3} \quad \text{There exists a pairing} \; \langle \cdot, \cdot \rangle_Y \; \text{such that the following diagram commutes for all classical weights} \; k \in Y.} \]

\[
\begin{array}{ccc}
\langle \cdot, \cdot \rangle_Y : H^1_p(D_Y)^{\leq \alpha} \otimes_{A(Y)} H^1_p(D_Y)^{\leq \alpha} & \longrightarrow & A(Y) \\
\downarrow & & \downarrow \\
\langle \cdot, \cdot \rangle_n : H^1_p(L_n(K))^{\leq \alpha} \otimes_K H^1_p(L_n(K))^{\leq \alpha} & \longrightarrow & K
\end{array}
\]

\[ (3.30) \]

\[ \text{Proof.} \; \text{This follows directly from the proposition.} \]

3.6 Symmetric Square \( p \)-adic \( L \)-function

Let \( D(V) \subset C_{p,N} \) be a local flat piece of the eigencurve arising from a \( V \in C \) such that \( D(V) = D(Y, \alpha^*) \). Let \( \lambda : \mathbb{T}(V) \otimes \mathcal{L} \to \mathcal{K} \). Again suppose we are in the situation where \( \lambda \) gives rise to the direct sum decomposition \( \mathbb{T}(V) \otimes \mathcal{L} \cong \mathcal{K} \oplus B \). Let \( A_K \) be the integral closure of \( A^0(Y) \) in \( K \). We suppose the component determined by \( \lambda \) is smooth. We call this component \( D(K) \subset D(V) \). Suppose \( Y \) is small enough so that the sheaf of overconvergent modular symbols \( \mathcal{E} \) is free of rank 2 on this component.

We can decompose \( \mathcal{E}(D(K)) \) into its \( \pm \)-components under the action of \( \iota \), \( \mathcal{E}(D(K))[\pm] \). We suppose \( K \) is a cuspidal component so \( \mathcal{E}(D(K)) \) is isomorphic to the \( \lambda \)-eigenspace of the maximal free quotient of the parabolic cohomology group with coefficients in \( D_Y^1 \otimes_{A(Y)} A_K \).

Let \( D_Y^1(A^0(Y)) \) denote the elements of \( D_Y^1 \) of norm \( \leq 1 \). Let \( D_Y^1(A_K) \) denote the elements of \( D_Y^1 \) of norm \( \leq 1 \). Let \( D_Y^1(A_K) \) denote the elements of \( D_Y^1 \) of norm \( \leq 1 \). Let \( M^{\lambda, \pm} \) denote the projection of \( H^1_p(D_Y^1(A_K))^{\leq \alpha} \) onto \( H^1_p(D_Y^1(K))^{\leq \alpha} \), and let \( M^{\lambda, \pm} \) denote the intersection of \( H^1_p(D_Y^1(A_K))^{\leq \alpha} \) with \( H^1_p(D_Y^1(K))^{\leq \alpha} \). Let \( M(K) = H^1_p(D_Y^1(K))[\lambda] \) and \( M(A_K) = H^1_p(D_Y^1(A_K))[\lambda] \). Let \( M^*(A_K) \subset M(K) \) be the lattice dual to \( M(A_K) \) via the pairing on
the big cohomology given by Theorem 3.5.3. Let \( \Phi_{\lambda, \pm} \in M_{\lambda, \pm} \) be generators. Let \( \Phi_{\lambda, \pm}^* \) be the element dual to \( \Phi_{\lambda, \pm} \). So we have \( \langle \Phi_{\lambda, \pm}, \Phi_{\lambda, \pm}^* \rangle = 1 \). Let \( \tilde{\Phi}_{\lambda, \pm} \) be the projection of \( \Phi_{\lambda, \pm}^* \) onto \( M(K)[\mathbb{T}] \).

**Definition 3.6.1** We define the symmetric square \( p \)-adic \( L \)-function on the component of the eigencurve determined by \( \lambda \) by

\[
L_\lambda = \langle \Phi_{\lambda, +}, \Phi_{\lambda, -} \rangle \in A_K
\]

(3.31)

We let \( n = k - 2 \). \( L_\lambda \) determines a function on the component of \( D(V) \) determined by \( \lambda \). By tensoring with \( K \) over the specialization to weight \( k \) map, we get a map \( \lambda_k : \mathbb{T}(V) \otimes_k K \rightarrow A(Y) \otimes_k K \). We let \( H^1_p(L_n(K))\langle \lambda_k \rangle \) denote the \( \lambda_k \)-eigenspace of the cohomology group. Let \( H^1_p(L_n(K))\langle \lambda_k, \pm \rangle \) denote the \( \pm \)-eigenspaces of the \( \lambda_k \)-eigenspace. Let \( M^{\lambda_k, \pm} \) denote the projection of \( H^1_p(L_n(K))\langle \lambda_k \rangle \leq \alpha \) onto \( H^1_p(L_n(K))\langle \lambda_k, \pm \rangle \), and let \( M_{\lambda_k, \pm} \) denote the intersection of \( H^1_p(L_n(O_K))\leq \alpha \) with \( H^1_p(L_n(K))\langle \lambda_k, \pm \rangle \). Let \( \xi_{\lambda_k, \pm} \) be generators of \( M_{\lambda_k, \pm} \).

**Proposition 3.6.2** The map on cohomology induced by \( \rho_k \)

\[
\rho_k : H^1_p(D^1(A_K))\leq \alpha \rightarrow H^1_p(L_n(O))\leq \alpha
\]

(3.32)

maps the generators \( \Phi_{\lambda_k, \pm} \) to the generators \( \xi_{\lambda_k, \pm} \) for \( k \geq \alpha + 1 \) with \( (\psi, k) \in Y \).

**Proof.** Let \( x = \rho_k(\Phi_{\lambda_k, \pm}) \in H^1_p(L_n(O_K))\leq \alpha \). We need to show that from some \( d \in D_0 \), one of the coefficients of \( x(d) \) is in \( O^X_K \). \( \Phi_{\lambda_k, \pm} \) has a \( d \in D_0 \) such that \( \Phi_{\lambda_k, \pm}(d) \) has norm 1. Let \( i \) be the moment where this norm 1 is realized. As long as \( k > i \), the generators map to generators. For other weights, the maximum amount we are off by is bounded independent of the weight. \( \square \)

Thus we have \( L_\lambda(\psi, k) = \langle \xi_{\lambda_k, +}, \xi_{\lambda_k, -} \rangle \in O_K \).

**Definition 3.6.3** We define the Gorenstein error by

\[
\epsilon_\lambda = \langle \Phi_{\lambda, +}, \Phi_{\lambda, -} \rangle.
\]

(3.33)

We remark that \( \epsilon_\lambda \) is a measure of how much the Hecke ring fails to be Gorenstein at the weight \( k \). In the case where the Hecke ring is Gorenstein, \( \epsilon_\lambda \) is a unit.
Theorem 3.6.4 The following equality holds when $\#C_k(\lambda_k)$ is large enough and $k \geq \alpha + 1$ with $(\psi, k) \in Y$:

$$|c^\lambda(\psi, k)|_p = |\#C_k(\lambda_k)|_p \cdot |\epsilon(\psi, k)|_p$$ \hspace{1cm} (3.34)

Proof. We have $\mathcal{L}_\lambda = \Phi_{\lambda,-}/\Phi_{\lambda,+}$. Also, $\epsilon(\psi) = \Phi_{\lambda,-}/\Phi_{\lambda,+}$. By the duality of Hecke rings and overconvergent cusp forms over $Y$, we have that the Hecke ring is isomorphic to the $\epsilon$-eigenspace of the cohomology group. So the order of the congruence module is given by $\#C_k(\lambda_k) = \Phi_{\lambda,-}/\Phi_{\lambda,+}$. For $k$ larger than the max of the weights determining that generators go to generators for all the symbols involved, the formula holds.

First we note that $|\#C_k(\lambda_k)|_p$ is well defined by finiteness proposition. We use duality of Hecke and forms. $\square$

3.7 Gorenstein Error

We recall the situation. We are considering a flat local piece of the eigencurve $D(V)$ given by an admissible set $V \in \mathcal{C}$ with weight space $Y$. $D(V)$ parameterizes all forms over $Y$ of slope $\leq t$. We are considering a component of $D(V)$ given by $\lambda : \mathcal{V}(V) \otimes \mathcal{L} \rightarrow \mathcal{K}$. The error defined in the previous section is a function attached to the family $\lambda$ on the integers $k$ occurring in $Y$ to give a classical specialization for the space for forms. This error is denoted $\epsilon(\lambda, k)$. The following technical theorem plays a major role in the proof of our main theorem:

Theorem 3.7.1 $\epsilon(\lambda) \chi \in Y$ is bounded as a function of the slope and is independent of the weight $k$.

We remark that the work in this section is a retracing of steps taken by Hida in [9] but keeping track of the error that arises.

There are two main steps in proving this theorem. The first is to show that for each $k$, $\epsilon(\lambda, \xi, k)$ is bounded as a function of $k$ as $\xi$ varies over finite order characters. For the second step we let $k_0$ denote the smallest classical weight occurring in $Y$. To show that the bound is independent of the weight, we approximate $\epsilon(\psi, k)$ by $\epsilon(\xi, k_0)$ with varying $\xi$. 49
In this section we work with cohomology for the group $\Gamma = \Gamma_1(Np)$ and module $L_n(\mathcal{O}_K)$. We remark that all the work done is also valid for $\Gamma_0(Np')$ and $L_n(\chi, \mathcal{O}_K)$ for a primitive character $\chi$ of conductor $Np'$.

### 3.7.1 Cohomology Groups

Here $R$ denote any commutative ring with identity. Let $X = \Gamma_1(Np) \setminus \mathcal{H}$, the uncompactified modular curve. We also suppose that $Np \geq 4$ to ensure that $\Gamma = \Gamma_1(Np)$ has no elements of finite order besides $\pm 1$. We recall the notation for the cohomology groups we will discuss. Let $H^q(X(\mathbb{C}), \widetilde{L_n(R)})$ denote the usual cohomology groups and $H^q_c(X(\mathbb{C}), \widetilde{L_n(R)})$ denote the cohomology groups with compact support. We have a canonical isomorphism $H^q(X(\mathbb{C}), \widetilde{L_n(R)}) \cong H^q(\Gamma, L_n(R))$ with the group cohomology groups, and we identify the two kinds of cohomology groups. We also let $H^q(X_{et}, \widetilde{L_n(R)})$ and $H^q_c(X_{et}, \widetilde{L_n(R)})$ denote the corresponding étale cohomology groups. The parabolic cohomology groups $H^q_p(X(\mathbb{C}), \widetilde{L_n(R)})$ and $H^q_p(X_{et}, \widetilde{L_n(R)})$ are defined to be the natural image of the $q$-th compactly supported cohomology groups in the usual $q$-th cohomology groups. We have a canonical isomorphism $H^q_p(X(\mathbb{C}), \widetilde{L_n(R)}) \cong H^q_p(\Gamma, L_n(R))$ where the parabolic cohomology for the group cohomology is defined by as the kernal of the restriction to the stabilizers map. We note that since the cohomological dimension of $X$ is less than 2, we have that

$$H^q_c(X, \widetilde{L_n(R)}) = 0, \quad H^q(X, \widetilde{L_n(R)}) = 0 \quad \text{for all } q \geq 3.$$  \hspace{1cm} (3.35)

In the situation when $R$ is a finite ring, we have the following isomorphisms between the étale cohomology groups and sheaf cohomology groups:

$$H^q(X_{et}, \widetilde{L_n(R)}) \cong H^q(X(\mathbb{C}), \widetilde{L_n(R)})$$  \hspace{1cm} (3.36)

$$H^q_c(X_{et}, \widetilde{L_n(R)}) \cong H^q_c(X(\mathbb{C}), \widetilde{L_n(R)})$$  \hspace{1cm} (3.37)

$$H^q_p(X_{et}, \widetilde{L_n(R)}) \cong H^q_p(X(\mathbb{C}), \widetilde{L_n(R)})$$  \hspace{1cm} (3.38)

Let $\mathcal{O}_K$ denote the integer ring of a $p$-adic field $K$, and let $\Lambda_p^m = \mathcal{O}_K/p^m\mathcal{O}_K$. We show the following useful lemma:

50

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**Lemma 3.7.2** We have the following exact sequence:

\[ 0 \to H^q(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^m} \to H^q(\Gamma, L_n(\Lambda_{p^m})) \to H^{q+1}(\Gamma, L_n(\mathcal{O}_K))_{p^m} \to 0 \]  

(3.39)

where \( H^{q+1}(X, L_n(\mathcal{O}_K))_{p^m} \) denotes the \( p^m \)-torsion of \( H^{q+1}(X, L_n(\mathcal{O}_K)) \). The sequence is also valid for the compactly supported cohomology and also for the dual module \( \hat{L}_n(\mathcal{O}_K) \).

