EVEN GALOIS REPRESENTATIONS AND THE FONTAINE–MAZUR CONJECTURE

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Abstract. We prove, under mild hypotheses, that there are no irreducible two-dimensional ordinary *even* Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with distinct Hodge-Tate weights. This is in accordance with the Fontaine–Mazur conjecture. If $K/\mathbf{Q}$ is an imaginary quadratic field, we also prove (again, under certain hypotheses) that $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ does not admit irreducible two-dimensional ordinary Galois representations of non-parallel weight.

1. Introduction

Potential modularity has proved to be a powerful tool for studying arithmetic questions in the Langlands program [29, 20]. In this note, we show how this circle of ideas can be employed in a novel way to deduce some new instances of the Fontaine–Mazur conjectures.

Let p be prime, let $\mathcal{O}$ be the ring of integers of a finite extension $L/\mathbf{Q}_p$ with residue field $\mathbf{F}$, and let

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{GL}_2(\mathcal{O})$$

be a continuous irreducible Galois representation unramified outside finitely many primes. Suppose, furthermore, that for a decomposition group $D_p \subset G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, the local representation $\rho|D_p$ is potentially semistable with distinct Hodge-Tate weights. If ρ is *odd*, namely, $\det(\rho(c)) = -1$ for a complex conjugation $c \in G_{\mathbf{Q}}$, then the conjecture of Fontaine and Mazur predicts that ρ is modular. This conjecture is now known in many cases, thanks to the combined efforts of many people. If ρ is *even*, however (that is, $\det(\rho(c)) = 1$), then the conjecture of Fontaine and Mazur predicts that ρ does not exist. We prove the following.

1.1. **Theorem.** *Let E be a totally real field, and let $\rho : G_E \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Suppose that $p \geq 7$, and, furthermore, that*

1. *$\rho|D_v$ is ordinary with distinct Hodge-Tate weights for all $v|p$.*
2. *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*
3. *E does not contain $\mathbf{Q}(\zeta_p)^+$.*

Then $\det(\rho(c)) = -1$ for any complex conjugation c.

In light of the recent proof of Serre's conjecture [20] and modularity lifting theorems for ordinary representations, we immediately deduce the following corollary.

1.2. **Corollary.** *Let $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Suppose that $p \geq 7$, and, furthermore, that*

1. *$\rho|D_p$ is ordinary, with distinct Hodge-Tate weights.*
2. *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*

Then ρ is modular.

Remark. It should be remarked that *some* p-adic Hodge theory condition is necessary for the proof Theorem 1.1. For example, Corollary 1(b) of Ramakrishna [24] shows that there exist infinitely many even surjective representations $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{SL}_2(\mathbf{Z}_7)$ unramified outside a finite set of primes.

The idea behind this theorem is simple: It suffices to show that ρ is *potentially* modular over some totally real field. Since modular representations are odd, the theorem follows immediately. (The actual argument is somewhat more circuitous.)

There are other circumstances in which one expects (following Fontaine-Mazur) the nonexistence of semistable Galois representations and for which the methods of this paper also apply. Let $K/\mathbf{Q}$ be an imaginary quadratic field, and let

$$\rho : G_K \rightarrow \mathrm{GL}_2(\mathcal{O})$$

be a continuous irreducible geometric Galois representation. The conjectures of Fontaine–Mazur [12] and Langlands predict the existence of a cuspidal automorphic representation π for $G = \mathrm{GL}(2)/K$ such that for all finite places $v \nmid p$ of K, π_v is determined by $\rho|_{K_v}$ via the local Langlands correspondence (and so, in particular, $L(\rho, s) = L(\pi, s)$). Suppose that p splits in K, and that the local representations $\rho|_{D_v}$ for $v|p$ have Hodge–Tate weights $(0, m)$ and $(0, n)$ respectively, for positive integers m, n. The Hodge–Tate weights determine the possible infinity types π_∞ of π. If $m \neq n$, however, a vanishing theorem of Borel and Wallach implies that no such cuspidal π can exist. We prove the following result in this direction.

1.3. **Theorem.** *Let $\rho : G_K \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible geometric Galois representation. Suppose that $p > 7$ splits in K, and, furthermore, that*

1. *$\rho|D_v$ is ordinary for $v|p$, with Hodge–Tate weights $(0, m)$ and $(0, n)$, where $m, n > 0$.*
2. *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$, and the projective representation $\mathrm{Proj}(\overline{\rho}) : G_K \rightarrow \mathrm{PGL}_2(\mathbf{F})$ does not extend to $G_{\mathbf{Q}}$.*

Then $m = n$.

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Theorem 1.1 is proven in section 4, and Theorem 1.3 is proven in section 3. Recall the abbreviations RAESDC and RACSDC for an automorphic form π for $\mathrm{GL}(n)$ stand for regular, algebraic, essentially-self-dual, and cuspidal and regular, algebraic, conjugate-self-dual, and cuspidal, respectively.

