0.1. **Introduction.** Let $F$ be a number field, and let $\Gamma_N = \text{SL}_N(\mathcal{O}_F)$. For an integer $M$, let $\Gamma_N(M)$ denote the principal congruence subgroup of level $M$, that is, the kernel of the mod-$M$ reduction map $\Gamma_N \to \text{SL}_N(\mathcal{O}_F/M)$. The cohomology and homology groups of $\Gamma_N$ in any fixed degree are well known to be stable as $N \to \infty$ [Cha80]. However, the cohomology and homology groups of $\Gamma_N(M)$ do not stabilize as $N \to \infty$ for trivial reasons (stability already fails for $H_1(\Gamma_N(p), \mathbb{F}_p)$). Given the important role that the cohomology of congruence subgroups plays in the Langlands programme [Ash92, Sch13, CG], it is of interest to see whether the failure of stability can be repaired in some way. Various alternatives have been suggested, including the notion of representation stability by Church and Farb [CF13] (see also [CEF, CEFN14], and [Put12]). The starting point for representation stability has been [CE12]:

\[ \text{Theorem 0.1. The modules } \tilde{H}_d(\text{SL}_N, \mathbb{Z}_p) \text{ stabilize as } N \to \infty; \text{ the corresponding limits:} \]

\[ \tilde{H}_d(\text{SL}_N, \mathbb{Z}_p) := \lim_{N \to \infty} \text{SL}_N(\mathcal{O}_F) \]

are finitely generated $\mathbb{Z}_p$-modules. Moreover, the action of $\text{SL}_N(\mathcal{O}_F) := \prod_{v | p} \text{SL}_N(\mathcal{O}_v)$ on $\tilde{H}_d(\text{SL}_N, \mathbb{Z}_p)$ is trivial for sufficiently large $N$.

The perspective of this paper is that the groups $\tilde{H}^*$ and $\tilde{H}_*$ are not merely a convenient way to package information concerning the classical cohomology and homology groups, but are instead the correct object of study. From an arithmetic perspective, the classical cohomology groups $H^*(\Gamma_N(p^r), \mathbb{F}_p)$ should carry interesting information about the field $\mathcal{O}_F$. 

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Let $G_N(p^r)$ be the kernel of the map $\text{SL}_N(\mathcal{O}_F) \to \text{SL}_N(\mathcal{O}/p^r)$; there is an inflation map $H^*(G(p^r), \mathbf{F}_p) \to H^*(\Gamma_N(p^r), \mathbf{F}_p)$. The ultimate reason for the failure of stability of classical congruence subgroups is that the source of this map is unstable. On the other hand, the source only contains local information concerning $F$, in particular, it only depends on the decomposition of $p$ in $\mathcal{O}_F$. Hence, by taking the direct limit over $r$ (or inverse limit in the case of homology), one excises all the local terms and arrives at a group that both contains all the interesting global information and is stable in $N$. From this optic, the relationship between representation stability (in this particular context) and arithmetic disappears and, in particular, representation stability seems more to be a phenomenon related to the cohomology of $p$-adic Lie groups rather than arithmetic groups.

Theorem 0.1 then says, in effect, that once one removes the uninteresting local terms from the cohomology of congruence subgroups, what remains is stable and finitely generated over $\mathbf{Z}_p$, and carries, as we shall demonstrate, interesting arithmetic information.

An instructive and elementary example of what information is contained in completed cohomology can already be seen in degree one, providing we consider the group $\text{GL}_N$ rather than $\text{SL}_N$. Suppose that $N$ is at least three. For convenience, let us also suppose that $F$ admits a real embedding, so that the congruence subgroup property holds (exactly) for $\text{SL}_N(\mathcal{O}_F)$ (see [BMS67], Theorem 3.6); more generally, by the same reference, if we assume that $p$ is prime to the order $w_F$ of the finite group of roots of unity in $F$, then the congruence kernel is abelian of order prime to $p$. The group $H_1(\text{SL}_N(\mathcal{O}_F), \mathbf{Z})$ may be identified with the abelianization of $\text{SL}_N(\mathcal{O}_F)$. The congruence subgroup property implies (under our assumptions) that that any finite quotient of order prime to $w_F$ must factor through the map $\text{SL}_N(\mathcal{O}_F) \to \prod_v \text{SL}_N(\mathcal{O}_{F,v})$ for some finite set of places $v \in S$. However, the abelianization of $\text{SL}_N(\mathcal{O}_{F,v})$ is trivial for any finite place $v$ (assuming that $N > 2$), and so $H_1(\text{SL}_N(\mathcal{O}_F), \mathbf{Z}_p)$ is trivial. The same argument shows that for the congruence subgroup $\Gamma_N(p^r) \subset \text{SL}_N(\mathcal{O}_F)$, there is an isomorphism

$$H_1(\Gamma_N(p^r), \mathbf{Z}_p) = \Gamma_N(p^r)/\Gamma_N(p^{2r}) \cong G_N(p^r)/G_N(p^{2r}).$$

It follows that the natural maps $H_1(\Gamma_N(p^{2r}), \mathbf{Z}_p) \to H_1(\Gamma_N(p^r), \mathbf{Z}_p)$ will be zero, and so, in particular, the completed homology group $\tilde{H}_1(\text{SL}_N, \mathbf{Z}_p)$ vanishes. This implies that the completed homology group $\tilde{H}_1(\text{GL}_N, \mathbf{Z}_p)$ may be identified under the determinant map with the completed first homology groups of the unit group $\mathcal{O}_F^\times = \text{GL}_1(\mathcal{O}_F)$. Let $\text{GL}_N(\mathcal{O}_F,p^r) \subset \text{GL}_N(\mathcal{O}_F)$ denote the principal congruence subgroup of level $p^r$. Then there is an equality

$$\tilde{H}_1(\text{GL}_N, \mathbf{Z}_p) := \lim_{\leftarrow} H_1(\text{GL}_N(\mathcal{O}_F,p^r), \mathbf{Z}_p) = \ker \left( \mathcal{O}_F^\times \otimes \mathbf{Z}_p \to \prod_{v|p} \mathcal{O}_v^\times \right).$$

The claim that the right hand side vanishes is exactly the statement of Leopoldt’s conjecture, which is well known to be a deep open problem concerning the arithmetic of the field $F$. More precisely, our assumptions on roots of unity in $F$ imply that $\mathcal{O}_F^\times \otimes \mathbf{Z}_p$ is torsion free; the usual

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1In this paper, we use local and global in the sense used by number theorists, namely, to distinguish quantities which only depend on the embeddings of $F$ into either $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ from those that depend on more subtle invariants of $F$. For example, the ranks of the $K$-groups $K_n(\mathcal{O}_F) \otimes \mathbf{Q}$ are local because they are determined by the number of real and complex embeddings of $F$ [Bar74]. In contrast, the torsion subgroups of $K_n(\mathcal{O}_F)$ are global.
statement of Leopoldt’s conjecture is that the map:

\[ \mathcal{O}_F^\times \otimes \mathbb{Q}_p \to \prod_{v \mid p} F_v^\times \]

is injective. For those unfamiliar with Leopoldt’s conjecture, it may be helpful to imagine replacing the prime \( p \) by the prime \( \infty \). The corresponding statement is then the injectivity of the map

\[ \mathcal{O}_F^\times \otimes \mathbb{R} \to \prod_{v \mid \infty} F_v^\times. \]

This map is exactly the classical regulator map \( (u \mapsto (\log |u|_v)) \) whose injectivity was established by Dirichlet as the first step in proving the unit theorem. In this paper, we shall argue how analogous questions and conjectures concerning the groups \( \tilde{H}_d(\text{SL}_n, \mathbb{Z}_p) \) for higher \( d \) concern the (conjectural) injectivity of higher \( p \)-adic regulator maps, and bear the same relation to Borel’s higher regulator maps as Leopoldt’s conjecture bears to the classical regulator map. (Why the \( p \)-adic versions of these maps seem much harder to control than their real analogues is somewhat of a mystery.)