**Proof.** From the short exact sequence

\[ 0 \to L_n(\mathcal{O}_K) \xrightarrow{p^m} L_n(\mathcal{O}_K) \to L_n(\Lambda_{p^m}) \to 0 \]  

(3.40)

we get the usual long exact sequence from cohomology which when we tensor with \( \Lambda_{p^m} \), we get our short exact sequence. The proof also goes through when we replace with compactly supported cohomology and/or the dual module \( \hat{L}_n(\mathcal{O}_K) \). \( \square \)

We now show two lemmas regarding the structure of the cohomology groups when tensored with \( \Lambda_{p^m} \).

**Lemma 3.7.3** We have the following isomorphisms:

\[ H^1(\Gamma, L_n(\Lambda_{p^m})) \cong H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \Lambda_{p^m} \]  

(3.41)

\[ H^1(\Gamma, \hat{L}_n(\Lambda_{p^m})) \cong H^1(\Gamma, \hat{L}_n(\mathcal{O}_K)) \otimes_{\mathcal{O}_K} \Lambda_{p^m} \]  

(3.42)

**Proof.** Using Lemma 3.7.2 when \( q = 2 \) and the fact that \( H^3 \) is 0, we have

\[ H^2(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^m} \cong H^2(\Gamma, L_n(\Lambda_{p^m})) \]  

(3.43)

\[ H^2(\Gamma, \hat{L}_n(\mathcal{O}_K)) \otimes \Lambda_{p^m} \cong H^2(\Gamma, \hat{L}_n(\Lambda_{p^m})) \]  

(3.44)

Because a global section of \( L_n(\Lambda_{p^m}) \) or \( \hat{L}_n(\Lambda_{p^m}) \) is constant, we have

\[ H^0(\Gamma, L_n(\Lambda_{p^m})) \cong H^0(\Gamma, \hat{L}_n(\Lambda_{p^m})) = 0 \]  

(3.45)

\[ H^0(\Gamma, \hat{L}_n(\Lambda_{p^m})) \cong H^0(\Gamma, \hat{L}_n(\Lambda_{p^m})) = 0 \]  

(3.46)

This with Lemma 3.7.2 shows that \( H^1(\Gamma, L_n(\mathcal{O}_K)) \) and \( H^1(\Gamma, \hat{L}_n(\mathcal{O}_K)) \) are both free over \( \mathcal{O}_K \). By Poincaré Duality for étale cohomology, the isomorphisms of cohomology stated at the beginning of this subsection, (3.45), and (3.46), we have that

\[ H^2(\Gamma, L_n(\Lambda_{p^m})) = 0 \text{ and } H^2(\Gamma, \hat{L}_n(\Lambda_{p^m})) = 0. \]  

(3.47)

Then using Lemma 3.7.2 with \( q = 1 \), (3.43), (3.44), and (3.47), we have our lemma. \( \square \)
Lemma 3.7.4

\[ 0 \to H^1_c(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^m} \to H^1_c(\Gamma, L_n(\Lambda_{p^m})) \quad (3.48) \]

\[ 0 \to H^1_c(\Gamma, \hat{L}_n(\mathcal{O}_K)) \otimes \Lambda_{p^m} \to H^1_c(\Gamma, \hat{L}_n(\Lambda_{p^m})) \quad (3.49) \]

where the cokernels are contained in the \( p^m \)-torsion subgroups of \( H^2_c(\Gamma, L_n(\mathcal{O}_K)) \) and \( H^2_c(\Gamma, \hat{L}_n(\mathcal{O}_K)) \) respectively.

Proof. This lemma follows directly from Lemma 3.7.2.

We explain a little in order to give some understanding as to why the maps in this lemma may not be surjective. The action of \( \Gamma \) on \( L_n(\Lambda_{p^m}) \) factors through \( \Gamma / \Gamma_{p^m} \) where

\[ \Gamma_{p^m} = \{ \gamma \in \Gamma | \gamma \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod p^m \}. \quad (3.50) \]

Depending on the structure of \( \Gamma / \Gamma_{p^m} \), \( L_n(\Lambda_{p^m}) \) may or may not be a simple \( \Gamma \)-module. If \( L_n(\Lambda_{p^m}) \) is not simple, the submodule of \( \Gamma \)-invariants may be nontrivial. Or in other words, \( H^0(\Gamma, L_n(\mathcal{O}_K)) \) may be nontrivial. By the same reasoning \( H^0(\Gamma, \hat{L}_n(\mathcal{O}_K)) \) may be nontrivial. By Lemma 3.7.2, the \( p \)-torsion of \( H^1(\Gamma, L_n(\mathcal{O}_K)) \) comes from quotients of \( H^0(\Gamma, L_n(\Lambda_{p^m})) \) for various \( m \). By Poincaré Duality for étale cohomology and the isomorphisms of cohomology stated at the beginning of this subsection, we have that \( H^1_c(\Gamma, L_n(\Lambda_{p^m})) \) and \( H^2_c(\Gamma, \hat{L}_n(\Lambda_{p^m})) \) have the same sizes as \( H^0(\Gamma, L_n(\Lambda_{p^m})) \) and \( H^0(\Gamma, \hat{L}_n(\Lambda_{p^m})) \) respectively.

3.7.2 Perfectness of the Cup Product

Let \( R \) be a commutative ring with identity. We recall that we have the natural duality pairing

\[ L_n(R) \otimes_R \hat{L}_n(R) \to R \quad (3.51) \]

This pairing gives rise to the cup product pairing

\[ H^1(X, L_n(R)) \otimes_R H^1_c(X, \hat{L}_n(R)) \to \to H^2_c(X, L_n(R) \otimes \hat{L}_n(R)) \to \to H^2_c(X, \hat{R}) \quad (3.52) \]

Using the canonical orientation of the complex manifold \( X(\mathbb{C}) \), we can specify an isomorphism \( H^2_c(X, \mathbb{Z}) \cong \mathbb{Z} \) which induces an isomorphism \( H^2_c(X, R) \cong R \) since \( H^2_c(X, R) \cong \mathbb{Z} \).
$H^2_c(X, \mathcal{L}) \otimes \mathbb{Z} R$. Thus by composing maps and identifying with the group cohomology groups we get the pairing:

$$B_R : H^1(\Gamma, L_n(R)) \otimes_R H^1_c(\Gamma, \hat{L}_n(R)) \to R \quad (3.53)$$

Our goal now is to show that the pairing $B_{\mathcal{O}_K}$ is close to being perfect. We first use the following lemma for finite quotients of $\mathcal{O}_K$:

**Lemma 3.7.5** $B_{\Lambda_{p^m}}$ is perfect. Namely, the following maps are isomorphisms:

$$H^1(\Gamma, L_n(\Lambda_{p^m})) \to Hom_{\Lambda_{p^m}}(H^1_c(\Gamma, \hat{L}_n(\Lambda_{p^m})), \Lambda_{p^m}) \quad (3.54)$$

$$H^1_c(\Gamma, \hat{L}_n(\Lambda_{p^m})) \to Hom_{\Lambda_{p^m}}(H^1(\Gamma, L_n(\Lambda_{p^m})), \Lambda_{p^m}) \quad (3.55)$$

**Proof.** This lemma follows from Poincaré duality for étale cohomology and the isomorphisms of the different cohomology groups stated at the beginning of the section.

\[ \square \]

We let $H^1, free(\Gamma, L_n(\mathcal{O}_K))$ denote the maximal free quotient of $H^1(\Gamma, L_n(\mathcal{O}_K))$. We use analogous notation of the superscript $free$ for maximal free quotients of other cohomology groups.

**Proposition 3.7.6** The pairing $B_{\mathcal{O}_K}$ induces a pairing

$$B^*_{\mathcal{O}_K} : H^1,free(\Gamma, L_n(\mathcal{O}_K)) \otimes H^1_c(\Gamma, \hat{L}_n(\mathcal{O}_K)) \to \mathcal{O}_K \quad (3.56)$$

This induced pairing $B^*_{\mathcal{O}_K}$ is perfect.

**Proof.** We will first show that

$$H^1,free(\Gamma, L_n(\mathcal{O}_K)) \to Hom_{\mathcal{O}_K}(H^1_c(\Gamma, \hat{L}_n(\mathcal{O}_K)), \mathcal{O}_K) \quad (3.57)$$

is an isomorphism. Let $m^*$ denote the maximum exponent of the $p$-torsion of $H^0(\Gamma, L_n(\mathcal{O}_K))$ and $H^0(\Gamma, \hat{L}_n(\mathcal{O}_K))$. We see from Lemma 3.7.2 that if $m_1 \geq m_2 > m^*$, then we have the following diagram:

$$
\begin{array}{ccc}
H^1,free(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^{m_1}} & \longrightarrow & H^1,free(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^{m_2}} \\
\cong \downarrow & & \cong \downarrow \\
H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^{m_1}} & \longrightarrow & H^1(\Gamma, L_n(\mathcal{O}_K)) \otimes \Lambda_{p^{m_2}}
\end{array} \quad (3.58)
$$

53
where the horizontal maps are surjective and the vertical maps are isomorphisms. From this we see that

\[ H^1,\text{free}(\Gamma, L_n(O_K)) \cong \lim_m H^1(\Gamma, L_n(O_K)) \otimes \Lambda_{p^m}. \]  
(3.59)

By (3.41) we have

\[ H^1,\text{free}(\Gamma, L_n(O_K)) \cong \lim_m H^1(\Gamma, L_n(\Lambda_{p^m})). \]  
(3.60)

Since the \( p \)-torsion of \( H^2_c(\Gamma, \tilde{L}_n(O_K)) \) has exponent bounded by \( m^* \), by (3.49), when \( m > m^* \), we have that

\[ H^1_c(\Gamma, \tilde{L}_n(\Lambda_{p^m})) \cong H^1_c(\Gamma, \tilde{L}_n(O_K)) \otimes \Lambda_{p^m} \]  
(3.61)

So for \( m > m^* \), using Poincare duality and (3.61), we have

\[ \lim_m H^1(\Gamma, L_n(\Lambda_{p^m})) \cong \lim_{m > m^*} \text{Hom}_{\Lambda_{p^m}}(H^1_c(\Gamma, \tilde{L}_n(\Lambda_{p^m})), \Lambda_{p^m}) \]  
(3.62)

\[ \cong \lim_{m > m^*} \text{Hom}_{\Lambda_{p^m}}(H^1_c(\Gamma, \tilde{L}_n(O_K)) \otimes \Lambda_{p^m}, \Lambda_{p^m}) \]  
(3.63)

\[ \cong \lim_{m > m^*} \text{Hom}_{O_K}(H^1_c(\Gamma, L_n(O_K)), O_K) \otimes \Lambda_{p^m} \]  
(3.64)

Thus we have

\[ H^1,\text{free}(\Gamma, L_n(O_K)) \cong \text{Hom}_{O_K}(H^1_c(\Gamma, \tilde{L}_n(O_K)), O_K). \]  
(3.65)

The isomorphism

\[ H^1_c(\Gamma, \tilde{L}_n(O_K)) \to \text{Hom}_{O_K}(H^1,\text{free}(\Gamma, L_n(O_K)), O_K) \]  
(3.66)

is proven similarly. \( \square \)

### 3.7.3 Error from the Parabolic Map

We recall that the following sequence defines the first parabolic cohomology group:

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1_p(\Gamma, M) \longrightarrow H^1(\Gamma, M) \xrightarrow{\text{res}} \prod_{s \in S} H^1(\Gamma_s, M) \\
\end{array}
\]  
(3.67)
where $M$ is a $\Gamma$-module, $S$ is the set of equivalence classes of cusps, and $\Gamma_s$ is the stabilizer of a cusp $s$ in $\Gamma$. It is well known that for a generator $\pi_s$ of $\Gamma_s$, we have

$$H^1(\Gamma_s, M) \cong M/(\pi_s - 1)M$$

(3.68)

Let $f_s(M)$ denote the least common multiple of the orders of the torsion elements of $H^1(\Gamma_s, M)$. Let $f(M)$ denote the least common multiple of the $f_s(M)$'s as $s$ ranges over the equivalence classes of the cusps. We also let

$$G^1(\Gamma, M) = \prod_{s \in S} H^1(\Gamma_s, M)$$

(3.69)

We see that the torsion of $H^1(\Gamma, M)/H^1_p(\Gamma, M)$ injects into the torsion of $G^1(\Gamma, M)$. If we let $T(M)$ denote the torsion of $H^1(\Gamma, M)$. Then we see that the following map is injective:

$$H^1,\text{free}(\Gamma, M)/H^1_p,\text{free}(\Gamma, M) \to G^1(\Gamma, M)/\text{res}(T(M)).$$

(3.70)

The pairing

$$B^{\ast}_{O_K} : H^1,\text{free}(\Gamma, L_n(O_K)) \otimes H^1_c(\Gamma, \hat{L}_n(O_K)) \to O_K$$

(3.71)

from the last subsection induces a pairing

$$A_{O_K} : H^1,\text{free}(\Gamma, L_n(O_K)) \otimes H^1_p,\text{free}(\Gamma, \hat{L}_n(O_K)) \to O_K.$$  

(3.72)

We show the following proposition:

**Proposition 3.7.7** The pairing $A_{O_K}$ is not perfect in that we have the following exact sequences:

$$0 \to H_p^1,\text{free}(\Gamma, L_n(O_K)) \to \text{Hom}_O(H_p^1,\text{free}(\Gamma, \hat{L}_n(O_K)), O_K) \to C_1 \to 0$$

(3.73)

$$0 \to H_p^1,\text{free}(\Gamma, \hat{L}_n(O_K)) \to \text{Hom}_O(H_p^1,\text{free}(\Gamma, L_n(O_K)), O_K) \to 0$$

(3.74)

where $C_1$ is torsion and annihilated by multiplication by $f(L_n(O_K))$.