2. The Induced Representation

In this section, we prove the following theorem:

2.1. **Theorem.** *Let $\rho : G_K \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible geometric Galois representation. Suppose that $p > 7$ splits in K, and, furthermore, that*

1. *$\rho|D_v$ is ordinary for $v|p$, with Hodge–Tate weights $(0, m)$ and $(0, n)$, where $m, n > 0$.*
2. *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$, and the projective representation $\mathrm{Proj}(\overline{\rho}) : G_K \rightarrow \mathrm{PGL}_2(\mathbf{F})$ does not extend to $G_{\mathbf{Q}}$.*
3. *The determinant $\det(\overline{\rho})$ of $\overline{\rho}$ extends to $G_{\mathbf{Q}}$.*
4. $m \equiv n \mod 2$.

Then $m = n$.

Although this is strictly weaker than Theorem 1.3, the proof is somewhat different and may still be of interest.

It suffices to assume that $m \neq n$ and derive a contradiction.

2.2. **Lemma.** *After replacing* ρ *by a twist* $\tilde{\rho}$, *we may assume that the four Hodge–Tate weights of* $\tilde{\rho}$ *at* $v|p$ *are distinct integers, and that* $\det(\tilde{\rho})$ *lifts to* $G_{\mathbf{Q}}$.

Proof. Let $\epsilon : G_K \rightarrow \mathcal{O}^{\times}$ be the cyclotomic character. Since $m \equiv n \mod 2$, we may find an algebraic Hecke character χ such that $\chi \equiv 1 \mod p$ and $\det(\rho \otimes \chi^{-1}) = \epsilon^m \tau$ for some finite order character τ. By construction, $\overline{\tau}$ lifts to $G_{\mathbf{Q}}$. If $\tilde{\tau}$ denotes the Teichmuller lift of $\overline{\tau}$, then $\tilde{\tau}\tau^{-1}$ is a finite order character of G_K that is congruent to $1 \mod p$. Since p is odd, there exists a character $(\tilde{\tau}\tau^{-1})^{1/2}$ whose square is $\tilde{\tau}\tau^{-1}$. Then, we may let $\tilde{\rho} = \rho \otimes \chi^{-1} \otimes (\tilde{\tau}\tau^{-1})^{1/2}$. The fact that the Hodge–Tate weights are all distinct is a trivial consequence of the assumption $m \neq n$. ☐

Let $\psi := \mathrm{Ind}^{\mathbf{Q}}_K \tilde{\rho}$; it defines a continuous representation $\psi : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_4(\mathcal{O})$.

2.3. **Lemma.** *There exists a symplectic pairing on the vector space underlying* ψ *which transforms under the action of* $G_{\mathbf{Q}}$ *by an odd multipler character* ν. *The corresponding symplectic representation into* $\mathrm{GSp}_4(\mathcal{O})$, *which we still denote by* ψ, *is ordinary and has distinct Hodge–Tate weights.*

Proof. Let η denote the quadratic character of $G_{\mathbf{Q}}$ associated to $K/\mathbf{Q}$. The existence of a symplectic pairing for ψ depends on finding a $G_{\mathbf{Q}}$ invariant line in $\wedge^2\psi$. Yet, recalling that $\det(\tilde{\rho})$ lifts to $G_{\mathbf{Q}}$, we find that

$$\wedge^2\psi = \det(\tilde{\rho}) \oplus (\det(\tilde{\rho}) \oplus \eta) \oplus (\rho \otimes \rho^c).$$

It follows that there exists *two* such pairings, and since exactly one of $\det(\tilde{\rho})$ and $\det(\tilde{\rho}) \otimes \eta$ is odd, we may take ν to be odd. The representation $\psi|D_p$ is ordinary by assumption, and has distinct Hodge–Tate weights by construction. ☐

2.4. **Lemma.** *There exists a totally real field* $F^+/\mathbf{Q}$ *such that the restriction* ψ_{F^+} *is modular. Explicitly, there exists a RAESDC form* Π_{F^+} *for* $\mathrm{GL}(4)/F^+$ *such that for each place* v *of* F^+, *the local factors* $\Pi_{F^+,v}$ *are determined by* $\psi|_{F^+_v}$ *via the local Langlands correspondence.*

Proof. This is an immediate consequence of [5], Theorem 7.5, providing that $\overline{\psi}$ has big image (in the technical sense), which requires a somewhat unpleasant computation that we relegate to Section 7. ☐