Theorem 0.1 gives a very good qualitative description of completed cohomology in the stable range. The main concern of this paper, which can be considered a sequel to [CE14], is then to address the groups \( \tilde{H}_s \) in a quantitative manner. One may also ask whether information concerning \( \tilde{H}_s \) or \( \tilde{H}_s^* \) can be translated into information concerning the classical cohomology groups. For example, Benson Farb asked the author whether Theorem 0.1 can be utilized for the computation of explicit cohomology groups. In order to answer the broader question of what it might mean to compute classical cohomology groups, we first recall what happens at full level. The stable homology of \( \text{GL}_N(\mathbb{Z}) \) may be identified with the homology of \( \text{BGL} (\mathbb{Z})^+ \), and the homotopy groups of \( \text{BGL} (\mathbb{Z})^+ \) for \( n \geq 1 \) are the algebraic \( K \)-groups \( K_n(\mathbb{Z}) \), which are finitely generated over \( \mathbb{Z} \) and are completely known, at least in terms of Galois cohomology groups (see Theorem 1.3). The ranks of the rational cohomology groups are determined by the signature of the field \( F \) by a theorem of Borel [Bor74]. In this paper, we show that:

- The groups \( \tilde{H}_s \) are the continuous homology groups of a certain homotopy fibre \( Y_\infty \) (Theorem 1.17) whose homotopy groups with coefficients in \( \mathbb{Z}_p \) we can calculate rationally (Lemma 1.4), and even integrally in many cases, in terms of Galois cohomology groups.
- Assuming, in addition, a generalization of Leopoldt’s conjecture, we can also give (Theorem 2.2) a complete description of \( \tilde{H}_s (\text{SL}, \mathbb{Q}_p) := \tilde{H}_s (\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q} \); note that this is different from the inverse limit with coefficients in \( \mathbb{Q}_p \), which, by Borel’s theorem, coincides with the stable rational homology at level one.
- For \( F = \mathbb{Q} \), we compute \( H_2(\Gamma_N(p), \mathbb{F}_p) \) for sufficiently large \( N \) (Corollary 3.4), answering a question of Farb).
- For very regular primes (Definition 4.1) \( p = p\mathfrak{p} \) in an imaginary quadratic field, we compute \( H_\ast (\Gamma_N(p^\alpha), \mathbb{Z}_p) \) in the stable range explicitly (Theorem 4.7).

We give an example of a precise theorem we can state now. Let \( G_S \) denote the Galois group of the maximal Galois extension of \( F \) unramified away from primes dividing \( p \) and \( \infty \). Let \( w_F \) denote the number of roots of unity in \( F \).

**Theorem 0.2.** If \( p \) does not divide \( w_F \cdot |K_2(\mathcal{O}_F)| \), then there is an isomorphism

\[ \tilde{H}^2 (\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1 (G_S, \mathbb{Q}_p/\mathbb{Z}_p (-1)). \]
For all $p$, the equality holds up to a finite group. If $F = \mathbb{Q}$ and $p \geq 3$, then $\tilde{H}^2(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p/\mathbb{Z}_p$ and $\tilde{H}_2(\text{SL}, \mathbb{Z}_p) \simeq \mathbb{Z}_p$.

We expect that the equality holds without any assumption on the order of $K_2(\mathcal{O}_F)$, and this would follow from Conjecture 1.5. (Note, however, that $K_2(\mathcal{O}_F)$ is always finite.) This theorem was inspired by the results of [CV12], particularly Chapter 8, which suggests a link between classes in the cohomology of $H^2(\Gamma_N(M), \mathbb{Q}_p/\mathbb{Z}_p)$ of congruence subgroups and classes in $H^1(F, \mathbb{Q}_p/\mathbb{Z}_p(-1))$ unramified outside $M$ (see, in particular, the discussion in §8.3 of ibid).

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1. Homotopy

1.1. Completed $K$-theory. In this section, we define the completed $K$-groups $\tilde{K}_*(\mathcal{O})$ of $\mathcal{O}$ with respect to a prime $p$, at least for the ring of integers $\mathcal{O} = \mathcal{O}_F$ of a number field $F$. Here the notion of “completion” refers (in principle) to replacing $\Gamma_N = \text{GL}_N(\mathcal{O})$ by the congruence subgroups $\Gamma_N(p^m)$ for arbitrarily large $m$, rather than other possible forms of completion — the notation is chosen via the analogy with the completed homology groups $\tilde{H}_*$. The groups $\tilde{K}_n(\mathcal{O})$ are essentially, depending on the parity of $n$, either the kernel or cokernel of Soulé’s $p$-adic regulator map [Sou81].

For any prime $p$ and integer $m$, define $X_m = X_m(\mathcal{O}, p^m)$ to be the homotopy fibre of the following map:

$$\text{BGL}(\mathcal{O})^+ \to \text{BGL}(\mathcal{O}/p^m)^+$$

The homotopy groups of $X_m$ live in a long exact sequence with the $K$-groups of $\mathcal{O}$ and of $\mathcal{O}/p^m$ (at least for $n \geq 1$). In particular, by a result of Quillen [Qui73], they are finitely generated (see also Corollary 3.8 of [Cha80]). The map $B\Gamma(p^m) \to X_m$ is not, however, a homological equivalence — this is the so called “failure of excision.” (This remark was made in the introduction to [Cha84].) The idea behind this paper is that, since the limiting groups $\tilde{H}_*$ have a trivial action of $\text{SL}(\mathcal{O}_p)$, then some analogue of excision will hold for $m = \infty$.

\[\text{2 Literally, of course, this is not true. Since the homotopy fibre of } \text{BGL}(\mathcal{O}) \to \text{BGL}(\mathcal{O}/p^m) \text{ is } B\Gamma(p^m), \text{ one way of rephrasing the difficulty above is that the + construction does not commute with taking homotopy fibres. The homotopy fibre of } B\Gamma(p^m) \to B\Gamma(p_{\mathcal{O}}), \text{ however, is essentially the discrete set } \text{GL}(\mathcal{O}_p)/\text{GL}(\mathcal{O}) \text{ — the map on } \pi_1 \text{ is injective and there are no higher homotopy groups. The plus construction applied to the discrete set } \text{GL}(\mathcal{O}_p)/\text{GL}(\mathcal{O}) \text{ (whatever the plus construction would mean here), however, is certainly not the homotopy fibre of } B\Gamma(p^m) \to B\Gamma(p_{\mathcal{O}})^+ \text{.}\]
Since $\pi_0(\text{BGL}(R)^+) = 0$ does not correspond to $K_0(R)$, it is technically more convenient to work with the homotopy fibrations of $K$-theory spaces:

$$K(\mathcal{O}, p^m) \longrightarrow K(\mathcal{O}) \longrightarrow K(\mathcal{O}/p^m),$$

$$K(\mathcal{O}, \mathcal{O}_p) \longrightarrow K(\mathcal{O}) \longrightarrow K(\mathcal{O}_p).$$

Note that $K(R, I)$ usually denotes the homotopy fibre of the map $K(R) \to K(R/I)$, and this is the meaning of $K(\mathcal{O}, p^m)$; by abuse of notation, we let $K(\mathcal{O}, \mathcal{O}_p)$ be the homotopy fibre of $K(\mathcal{O}) \to K(\mathcal{O}_p)$; hopefully no confusion will arise.