**Proof.** We have the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & H^1,\text{free}(\Gamma, L_n(O_K)) \\
\phi \downarrow & & \text{Hom}_O(H^1_p(\Gamma, \hat{L}_n(O_K)), O_K) \\
H^1,\text{free}(\Gamma, L_n(O_K)) & \cong & \text{Hom}_O(H^1_c(\Gamma, \hat{L}_n(O_K)), O_K)
\end{array}
$$

(3.75)
The left vertical map is an injection. The right vertical map is in fact an isomorphism. It follows that ϕ is injective and the cokernel $C_1$ is isomorphic to a subgroup of $G^1(\Gamma, M)/\text{res}(T(M))$. Thus $C_1$ is annihilated by $f(L_n(\mathcal{O}_K))$.

For the other sequence, we have the following commutative diagram:

$$
egin{array}{ccc}
H^1_{\text{free}}(\Gamma, \hat{L}_n(\mathcal{O}_K)) & \xrightarrow{\psi} & \text{Hom}_{\mathcal{O}_K}(H^1_{\text{free}}(\Gamma, L_n(\mathcal{O}_K)), \mathcal{O}_K) \\
\cong & & \downarrow \\
H^1_c(\Gamma, \hat{L}_n(\mathcal{O}_K)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{O}_K}(H^1(\Gamma, L_n(\mathcal{O}_K)), \mathcal{O}_K)
\end{array}
$$

(3.76)

The left vertical map is an isomorphism. The right vertical map is an injection since we have the splitting that gives a surjective map from $H^1$ to $H^1_{\text{free}}$. It follows that $\psi$ is in fact an isomorphism.

\[\square\]

**Proposition 3.7.8** We have the following bound:

$$f(L_n(\mathcal{O}_K)) \leq Np \cdot \text{lcm}\{i\}_{i=1}^n$$

(3.77)

**Proof.** The group $\Gamma_s$ is generated by an element conjugate to an element of the form

$$\pi = \pm \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

(3.78)

where $m$ is a divisor of the level $N$. The matrix of the action of $\pi$ on $L_n(\mathbb{Z})$ is conjugate to a matrix of the form

$$
\begin{bmatrix}
1 & \binom{n}{1}m & \binom{n}{2}m^2 & \ldots & m^n \\
0 & 1 & \binom{n-1}{1}m & \ldots & m^{n-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & m \\
0 & \ldots & \ldots & \ldots & 1
\end{bmatrix}
$$

(3.79)

One sees by induction that $Np \cdot \text{lcm}\{i\}_{i=1}^n$ times each standard basis vector of $L_n(\mathcal{O}_K)$ is contained in $(\pi - 1)L_n(\mathcal{O}_K)$.

\[\square\]
3.7.4 Error from the Theta Map

Recall the map

$$\theta_n : L_n(O_K) \to \hat{L}_n(O_K).$$  \hspace{1cm} (3.80)

This induces a map on cohomology which we denote again by $\theta_n$:

$$\theta_n : H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \to H_p^{1,\text{free}}(\Gamma, \hat{L}_n(O_K))$$  \hspace{1cm} (3.81)

Now we define a pairing $A_{O_K}^*$ on $H_p^{1,\text{free}}(\Gamma, L_n(O_K))$ by the following commutative diagram:

$$A_{O_K}^* : H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \otimes H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \longrightarrow O_K$$

$$\downarrow \quad \downarrow$$

$$A_O : H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \otimes H_p^{1,\text{free}}(\Gamma, \hat{L}_n(O_K)) \longrightarrow O_K$$  \hspace{1cm} (3.82)

The following theorem gives us a bound on how far this pairing on the parabolic cohomology fails to be perfect.

**Theorem 3.7.9** The pairing $A_{O_K}^*$ is not perfect in that we have the following exact sequences:

$$0 \to C_2 \to H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \xrightarrow{z_{\Gamma}(x,:)} \text{Hom}_{O_K}(H_p^{1}(\Gamma, L_n(O_K)), O_K) \to C_1 \to 0$$  \hspace{1cm} (3.83)

and

$$0 \to H_p^{1,\text{free}}(\Gamma, L_n(O_K)) \xrightarrow{z_{\Gamma}(x,:)} \text{Hom}_{O_K}(H_p^{1}(\Gamma, L_n(O_K)), O_K) \to C_3 \longrightarrow 0$$  \hspace{1cm} (3.84)

where $C_2$ and $C_3$ are annihilated by $lcm\{(\frac{n}{i})\}_{i=1}^n$, and $C_1$ is annihilated by $Np \cdot lcm\{i\}_{i=1}^n$.

**Proof.** Let $C$ denote the cokernel of the $\theta_n$ map:

$$0 \longrightarrow L_n(O_K) \xrightarrow{\theta_n} \hat{L}_n(O_K) \longrightarrow C \longrightarrow 0$$  \hspace{1cm} (3.85)

The $\theta_n$ map on cohomology satisfies the following sequence:

$$H_c^0(\Gamma, C) \longrightarrow H_c^1(\Gamma, L_n(O_K)) \xrightarrow{\theta_n} H_c^1(\Gamma, \hat{L}_n(O_K)) \longrightarrow H_c^1(\Gamma, C)$$  \hspace{1cm} (3.86)

We have that $H_c^0(\Gamma, C) = 0$ because $H_c^1(\Gamma, L_n(O_K))$ is free. This sequence implies that the induced $\theta_n$ map on parabolic cohomology has cokernel that is annihilated by each
annihilator of $C$. We note here that $lcm\{\binom{n}{i}\}_{i=1}^{n}$ annihilates $C$. We have the following commutative diagrams:

$$
\begin{array}{c}
0 \\
\downarrow \\
C_4 \\
\downarrow \\
0 \\
\end{array}
\quad \xrightarrow{\varpi_l(z, x)} 
\begin{array}{c}
0 \\
\longrightarrow H^1_p(\Gamma, L_n(\mathcal{O}_K)) \\
\downarrow \theta_n \\
\longrightarrow \text{Hom}_\mathcal{O}(H^1_p(\Gamma, \hat{L}_n(\mathcal{O}_K)), \mathcal{O}) \\
\downarrow \\
0
\end{array} 
\quad C_1 \to 0 \quad (3.87)

$$

$$
\begin{array}{c}
0 \\
\uparrow \\
C_5 \\
\uparrow \\
0 \\
\end{array}
\quad \xrightarrow{\varpi_l(z, x)} 
\begin{array}{c}
0 \\
\longrightarrow H^1_{free}(X, \hat{F}_n(\mathcal{O})) \\
\uparrow \theta_n \\
\longrightarrow \text{Hom}_\mathcal{O}(H^1_{free}(X, F_n(\mathcal{O})), \mathcal{O}) \\
\uparrow \\
0
\end{array} 
\quad H^1_{free}(X, F_n(\mathcal{O})) \to 0 \quad (3.88)

From the first diagram, we see that $C_2$ inject into $C_4$. From the second diagram, we see that $C_3$ injects into $C_5$.

3.7.5 Independent of weight

**Theorem 3.7.10** For fixed $k$, $\epsilon_\lambda(\xi, k)$ is bounded as a function of $k$ as $\xi$ varies.

**Proof.** The bounds in the previous subsection are independent of whether we use $L_n(\mathcal{O}_K)$ or $L_n(\xi, \mathcal{O}_K)$. This gives the theorem.

58
Theorem 3.7.11 $\epsilon_\lambda(\chi)$ is bounded away from 0 and $\infty$ as a function of the slope $\alpha$ and is independent of the weight character $\chi$. Further, $\epsilon_\lambda$ has no zeros or poles.

Proof. Let $k_0$ be the smallest integer weight $\geq \alpha + 1$ with $(\psi, k_0) \in Y$. $(\psi, k)$ can be approximated with $(\xi, k_0)$ with varying $\xi$. Thus $\epsilon_\lambda(\psi, k)$ can be approximated with $\epsilon_\lambda(\xi, k_0)$ with varying $\xi$. The $\epsilon_\lambda(\xi, k_0)$ are bounded away from 0 and $\infty$ as a function of $\alpha$ independent of $k$. Thus we have the theorem.

3.8 Ramification Points: Part II

In this section we state and show the main result which relates the zeros of the $p$-adic $L$-function $L$ defined on the eigencurve $C_{p,N}$ and the ramification points of the weight projection map $\pi : C_{p,N} \to \mathcal{W}_N$.

Theorem 3.8.1 $L$ does not vanish on the étale points of the eigencurve.

Proof. At an étale point, the congruence module is of finite size and by the corollary $L$ is not 0.

We state a proposition about analytic continuation from [5].

Proposition 3.8.2 Let $S$ be a compact subset of $\mathbb{P}^1$, contained in an affinoid set $X$. Define $\Omega(G) = \Omega = X \setminus S$ by the family $G$ of the affinoid subsets of $X$, not meeting $S$. A meromorphic function $f$ on $\Omega$ has a continuation to a meromorphic function on $X$ is the set $f(X \setminus S)$ is not dense in $K \cup \{\infty\}$.

Theorem 3.8.3 $L$ extends to a locally analytic function on the eigencurve and is zero on the ramification points.

Proof. This follows from the proposition. $L$ takes values in $\mathcal{O}_K$ so it is not dense in $K$.

Theorem 3.8.4 The zeros of $L$ are precisely the ramification points on the eigencurve.

Proof. This follows from the corollary and the theorem in part 1.
3.9 Special Values

3.9.1 Symmetric square $L$-function

Let $f \in S_k(\Gamma_1(Np),K)^{\leq \alpha}$ be a classical normalized eigenform. Suppose also that $f$ is primitive. Let $\psi : (\mathbb{Z}/Np\mathbb{Z})^* \to \mathcal{O}^*$ be the character of $f$. Suppose this character is primitive also. We can write the $q$-expansion of $f$

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \quad \text{where} \quad q = e^{2\pi i z} \quad (3.89)$$

For each prime $l$, let $\alpha_l$ and $\beta_l$ denote the roots of $1 - \alpha_l X + \psi(l)l^{k-1}X^2$. We define the twisted symmetric square $L$-function for $f$ and a primitive character $\omega$:

$$L(s,f,\omega) = \prod_l \left[ 1 - \omega(l) \alpha_l^{-s} l^{-s} \right]^{-1} \left[ 1 - \omega(l) \beta_l^{-s} l^{-s} \right]^{-1} \left[ 1 - \omega(l) \beta_l^{-2} l^{-s} \right]^{-1} \quad (3.90)$$

We define the symmetric square $L$-function of $f$ to be

$$L(s,f) = L(s,f,\bar{\psi}) \quad (3.91)$$

We note by Shimura, we know that this $L$-function is holomorphic at $s = k$. Recall the Petersson inner product on $S_k(\Gamma)$ for a congruence subgroup $\Gamma$:

$$(f,g)_\Gamma = \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} y^{k-2} dx dy \quad (3.92)$$

**Proposition 3.9.1**

$$L(k,f) = \frac{N^2 p^{2k} \delta_{k+1}}{(k-1)!} (f,f)_{\Gamma_1(Np)} \quad (3.93)$$

**Proof.** To prove this proposition, we first define the Rankin product Dirichlet series. Let $g \in S_m(\Gamma_0(M),\omega)$ be a normalized eigenform. Let $g = \sum_{n=1}^{\infty} b_n q^n$ denote the $q$-expansion of $g$, and let $\alpha'_l$ and $\beta'_l$ be the analogous roots of $1 - b_l X + \omega(l)m^{-1}X^2$ for $g$.