Let $F = F^+.K$, let $d = [F^+ : \mathbf{Q}]$, and let η denote the quadratic character of F/F^+. Since $\psi \otimes \eta \simeq \psi$, we deduce from Theorem 4.2 of [2] that Π_{F^+} arises via automorphic induction from some automorphic representation π_F for $G := \mathrm{GL}(2)/F$. Moreover, we may identify the possibly infinity types $\pi_{F,\infty}$ from $\Pi_{F^+,\infty}$ and hence from the Hodge–Tate weights of ψ. Since m and n are positive, $\pi_{F,\infty}$ is a regular algebraic cohomological representation of $G_{\infty} := G(\mathbf{R})$. Let $L(\pi_{F,\infty})$ denote the corresponding irreducible finite dimensional representation of G_{∞}. Then π contributes to the interior cohomology of a suitable arithmetic quotient of the symmetric space associated to G with coefficients in $L(\pi_{F,\infty})$. Since $m \neq n$, $L(\pi_{F,\infty})$ is not preserved by the Cartan involution. Hence, by a Theorem 6.7, VII, p.226 [7] of Borel and Wallach, this cohomology vanishes, and π_F does not exist. (Technically, the result of [7] only applies in the compact quotient case. For the precise statement for non-compact quotients, see Harder [16], § 3.6.1, p.74. In the compact case, one also has the prior work of Raghunathan [27], Theorem 1.) This is a contradiction, and hence Theorem 1.3 is proven.

Remark. The assumption that $\det(\overline{\rho})$ lifts to $G_{\mathbf{Q}}$ is essential to the methods of this section, since $\mathrm{Ind}_K^{\mathbf{Q}}(\rho \otimes \chi)$ is not self-dual in general for any character χ.

Remark. Theorem 1.3 admits a natural generalization to ordinary representations over a CM field K (assuming that K^+ does not contain $\mathbf{Q}(\zeta_p)^+$). One caveat is that the conclusion is weaker — one can only deduce that ρ has parallel weight at one infinite place rather than all as predicted by the Fontaine–Mazur conjecture.

3. The Tensor Representation

In this section, we prove Theorem 1.3 by eliminating the parity condition on m and n and the lifting condition on the determinant. The proof of Theorem 1.3 was inspired by the construction by Harris, Soudry and Taylor [15, 30] of Galois representations associated to cohomological π_K for $\mathrm{GL}(2)/K$ with Galois invariant central characters. In contrast, the following proof schema uses an idea of Ramakrishnan [25] to construct the Galois representations associated to $\mathrm{Sym}^2\pi_K$ without assuming any conditions on the central character.

Let ρ satisfy the first condition of Theorem 1.3. Let us consider the representation $\psi := \rho \otimes \rho^c : G_K \to \mathrm{GL}_4(\mathcal{O})$, it lifts to a representation of $G_{\mathbf{Q}}$ that is unique up to twisting by the quadratic character η of $\mathrm{Gal}(K/\mathbf{Q})$.

3.1. **Lemma.** *The extension ψ of $\rho \otimes \rho^c$ to $G_{\mathbf{Q}}$ satisfies $\psi \simeq \psi^\vee \chi$, where $\chi|_K = \det(\rho)\det(\rho^c)$ and $\chi(c) = \varepsilon = +1$ for any complex conjugation $c \in G_{\mathbf{Q}}$.*

Proof. Since $\psi|_K$ and $\psi^\vee \det(\rho)\det(\rho^c)|_K$ are isomorphic and extend to $\mathbf{Q}$ uniquely up to twisting by η, the lemma is obvious up to the sign of ε. If $\varepsilon = -1$, then $\psi^\vee(c) = -\psi(c)$ and hence $\mathrm{Tr}(\psi(c)) = 0$. From the definition of ψ, however,

$$\psi(c) \sim \pm \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix},$$

and hence $\varepsilon = +1$. □

3.2. **Theorem.** *The representation ψ is modular for $\mathrm{GL}(4)/F^+$ for some totally real field F^+.*

Proof. The representation ψ is ordinary with distinct Hodge-Tate weights. The isomorphism $\psi \simeq \psi^\vee\chi$ gives rise to a pairing $\langle x, y\rangle$ on the vector space L^4 associated to ψ such that $\langle \sigma x, \sigma y\rangle = \chi(\sigma)\langle x, y\rangle$. Because $\wedge^2\psi$ is irreducible, this pairing is symmetric. Thus $\langle y, x\rangle = \varepsilon\langle x, y\rangle$, where $\varepsilon = +1 = \chi(c)$. Hence, we may apply Theorem 7.5 of [5] to deduce the existence of a RAESDC representation Π_{F^+} for $\mathrm{GL}(4)/F^+$ associated to ψ. In order to apply this theorem, we also need to assume that $p > 7$, and that $\overline{\psi}$ has 2-big image, a calculation that we relegate to section 7. □