We make the following definition:

**Definition 1.1.** The completed $K$-groups $\tilde{K}_n(\mathcal{O}) := K_n(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}_p)$ with respect to a prime $p$ are, for $n \geq 1$, the homotopy groups $\pi_n(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}_p)$.

We make below (Definition 1.7) an ad hoc definition of $\tilde{K}_0(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}_p)$.

**Lemma 1.2.** Let $\mathcal{O} = \mathcal{O}_F$ for a number field $F$. There is a long exact sequence:

$$\cdots \longrightarrow K_n(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}_p) \longrightarrow K_n(\mathcal{O}) \otimes \mathbb{Z}_p \longrightarrow K_n(\mathcal{O}; \mathbb{Z}_p) \longrightarrow K_{n-1}(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}_p) \longrightarrow \cdots$$

**Proof.** We first note that all the relevant spaces $K(\mathcal{O}, \mathcal{O}_p)$, $K(\mathcal{O})$, $K(\mathcal{O}_p)$, etc. we consider are infinite loop spaces. We now recall some basic constructions, definitions, and theorems concerning $K$-theory with coefficients.

1. ([Wei13], IV.2.1): The homotopy groups with coefficients in $\mathbb{Z}/p^m$ are defined to be

$$\pi_n(X; \mathbb{Z}/p^m) := [P^n(\mathbb{Z}/p^m), X],$$

where $P^n(\mathbb{Z}/p^m)$ is the space formed from the sphere $S^{n-1}$ by attaching an $n$-cell via a degree $p^m$ map. In general, this space is a group for $n \geq 3$ and a set for $n = 2$, however, it is well defined as a group for all $n \geq 1$ when $X$ is an infinite loop space. Given a Serre fibration, there is a naturally associated long exact sequence.

2. ([Wei13], IV.2.9): The $p$-adically completed homotopy groups $\pi_n(\mathcal{E}; \mathbb{Z}_p)$ of a spectrum $\mathcal{E}$ are defined to be the homotopy groups of the homotopy limit $\tilde{\mathcal{E}}$ of $\mathcal{E} \wedge P^\infty(\mathbb{Z}/p^m)$.

If the $\pi_n(\mathcal{E}; \mathbb{Z}/p^m)$ are finite for all $n$, then

$$\pi_n(\mathcal{E}; \mathbb{Z}_p) \simeq \text{proj} \lim \pi_n(\mathcal{E}; \mathbb{Z}/p^m).$$

3. ([Wei13], IV.2.2): There is a universal coefficient sequence:

$$0 \to \pi_n(X) \otimes \mathbb{Z}/p^m \to \pi_n(X; \mathbb{Z}/p^m) \to \pi_{n-1}(X)[p^m] \to 0.$$

Note that $K_n(\mathcal{O}) := \pi_n(K(\mathcal{O}))$ for all $n$. The groups $K_n(\mathcal{O})$ for $\mathcal{O} = \mathcal{O}_F$ the ring of integers of a number field are known to be finitely generated abelian groups, and hence the projective limit $\text{proj} \lim K_{n-1}(\mathcal{O})[p^m]$ vanishes. It follows from the universal coefficient sequence that

$$K_n(\mathcal{O}_p; \mathbb{Z}_p) = \text{proj} \lim K_n(\mathcal{O}; \mathbb{Z}/p^m) = K_n(\mathcal{O}) \otimes \mathbb{Z}_p.$$

The groups $K_n(\mathcal{O}_p; \mathbb{Z}/p^m)$ are similarly finite (by Theorem A of [HM03]), and so

$$K_n(\mathcal{O}_p; \mathbb{Z}_p) = \text{proj} \lim K_n(\mathcal{O}_p; \mathbb{Z}/p^m).$$

The Serre long exact sequence gives a sequence

$$\cdots \longrightarrow K_n(\mathcal{O}, \mathcal{O}_p; \mathbb{Z}/p^m) \longrightarrow K_n(\mathcal{O}; \mathbb{Z}/p^m) \longrightarrow K_n(\mathcal{O}_p; \mathbb{Z}/p^m) \longrightarrow \cdots$$
By comparison with the surrounding terms, we see that the $\mathbb{Z}_p$-modules $\pi_n(K(\ell, \ell_p), \mathbb{Z}/p^n)$ are finite and also have bounded rank (as $\mathbb{Z}_p$-modules) for all $m$. Hence, taking an inverse limit in $m$ and replacing $K_n(\ell; \mathbb{Z}_p)$ by $K_n(\ell) \otimes \mathbb{Z}_p$, we obtain the desired exact sequence. (The finiteness properties guarantee that the relevant Mittag–Leffler conditions are automatically satisfied, and so there are no issues concerning $\lim^1$. ) For the tail of the long exact sequence involving terms with $n = 0$, see the remarks after Definition 1.7.

It will also be useful to work with the homotopy fibrations

$$Y_m = SK(\ell, p^n) \rightarrow SK(\ell) \rightarrow SK(\ell/p^n).$$

(1) $$Y_\infty = SK(\ell, \ell_p) \rightarrow SK(\ell) \rightarrow SK(\ell_p).$$

Since $\pi_n(BSL(R^+)) \simeq \pi_n(BGL(R^+)) = \pi_n(SK(R))$ for $n \geq 2$, one obtains isomorphisms $\pi_n(Y_\infty; \mathbb{Z}_p) \simeq K_n(\ell, \ell_p; \mathbb{Z}_p)$ for $n \geq 2$ and $m \in \mathbb{N} \cup \infty$. Working with $Y_\infty$ allows us to apply the Hurewicz Theorem more usefully, since, for many fields $F$, the fibre $Y_\infty$ will be simply connected.