We define a Rankin product Dirichlet series of $f$ and $g$ by

$$D(s,f,g) = \sum_{n=1}^{\infty} a_n b_n n^{-s} \quad (3.94)$$

This series has the following Euler product:

$$D(s,f,g) = \prod_l \left[ 1 - \alpha_l \beta_l \alpha'_l \beta'_l l^{-2s} \right]^{-1} \left[ 1 - \alpha_l \alpha'_l l^{-s} \right]^{-1} \left[ 1 - \beta_l \beta'_l l^{-s} \right]^{-1} \left[ 1 - \beta_l \beta'_l l^{-s} \right]^{-1} \quad (3.95)$$
For this proof we take
\[ g(z) = \sum_{n=1}^{\infty} \overline{\psi}(n) a_n q^n \in S_k(N^2 p^2) \quad (3.96) \]
we have
\[ \zeta_{NP}(2s - 2k + 2)D(s, f, g) = \zeta_{NP}(s - k + 1)L(s, f) \quad (3.97) \]
where \( \zeta_{NP} \) is the Riemann zeta function with the Euler factors for primes dividing \( NP \) excluded. Now we look at the residues of the poles at \( s = k \) on both sides of the equation to get
\[ \zeta_N(2) Res_{s=k} D(s, f, g) = L(k, f) Res_{s=k} \zeta_N(s) \quad (3.98) \]
We use that facts \( \zeta(2) = \pi^2/6 \) and \( Res_{s=k} \zeta(s) = 1 \) to get
\[ \frac{\pi^2}{6} \left( \prod_{l | NP} (1 - l^{-2}) \right) Res_{s=k} D(s, f, g) = \left( \prod_{l | NP} (1 - l^{-1}) \right) L(k, f) \quad (3.99) \]
For \( n \in \mathbb{Z} \) where \( (n, NP) = 1 \), it is know that \( \overline{a_n} = \overline{\psi}(n) a_n \), thus giving us \( D_{NP}(s, f, f_p) = D_{NP}(s, f, g) \) where \( f_p(z) = \sum_{n=1}^{\infty} \overline{a_n} q^n \) and the subscript \( NP \) means that we exclude the Euler factors for primes dividing \( NP \). Further, we have
\[ D(s, f, f_p) = \left( \prod_{l | NP} (1 - l^{-1-s})^{-1} \right) D(s, f, g) \quad (3.100) \]
By work of Petersson, it is known that
\[ Res_{s=k} D(s, f, f_p) = (k - 1)!^{-1} \left( \frac{\pi}{6} N^2 p^2 \right) \prod_{l | NP} (1 - l^{-2})^{-1} (4n)^k (f, f)_{\Gamma_1(N)} \quad (3.101) \]
Thus putting together (3.99), (3.100), and (3.101), we have the proposition. \( \square \)

3.9.2 Eichler-Shimura isomorphism

Let \( f \in S_k(\Gamma_1(NP), K)^{\leq \alpha} \) be a classical normalized eigenform. We have the Eichler-Shimura isomorphism:
\[ \delta : S_k(\Gamma_1(NP)) \oplus \overline{S}_k(\Gamma_1(NP)) \xrightarrow{\cong} H^1_p(\Gamma_1(NP), L_n(\mathbb{C})) \quad (3.102) \]
where \( k = n + 2 \). Let \( H_p^1(\Gamma_1(N), L_n(\mathcal{O}))[f, \pm] \) be the \( \pm \) eigenspace of the eigenspace of \( f \). Let \( \xi_{f, \pm} \) be a generator of this space. We define periods \( \Omega_{f,+} \) and \( \Omega_{f,-} \) by the following formulas

\[
\delta(f) + \overline{\delta(f)} = \Omega_{f,+} \cdot \xi_{f,+} \tag{3.103}
\]

\[
\delta(f) - \overline{\delta(f)} = \Omega_{f,-} \cdot \xi_{f,-} \tag{3.104}
\]

We relate the pairing of the generators to the Petersson inner product of \( f \) with itself with the following proposition:

**Proposition 3.9.2**

\[
\langle \xi_{f,+}, \xi_{f,-} \rangle = \frac{(f, f)_{\Gamma_1(Np)} 2^k k^{k+1} W(f)}{\Omega_{f,+} \Omega_{f,-}} \tag{3.105}
\]

**Proof.**

\[
\Omega_{f,+} \cdot \Omega_{f,-} \langle \xi_{f,+}, \xi_{f,-} \rangle = \langle \delta(f) + \overline{\delta(f)}, \delta(f) - \overline{\delta(f)} \rangle \tag{3.106}
\]

\[
= 4i \int_{\Gamma_1(Np)} [\delta(f)|_{W_{n,p}}, \delta(f)] \, dx \wedge dy \tag{3.107}
\]

\[
= 2^k k^{k+1} W(f) \int_{\Gamma_1(Np)} |f|^2 y^{-k+2} \, dx \, dy \tag{3.108}
\]

\[
= 2^k k^{k+1} W(f)(f, f)_{\Gamma_1(Np)} \tag{3.109}
\]

\[
\square
\]

### 3.9.3 Interpolation of special values

The following theorem gives the interpolation property of the \( p \)-adic \( L \)-function \( \mathcal{L}(X) \).

**Theorem 3.9.3**

\[
\mathcal{L}(k) = \frac{i^k k^{k+1} W(f)(k - 1)!}{\Omega_{f,+} \cdot \Omega_{f,-} 2^k \pi^{k+1} N^2 p^2 L(k, f)} \tag{3.110}
\]

**Proof.** This follows by putting together Proposition and Theorem. \( \square \)
Chapter 4

Two Variable Symmetric Square $p$-adic $L$-function

4.1 Introduction

In this chapter we construct the two variable symmetric square $p$-adic $L$-function on the eigencurve. Let $D(V) \subset C_{p,N}$ be a local flat piece of the eigencurve arising from a $V \in \mathcal{C}$ such that $D(V) = D(Y, \alpha^*)$ for an affinoid disk $Y \subset \mathcal{W}_N$. We make the additional supposition that $D(V)$ is smooth over $Y$. Recall that $\mathcal{T}(V)$ is the Hecke ring associated to $D(V)$, and $\mathcal{T}^0(V)$ is its subring of elements with norm $\leq 1$. Recall that $A(Y)$ denotes the affinoid ring attached to $Y$, and $A^0(Y)$ denotes its subring of elements with norm $\leq 1$. Let $\mathcal{L}$ denote the quotient field of $A^0(Y)$, and let $\mathcal{K}$ be a finite extension of $\mathcal{L}$. Let $\mathcal{I}$ be the integral closure of $A^0(Y)$ in $\mathcal{K}$. An overconvergent family of modular forms in $D(V)$ given by a map

$$\lambda : \mathcal{T}^0(V) \to \mathcal{I}$$

(4.1)

We let $D(V_\lambda)$ denote the component of $D(V)$ determined by $\lambda$.

We will deal with three different tame levels which are denoted $N$, $J$, and $L$, all prime to $p$. $N$ is the tame level of the eigencurve that our family of modular forms are a part of. $J$ is the tame level for certain $\theta$-series that we will use which gives rise to the “other” variable in the two variable $p$-adic $L$-function that we will construct. $L$ will
be the least common multiple of $N$ and $J$ and will be used for the Eisenstein family of half integral weight. The two variable symmetric square $p$-adic $L$-function on this family $\lambda$ is defined as an element $\mathcal{L} \in \text{Meas}(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) \hat{\otimes} \mathcal{O}_K \mathcal{I}$. For a height 1 prime ideal $P$ of $\mathcal{I}$, let $\mathcal{L}_P$ denote the image of $\mathcal{L}$ under the map

$$\text{Meas}(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) \hat{\otimes} \mathcal{O}_K \mathcal{I} \rightarrow \text{Meas}(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) \hat{\otimes} \mathcal{O}_K \mathcal{I}/P \cong \text{Meas}(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) \hat{\otimes} \mathcal{O}_K \mathcal{O}' \hspace{1cm} (4.2)$$

for some extension $\mathcal{O}'$ of $\mathcal{O}_K$. Given a prime ideal $P = P_f$ of $\mathcal{I}$ determined by a classical new eigenform $f \in S_k(\Gamma_0(Np), \psi, K)$, a finite order character $\chi: \mathbb{Z}_{p,J}^X \rightarrow \mathcal{O}_K^\times$, and an integer $n$ such that $1 \leq n < k$, $\mathcal{L}$ satisfies and is uniquely determined by the interpolation property

$$\int_{\mathbb{Z}_{p,J}^X} \chi(z) z_n^p d\mathcal{L}_{P_f} \sim \frac{L(n, f, \psi \chi, \omega^n)}{\Omega_f^+ \Omega_f^-} \hspace{1cm} (4.3)$$

where $\sim$ is equality up to certain explicitly given factors and constants, $\Omega_f^\pm$ are periods associated to $f$, and $L(s, f, \chi)$ is the complex symmetric square $L$-function associated to $f$ and twisted by the character $\chi$. These $p$-adic $L$-functions defined locally on the eigencurve glue together to give the two variable symmetric square $p$-adic $L$-function $\mathcal{L}(w, s): \mathcal{W}_f \times C_{p,N}^{n,s} \rightarrow \mathbb{C}_p$ where $C_{p,N}^{n,s}$ is the nonsingular locus of $C_{p,N}^{0,s}$ and $\mathcal{L}$ is analytic in $w$ and locally analytic in $s$.

To construct $\mathcal{L}$, we extend the work of Hida in [8] following his work very closely. The construction of the local two variable symmetric square $p$-adic $L$-functions involves first constructing a measure that convolutes a half integral weight Eisenstein measure with a theta measure and takes values in the space of $p$-adic modular forms. This measure was constructed by Hida but only used to construct the above described $L$-function on families of ordinary $p$-adic modular forms. Hida completed his construction by using his ordinary projector to project the convoluted measure to the ordinary family. This projection coincides with a certain integral representation of the classical complex Rankin product $L$-function at the arithmetic points which in turn can be related to the classical complex symmetric square $L$-function. To extend the construction to the non-ordinary case, we develop a new projector that generalizes Hida's ordinary projector in this situation.
4.2 Modular Forms of Half Integral Weight

We begin by recalling definitions and properties of modular forms of half integral weight. This theory was originally developed by Shimura [17], and is foundational material for the construction of the $\theta$-measure and the Eisenstein measure of half integral weight.

4.2.1 Classical Modular Forms of Half Integral Weight

The first example of a modular form of half integral weight is the theta series

$$\Theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{where } q = e^{2\pi i z} \quad (4.4)$$

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(4)$. Define

$$j(\gamma, z) = \Theta(\gamma(z))/\Theta(z) \quad (4.5)$$

Take the congruence subgroup $\Gamma_1(N)$ with $4|N$. Here $p$ may divide $N$. For each positive odd integer $k$ we define the space $\mathcal{G}_{k/2}(\Gamma_1(N), \mathbb{C})$ of modular forms of weight $k/2$ to be all holomorphic functions on the complex upper half plane satisfying

$$f|_{k/2}\gamma(z) = f(\gamma(z))j(\gamma, z)^{-1}(cz + d)^{-k/2} = f(z) \quad (4.6)$$

for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N)$ and are holomorphic at all the cusps of $\Gamma_1(N)$. We define the space of cusp forms $\mathcal{P}_{k/2}(\Gamma_1(N), \mathbb{C})$ to be the subspace of $\mathcal{G}_{k/2}(\Gamma_1(N), \mathbb{C})$ of forms that vanish at the cusps.

4.2.2 $p$-adic Modular Forms of Half Integral Weight

For a subring $A \subset \mathbb{C}$, we define

$$\mathcal{G}_{k/2}(\Gamma_1(N), A) = \mathcal{G}_{k/2}(\Gamma_1(N), \mathbb{C}) \cap A[[q]] \quad (4.7)$$

$$\mathcal{P}_{k/2}(\Gamma_1(N), A) = \mathcal{P}_{k/2}(\Gamma_1(N), \mathbb{C}) \cap A[[q]] \quad (4.8)$$

65
For a subring $A \subset \overline{Q}_p$, we define

\[ G_{k/2}(\Gamma_1(N), A) = G_{k/2}(\Gamma_1(N), Q) \otimes A \]  
(4.9)

\[ P_{k/2}(\Gamma_1(N), A) = P_{k/2}(\Gamma_1(N), Q) \otimes A \]  
(4.10)

Let $K$ be a finite extension of $Q_p$, and let $O_K$ denote its ring of integers. Now we make the assumption that $(N, p) = 1$. For each integer $r \geq 1$, define

\[ G(\Gamma_1(Np^r), K) = \bigoplus_{m=0}^{\infty} G_{m+1/2}(\Gamma_1(Np^r), K) \]  
(4.11)

\[ G(\Gamma_1(Np^r), O_K) = G(\Gamma_1(Np^r), K) \cap O_K[[q]] \]  
(4.12)

These spaces are analogous to the spaces of divided congruences for integral weight modular forms which arise in defining spaces of $p$-adic modular forms as in the exposition by Gouvêa [6]. We let $\overline{G}(\Gamma_1(Np^r), K)$ and $\overline{G}(\Gamma_1(Np^r), O_K)$ be the completions of $G(\Gamma_1(Np^r), K)$ and $G(\Gamma_1(Np^r), O_K)$ with respect to the $p$-adic topology on $q$-expansions.

We also have analogous spaces of cusp forms which we denote similarly but with a $P$ instead of a $G$. The following theorem is by Hida.

**Theorem 4.2.1 (Hida)** The spaces $G(\Gamma_1(Np^r), K)$, $G(\Gamma_1(Np^r), O_K)$, $P(\Gamma_1(Np^r), K)$, and $P(\Gamma_1(Np^r), O_K)$ are independent of $r \geq 1$ and thus we denote them $G(N, K)$, $G(N, O_K)$, $P(N, K)$, and $P(N, O_K)$ respectively. We call elements of these spaces $p$-adic modular forms of half integral weight.