We now show why Π_{F^+} does not exist. A result of Kim [23] implies the existence of the exterior square $\wedge^2\Pi_{F^+}$ for $\mathrm{GL}(6)/F^+$, whose construction is compatible with functoriality at all places except possibly those dividing 2 and 3. In particular, the infinity type of $\wedge^2\Pi_{F^+}$ arises via functoriality from the infinity type of Π_{F^+}. Since the Galois representation $\wedge^2\psi_{F^+}$ is absolutely irreducible, $\wedge^2\Pi_{F^+}$ is cuspidal. On the other hand, on the level of Galois representations, if we let $F = F^+.K$ then

$$\wedge^2\psi|_F = \mathrm{Sym}^2(\rho)\det(\rho^c) \oplus \mathrm{Sym}^2(\rho^c)\det(\rho).$$

By multiplicity one for $\mathrm{GL}(6)$ [19], it follows that $\wedge^2\Pi_{F^+} \simeq \wedge^2\Pi_{F^+} \otimes \eta$, where η is the quadratic character of F/F^+. In particular, from Theorem 4.2 of [2], $\wedge^2\Pi_{F^+}$ is the automorphic induction of an automorphic form for $\mathrm{GL}(3)/F$, which we shall denote by $S(\pi_F)$. The notation is meant to suggest the existence of an automorphic form π_F for $\mathrm{GL}(2)/F$ associated to $\rho|_F$; if such a π_F existed then $\mathrm{Sym}^2\pi_F$ would be isomorphic (up to twist) to $S(\pi_F)$. It is not necessary for our arguments, however, to establish the existence of such a π_F. Instead, we proceed as follows. By construction, the infinity type of $S(\pi_F)$ is determined via local Langlands by the Hodge–Tate weights of $\mathrm{Sym}^2(\rho)\otimes\omega^c$, and is hence regular algebraic and cohomological. (Explicitly, it corresponds (up to twist) to the symmetric square of the infinity type of the conjectural π_F for $GL(2)/F$). In particular, we conclude (as in section 2) that $L(S(\pi_F)_\infty)$ is not preserved by the Cartan involution, and thus $S(\pi_F)$ cannot exist, by Borel–Wallach (Theorem 6.7, VII, p.226 [7]).

4. EVEN REPRESENTATIONS

Let $E/\mathbf{Q}$ be a totally real field. Recall that a representation $r : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ is *odd* if, for any complex conjugation $c \in G_E$, $\mathrm{Tr}(\rho(c)) = -1$, 0, or 1. We start by noting the following:

4.1. **Theorem.** *Let n be odd. Let E be a totally real field, and let $r : G_E \rightarrow \mathrm{GL}_n(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Let χ be a finite order character of G_E that is unramified at all $v|p$. Suppose that $p > 2n$, and, furthermore, that*

1. *r is self dual up to twist: $r \simeq r^\vee \epsilon^{1-n}\chi$.*
2. *r is ordinary for all $v|p$ with distinct Hodge-Tate weights.*
3. *$\overline{r}$ has 2-big image, in the sense of* [5]*, Definition 7.2.*
4. *The fixed field of $\mathrm{ad}(\overline{r})$ does not contain $E(\zeta_p)^+$.*
5. *$(\det \overline{r})^2 = \epsilon^{n(1-n)} \mod p$.*

Then there exits a totally real field $F^+/\mathbf{Q}$ such that the restriction $\psi|_{F^+}$ is modular, i.e., associated to a RAESDC form Π_{F^+}.

Proof. Taking determinants of both sides of the relation $r \simeq r^\vee\epsilon^{1-n}\chi$, we deduce that $\epsilon^{1-n}\chi(c) = +1$ for any complex conjugation c. Since r is irreducible and self-dual up to twist, we may deduce that r preserves a non-degenerate pairing $\langle x, y\rangle$. Since n is odd, moreover, this pairing must be symmetric. Hence we may apply Theorem 7.5 of [5] (with $\epsilon = +1$) to deduce the result. ☐

Remark. It should be possible to prove this theorem for $p > n$ as follows. Suppose that r corresponds to a RACSDC automorphic Π_F for $\mathrm{GL}(n)/F$ for some CM field F. By Lemma 4.3.3 of [10], we may deduce that Π_F descends to a RAESDC form Π_{F^+} for $\mathrm{GL}(n)/F^+$. Hence, in light of modularity lifting theorems of Geraghty [14] (in particular, Theorem 5.3.2), it suffices to prove that $\overline{r}|_{G_F}$ is modular for some CM field F which is sufficiently disjoint from $E(\zeta_p)^+$. One approach to proving modularity theorems of this type arises in the work of Barnet-Lamb ([4], Proposition 7). As written, this theorem requires some extra assumptions, in particular, that $\overline{r}$ is crystalline and that the residue field of $\mathcal{O}$ is $\mathbf{F}_p$ (the fact that n is odd guarantees that all sign conditions are satisfied.) However, combining this approach with recent advances (particularly, generalizing the results about the monodromy of the Dwork family proved in ([5] §4, §5) to the more general setting of [4]), these conditions can presumably be removed in the ordinary case. The reason that one obtains a better bound on p is that this method requires modularity lifting theorems for GL_n (over a CM field) rather than modularity theorems for GL_{2n} representations (over a totally real field) as in the proof of Theorem 7.5 of [5].