1.2. Tate Global Duality. Let $S = \{v|p\} \cup \{v|\infty\}$, and let $G_S$ denote the Galois group of the maximal extension of $F$ outside $S$. If $M$ is a finite $G_S$-module of odd order, then $H^*(G_S, M) \simeq H^*(\text{Spec}(\ell[1/p]), M)$ is finite. If $n$ is a positive integer, let $\mathbb{Z}_p(n)$ denote the $n$th Tate twist of $\mathbb{Z}_p$, that is, the module $\mathbb{Z}_p$ such that the action of $G_{\mathbb{Q}}$ is via the $n$th power of the cyclotomic character. Similarly, the module $F_p(n)$ denotes the corresponding twist of the module $F_p$. For a compact or discrete $\mathbb{Z}_p$-module $A$, let $A^\vee = \text{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ denote the Pontryagin dual of $A$ (so canonically $A^{\vee\vee} \simeq A$). For a compact or discrete $G_S$-module $M$, let $M^\ast$ denote the twisted Pontryagin dual $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(1))$. For a finite $G_S$-module $M$, recall ([Mil06, Tat63]) that by Poitou–Tate duality there is an exact sequence as follows:

$$\begin{array}{c}
0 \rightarrow H^0(G_S, M) \rightarrow \prod_S H^0(G_v, M) \rightarrow H^2(G_S, M^\ast)^\vee \\
\uparrow \\
H^1(G_S, M^\ast)^\vee \leftarrow \prod_S H^1(G_v, M) \leftarrow H^1(G_S, M) \\
\downarrow \\
H^2(G_S, M) \rightarrow \prod_S H^2(G_v, M) \rightarrow H^0(G_S, M^\ast)^\vee \rightarrow 0
\end{array}$$

(2) By taking limits, one also obtains a corresponding sequence for compact $M$, where cohomology of compact or discrete modules is taken in the usual continuous sense. Specifically, let $n$ be a positive integer and let $M = \mathbb{Z}_p(n)$. Then $M^\ast = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathbb{Q}_p/\mathbb{Z}_p(1 - n)$. Moreover, $H^0(G_S, M) = H^0(G_v, M) = 0$. We thus obtain the following exact sequence:
Theorem 1.3. Let $p > 2$. Then one has isomorphisms for $n > 1$ as follows:

\[
K_{2n-1}(\mathcal{O}) \otimes \mathbb{Z}_p \simeq H^1(G_S, \mathbb{Z}_p(n)), \quad K_{2n-2}(\mathcal{O}) \otimes \mathbb{Z}_p \simeq H^2(G_S, \mathbb{Z}_p(n)),
\]

\[
K_{2n-1}(\mathcal{O}_p; \mathbb{Z}_p) \simeq \prod_S H^1(G_v, \mathbb{Z}_p(n)), \quad K_{2n-2}(\mathcal{O}_p; \mathbb{Z}_p) \simeq \prod_S H^2(G_v, \mathbb{Z}_p(n)).
\]

Moreover, the maps $K_n(\mathcal{O}) \otimes \mathbb{Z}_p \to K_n(\mathcal{O}_p; \mathbb{Z}_p)$ induce the natural maps on cohomology.

For specific references, we refer to Theorem A of [HM03] for the $K$-theory of (rings of integers in) local fields (see also Theorem 61 of [Wei05]), and Theorem 70 of Weibel’s survey [Wei05] for the $K$-theory of global fields. We note that, in the latter reference, the Galois cohomology groups have been excised from the statement for the benefit of topologists, but they are easily extracted from the argument. The compatibility of the these isomorphisms follows from the compatibility of the corresponding motivic spectral sequences. Note that for $p = 2$, the equalities hold up to a (known) finite group, and hold on the nose if $F$ is totally imaginary. For the groups $K_1$, which are known classically, one must make a minor adjustment to these descriptions in terms of Galois cohomology: this amounts (for experts) to replacing the Galois cohomology groups $H^1$ with the Bloch–Kato groups $H^1$. More prosaically, there are Kummer isomorphisms

\[
H^1(G_S, \mathbb{Z}_p(1)) = (\mathcal{O}_F[1/p])^\times \otimes \mathbb{Z}_p, \quad H^1(G_v, \mathbb{Z}_p(1)) = F_v^\times \otimes \mathbb{Z}_p,
\]

whereas the $K_1$ groups we are interested in should be identified with the groups

\[
H^1_f(G_S, \mathbb{Z}_p(1)) = (\mathcal{O}_F^\times) \otimes \mathbb{Z}_p, \quad H^1_f(G_v, \mathbb{Z}_p(1)) = \mathcal{O}_v^\times \otimes \mathbb{Z}_p
\]

respectively. This discrepancy is related to the fact that $K_*(\mathcal{O}_F[1/p])$ and $K_*(\mathcal{O}_F)$ coincide in higher degrees but not in degree one. We now have:

Lemma 1.4. Define the groups $\bar{K}^2_n(\mathcal{O})$ as follows:

\[
\bar{K}^2_{2n-1}(\mathcal{O}) := H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee + H^0(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-n))^\vee,
\]

\[
\bar{K}^2_{2n-2}(\mathcal{O}) := H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee.
\]

Then there is an exact sequence as follows:

\[
\ldots \to \bar{K}^2_n(\mathcal{O}) \to K_n(\mathcal{O}) \otimes \mathbb{Z}_p \to K_n(\mathcal{O}_p; \mathbb{Z}_p) \to K^2_{n-1}(\mathcal{O}) \to \ldots
\]

There are rational isomorphisms $\bar{K}_n(\mathcal{O}) \otimes \mathbb{Q} = \bar{K}^2_n(\mathcal{O}) \otimes \mathbb{Q}$. If either $K_n(\mathcal{O})$ is finite of order prime to $p$ or $K_{n+1}(\mathcal{O}_p; \mathbb{Z}_p) = 0$, then there is an isomorphism $\bar{K}_n(\mathcal{O}) = \bar{K}^2_n(\mathcal{O})$. 

\[
0 \to H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee \to H^1(G_S, \mathbb{Z}_p(n)) \to \prod_S H^1(G_v, \mathbb{Z}_p(n))
\]

\[
\to H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee \to H^2(G_S, \mathbb{Z}_p(n)) \to \prod_S H^2(G_v, \mathbb{Z}_p(n))
\]

\[
\to H^0(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee \to 0.
\]
Proof. The exact sequence follows directly from the Poitou–Tate sequence [2] above (one also has to make an easy check for $n = 1$). It follows that if there existed maps $\tilde{K}_n(\mathcal{O}) \to \tilde{K}_n^2(\mathcal{O})$ inducing a commutative map between long exact sequences, then $\tilde{K}_n(\mathcal{O}) \simeq \tilde{K}_n^2(\mathcal{O})$ by the 5-lemma. Such a map exists after tensoring with $\mathbb{Q}$ since $\mathbb{Q}_p$ is projective. If $K_n(\mathcal{O}) \otimes \mathbb{Z}_p = 0$, then both $\tilde{K}_n(\mathcal{O})$ and $\tilde{K}_n^2(\mathcal{O})$ are isomorphic to the quotient of $K_{n+1}(\mathcal{O}/\mathbb{Z}_p) = 0$, then both $\tilde{K}_n(\mathcal{O})$ and $\tilde{K}_n^2(\mathcal{O})$ are isomorphic to the kernel of the map $K_n(\mathcal{O}) \otimes \mathbb{Z}_p \to K_n(\mathcal{O}/\mathbb{Z}_p)$.

We make the following conjecture:

**Conjecture 1.5.** There is an isomorphism $\tilde{K}_n(\mathcal{O}) \simeq \tilde{K}_n^2(\mathcal{O})$.

This seems natural enough for $n$ even. For $n$ odd, the conjecture also seems natural in light of the fact that we expect the term $H^3(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))^\vee$ to vanish (see also Remark 1.10).

**Remark 1.6.** There is some hope to prove this statement by constructing a natural map $\tilde{K}_n(\mathcal{O}) \to \tilde{K}_n^2(\mathcal{O})$. The author has had some discussions with Matthew Emerton regarding this question, and we hope to return to it in the future.