**Proof.** See [8] Theorem 2.1. \[\square\]

### 4.2.3 Properties of Modular Forms of Integral Weight

**The action of $Z^X_{p,N}$.** Let $f \in M_k(\Gamma_1(Np^r), \overline{Q}_p)$, and let $Z^X_{p,N} = \lim_{r \to \infty} (Z/NZ)^X$. We define the action of $Z^X_{p,N} = Z^X_p \times (Z/NZ)^X$ on $M_k(\Gamma_1(Np^r), \overline{Q}_p)$ as follows: Let $(z, a) \in Z^X_p \times (Z/NZ)^X$. We define the action by

\[ f|z = z^k f|k\sigma \]  
(4.13)

where $\sigma \in \Gamma_0(Np^r)$ with $\sigma \equiv \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \mod p^r$ and $\sigma \equiv \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \mod N$. 

66
4.2.4 Properties of Modular Forms of Half Integral Weight

For \( f \in \mathcal{G}(N, A) \) or \( \mathcal{M}(N, A) \) where \( A \) is a subring of \( \mathbb{C} \) or \( \mathbb{C}_p \), we let \( \sum_{n=0}^{\infty} a(n, f)q^n \) denote the \( q \)-expansion of \( f \). Some of the actions below will be defined in terms of their \( q \)-expansions.

**The action of** \( \mathbb{Z}^x_{p,N} \). Let \( f \in \mathcal{G}_{k/2}(\Gamma_1(N), \overline{q}_p) \), and \((z, a) \in \mathbb{Z}^x_{p,N} \times (\mathbb{Z}/NZ)^x \). We define two actions of \( \mathcal{G}_N^x = \mathbb{Z}^x_{p,N} \times (\mathbb{Z}/NZ)^x \) on \( \mathcal{G}_{k/2}(\Gamma_1(Np^r), \overline{q}_p) \) given by

\[
f||z, a) = \chi(a)z^{(k+1)/2}f|_{k/2}\sigma
\]
\[
f|z = \chi(a)z^{(k-1)/2}f|_{k/2}\sigma = \chi(a)z^{-1}f||z
\]

where \( \sigma \in \Gamma_0(4) \) with \( \sigma \equiv \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \mod p^r \), \( \sigma \equiv \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \mod N \), and \( \chi : (\mathbb{Z}/NZ)^x \to \{\pm 1\} \) is the Legendre symbol for \( \mathbb{Q}(\sqrt{-1})/\mathbb{Q} \).

**Multiplication.** For and integer \( k \) and odd integer \( l \), there is a product

\[
\mathcal{M}_k(\Gamma_1(Np^r), K) \times \mathcal{G}_{l/2}(\Gamma_1(Np^r), K) \to \mathcal{G}_{k+l/2}(\Gamma_1(Np^r), K)
\]

(4.16) given by multiplication of \( q \) expansions in \( K[[q]] \). This multiplication satisfies

\[
(fg)|z = (f|z)(g||z)
\]

(4.17)

**The \([t]\) map.** For \( 0 < t \in \mathbb{Z} \), we define

\[
[t] : \mathcal{G}(N, K) \to \mathcal{G}(Nt, K)
\]

(4.18)

by

\[
f|[t] = \sum_{n=1}^{\infty} a(n, f)q^{nt}
\]

(4.19)

More specifically, \([t]\) induces

\[
[t] : \mathcal{G}_{k/2}(\Gamma_0(Np^r), \xi, K) \to \mathcal{G}_{k/2}(\Gamma_0(Ntp^r), \xi\chi_t, K)
\]

(4.20)

where \( \chi_t \) is the Dirichlet character corresponding to the extension \( \mathbb{Q}(\sqrt{t})/\mathbb{Q} \).

**The \( \tau \) involution.** We define the involution \( \tau = \tau(Np^r) \) on \( \mathcal{G}_{l/2}(\Gamma_1(Np^r), \mathbb{C}) \) and \( \mathcal{M}_k(\Gamma_1(Np^r), \mathbb{C}) \) by

\[
f|\tau = \begin{cases} f(-1/Np^r z)(Np^r)^{-l/4}(-iz)^{l/2}, & f \in \mathcal{G}_{l/2} \\ f(-1/Np^r z)(Np^r)^{-k/2}(-iz)^k, & f \in \mathcal{M}_k \end{cases}
\]

(4.21)

67
We have that
\[ r^2 = \begin{cases} 
1 & \text{on } \mathcal{G}_{l/2} \\
(-1)^k & \text{on } \mathcal{M}_k
\end{cases} \quad (4.22) \]

**The twisting operator.** For a function \( \phi : (\mathbb{Z}/Mp'\mathbb{Z}) \rightarrow \mathcal{O}_K^\times \), there is a twisting operator
\[ \phi : \overline{G}(N, K) \rightarrow \overline{G}(NM^2, K) \quad (4.23) \]
given by
\[ a(n, f|\phi) = \phi(n)a(n, f) \quad (4.24) \]
for all \( n \geq 0 \).

**The differential operator.** The differential operator
\[ d : \overline{G}(N, K) \rightarrow \overline{G}(N, K) \quad (4.25) \]
is given by
\[ a(n, df) = na(n, f) \quad (4.26) \]
for all \( n \geq 0 \).

### 4.3 Theta Measure

In this section, we construct the \( \theta \)-measure that will be used in constructing the \( p \)-adic \( L \)-function. We construct the \( \theta \)-measure in more generality than we need, but state interpolation property of the specific \( \theta \)-measure that we need at the end of the section.

Let \( V \) be a vector space over \( \mathbb{Q} \). Let \( l \) denote the dimension of \( V \) over \( \mathbb{Q} \), and let \( n : V \rightarrow \mathbb{Q} \) be a positive definite quadratic form on \( V \). Let \( S : V \times V \rightarrow \mathbb{Q} \) be the symmetric bilinear form defined by
\[ S(u, v) = n(u + v) - n(u) - n(v). \quad (4.27) \]

Let \( I \) be a lattice of \( V \) satisfying \( n(I) \subset \mathbb{Z} \), and define
\[ I^* = \{ v \in V | S(v, I) \subset \mathbb{Z} \}. \quad (4.28) \]
Note that we have $I \subset I^*$. Let $M$ be the smallest positive integer such that $Mn(I^*) \subset \mathbb{Z}$, and let $\Delta = [I^* : I]$.

**Definition 4.3.1** \(\eta : V \to \mathbb{C}\) is a spherical function on $V$ of degree $\alpha$ if $\eta$ is a homogeneous function of degree $\alpha$ for $\alpha = 0$ or 1, or there exist finitely many $w \in V \otimes_{\mathbb{Q}} \mathbb{C}$ with $n(w) = 0$ such that $\eta$ can be written as

\[
\eta(v) = \sum_w c(w)S(w, v)^\alpha
\]  

(4.29)

where $c(w) \in \mathbb{C}$ and $\alpha \geq 2$.

In this thesis, we will only deal with spherical functions of degree 1 or 0 so one can ignore the more complicated case of the definition. But one can produce some generalizations of the results in this thesis by considering spherical functions of degree $\geq 2$.

For any function $h : I^* \to \mathbb{C}$, one can formally define the series

\[
\theta(h)(z) = \sum_{v \in I^*} h(v)q^{n(v)}
\]  

(4.30)

where $q = e^{2\pi iz}$. This series converges and defines a holomorphic function on the complex upper-half plane if $h$ can be written in the form $h = \Phi\eta$ for a complex-valued function, $\Phi$, on $I^*/I$ and a spherical function $\eta$.

Now we will define some spaces on which our measures will be defined. Let $\mathcal{W} = \{v \in I^* | n(v) \in \mathbb{Z}\}$. We define

\[
W = \lim_{\nu} \mathcal{W}/p^\nu I
\]  

(4.31)

We also define

\[
X = \lim_{\nu} I^*/p^\nu I
\]  

(4.32)

We note that $W \subset X$.

We now define the $\theta$-series. Let $\eta : V \to \overline{\mathbb{Q}}$ be a spherical function of degree $\alpha \geq 0$. We note that there is a natural action of $\mathbb{Z}_{p,M}^\times$ on $X$ that leaves $W$ stable. Let $\phi : W \to \overline{\mathbb{Q}}$ be a locally constant function for which there exists a finite order character $\chi$ of $\mathbb{Z}_{p,M}^\times$ such that

\[
\phi(zw) = \chi(z)\phi(w) \quad \text{for } z \in \mathbb{Z}_{p,M}^\times, \quad w \in W
\]  

(4.33)
We define
\[ \theta(\phi\eta)(z) = \sum_{w \in \mathcal{W}} \phi(w)\eta(w)q^{n(w)}, \quad q = e^{2\pi iz} \]  \hspace{1cm} (4.34)

Let
\[ \chi^r(a) = \chi(a) \left( \frac{-1}{a} \right)^{1/2} \left( \frac{\Delta}{a} \right) \]  \hspace{1cm} (4.35)

Let \( \beta \) be such that the conductor of \( \chi^r \) divides \( M_p^\beta \). Then
\[ \theta(\phi\eta) \in \begin{cases} \mathcal{M}_{l/2+\alpha}(\Gamma_0(Mp^\beta), \chi^r) & \text{for even } \ell \\ \mathcal{G}_{l/2+\alpha}(\Gamma_0(Mp^\beta), \chi^r) & \text{for odd } \ell \end{cases} \]  \hspace{1cm} (4.36)

This family gives rise to the \( \theta \)-measure once we extend to \( W \) by continuity: We define the \( \theta \)-measure
\[ \theta : C(W^X, \mathcal{O}_K) \to \begin{cases} \overline{\mathcal{M}}(M, \mathcal{O}_K) & \text{for even } \ell \\ \overline{\mathcal{G}}(M, \mathcal{O}_K) & \text{for odd } \ell \end{cases} \]  \hspace{1cm} (4.37)

In this thesis, we will be concerned with the situation when \( V = \mathbb{Q}, \ n(x) = x^2, \) and \( I = J\mathbb{Z} \) for a positive integer \( J \). In this case, \( M = 4J^2, \ \Delta = 2J^2, \ W = \mathbb{Z}^X_{p,J}, \) and \( \ell = 1 \).

We summarize the interpolation property of the \( \theta \)-measure that we will be using with the following proposition:

**Proposition 4.3.2** There exists a unique measure
\[ \theta : C(Z^X_{p,J}, \mathcal{O}_K) \to \overline{\mathcal{G}}(4J^2, \mathcal{O}_K) \]  \hspace{1cm} (4.38)

satisfying the following interpolation property: If \( \phi : Z^X_{p,J} \to \overline{Q} \) is a locally constant function, and \( \eta : V \to \overline{Q} \) is a spherical function of degree \( \alpha \), then
\[ \theta(\phi\eta)(z) = \sum_{w \in \mathcal{W}} \phi(w)\eta(w)q^{w^2} \]  \hspace{1cm} (4.39)

### 4.4 Eisenstein Measure of Half Integral Weight

In this section we will construct the Eisenstein measure of half integral weight. Before constructing the measure, we will define the classical Eisenstein series of half integral weight.
Let $k > 0$ be an odd integer, and let $\xi : (\mathbb{Z}/Lp^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be a character with $\xi(-1) = (-1)^{(k-1)/2}$. Let $j(\gamma, z) = \Theta(\gamma(z))/\Theta(z)$ be the automorphic factor for forms of half integral weight. We make the following definition:

**Definition 4.4.1** The Eisenstein series of half integral weight is given by

$$E_{k/2}^\bullet(z, s, \xi) = L_{Lp}(2s + k - 1, \xi^2) \sum_{\gamma \in \Gamma_0 \setminus \Gamma_0(Lp^r)} \xi \chi_{Lp^r}^{(k-1)/2}(\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-2s}$$

Also define

$$E_{k/2}(z, m, \xi) = (2\pi)^{m-k} (L_{Lp})^{k-2m/2} \Gamma\left(\frac{k-m}{2}\right) (2y)^{-m} E_{k/2}^\bullet(z, -m, \xi) \mid_{k/2} \tau(L_{Lp})$$

and

$$E_{k/2}(\xi) = E_{k/2}(z, 2 - k, \xi) \in G_{k/2}(\Gamma_0(L_{Lp}), \xi, \bar{\mathbb{Q}}) \tag{4.40}$$

We have that $E_{k/2}(\xi) \in G_{k/2}(\Gamma_0(L_{Lp}), \xi, \bar{\mathbb{Q}})$, and we refer to $E_{k/2}(\xi)$ as the Eisenstein series of weight $k/2$ with character $\xi$. The $q$-expansion of $E_{k/2}(\xi)$ is given by

**Proposition 4.4.2**

$$E_{k/2}(\xi) = L_{Lp}(2 - k, \xi^2) + \sum_{n=1}^{\infty} \left( L_{Lp}(\frac{3-k}{2}, \xi \chi_n) \cdot \sum_{\substack{u^2t^2 | n \\ \ u^2t^2 | Lp \\ u > 0}} \mu(u) \xi(u^2) \chi_n(u) t(u^2) \frac{k-3}{2} \right) q^n$$

where $\mu$ is the Möbius function and $\chi_n(m) = \left( \frac{n}{m} \right)$ is the Legendre symbol.