Proof of Theorem 1.1. Suppose that ρ satisfies the conditions of Theorem 1.1. Then $r = \mathrm{Sym}^2(\rho) \otimes (\epsilon^{-1}\det(\rho)^{-1}) = \mathrm{Ad}^0(\rho)\otimes\epsilon^{-1}$ satisfies the conditions of Theorem 4.1, and hence r is modular. (The only non-trivial condition to verify is 3, which follows from Corollary 2.5.4 of [10].) By multiplicity one for GL(3) ([19]), we deduce that $\Pi^\vee_{F^+} \simeq \Pi_{F^+}$, and Π_{F^+} has trivial central character. If follows from Theorem A and Corollary B of [26] that Π_{F^+} is the symmetric square of a RAESDC automorphic form π_{F^+} for $\mathrm{GL}(2)/F^+$, that is, a Hilbert modular form. Such automorphic forms are known to admit a p-adic Galois representation, which must be equal to $\rho|_{F^+}$ up to twist. Yet the representation associated to π_{F^+} is odd, and thus ρ is odd. $\square$

An alternative approach to deriving a contradiction from the automorphy of $\mathrm{Sym}^2(\rho)$ may be given as follows.

4.2. **Expected Lemma.** *Let F^+ be a totally real field, let Π_{F^+} be a RAESDC automorphic form for $\mathrm{GL}(n)/F^+$ for any n, and let r be a p-adic Galois representation associated to Π_{F^+} Then r is odd.*

Sketch. Let M be a pure Grothendieck motive over a totally real field E with (pure) weight w. For a real embedding σ of E, M admits a Hodge decomposition

$$M_{\sigma,B} \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=w} H^{p,q}.$$

The action of complex conjugation sends $H^{p,q}$ to $H^{q,p}$. Hence, if M is regular ($\dim(H^{p,q}) \leq 1$), then the trace of the action of complex conjugation on $M_{\sigma,B} \otimes \mathbf{C}$ lies in $\{-1, 0, 1\}$. The compatibility of étale and de Rham realizations of M implies that the associated p-adic Galois representations associated to M are also odd (compare [28]). If Π_{F^+} is of sufficiently regular weight, then the p-adic Galois representations arise from a motive M/F for some (any) CM-extension F/F^+. The Tate conjecture implies that the motive M descends to F^+. In order to avoid an appeal to the Tate conjecture, one must construct the appropriate cycles necessary for descent to F^+. This has been done by Taylor [31] by making explicit a certain isomorphism between $\mathrm{U}(n-1,1)$ and $\mathrm{U}(1,n-1)$. For RAESDC forms in low weight, it suffices to show that the associated Galois representations are *congruent* to RAESDC forms of sufficiently regular weight. The construction of Galois representations for such forms uses congruences between RACSDC forms on unitary groups. This is not sufficient for our purposes, because, for a typical such form, the Galois representation does not descend to F^+. The solution is to construct congruences between the Galois representation r and Galois representations coming from RACSDC forms which descend to $\mathrm{GL}(n)/F^+$. This requires a minor technical adjustment, namely, that we must work with symplectic and orthogonal groups rather than unitary groups.

It is clear that r is odd if and only if it is odd over any totally real extension of F^+. Hence, we may assume (after a solvable totally real base change) that $\Pi_{F^+,v}$ is an unramified principal series for all $v|p$. If the Galois representation r is of symplectic type, then, since the multiplier character χ is odd, $\mathrm{trace}(r(c)) = \chi(c)\mathrm{trace}(r(c)) = 0$, and r is odd. Hence, we may further assume that r is of orthogonal type. We prove below that, after a solvable totally real base change, that there exists a group G/F^+ which is compact modulo center at every real place, quasi-split at all finite places, and such that ${}^LG = \mathrm{GO}(n)$. Such a group is an inner form of a classical group, and thus, by forthcoming work of Arthur [1], Π_{F^+} arises by transfer from G. By the construction of [11] (the paragraph before § 3.3), this form deforms on an equidimensional eigencurve that is locally finite over weight space. Passing to a sufficiently regular weight and transferring back to $\mathrm{GL}(n)/F^+$, we obtain the desired congruence with a RAESDC form which is motivic.