**Definition 1.7** (The group $\tilde{K}_0(\mathcal{O})$). We now extend our definitions to $n = 0$. For $n = 0$, let $\tilde{K}_0(\mathcal{O}) := \tilde{K}_0^2(\mathcal{O}) = H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p)^\vee$.

There is an isomorphism $H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p)^\vee = \text{Hom}(G_S, \mathbb{Q}_p/\mathbb{Z}_p)^\vee = (G_S)^{ab} \otimes \mathbb{Z}_p$. There are isomorphisms $K_1(\mathcal{O}) \simeq \mathcal{O}^\times$ and $K_1(\mathcal{O}/\mathbb{Z}_p) = \mathcal{O}_p \otimes \prod_{v \mid p} \mathcal{O}_p^\times$. Moreover, $K_0(\mathcal{O}) \simeq \text{Pic}(\mathcal{O}) = \text{Cl}(\mathcal{O}) \oplus \mathbb{Z}$. From class field theory, there is an exact sequence:

$$\mathcal{O}_p^\times \otimes \mathbb{Z}_p \to \left( \prod_{v \mid p} \mathcal{O}_p^\times \right) \otimes \mathbb{Z}_p \to (G_S)^{ab} \otimes \mathbb{Z}_p \to \text{Cl}(\mathcal{O}) \otimes \mathbb{Z}_p \to 0.$$

Hence the long exact sequence of Lemma 1.2 continues as far as:

$$\ldots \to K_1(\mathcal{O}) \otimes \mathbb{Z}_p \to K_1(\mathcal{O}/\mathbb{Z}_p) \to \tilde{K}_0(\mathcal{O}) \to K_0(\mathcal{O}) \otimes \mathbb{Z}_p \to \mathbb{Z}_p \to 0.$$

After tensoring with $\mathbb{Q}$, the long exact sequence of Poitou–Tate and hence of $K$-groups breaks up into exact sequences of length 6 (the $H^0 \otimes \mathbb{Q}$ term vanishes for $n > 1$). One immediately obtains the following:

**Lemma 1.8.** There is an equality:

$$\dim \tilde{K}_{2n-2}(\mathcal{O}) \otimes \mathbb{Q} - \dim \tilde{K}_{2n-1}(\mathcal{O}) \otimes \mathbb{Q} = \left\{\begin{array}{ll} r_1 + r_2, & n > 1 \text{ even}, \\ r_2, & n > 1 \text{ odd}, \\ r_2 + 1, & n = 1. \end{array}\right.$$

**Proof.** By computing the Euler characteristic of the six term exact sequence, the result follows from Theorems 1.12 and 1.13 below. Alternatively, the result follows for $\tilde{K}_*^2$ by the global Euler characteristic formula [Mil06 Tat63].

A complete evaluation of the rank of $\tilde{K}_*$ would follow (and is equivalent to) from the following conjecture, which is already implicit in the work of Soulé:

**Conjecture 1.9.** For all $n$, $\dim \tilde{K}_{2n-1}(\mathcal{O}) \otimes \mathbb{Q} = 0$. 

For \( n = 1 \), this is Leopoldt’s conjecture. For totally real fields, Conjecture 1.9 is equivalent to the non-vanishing of a certain \( p \)-adic zeta function. This equivalence follows from Theorem 3 of [Sou81] (see also Remark 3.4 p.399 of ibid). Actually, Soulé assumes that \( F \) is abelian, but the general case follows from the proof of the Quillen–Lichtenbaum conjecture (giving a cohomological description of the global and local \( K \) groups) together with the main conjecture for totally real fields proved by Wiles [Wil90]. In the general case, it is equivalent to showing that the kernel of the \( p \)-adic regulator map on \( K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q}_p \) is zero. The first non-trivial case of this conjecture for \( F = \mathbb{Q} \) is \( n = 3 \), where (as noted above) it “reduces” to the question of the non-vanishing of the Kubota–Leopoldt zeta function \( \zeta_p(3) \). One might actually make the stronger conjecture that this number is irrational (see [Cal05]). For more discussion of injectivity of localization maps, see [BBdJR09].

**Remark 1.10.** Since \( G_S \) has cohomological dimension 2 (with coefficients in \( \mathbb{Z}_p \) for \( p \neq 2 \)), the group \( H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n)) \) is divisible and hence its dual is torsion free. In particular, if Conjecture 1.9 is true, then this \( H^2 \) term vanishes, and

\[
\tilde{K}_{2n-1}^2(\mathcal{O}) = H^0(G_S, \mathbb{Q}_p/\mathbb{Z}_p(1-n))' \simeq H_0(G_S, \mathbb{Z}_p(n)) = \mathbb{Z}_p/(a^n - 1)\mathbb{Z}_p,
\]

where \( a \) is any topological generator of \( \mathbb{Z}_p^\times \). In particular, \( \tilde{K}_{2n-1}^2(\mathcal{O}) = 0 \) unless \( n \equiv 0 \mod p - 1 \).

More generally, we have the following estimates:

**Lemma 1.11.** Suppose that \( F \) is a number field of degree \( d \) and signature \((r_1, r_2)\). If \( p \) does not divide the order of the class group of \( F(\zeta_p) \), then, for \( n > 0 \), we have equalities:

\[
\dim \tilde{K}_n(\mathcal{O}) \otimes \mathbb{Q} = \begin{cases} 
 0, & n \equiv 0 \mod 2 \\
 1, & n \equiv 1 \mod 2 \\
 r_2, & n \equiv 3 \mod 4 \\
 r_1 + r_2, & n \equiv 2 \mod 4 \\
r_1 + r_2 - 1, & n = 1 \mod 4 \text{ and } n > 1
\end{cases}
\]

If \( r_2 = 0 \), so \( F \) is a totally real field of degree \( d \), then unconditionally

\[
\dim \tilde{K}_{4n-1}(\mathcal{O}) \otimes \mathbb{Q} = 0, \quad \dim \tilde{K}_{4n-2}(\mathcal{O}) \otimes \mathbb{Q} = d.
\]

In particular, \( \tilde{K}_2(\mathcal{O}) \otimes \mathbb{Q} = \tilde{H}_2 \otimes \mathbb{Q} = \mathbb{Q}_p^d \).

**Proof.** We begin by recalling Borel’s theorem (Prop 12.2 of [Bor74]):

**Theorem 1.12** (Borel). Suppose that \( F \) is a number field of degree \( d \) and signature \((r_1, r_2)\). For \( n > 0 \), we have equalities:

\[
\dim K_n(\mathcal{O}) \otimes \mathbb{Q} = \begin{cases} 
 0, & n \equiv 0 \mod 2 \\
 1, & n \equiv 1 \mod 4 \text{ and } n > 1 \\
 r_2, & n \equiv 3 \mod 4 \\
 r_1 + r_2, & n \equiv 2 \mod 4 \\
r_1 + r_2 - 1, & n = 1 \mod 4
\end{cases}
\]

We also have the following result (Theorem 61 of [Wei05]), which (as noted in ibid.) is essentially due to Wagoner and Milgram [Wag76] and Panin [Pan86]. (Alternatively, this result follows directly from local Tate duality and the Euler characteristic formula given the identification of these groups with Galois cohomology in Theorem 1.3.)
Theorem 1.13. Let $p$ be a rational prime, and let $F$ be a number field of degree $d$ and signature $(r_1, r_2)$. For $n > 0$, we have equalities:

$$\dim K_n(\mathcal{O}; \mathbb{Z}_p) \otimes \mathbb{Q} = \begin{cases} 
  d = r_1 + 2r_2, & n \equiv 1 \mod 2 \\
  0, & n \equiv 0 \mod 2.
\end{cases}$$