**Proof.** See [8] equation (3.2b). \(\Box\)

Now we will state the interpolation property of the Eisenstein measure of half integral weight and recall the proof of its existence by sketching the construction of the measure.

**Proposition 4.4.3** For each $b$ relatively prime to $Lp$, there exists a unique measure

$$E_b : C(\mathbb{Z}_{p,L}, O) \to \overline{\mathcal{C}}(L, O) \tag{4.41}$$
satisfying the following interpolation property: If \( \xi : (\mathbb{Z}/L\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times \) is a character, then

\[
\int_{\mathbb{Z}_L} \xi(z)z_p^{(k-3)/2}dE_b = \begin{cases} 
(1 - \xi_L(b)b^{(k-1)/2})E_{k/2}(\xi)|_{t_p} & \text{if } \xi(-1) = (-1)^{(k-1)/2} \\
0 & \text{if } \xi(-1) = (-1)^{(k+1)/2}
\end{cases} (4.42)
\]

where \( \xi_L \) is the restriction of \( \xi \) to \((\mathbb{Z}/L\mathbb{Z})^\times \).

**Proof.** First recall the measures \( \zeta^b : C(\mathbb{Z}_{p,L}^\times, \mathcal{O}_K) \to \mathcal{O}_K \) defined for each \( b > 1 \) with \( (b, Lp) = 1 \) and character \( \alpha : (\mathbb{Z}/Cp^\times)^\times \to \mathcal{O}_K \). They are defined by the interpolation property

\[
\int_{\mathbb{Z}_L} \xi(z)z_p^{n-1}d\zeta^b = (1 - \xi_L\alpha_C(b)b^n)L_{Lp}(1 - n, \xi\alpha) \tag{4.43}
\]

for a finite order character \( \xi : \mathbb{Z}_{p,L}^\times \to \mathcal{O}_K \). We define the Eisenstein measure with the following equation

\[
\int_{\mathbb{Z}_{p,L}^\times} \phi(z)dE_b = \sum_{\substack{n=1 \\ (n,p)=1}}^\infty \left( \sum_{\substack{u^2t^n \equiv 1 \\ (ut,Lp)=1 \\ u>0 \\ t>0}} \mu(u)\chi_n(u)t \int_{\mathbb{Z}_L} (\phi|ut^2)d\zeta^{b_n} \right)q^n \tag{4.44}
\]

where \( \phi \in C(\mathbb{Z}_{p,L}^\times, \mathcal{O}_K) \). This gives the interpolation property that \( E_b \) satisfies in that statement of the proposition. \( \square \)

### 4.5 Shimura Differential Operators

We state the definitions of the differential operators on \( \mathbf{H} \) studied by Shimura in [18]:

\[
\delta_s = \frac{1}{2\pi i} \left( \frac{s}{2\pi y} + \frac{\partial}{\partial z} \right), \tag{4.45}
\]

\[
d = \frac{1}{2\pi i} \frac{\partial}{\partial z} = q \frac{d}{dq} \quad (q = e^{2\pi iz}, \ z = x + iy), \tag{4.46}
\]

\[
\delta_s^r = \delta_{s+2r-2} \ldots \delta_{s+2} \delta_s \quad \text{for } 0 \leq r \in \mathbb{Z} \tag{4.47}
\]

where we take \( \delta_s^0 = 1 \). These operators satisfy the following equations:

\[
\delta_{s+t}(fg) = g\delta_t(f) + f\delta_t(g) \tag{4.48}
\]

\[
\delta^r_s(f|k\gamma) = (\delta^r_s f)|_{k+2r\gamma} \tag{4.49}
\]
for $\gamma \in GL_2^+(\mathbb{R})$. The relation between $\delta$ and $d$ is given by

$$\delta^r = \sum_{0 \leq t \leq r} \binom{r}{t} \frac{\Gamma(s + r)}{\Gamma(s + t)} (-4\pi y)^{t-r} d^t \quad (4.50)$$

We state a few lemmas of Shimura from [18].

Lemma 4.5.1 Let $g \in G_{l/2}(\Gamma_1(N), \mathbb{C})$ and $h \in G_{m/2}(\Gamma_1(N), \mathbb{C})$. Let $N$ be a positive integer. Then we have

$$g\delta_m^r h = \sum_{n=0}^r \delta_{k-2n}^n g_n \quad (4.51)$$

with $g_n \in M_{k-2n}(\Gamma_1(N), \mathbb{C})$ for $k = (l + m)/2 + 2r$. The $g_n$ are uniquely determined by $g$ and $h$.

Definition 4.5.2 Shimura's holomorphic projector $H$ is defined by $H(g\delta_m^r h) = g_0$.

Lemma 4.5.3 Let $g \in G_{l/2}(\Gamma_1(N), \mathbb{C})$ and $h \in G_{m/2}(\Gamma_1(N), \mathbb{C})$. Let $r$ be a positive integer. Let $f \in M_k(\Gamma_1(N), \mathbb{C})$ with $k = (l + m)/2 + 2r$. Then

$$\langle f, g\delta_m^r h \rangle_N = \langle f, H(g\delta_m^r h) \rangle_N \quad (4.52)$$

Lemma 4.5.4 Suppose that $g \in G_{l/2}(\Gamma_1(N), \overline{\mathbb{Q}})$ and $h \in G_{m/2}(\Gamma_1(N), \overline{\mathbb{Q}})$, and define $g_n \in M_{k-2n}(\Gamma_1(N), K_0)$ for $0 \leq n \leq r$ and $k = (l + m)/2 + 2r$, for a positive integer $r$ as in lemma 4.5.1. Put $g' = -\sum_{n=0}^{r-1} d^n g_{n+1}$. Then the $p$-adic norm $|a(n, g')|_p$ of the Fourier coefficients of $g'$ for all $n$ are bounded, and we have that

$$H(g\delta_m^r h) = gd^r h + dg' \quad (4.53)$$

Lemma 4.5.5 If $g \in G_{k/2}(\Gamma_1(N), \psi, \mathbb{C})$ and $h \in G_{l/2}(\Gamma_1(N), \psi, \mathbb{C})$, then

$$H(h\delta_{k/2}^r g) = (-1)^r H(g\delta_{l/2}^r h). \quad (4.54)$$

4.6 Two variable symmetric square $p$-adic $L$-function

In this section we state the main theorem concerning the two variable symmetric square $p$-adic $L$-function defined on the eigencurve and give the ideas behind how it is
proved. This $p$-adic $L$-function will be given locally, and we start by recalling the notation for the local situation. Let $D(V) \subset C_{p,N}$ be a local flat piece of the eigencurve arising from a $V \in C$ such that $D(V) = D(Y, \alpha^*)$ for an affinoid disk $Y \subset \mathcal{W}_N$. We make the additional supposition that $D(V)$ is smooth over $Y$.

Recall that $\mathcal{T}(V)$ is the Hecke ring associated to $D(V)$, and $\mathcal{T}^0(V)$ is its subring of elements with norm $\leq 1$. Recall that $A(Y)$ denotes the affinoid ring attached to $Y$, and $A^0(Y)$ denotes its subring of elements with norm $\leq 1$. Let $\mathcal{L}$ denote the quotient field of $A^0(Y)$, and let $K$ be a finite extension of $\mathcal{L}$. Let $\mathcal{I}$ be the integral closure of $A^0(Y)$ in $K$. An overconvergent family of modular forms in $D(V)$ given by a map

$$\lambda : \mathcal{T}^0(V) \to \mathcal{I}$$  \hfill (4.55)

We let $D(V_\lambda)$ denote the component of $D(V)$ determined by $\lambda$. We suppose that $Y$ is small enough that it is contained in a connected component of weight space, and let $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}_K^\times$ denote the character determined by this component of weight space. By the character of $\lambda$ we mean $\psi$. When we specialize the family $\lambda$ to a height 1 prime ideal $P_{\psi,k}$ of $\mathcal{T}^0(V)$ given by the character $(\psi, k) \in Y$ where $\psi$ is of conductor $Np$ and $k/\text{geq} \alpha$, we get a cusp form $f$ in

$$S_k(\Gamma_0(Np), \psi, \mathcal{O}_K).$$  \hfill (4.56)

We suppose for the ease of exposition that the classical cusp forms that $\lambda$ specializes to are new.

We now define the complex symmetric square $L$-function. First, we can write the $q$-expansion of $f$

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \quad \text{where} \quad q = e^{2\pi i z}$$  \hfill (4.57)

For each prime $l$, let $\alpha_l$ and $\beta_l$ denote the roots of $1 - a_lX + \psi(l)X^2$. Let $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}$ be a primitive character. The definition of the twisted complex symmetric square $L$-function is given by

$$L(s, f, \chi) = \prod_l [(1 - \bar{\psi}(l)\chi(l)\alpha_l^{-s})(1 - \bar{\psi}(l)\chi(l)\alpha_l\beta_l^{-s})(1 - \bar{\psi}(l)\chi(l)\beta_l^{2l^{-s}})]^{-1}$$  \hfill (4.58)

where $l$ varies over all primes.
Let $\text{Meas}(\mathbb{Z}_{p,J}^\times, O_K)$ denote that space of $O_K$-values measures on $\mathbb{Z}_{p,J}^\times$. The two variable symmetric square $p$-adic $L$-function that we will construct will be given as an element of

$$\text{Meas}(\mathbb{Z}_{p,J}^\times, O_K) \otimes O_K \mathcal{I}$$

(4.59)

and interpolate the special values $L(n, f, \chi)$ for finite order characters $\chi : \mathbb{Z}_{p,J}^\times \to O_K$, classical cusp forms $f$ arising from specializing the family $\lambda$ to arithmetic points of weight $k$, and integers $n$ satisfying $1 \leq n \leq k - 1$.

We define some constants that show up in the interpolation formula. Given a character $\psi$ on $(\mathbb{Z}/Np\mathbb{Z})^\times$, we let $\psi_N$ denote the restriction of $\psi$ to $(\mathbb{Z}/N\mathbb{Z})^\times$ and $\psi_p$ the restriction of $\psi$ to $(\mathbb{Z}/p\mathbb{Z})^\times$. We also denote $G(\chi)$ for the Gauss sum of the character $\chi$ and $W_N(f)$ for the $N$-part of the root number of a form $f$. For a character $\chi$, we let $\chi_n = \chi^{p^{-n}}$. Here is the precise statement of the theorem:

**Theorem 4.6.1** There exists a unique $p$-adic analytic function on $\mathcal{W}_J \times D(V_\lambda)$:

$$\mathcal{L}(w, s) : \mathcal{W}_J \times D(V_\lambda) \to \mathcal{C}_p$$

(4.60)

satisfying the following interpolation property: If $w_{\chi, n} \in \mathcal{W}_J$ is the character on $\mathbb{Z}_J^\times$ given by $z \mapsto \chi(z)z^n$ for a finite order character $\chi$, $c' \in D(V_\lambda)$ corresponds to a classical form $f_{k'}$ of weight $k'$ and character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to O_K$ (so we have $k' \geq \alpha + 1$), and $1 \leq n \leq k' - 1$ then

$$\mathcal{L}(w_{\chi, n}, c') = c(w_{\chi, n}, c') S(f_{k'}) E(w_{\chi, n}, c') \times \frac{L(n, f, \psi_{\chi_n})}{\Omega_{f_{k'}, \mu_{f_{k'}}}^+}$$

(4.61)

where

$$c(w_{\chi, n}, c') = \frac{4\pi(1 + 2m)/2(-1)^{k+m}(Jp)^{1/2}G(\chi_n)}{\Gamma(1 + 2m)}$$

(4.62)

$$S(f_k) = \left(\frac{\psi_N(p)\alpha_{p^{-2}}}{p^k}\right)\left(1 - \frac{\psi(p)p^{k-1}}{a_p^2}\right)\left(1 - \frac{\psi(p)p^{k-2}}{a_p^2}\right)$$

(4.63)

$$E(w_{\chi, n}, c') = (1 - \chi_n(p)\alpha_{p^{-2}}p^{n-1})(\psi_N(p)^{-1}\alpha_{p^{-2}})$$

(4.64)

and where $\Omega_{f_{k'}}^\pm$ are a compatible set of periods for varying specializations $f_{k'}$ of $\lambda$ that depend on a choice of periods $\Omega_{f_{k}}^\pm$ for a single specialization $f_k$. 