Let us now prove the existence of G. The last condition implies that, up to isogeny, G is either a generalized symplectic group or orthogonal group (depending on whether n is odd or even). Since the first two conditions are invariant under isogeny, it suffices to prove the first two statements for these classical groups. There exist forms compact at ∞ over F^+ for both groups, given by the quaternionic unitary group associated to any quaternion algebra D/F^+, or the orthogonal group corresponding to any positive definite quadratic form. Hence, it suffices to note that one can make any such group quasi-split at all finite places after a totally real solvable base change. ☐

Remark. If one is willing to accept Lemma 4.2, then one may deduce immediately from the modularity of r that $\mathrm{Sym}^2(\rho)$ and hence ρ are odd.

Remark. Although the Fontaine–Mazur conjecture predicts that Theorem 4.1 should continue hold when n is even, the argument above cannot (directly) be made to work. For n even, the representations $\overline{r}$ will *not*, in general, be potentially modular (in the sense we are using) over any CM field F, because the Bellaïche–Chenevier sign may be -1. Indeed, when $n = 2$, all representations are self-dual up to twist, and so an even representation $\overline{\rho} : G_E \rightarrow \mathrm{GL}_2(\mathbf{F})$ is *never* potentially modular over a CM extension F, in the sense we are using. (On the other hand, both conjecturally and experimentally (see [13]), $\overline{\rho}$ *is* modular for $\mathrm{GL}(2)/F$ if we omit the self-dual requirement.)

Remark. For $n = 4$, suppose that E be a totally real field, and $\rho : G_E \rightarrow \mathrm{GSp}_4(\mathcal{O})$ is a continuous irreducible Galois representation unramified outside finitely many primes. Assume, otherwise, that r satisfies all the conditions of Theorem 4.1. Since ρ is symplectic, $\wedge^2\rho$ has a one dimensional summand. Let r be the complementary summand. It is a simple exercise to see that r is self-dual up to twist by a character that is either totally odd or totally even, and that r is ordinary with distinct Hodge–Tate weights. If $\overline{\rho}$ has image containing $\mathrm{GSp}_4(\mathbf{F})$, then the image of $\overline{r}$ in $\mathrm{GL}_5(\mathbf{F})$ is presumably large (this is an unpleasant calculation that the author has no interest in attempting). If so, then one may apply Theorem 4.1 to deduce that r is modular, and deduce from Lemma 4.2 that r is odd. Consequently $\overline{\rho}$ cannot be totally even, that is, $\overline{\rho}(c)$ is not a scalar for any complex conjugation $c \in G_E$. This result seems to be about the natural limit for such arguments — there does not seem to be a way to deduce anything about totally even representations with image in $\mathrm{GSp}_6(\mathcal{O})$, for example.

5. Compatible Families

Let us recall from [32] the notion of a weakly compatible family of Galois representations. (In this section only, $\mathcal{O}$ will be denote a global ring of integers, not a local one.)

5.1. **Definition.** *A weakly compatible family* $R = (L, \{\rho_\lambda\}, P_\ell(T), S, \{m, n\})$ *of two dimensional Galois representations over* $\mathbf{Q}$ *consists of:*

1. *A number field* L *with ring of integers* $\mathcal{O}$*,*
2. *A finite set of rational primes* S*,*
3. *For each prime* $\ell \notin S$*, a monic polynomial* $P_\ell(T)$ *of degree* 2 *with coefficients in* L*.*
4. *For each prime* λ *of* $\mathcal{O}$ *with residue characteristic* p*,*

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{GL}_2(\mathcal{O}_\lambda)$$

   is a continuous representations such that, if $p \notin S$*, then* $\rho_\lambda | D_p$ *is crystalline, and if* $\ell \notin S \cup \{p\}$ *then* ρ_λ *is unramified at* ℓ *and* $\rho_\lambda(\mathrm{Frob}_\ell)$ *has characteristic polynomial* $P_\ell(T)$*,*

5. m *and* n *are integers such that for all primes* λ *of* $\mathcal{O}_F$ *above* p*, the representation* $\rho_\lambda | D_p$ *is Hodge-Tate with Hodge-Tate weights* m *and* n*.*

Say that R is irreducible if one (respectively, any) ρ_λ is irreducible. We prove the following result (contrast the result of Kisin [21], Corollary (0.5)).