All the equalities now follow from a diagram chase, assuming that the maps $H^1(G_S, \mathbb{Z}_p(n)) \rightarrow \prod_{v|p} H^1(G_S, \mathbb{Z}_p(n))$ are injective after tensoring with $\mathbb{Q}$. Specifically, the groups $\tilde{K}_n(\mathcal{O})$ will then identified with the cokernel of the map from $K_{n+1}(\mathcal{O}; \mathbb{Z}_p) \otimes \mathbb{Q}$ to $K_n(\mathcal{O}; \mathbb{Z}_p) \otimes \mathbb{Q}$, and hence have dimension zero when $n$ is odd and, for $n > 1$, dimension $(r_1 + 2r_2) - (r_1 + r_2), \quad n \equiv 0 \mod 4,$

$(r_1 + 2r_2) - r_2, \quad n \equiv 2 \mod 4.$

The injectivity of the map of Galois cohomology groups is trivial when the group on the left is actually zero after tensoring with $\mathbb{Q}$, which accounts for the unconditional cases when $F$ is totally real and $r_2 = 0$. But any element in the kernel of the map on cohomology will give rise to unramified non-trivial extension classes of $\mathbb{Z}_p$ by $\mathbb{Z}_p(n)$. In particular, the first layer will generate an unramified extension of $F(\mathbb{F}_p(n)) \subseteq F(\zeta_p)$ of degree $p$, which would contradict the assumption on the class number. □

Note that Conjecture 1.9 would imply that the equality of ranks holds for all primes, not just regular ones. We also note that these regulator maps in the optic of Galois cohomology were also studied by Schneider [Sch79, Sch80]. In particular, by Satz 3 of §6 of [Sch79], one has:

Proposition 1.14. For any fixed number field field $F$ and prime $p$, Conjecture 1.9 is true for all but finitely many $n$.

1.3. Comparison of Homologies.

Lemma 1.15. Let $A_j$ and $B_j$ be a sequence of $\mathbb{F}_p$-modules with trivial $G$-action, and suppose that $A_0 = B_0 = \mathbb{F}_p$. Suppose that one has spectral sequences over $\mathbb{F}_p$:

$$EA^2_{i,j} := H_i(G, A_j) \Rightarrow C_{i+j}, \quad EB^2_{i,j} = H_i(G, B_j) \Rightarrow C_{i+j}$$

are induced from the maps

$$A_{i,j} = H_i(G, A_j) = H_i(G, \mathbb{F}_p) \otimes A_j = H_i(G, A_0) \otimes H_i(G, A_j) = EA^2_{i,j}. \quad \text{for all } m \geq 2$$

inducing an automorphism of $C_{i+j}$. Suppose that the maps

$$H_i(G, A_j) = EA^2_{i,j} \rightarrow EB^2_{i,j} = H_i(G, B_j)$$

are induced from the maps

$$A_j = EA^2_{0,j} \rightarrow EB^2_{0,j} = B_j.$$ 

Then there is an isomorphism $A_j \simeq B_j$ for all $j$.

Proof. Since $A_0$ is a trivial $G$-module, there is a canonical isomorphism

$$EA^2_{p,q} = H_p(G, A_q) = H_p(G, \mathbb{F}_p) \otimes A_q = H_p(G, A_0) \otimes H_0(G, A_q) = EA^2_{0,q} \otimes EA^2_{p,0}.$$
The assumed compatibility implies that there is a commutative diagram as follows:

\[
\begin{array}{ccc}
EA^2_{0,q} \otimes EA^2_{p,0} & \cong & EA^2_{p,q} \\
\downarrow & & \downarrow \\
EB^2_{0,q} \otimes EB^2_{p,0} & \cong & EB^2_{p,q}
\end{array}
\]

The result then follows from the Zeeman comparison theorem (Theorem 3.26 of [McC01]). (Note that, since the coefficient ring is a field, one may disregard the \(\text{Tor}_1\) terms.) \(\square\)

**Definition 1.16.** For a space \(X\), define the continuous homology groups \(H^\text{cont}_*(X, \mathbb{Z}_p)\) to be \(H^\text{cont}_*(X, \mathbb{Z}_p) := \text{proj lim} H_*(X, \mathbb{Z}/p^n)\).

**Theorem 1.17.** There are isomorphisms

\[
\tilde{H}_*(\text{SL}, \mathbb{F}_p) \cong \lim_{\leftarrow} H_*(Y_m, \mathbb{F}_p) \cong H_*(Y_\infty, \mathbb{F}_p),
\]

\[
\tilde{H}_*(\text{SL}, \mathbb{Z}/p^r \mathbb{Z}) \cong \lim_{\leftarrow} H_*(Y_m, \mathbb{Z}/p^r \mathbb{Z}) \cong H_*(Y_\infty, \mathbb{Z}/p^r \mathbb{Z}),
\]

\[
\tilde{H}_*(\text{SL}, \mathbb{Z}_p) \cong \lim_{\leftarrow} H^\text{cont}_*(Y_m, \mathbb{Z}_p) \cong H^\text{cont}_*(Y_\infty, \mathbb{Z}_p).
\]

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
B\Gamma(p^m) & \longrightarrow & B\text{SL}(\mathcal{O}) \\
\downarrow & & \downarrow \\
Y_m & \longrightarrow & B\text{SL}(\mathcal{O})^+ \\
\uparrow & & \uparrow \\
Y_\infty & \longrightarrow & B\text{SL}(\mathcal{O})^+
\end{array}
\]

We obtain corresponding maps:

\[
H_*(\Gamma(p^m), \mathbb{F}_p) \rightarrow H_*(Y_m, \mathbb{F}_p) \leftarrow H_*(Y_\infty, \mathbb{F}_p).
\]

Moreover, we have a natural map of spectral sequences:

\[
H_*(\text{SL}(\mathcal{O}/p^m), H_j(\Gamma(p^m), \mathbb{F}_p)) \Longrightarrow H_{i+j}(\Gamma, \mathbb{F}_p)
\]

(3)

**Sublemma 1.18.** The \(E^2_{i,j}\) and hence \(E^n_{i,j}\) terms of the spectral sequence

\[
H_*(\text{SL}(\mathcal{O}/p^m), H_j(Y_m, \mathbb{F}_p)) \Longrightarrow H_{i+j}(\Gamma, \mathbb{F}_p).
\]

have uniformly bounded dimension as \(m \rightarrow \infty\).
We suspect that a stronger claim holds, namely that the spectral sequence is constant for sufficiently large \( m \), but the boundedness is sufficient for our purposes.