75
The construction of this two variable symmetric square $p$-adic $L$-function is classically motivated by the formulas involving the Rankin product of two forms. Let $f \in S_k(\Gamma_0(Np^\ell), \psi)$ and $g \in G_{l/2}(\Gamma_0(Jp^\ell), \xi)$. Let $L$ be the least common multiple of $N$ and $J$. The Rankin product $L$-function is given by

$$D_{Lp}(s, f, g) = L_{Lp}(2s - 2k - l + 3, \psi^2 \xi^2) \sum_{n=1}^{\infty} a_n b_n n^{-s/2}$$  \hspace{1cm} (4.65)$$

where $L_{Lp}(s, \chi)$ denotes the Dirichlet $L$-function for the character $\chi$ with Euler factors for primes dividing $Lp$ removed. In the situation where $g$ is the theta series

$$\theta(\xi) = \sum_{n=0}^{\infty} \xi(n)q^n$$  \hspace{1cm} (4.66)$$

we have by an easy calculation

$$L(s, f, \xi) = D_{Lp}(s, f, \theta(\xi))$$  \hspace{1cm} (4.67)$$

Our goal is to construct a measure, or more precisely an element of $L \in \text{Meas}(\mathbb{Z}_p^x, \mathcal{O}_K) \hat{\otimes} \mathcal{O}_K \mathcal{L}$ with the property that given a classical form $f$ in the overconvergent family, the measure $L$ specializes to an an element $L_f \in \text{Meas}(\mathbb{Z}_p^x, \mathcal{O}_K)$. This specialized measure should in turn satisfy the following interpolation property: Given a finite order character $\chi : \mathbb{Z}_p^x \rightarrow \mathcal{O}_K^x$ and an integer $m$ satisfying $0 \leq 2m < k - 1$,

$$\int_{\mathbb{Z}_p^x} \chi(z) z^{2m+\nu} d\Phi_f \sim L(n + \nu + 1, f, \psi \overline{\chi})$$  \hspace{1cm} (4.68)$$

where the $\sim$ is equality up to some explicitly given constants and Euler factors, $n = 2m$, and $\nu = 0$ or 1.

In order to do this, we use the following formula that relates the symmetric square $L$-function with a Rankin product $L$-function:

$$D_{Lp}(1 + n, f, \theta(\chi)|1/2 \tau(4J^2)) = (1 - (\psi(2)\overline{\chi}(2))^2 2^{2k-4-2m}) \cdot \frac{L(n+1, f, \psi \overline{\chi})}{\Omega_f \overline{\Omega_f}}$$  \hspace{1cm} (4.69)$$

The Rankin product has a nice integral representation that can be related to an expression involving the Petersson inner product that has nice arithmetic properties that allow for $p$-adic interpolation. This is explicitly given by

$$D_{Lp}(m, f, \theta(\chi)|\tau) \sim \langle f^p, \theta(\chi)|\tau \cdot E_{k-1/2}^*(m + 2 - 2k, \psi \overline{\chi}|\tau L_p)y^{m/2+1-k} \rangle.$$  \hspace{1cm} (4.70)$$

76
By the invariance of the inner product under the action on \( \tau \), the right hand side is equal to
\[
\langle f^p|\tau, \theta(\chi) \rangle \cdot E_{k-1/2}(2k-m-2, \overline{\psi_{\chi^*} \gamma_{L_p}}).
\] (4.71)
We have that our Eisenstein measure of half integral weight is related to the Eisenstein series in the above formula by
\[
E_{k/2}(\psi_{\overline{\chi^*} \gamma_{L_p}}) = E_{k/2}(2 - k, \psi_{\overline{\chi^*} \gamma_{L_p}}).
\] (4.72)

Using Shimura’s differential operator, we have
\[
E_{k-1/2}(2k-m-2, \overline{\psi_{\chi^*} \gamma_{L_p}}) = \delta_{k+1/2-m}^m E_{k+1/2-m}(\psi_{\overline{\chi^*} \gamma_{L_p}}).
\] (4.73)
Thus we have
\[
D_{L_p}(m, f, \theta(\chi)|\tau) \sim \langle f^p|\tau, \theta(\chi) \rangle \cdot \delta_{k+1/2-m}^m E_{k+1/2-m}(\psi_{\overline{\chi^*} \gamma_{L_p}}).
\] (4.74)

The problem with \( p \)-adically interpolating the right hand side of this formula is that the \( m \) should be incorporated with the \( \chi \) for the \( \theta \)-measure to evaluate and that \( \delta \) is something that is complex, not \( p \)-adic. We fix the issue with the \( p \)-adic characters by using formulas involving the differential operators. We can remove the \( \delta \) by using formulas that relate \( \delta \) to \( d \).

4.7 Construction of the Convoluted Measure

In this section we will construct the convoluted measure that will be used in the construction of the two variable symmetric square \( p \)-adic \( L \)-function. We first recall the notation. Let \( K \) be a \( p \)-adic field and \( \mathcal{O}_K \) denote its ring of integers. Let \( \mathbb{Z}_{p,J}^X = \lim_r (\mathbb{Z}/Jp^r\mathbb{Z})^X \) where \( (J,p) = 1 \). We also define \( \mathcal{A}_J = \mathcal{O}_K[[\mathbb{Z}_{p,J}^X]] \). We will also use another tame level \( L \) with \( J|L \) and \( (L,p) = 1 \). Let \( Cont(J) = C(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) \) denote the space of continuous functions from \( \mathbb{Z}_{p,J}^X \) to \( \mathcal{O}_K \). \( Cont(J) \) is an \( \mathcal{A}_J \)-module. Let \( Meas(J) \) be the following \( \mathcal{A}_J \)-module:
\[
Meas(J) = Meas(\mathbb{Z}_{p,J}^X, \mathcal{O}_K) = \text{Hom}_{\mathcal{O}_K}(C(\mathbb{Z}_{p,J}^X, \mathcal{O}_K), \mathcal{O}_K).
\] (4.75)
Also, recall that \( \overline{G}(L, {\mathcal{O}_K}) \) is the space of \( p \)-adic half integral weight modular forms of tame level \( L \), and \( \overline{F}(L, {\mathcal{O}_K}) \) is the space of \( p \)-adic half integral weight cusp forms of tame level \( L \). We note these spaces of modular forms are \( A_L \)-modules.

Now we define notation for the convoluted measure we will construct. Let

\[
E: \text{Cont}(L) \to \overline{G}(L, {\mathcal{O}_K})
\]

be the \( A_L \)-linear map given by the Eisenstein measure of half integral weight defined in proposition 4.4.3. Let

\[
\theta: \text{Cont}(J) \to \overline{F}(L, {\mathcal{O}_K})
\]

be an \( A_L \)-linear map given by the \( \theta \)-measure defined in proposition 4.3.2 where \( A_L \) acts on \( \text{Cont}(J) \) by projecting \( Z_{p,L}^X \) to \( Z_{p,J}^X \). Our first goal is to construct a two variable measure that convolutes the Eisenstein measure of half integral weight and the theta measure:

\[
E \ast \theta: \text{Cont}(J) \otimes_{\mathcal{O}_K} \text{Cont}(L) \to \overline{S}(L, {\mathcal{O}_K})
\]

We now begin the construction of this convoluted measure. We first define a map to multiply our forms. We define the \( {\mathcal{O}_K} \)-linear map:

\[
m: \overline{G}(L, {\mathcal{O}_K}) \otimes_{\mathcal{O}_K} \overline{F}(L, {\mathcal{O}_K}) \to \overline{S}(L, {\mathcal{O}_K})
\]

(4.79)

to be the usual multiplication of forms. We extend the theta measure and then compose it with this multiplication map. Let

\[
id \otimes \theta: \overline{G}(L, {\mathcal{O}_K}) \otimes_{\mathcal{O}_K} \text{Cont}(J) \to \overline{G}(L, {\mathcal{O}_K}) \otimes_{\mathcal{O}_K} \overline{F}(L, {\mathcal{O}_K})
\]

(4.80)

Then define

\[
\tilde{\theta}: \overline{G}(L, {\mathcal{O}_K}) \otimes_{\mathcal{O}_K} \text{Cont}(J) \to \overline{S}(L, {\mathcal{O}_K})
\]

(4.81)

with \( \tilde{\theta} = m \circ (id \otimes \theta) \). This \( \tilde{\theta} \) map takes a form and a character and multiplies the form by the theta series with that character.

Now we look at the space \( C(\text{Meas}(J) \times Z_{p,L}^X, {\mathcal{O}_K}) \) of continuous functions from \( \text{Meas}(J) \times Z_{p,L}^X \) to \( {\mathcal{O}_K} \). This will be an ambient space for the domain of the two variable measure \( E \ast \theta \). Let \( F \in C(\text{Meas}(J) \times Z_{p,L}^X, {\mathcal{O}_K}) \). We define an action of \( Z_{p,L}^X \) on this
space by $F(z)(m, x) = F(z^{-1}m, zx)$. We define a map $E\ast(F) : \text{Meas}(J) \to \overline{G}(L, O_K)$. by

$$E\ast(F)(m) = \int_{\mathcal{L}_{p,L}^J} (F(z)(m, 1) dE(z) \quad (4.82)$$

One verifies that $E\ast(F) \in C(\text{Meas}(J), \overline{G}(L, O_K))$.

Now we look at the subspace $\text{Cont}(L) \hat{\otimes}_{O_K} \text{Cont}(J) \subset C(\text{Meas}(J) \times \mathcal{L}_{p,L}^J, O_K)$. This will be the domain for the two variable measure $E \ast \theta$. When we restrict $E\ast$ to this space, we get a map

$$E\ast : \text{Cont}(L) \hat{\otimes}_{O_K} \text{Cont}(J) \to \text{Hom}_c(\text{Meas}(J), \overline{G}(L, O_K)) \cong \overline{G}(L, O_K) \hat{\otimes}_{O_K} \text{Cont}(J) \quad (4.83)$$

where $\text{Hom}_c$ denote the space of continuous homomorphisms.

Now we put together the $E\ast$ and the $\hat{\theta}$. We define the map

$$E \ast \hat{\theta} : \text{Cont}(L) \hat{\otimes}_{O_K} \text{Cont}(J) \to \overline{S}(L, O_K) \quad (4.84)$$

by $(E \ast \hat{\theta})(F) = \hat{\theta}(E\ast(F))$. We define an action of $\mathcal{L}_{p,L}^J$ on $C(\text{Meas}(J) \times \mathcal{L}_{p,L}^J, O_K)$ by

$$(\phi||z)(m, x) = \phi(m, zx) \quad (4.85)$$

This action makes $E \ast \theta$ into a map of $\mathcal{A}_L$-modules.

Tracing through the above construction, we have the following theorem:

**Theorem 4.7.1** There exists a unique measure $E \ast \theta : \text{Cont}(L) \times \text{Cont}(J) \to \overline{S}(L, O_K)$ satisfying the following interpolation property: Given finite order characters $\psi : \mathcal{L}_{p,N}^J \to O_K$ and $\chi : \mathcal{L}_{p,L}^J \to O_K$, and positive integers $k$ and $m$,

$$E \ast \theta(y_{\psi,k}, w_{\chi,m}) = \theta(w_{\chi,m})E(\psi \chi) \quad (4.86)$$

### 4.8 The $p$-adic Holomorphic Projector

**Proposition 4.8.1** There exists a projection map:

$$e_\psi^0 : M_Y^1(N) \to N(V_\alpha) \quad (4.87)$$

**Proof.** See [14] \hfill $\square$
Theorem 4.8.2 There exists a unique measure $\tilde{H} : Y \times \text{Cont}(J) \to N(V)$ satisfying the following interpolation property: Given finite order characters $\psi : \mathbb{Z}_{p,N}^\times \to \mathcal{O}_K$ and $\chi : \mathbb{Z}_{p,J}^\times \to \mathcal{O}_K$, and positive integers $k$ and $m$,

$$\tilde{H}(y_{\psi,k}, w_{\chi,m}) = e_{\psi}^{\chi} \circ H(E_{k-1/2}(\psi) \delta^m \theta(\chi))$$  \hspace{1cm} (4.88)

**Proof.** First, we can write

$$E_{k-1/2}(\psi) \delta^m \theta(\chi) = E_{k-1/2}(\psi) d^{m} \theta(\chi) + \sum_{i=1}^{m} d^{i} g_{i}$$ \hspace{1cm} (4.89)

for some classical modular forms $g_{i}$. Let $E_2$ denote the weight 2 Eisenstein series. We can decompose the right hand side into a sum

$$E_{k-1/2}(\psi) d^{m} \theta(\chi) + \sum_{i=1}^{m} d^{i} g_{i} = G(y_{\psi,k}, w_{\chi,m}) + H(y_{\psi,k}, w_{\chi,m})$$ \hspace{1cm} (4.90)

where $G(y_{\psi,k}, w_{\chi,m}) \in M^\dagger$ and $H(y_{\psi,k}, w_{\chi,m}) \in E_2(M^\dagger[E_2])$ in the direct sum decomposition

$$M^\dagger[E_2] = M^\dagger \oplus E_2(M^\dagger[E_2]).$$ \hspace{1cm} (4.91)

One can verify that $G(y_{\psi,k}, w_{\chi,m})$ is $p$-adic analytic, thus giving the theorem. \hfill \Box

4.9 Linear Form and Periods

We recall the definition of the Petersson inner product: For $f \in S_k(\Gamma_1(M), \mathbb{C})$ and $g \in M_k(\Gamma_1(M), \mathbb{C})$ we let

$$\langle f, g \rangle_M = \int_{\Gamma_1(M) \backslash \mathcal{H}} \overline{f(z)} g(z) y^{k-2} \, dx \, dy$$ \hspace{1cm} (4.92)

Let $f_k \in M_k(\Gamma_0(Np), \psi, \overline{\mathbb{Q}})$. We define a linear form $\tilde{i}_f : S_k(\Gamma_0(Np), \psi, \overline{\mathbb{Q}}) \to \overline{\mathbb{Q}}$ by the following:

$$\tilde{i}_{f_k}(g) = \frac{\langle h_k, g \rangle_{Np}}{\langle h_k, f_k \rangle_{Np}}$$ \hspace{1cm} (4.93)

where $h_k = f_k |k \left[ \begin{array}{cc} 0 & -1 \\ Np & 0 \end{array} \right]$ with $f_k^p$ being the $q$-expansion obtained by taking the complex conjugates of the Fourier coefficients of $f_k$.