5.2. **Theorem.** *Let R be an irreducible weakly compatible family of two dimensional Galois representations of $\mathbf{Q}$. Then R arises from a rank 2 motive M with coefficients in L, where, up to twist, either:*

1. *M is the Grothendieck motive $M(f)$ attached to a classical modular form f of weight ≥ 2.*
2. *M is an Artin motive.*

Proof. If $\{\rho_\lambda\}$ is odd (for any λ), this is a consequence of [21], Corollary (0.5). If $m = n$, this is a consequence of the main result of [22]. Hence, we may assume that $k = n - m > 0$ and that ρ_λ is even for all λ. Without loss of generality, we may assume that $m = 0$. If the projective image of $\overline{\rho}_\lambda$ is either $\{A_4, S_4, A_5\}$ for infinitely many λ, then the projective image of ρ_λ is also finite and we are in case 2 above. If the projective image of $\overline{\rho}_\lambda$ is dihedral for infinitely many λ, then ρ_λ is induced from a one dimensional character of a quadratic extension of $\mathbf{Q}$. In the latter case, the modularity of ρ_λ is an easy consequence of class field theory (see [12]). Hence, we may assume that there exist infinitely many primes $\lambda \in \mathcal{O}$ such that $\mathcal{O}_\lambda = \mathbf{Q}_p$, and such that the image of $\overline{\rho}_\lambda$ contains $\mathrm{SL}_2(\mathbf{F}_p)$. Consider a sufficiently large such prime λ. If $\rho_\lambda | D_p$ is ordinary, then Theorem 1.1 implies that ρ_λ is odd, a contradiction. Otherwise, since we may assume that $p \notin S$, the representation $\rho_\lambda | D_p$ is crystalline at p, and (assuming that p is sufficiently large with respect to k) that $\overline{\rho}_\lambda | D_p \simeq \mathrm{Ind}(\omega_2^k)$ for the fundamental tame character ω_2 of level 2. Consider the representation $r = \mathrm{ad}^0(\overline{\rho}_\lambda) \otimes \epsilon$, where ϵ is the cyclotomic character. The proof of Theorem 4.1 can be modified to show that for p sufficiently large, r is modular and hence $\overline{\rho}_\lambda$ is odd, completing the proof of the theorem. The key adjustment required to prove the potential modularity of r is to replace the appeal to Theorem 7.5 of [5] with Theorem 7.6 of *ibid*, noting that the coefficients of the representation $\overline{\rho}_\lambda$ are $\mathbf{Q}_p$, and that we may take λ sufficiently large so as to deduce from Lemma 7.4 of [5] that $\overline{r}$ has $2k$-big image. □

6. COMPLEMENTS

One idea of this paper is to use potential modularity and functoriality to rule out the existence of Galois representations whose infinity type has the same infinitesimal character as a non-unitary finite dimensional representation. This method, however, cannot be applied in all such situations. Consider a representation $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_3(E)$ with three distinct Hodge–Tate weights that are *not* in arithmetic progression. The 8 dimensional irreducible subrepresentation ψ of $\rho \otimes \rho^\vee$, which one might hope to prove is automorphic for $\mathrm{GL}(8)/F^+$, does not have distinct Hodge–Tate weights. Moreover, even if one knew that ψ arose from some automorphic form Π_{F^+}, functoriality is not sufficiently developed to reconstruct the π_{F^+} (associated to ρ) from Π_{F^+}. Another natural question that falls outside the scope of our methods is the following.

6.1. **Question.** *Let K be an imaginary quadratic field, and let E/K be an elliptic curve. Does there exist a totally real field F^+ such that E is potentially modular over $F := F^+.K$? That is, does there exist an automorphic representation π for $\mathrm{GL}(2)/F$ such that $L(\pi_F, s) = L(E/F, s)$?*

7. $\overline{\psi}$ HAS BIG IMAGE

In this final, technical section, we verify that $\overline{\psi}$ has big image, as required in the proof of Lemma 2.4.

7.1. **Lemma.** *The residual representation $\overline{\psi}$ of section 2 has 2-big image in the sense of* [5]*, Definition 7.2.*

Proof. Write V for the 4-dimensional vector space underlying $\overline{\psi}$. Recall that $G = \mathrm{im}(\psi) \subseteq \mathrm{GL}_4(\mathbf{F})$ has 2-big image if:

1. $H^k(G, \mathrm{Ad}^0(V)) = 0$ for $k = 0$ and $k = 1$.

2. For every irreducible G-submodule W of $\mathrm{Ad}^0(V) \subset \mathrm{Hom}(V,V)$, there exists an $g \in G$ such that:
    1. The α-generalized eigenspace $V_{g,\alpha}$ for g is one dimensional for some $\alpha \in \mathbf{F}$.
    2. For every other eigenvalue β of g, $\beta^2 \neq \alpha^2$.
    3. If $v \in V_{g,\alpha}$, there is an inclusion $v \in W(v)$.

If this property holds for a finite index subgroup $H \subseteq G$ of index co-prime to p, then it holds for G. Note that if g has odd order, then $g^{2^n} = g$ for some n, and hence condition (a) automatically implies condition (b).