**Proof.** We first assume that \( E^{2}_{i,0} = H_{i}(\text{SL}(\mathcal{O}/p^{m}), \mathbb{F}_{p}) \) is bounded for sufficiently large \( m \). The claimed result then follows immediately for the first row. We now proceed by induction on the rows. The general row consists of \( \dim H_{j}(Y_{m}, \mathbb{F}_{p}) \) copies of the first row, so it suffices to show that this dimension is uniformly bounded. However, if \( E^{2}_{j,0} = H_{j}(Y_{m}, \mathbb{F}_{p}) \) is unbounded and all the terms in lower rows are uniformly bounded, then \( E^{\infty}_{j,0} \) and hence \( H_{j}(\Gamma, \mathbb{F}_{p}) \) will also be unbounded, which is a contradiction. It thus remains to show that \( H_{i}(\text{SL}(\mathcal{O}/p^{m}), \mathbb{F}_{p}) \) is bounded as \( m \) increases. By classical stability \([\text{Cha80}]\), we may replace \( \text{SL}(\mathcal{O}/p^{m}) \) by \( \text{SL}_{N}(\mathcal{O}/p^{m}) \) for some \( N \) depending only on \( i \). Let \( \text{SL}_{N}(\mathcal{O}/p^{m}, p) \subset \text{SL}_{N}(\mathcal{O}/p^{m}) \) denote the kernel of the reduction map modulo \( p \). There is a spectral sequence:

\[
H_{i}(\text{SL}_{N}(\mathbb{F}_{p}), H_{j}(\text{SL}_{N}(\mathcal{O}/p^{m}, p), \mathbb{F}_{p})) \Rightarrow H_{i+j}(\text{SL}_{N}(\mathcal{O}/p^{m}), \mathbb{F}_{p}).
\]

It suffices to note that the coefficient system \( H_{j}(\text{SL}_{N}(\mathcal{O}/p^{m}, p), \mathbb{F}_{p}) \) is independent of \( m \) for \( m \geq 1 \) by Corollary 2.34 of \([\text{BP00}]\). \( \square \)

We can not deduce that \( H_{j}(\Gamma(p^{m}), \mathbb{F}_{p}) \simeq H_{j}(Y_{m}, \mathbb{F}_{p}) \) from equation 3, exactly because the action of \( \text{SL}(\mathcal{O}/p^{m}) \) on \( H_{j}(\Gamma(p^{m}), \mathbb{F}_{p}) \) is non-trivial. The key point is thus that, in the limit, the action of \( \text{SL}(\mathcal{O}_{p}) \) on \( \bar{H}_{j} \) is trivial by Theorem 0.1. The diagram above gives rise to a compatible map of spectral sequences

\[
\begin{array}{ccc}
EA^{2}_{i,j} = H_{i}(\text{SL}(\mathcal{O}_{p}), \bar{H}_{j}(\text{SL}, \mathbb{F}_{p})) & \Rightarrow & H_{i+j}(\Gamma, \mathbb{F}_{p}) \\
\downarrow & & \downarrow \\
EB^{2}_{i,j} = H_{i}(\text{SL}(\mathcal{O}_{p}), \lim_{\leftarrow} H_{j}(Y_{m}, \mathbb{F}_{p})) & \Rightarrow & H_{i+j}(\Gamma, \mathbb{F}_{p}) \\
\downarrow & & \downarrow \\
EC^{2}_{i,j} = H_{i}(\text{SL}(\mathcal{O}_{p}), H_{j}(Y_{\infty}, \mathbb{F}_{p})) & \Rightarrow & H_{i+j}(\Gamma, \mathbb{F}_{p})
\end{array}
\]

The inverse limit on the second term commutes with the construction of the spectral sequence because all the terms involved are uniformly bounded vector spaces over \( \mathbb{F}_{p} \) by Sublemma \([1.15]\), and so all inverse limits satisfy the Mittag–Leffler condition. The action of \( \text{SL}(\mathcal{O}_{p}) \) on \( H_{s}(Y_{m}, \mathbb{F}_{p}) \) and \( H_{j}(Y_{\infty}, \mathbb{F}_{p}) \) is trivial by construction, and the action on \( \bar{H}_{s} \) is trivial by Theorem 0.1. Hence, by Lemma \([1.15]\), one obtains isomorphisms \( \bar{H}_{j}(\text{SL}, \mathbb{F}_{p}) \simeq H_{j}(Y_{\infty}, \mathbb{F}_{p}) \). By dévissage, we obtain isomorphisms

\[
\bar{H}_{j}(\text{SL}, \mathbb{Z}/p^{r}\mathbb{Z}) \simeq \lim_{\leftarrow} H_{j}(Y_{m}, \mathbb{Z}/p^{r}\mathbb{Z}) \simeq H_{j}(Y_{\infty}, \mathbb{Z}/p^{r}\mathbb{Z})
\]

for all \( r \). Namely, we apply induction and compare the long exact sequences of homology associated to the short exact sequence

\[
0 \to \mathbb{Z}/p^{r-1}\mathbb{Z} \to \mathbb{Z}/p^{r}\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0,
\]

and then apply the 5-lemma. The groups \( \bar{H}_{j}(\text{SL}, \mathbb{Z}_{p}) \) are finitely generated over \( \mathbb{Z}_{p} \) (by Theorem 0.1), and hence coincide with the inverse limit of \( \bar{H}_{j}(\text{SL}, \mathbb{Z}/p^{r}\mathbb{Z}) \). \( \square \)
Note that the homology groups \( H_*(Y_{\infty}, \mathbb{Z}) \) are presumably quite badly behaved, thus \( H_*(Y_{\infty}, \mathbb{Z}) \otimes \mathbb{Z}_p \) presumably differs from \( H^*_{\text{cont}}(Y_{\infty}, \mathbb{Z}_p) \) in general.

**Remark 1.19.** Combining Conjecture \[1.5\] with Lemmas \[1.4\], \[1.8\] and Theorem \[1.17\], we see that we have constructed an infinite loop space whose homotopy groups with coefficients in \( \mathbb{Z}_p \) are given by \( \tilde{K}_*(\theta) \) and whose (continuous) homology groups are given by \( \tilde{H}_*(\text{SL}, \mathbb{Z}_p) \). This should be thought of as completely analogous to the classical story, where the infinite loop space \( K(\theta) \) has homotopy groups \( K_*(\theta) \) and homology groups \( H_*(\text{SL}, \mathbb{Z}) \).

**Remark 1.20.** Our methods may be extended in various natural ways. For example, one can take the completed cohomology groups \( \tilde{H}^* \) with respect to some subset of the primes dividing \( p \) in \( F \) (see \S\[4\]). One may also add a tame level structure \( M \) for \( (M, p) = 1 \), that is, take the limit over the congruence subgroups \( \Gamma(Mp^r) \). In the latter case, the answer will only depend on the radical of \( M \) (that is, the product of distinct primes dividing \( M \)), for reasons we now explain.

Now suppose that \( \text{by Herbrand's Theorem and class field theory. It follows that} \) \( K \) now follows rationally by Lemma \[1.4\], and also integrally under the assumption that \( \text{Lemma} \[1.4\]. \)

**Proof of Theorem \[0.2\].** As explained in the introduction, the assumption that \( p \) does not divide \( u_F \) implies, by Theorem 3.6 of \[BMS67\], that \( \tilde{H}_1(\text{SL}, \mathbb{Z}/p^r\mathbb{Z}) = 0 \) for all \( r \).