The following theorem states that these linear forms can be interpolated as $f_k$ varies in a overconvergent family.
Theorem 4.9.1 There exists a unique linear form $\tilde{l}_\lambda : M^\dagger_Y(N) \to A(Y)$ satisfying the following commutative diagram for all $k \geq \alpha + 1$ with $(\psi, k) \in Y$:

$$
\begin{array}{c}
M^\dagger_Y(N) \\
\downarrow
\end{array}
\xrightarrow{\tilde{l}_\lambda}
\begin{array}{c}
A(Y) \\
\downarrow
\end{array}
\begin{array}{c}
M_k(\Gamma_0(Np), \psi, K) \\
\downarrow
\end{array}
\xrightarrow{\tilde{l}_{f_k}}
\begin{array}{c}
K
\end{array}

(4.94)

where the vertical arrows are the specialization maps associated to specializing $\lambda$ to $f_k$.

Proof. See Proposition 6.7 in [14] □

We also have the following lemma:

Lemma 4.9.2 Let $f_k \in M_k(\Gamma_0(Np), \psi, \overline{Q})$ be a new form and $\psi$ of conductor $Np$.

$$
\frac{\langle h_k, f_k \rangle_{Np}}{\langle f_k, f_k \rangle_{Np}} = (-1)^k W_N(f_k)(a_p(f_k))^p)^{-1} S(f_k) p^{k/2} G(\psi)
$$

(4.95)

where

$$
S(f_k) = \left( \frac{\psi_N(p)a_p^2}{p^k} \right) \left( 1 - \frac{\psi(p)p^{k-1}}{a_p^2} \right) \left( 1 - \frac{\psi'(p)p^{k-2}}{a_p^2} \right)
$$

(4.96)


Let $\lambda$ be an overconvergent family. Fix a form $f_k$ in the family. Choose periods $\Omega^+_{f_k}$. By theorem 3.9.3, we have a function interpolating $\frac{\langle f_{k'}, f_k \rangle_{\Omega^+_{f_k}}}{\Omega^+_{f_{k'}} \Omega^+_{f_k}}$ for specializations $f_{k'}$ of $\lambda$ at various weights $k'$ and a compatible set of periods $\Omega^+_{f_{k'}}$. Let

$$
l_{f_k}(g) = \frac{\langle h_k, g \rangle_{Np}}{\Omega^+_{f_k} \Omega_{f_{k'}}} \cdot \left( (-1)^k W_N(f_k)(a_p(f_k))^p)^{-1} S(f_k) p^{k/2} G(\psi) \right)^{-1}.
$$

(4.97)

Then we have the following theorem:

Theorem 4.9.3 There exists a unique linear form $l_\lambda : M^\dagger_Y(N) \to A(Y)$ satisfying the following commutative diagram for all $k \geq \alpha + 1$ with $(\psi, k) \in Y$:

$$
\begin{array}{c}
M^\dagger_Y(N) \\
\downarrow
\end{array}
\xrightarrow{l_\lambda}
\begin{array}{c}
A(Y) \\
\downarrow
\end{array}
\begin{array}{c}
M_k(\Gamma_0(Np), \psi, K) \\
\downarrow
\end{array}
\xrightarrow{l_{f_k}}
\begin{array}{c}
K
\end{array}

(4.98)

where the vertical arrows are the specialization maps associated to specializing $\lambda$ to $f_k$. 

81
4.10 Proof of the Main Theorem

We restate the main theorem for the convenience of the reader:

**Theorem 4.10.1** There exists a unique $p$-adic analytic function on $W_J \times D(V_\lambda)$:

$$L(w, s) : W_J \times D(V_\lambda) \to \mathbb{C}_p$$  \hspace{1cm} (4.99)

satisfying the following interpolation property: If $w_{X,n} \in W_J$ is the character on $\mathbb{Z}_J^+$ given by $z \mapsto \chi(z)(z)^n$ for a finite order character $\chi$, $c' \in D(V_\lambda)$ corresponds to a classical form $f_{k'}$ of weight $k'$ and character $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}_K$ (so we have $k' \geq \alpha + 1$), and $1 \leq n \leq k' - 1$ then

$$L(w_{X,n}, c') = c(w_{X,n}, c')S(f_{k'})E(w_{X,n}, c') \times \frac{L(n, f, \psi \chi_n)}{\Omega_{f_{k'}}^+ \Omega_{f_{k'}}^-} \hspace{1cm} (4.100)$$

where

$$c(w_{X,n}, c') = \frac{(4\pi)^{(1+2m)/2}(-1)^{k+m}(Jp)^{1/2}G(\chi_n)}{\Gamma(\frac{1+2m}{2})} \hspace{1cm} (4.101)$$

$$S(f_k) = \left( \frac{\psi_N(p)\alpha_p^{-2}}{p^k} \right) \left( 1 - \frac{\psi(p)p^{k-1}}{a_p^2} \right) \left( 1 - \frac{\psi(p)p^{k-2}}{a_p^2} \right) \hspace{1cm} (4.102)$$

$$E(w_{X,n}, c') = (1 - \chi_n(p)\alpha_p^{-2}p^{n-1})(\psi_N(p))^{-1}\alpha_p^{-2} \hspace{1cm} (4.103)$$

and where $\Omega_{f_k}^\pm$ are a compatible set of periods for varying specializations $f_{k'}$ of $\lambda$ that depend on a choice of periods $\Omega_{f_k}^\pm$ for a single specialization $f_k$.

**Proof.** We consider $n$ with $0 \leq n < k - 1$ and then adjust the formula we prove by 1 to match up with the statement of the theorem. Let $\eta$ be the spherical function $\eta(a) = a^\delta$ where $\delta = 0$ or 1. To access the special values for even $n$, we take $\delta = 0$ and let define $m$ by $n = 2m$. To access the special values for odd $n$, we take $\delta = 1$ and define $m$ by $n = 2m + 1$. We have the following formula that relates the classical symmetric square $L$-function to the Rankin product:

$$D_{L,p}(1 + 2m + \delta, f, \theta(\chi_n\eta)) = (1 - (\psi \chi(2))^2(2)^{2k-4-2n}) \cdot L(n + 1, f, \psi \chi_n) \hspace{1cm} (4.104)$$

We note that the factor $(1 - (\psi \chi(2))^2(2)^{2k-4-2n})$ can be interpolated $p$-adic analytically. So we are are left with interpolating the Rankin product on the left hand side. We will work out the details in the case where $\delta = 0$. 

82
The next step is to relate the Rankin product to an expression involving the Petersson inner product. We also suppose for the ease of exposition that $\chi_n$ has conductor $Jp$. By lemma 4.10.2 we have that

$$D_{Lp}(1+2m, f, \theta(\chi_n))$$

is equal to

$$(4\pi)^{(1+2m)/2} \frac{\Gamma(\frac{1+2m}{2})}{\Gamma(\frac{1+2m}{2})} (f^p, \theta(\chi_n) \cdot E_{k-1/2}^*(z, 2k - m - 2, \psi_{\chi_n^*} \gamma Lp)) y^{m/2+1-k}$$  \hspace{1cm} (4.105)$$

The we have again by lemma 4.10.2 that this is equal to:

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, \theta(\chi_n) | \tau \cdot (E_{k-1/2}^*(z, 2k - m - 2, \psi_{\chi_n^*} \gamma Lp)) y^{m/2+1-k} | \tau)$$ \hspace{1 cm} (4.106)$$

where $\tau = \tau(Lp)$. Then by definition 4.4.1 and lemma 4.10.3, this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, \theta(\chi_n) \cdot E_{k-1/2}^*(z, m + 2 - 2k, \psi_{\chi_n^*} \gamma Lp))$$ \hspace{1cm} (4.107)$$

By lemma 4.10.4 this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, \theta(\chi_n) \cdot \delta_{k+1/2-m}^m E_{k-1/2-m}(z, 2m - 2k + 1, \psi_{\chi_n^*} \gamma Lp))$$ \hspace{1cm} (4.108)$$

By definition 4.4.1 this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, \theta(\chi_n) \cdot \delta_{k+1/2-m}^m E_{k-1/2-m}(\psi_{\chi_n^*} \gamma Lp))$$ \hspace{1cm} (4.109)$$

By lemma 4.5.3 this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, H(\theta(\chi_n) \cdot \delta_{k+1/2-m}^m E_{k-1/2-m}(\psi_{\chi_n^*} \gamma Lp)))$$ \hspace{1cm} (4.110)$$

By lemma 4.5.5 this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, e^{g \circ H(E_{k-1/2-m}(\psi_{\chi_n^*} \gamma Lp) \cdot \delta_{1/2}^m \theta(\chi_n))))$$ \hspace{1cm} (4.111)$$

We then have that this is equal to

$$\frac{(4\pi)^{(1+2m)/2}(-1)^k (Jp)^{1/2} G(\chi_n)}{\Gamma(\frac{1+2m}{2})} (f^p | \tau, e^{g \circ H(E_{k-1/2-m}(\psi_{\chi_n^*} \gamma Lp) \cdot \delta_{1/2}^m \theta(\chi_n))))$$ \hspace{1cm} (4.112)$$

83
By theorem 4.8.2 this is equal to
\[
\frac{(4\pi)^{(1+2m)/2}(-1)^{k+m}(Jp)^{1/2}G(x_n)}{\Gamma(\frac{1+2m}{2})} \langle f^\theta | \tau, \tilde{H}(E_{k-1/2-m}(\psi x_n^2 \gamma Lp) \cdot \delta_{1/2}^m \theta(x_n)) \rangle
\]  \hspace{1cm} (4.114)

The following expression can be interpolated $p$-adically:
\[
\langle f^\theta | \tau, \tilde{H}(E_{k-1/2-m}(\psi x_n^2 \gamma Lp) \cdot \delta_{1/2}^m \theta(x_n)) \rangle
\]  \hspace{1cm} (4.115)

This interpolation is given by $l_{\tilde{k}}(\tilde{H}(w_{\chi,n}, c'))$ which we have by putting together theorem 4.8.2 and theorem 4.9.1.

\[\square\]

**Lemma 4.10.2** Let $f \in S_k(\Gamma_0(\Gamma_0(Lp)^{\beta}), \psi)$ and $g \in G_{l/2}(\Gamma_0(Lp)^{\beta}, \xi)$ with $k > l/2$. Then
\[
(4\pi)^{-s/2} \Gamma(s/2) D_{Lp}(s, f, g)
\]  \hspace{1cm} (4.116)

\[
= \langle f^\theta, g \cdot E^*_{k-l/2}(z, s + 2 - 2k, \xi \psi \chi - Lp^\beta) y^{s/2 + 1 - k} \rangle_{Lp^\beta}
\]  \hspace{1cm} (4.117)

\[
= (-i)^k \langle f^\theta | k \tau, g | l/2 \tau \cdot (E^*_{k-l/2}(z, s + 2 - 2k, \xi \psi \chi - Lp^\beta) y^{s/2 + 1 - k} \rangle_{k-l/2 \tau}
\]  \hspace{1cm} (4.118)

where $\tau = \tau(Lp^\beta)$.

**Proof.** See [8] lemma 4.5. \[\square\]

**Lemma 4.10.3** Let $\eta(z) = z^\alpha$ where $\alpha = 0$ or 1, and let $\chi : (\mathbb{Z}/Jp\mathbb{Z})^\times \to \mathcal{O}$ be a primitive character. Then
\[
\theta(\chi \eta) | \tau (4J^2p^\beta) = (-i)^a (Jp)^{-1/2} G(\chi \theta(\chi \eta))
\]  \hspace{1cm} (4.119)

**Proof.** See [8] equation (5.1c). \[\square\]

**Lemma 4.10.4**
\[
E_{k/2}(z, m, \xi) = \delta_{j/2}^l E_{j/2}(z, 2 - j, \xi)
\]  \hspace{1cm} (4.120)

if $m \in [(k - 1)/2, k]$ and $m$ is odd with $i = (k - m)/2 - 1$ and $j = 2m + 4 - k$.

**Proof.** This is an easy verification from the definitions. \[\square\]
Bibliography


Bibliography