By assumption, if G is the image of $\overline{\psi}$, then G contains the group $H = \mathrm{SL}_2(\mathbf{F})^2 \ltimes \mathbf{Z}/2\mathbf{Z}$. There is a natural decomposition $V = X \oplus Y$ where the first $\mathrm{SL}_2(\mathbf{F})$ factor acts on X and the second on Y. We find that

$$\mathrm{Ad}^0(V) = W_1 \oplus W_6 \oplus W_8$$

for irreducible representations W_1, W_6 and W_8 of dimensions 1, 6, and 8 respectively. Explicitly,

1. $W_1 \subset \mathrm{Hom}(V,V)$ consists of linear maps that act as the scalar λ on X and $-\lambda$ on Y,
2. $W_6 = \mathrm{Ad}^0(X) \oplus \mathrm{Ad}^0(Y)$,
3. $W_8 = \mathrm{Hom}(X,Y) \oplus \mathrm{Hom}(Y,X)$.

We first verify condition (2). Any $h \in \mathrm{SL}_2(\mathbf{F})^2$ with distinct eigenvalues has two eigenvectors in X and two eigenvectors in Y. Conversely, Any $h \in H$ with four distinct eigenvalues either lies in $\mathrm{SL}_2(\mathbf{F})^2$, in which case its eigenvectors lie in X or Y, or it does not, in which case it has at least one eigenvector that lies neither in X nor Y. Elements of both kind (of odd order) exist if $p > 3$. If $v \in X$ then $v \in W_1(v)$. For all $v \in V$, $v \in W_6(v)$. If v is neither in X nor Y then $v \in W_8(v)$. Hence condition (2) is satisfied.

We now verify condition (1). Clearly $H^1(\mathrm{SL}_2(\mathbf{F})^2, \mathbf{F}) = 0$, and hence the cohomology vanishes for $W = W_1$. If $W = W_6$, it suffices to consider (by symmetry) the cohomology of $\mathrm{Ad}^0(X)$. The action of the second $\mathrm{SL}_2(\mathbf{F})$ on $\mathrm{Ad}^0(X)$ is trivial, and hence any cocycle must act trivially. Restricting to the first factor we obtain the group $H^1(\mathrm{SL}_2(\mathbf{F}), \mathrm{Ad}^0(\mathbf{F}^2))$, which vanishes if $p > 5$. Let $W = W_8$. To compute $H^1(\mathrm{SL}_2(\mathbf{F})^2, \mathrm{Hom}(X,Y))$ let $U \subset B \subset \mathrm{SL}_2(\mathbf{F})^2$ denote the unipotent matrices inside the Borel subgroup of $\mathrm{SL}_2(\mathbf{F})^2$ Then

$$H^1(\mathrm{SL}_2(\mathbf{F})^2, \mathrm{Hom}(X,Y)) \subset H^1(B, \mathrm{Hom}(X,Y)) = H^1(U, \mathrm{Hom}(X,Y))^{B/U} = 0,$$

providing $p > 3$. □

7.2. **Lemma.** *The residual representation $\overline{\psi}$ of section 3 has 2-big image in the sense of* [5]*, Definition 7.2.*

Proof. Write V for the 4-dimensional vector space underlying $\overline{\psi}$. As above, $G = \mathrm{im}(\overline{\psi})$ contains (with index coprime to p) the group $H = \mathrm{SL}_2(\mathbf{F})^2 \ltimes \mathbf{Z}/2\mathbf{Z}$. In this case, there is a natural decomposition $V = X \otimes Y$ where the first $\mathrm{SL}_2(\mathbf{F})$ factor acts on X and the second on Y. We find that

$$\mathrm{Ad}^0(V) = W_6 \oplus W_9$$

for irreducible representations W_6 and W_9 of dimensions 6 and 9, respectively. Explicitly,

1. $W_6 = \mathrm{Ad}^0(X) \oplus \mathrm{Ad}^0(Y)$,
2. $W_9 = \mathrm{Ad}^0(X) \otimes \mathrm{Ad}^0(Y)$.

Since $w \in \mathrm{Ad}^0(W)w$ for any $w \in W$, to verify the existence of an appropriate pair (h, α) for W_6 and W_9, it suffices to find an $h \in H$ of odd order with distinct eigenvalues, which is possible if $p > 3$.

We shall prove that $H^1(\mathrm{SL}_2(\mathbf{F})^2, W) = 0$ for $W = W_6, W_9$. The case of W_6 has been established above. The module $W_9 = \mathrm{Ad}^0(X) \otimes \mathrm{Ad}^0(Y)$ has a unique U-invariant submodule Z, which, for $p > 5$, is the unique factor in the U-filtration such that $H^1(U, Z)^{B/U} \neq 0$. Yet one computes directly that the map $H^1(U, Z) \to H^1(U, W_9)$ is zero. □

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