It follows that \( \pi_1(Y_{\infty}; \mathbb{Z}/p^r\mathbb{Z}) = 0 \) for all \( r \), and hence, via the Hurewicz map (for Hurewicz with coefficients, see \[Nei10\], Thm 9.7), we obtain isomorphisms

\[
\tilde{K}_2(\theta) := \pi_2(Y_{\infty}; \mathbb{Z}_p) = \lim_\leftarrow \pi_2(Y_{\infty}; \mathbb{Z}/p^r\mathbb{Z}) = \lim_\leftarrow H_2(Y_{\infty}, \mathbb{Z}/p^r\mathbb{Z}) = \tilde{H}_2(\text{SL}, \mathbb{Z}_p),
\]

where the last equality follows from Theorem \[1.17\] and the second equality was established in the proof of Lemma \[1.2\]. The isomorphism

\[
\tilde{H}_2(\text{SL}, \mathbb{Z}_p) = \tilde{K}_2(\theta) = \tilde{K}_2(\theta)^\vee := H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1))^\vee
\]

now follows rationally by Lemma \[1.4\] and also integrally under the assumption that \( p \) does not divide the order of \( K_2(\theta) \). The main statement of Theorem \[0.2\] is the Pontryagin dual of this equality. Now suppose that \( F = \mathbb{Q} \) and \( p \geq 3 \). Then \( |K_2(\mathbb{Z})| = 2 \) and \( H^1(G_S, \mathbb{F}_p(-1)) = \mathbb{F}_p \) by Herbrand’s Theorem and class field theory. It follows that

\[
\mathbb{F}_p = H^1(G_S, \mathbb{F}_p(-1)) = H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1))[p],
\]

and thus, by Pontryagin duality, that \( H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p)^\vee/p = \mathbb{F}_p \). By Lemma \[1.11\] the group \( H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1))^\vee \) is infinite (it has rank \( d = 1 \)). Hence, by Nakayama's Lemma, we deduce that \( H_1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1))^\vee = \mathbb{Z}_p \), and thus \( \tilde{H}_2(\text{SL}, \mathbb{Z}_p) = \mathbb{Z}_p \). □

**Remark 1.21. Homology with coefficients.** Let \( \mathcal{L}_N \) be any algebraic local system (with \( \mathbb{F}_p \) or \( \mathbb{Z}_p \) coefficients) for \( \text{SL}_N(\theta_q) \). Then one may also consider the completed homology groups \( \tilde{H}_*(\text{SL}_N, \mathcal{L}_N) \) = \( \lim H_*(\Gamma(p^r), \mathcal{L}_N) \). Since \( \mathcal{L}_N/p^m \) is trivial as a \( \Gamma(p^m) \)-module, the standard weight–level argument (Shapiro’s Lemma) implies that \( \tilde{H}_*(\text{SL}_N, \mathcal{L}_N) \simeq \).
Corollary 2.1. There are inequalities:

\[ d = r_1 + 2r_2 \geq \dim \tilde{H}_2(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q}, \]
\[ \dim \tilde{H}_2(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q} \geq r_1 + r_2 = d - \dim K_3(\ell) \otimes \mathbb{Q}. \]
By Theorem 0.2 (or rather the proof in §1.4), we obtain an isomorphism $\tilde{H}_2(\mathbb{Z}_p) = \tilde{K}_2(\mathcal{O})$, which therefore identifies $\tilde{H}_2(\text{SL}, \mathbb{Z}_p)$ with (the dual of) a certain Galois cohomology group. The reader unfamiliar with the complications in computing Galois cohomology groups may be surprised that this equality does not allow us to (greatly) improve the estimate of Corollary 2.1. The difficulty is that determining the ranks of these groups is effectively a generalization of Leopoldt’s conjecture, which appears to be very difficult.

2.2. The Hochschild–Serre Spectral Sequence II: higher terms. Let us suppose that $F = \mathbb{Q}$. One may play the spectral sequence game to obtain information concerning $\tilde{H}^d$ for higher $d$. Let $G = \text{SL}(\mathbb{Z}_p)$, so

$$H^*(G, \mathbb{Q}_p/\mathbb{Z}_p) \approx \mathbb{Q}_p/\mathbb{Z}_p \otimes \Lambda_{\mathbb{Z}_p}[x_3, x_5, x_7, \ldots].$$

We first consider $\tilde{H}^3$. Let us work (as in the last section) in the Serre category of co-finitely generated $\mathbb{Z}_p$-modules up to co-torsion modules (so every term is equivalent to a finite number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$). Since $\tilde{H}^2 \approx \mathbb{Q}_p/\mathbb{Z}_p$, the relevant terms of the spectral sequence are

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</tbody>
</table>

Yet $H^3(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$ is trivial (since $K_3(\mathbb{Z}) = \mathbb{Z}/48\mathbb{Z}$ is finite [LS76] — or alternatively we can see this from the computation of stable cohomology given by Theorem 1.12), and thus $\tilde{H}^3(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx 0$. Consider the next few terms. We obtain the following:

<table>
<thead>
<tr>
<th>$q$</th>
<th>(\tilde{H}^5)</th>
<th>(\mathbb{Q}_p/\mathbb{Z}_p)</th>
<th>(\mathbb{Q}_p/\mathbb{Z}_p)</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>$\mathbb{Q}_p/\mathbb{Z}_p$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Since \( K_1(\mathbb{Z}) \otimes \mathbb{Q} \) is trivial and \( K_5(\mathbb{Z}) \otimes \mathbb{Q} \) has rank one, we deduce that either \( \tilde{H}^4(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx \mathbb{Q}_p/\mathbb{Z}_p \) and \( \tilde{H}^5(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx 0 \) or \( \tilde{H}^4(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx (\mathbb{Q}_p/\mathbb{Z}_p)^2 \) and \( \tilde{H}^5(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx \mathbb{Q}_p/\mathbb{Z}_p \). In particular, we have that

\[
\dim \tilde{H}_4(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q} - \dim \tilde{H}_5(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q} = 1.
\]

We can not rule out either possibility, just as we cannot rule out that \( \tilde{K}_4(\mathbb{Z}) \otimes \mathbb{Q} \neq 0 \) or \( \tilde{K}_5(\mathbb{Z}) \otimes \mathbb{Q} \neq 0 \); however, assuming the conjectural part of Lemma 1.11, we deduce that \( \tilde{H}^4(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \approx \mathbb{Q}_p/\mathbb{Z}_p \) and \( \tilde{H}^5 \approx 0 \).

### 2.3. Rational completed cohomology groups.

We may define rational completed cohomology groups \( \tilde{H}^* (\text{SL}, \mathbb{Q}_p) \) as follows. Let \( \tilde{H}^*(\text{SL}, \mathbb{Z}/p^n\mathbb{Z}) := \lim_\longrightarrow H^*(\Gamma(p^r), \mathbb{Z}/p^n\mathbb{Z}) \) and

\[
\tilde{H}^*(\text{SL}, \mathbb{Z}_p) = \lim_\longrightarrow \lim_\longleftarrow H^*(\Gamma(p^r), \mathbb{Z}/p^{s}\mathbb{Z}) = \lim_\longrightarrow \tilde{H}^*(\text{SL}, \mathbb{Z}/p^s\mathbb{Z}).
\]

This allows us to define rational completed cohomology groups as follows:

\[
\tilde{H}^*(\text{SL}, \mathbb{Q}_p) := \tilde{H}^*(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q}_p, \quad \tilde{H}_*(\text{SL}, \mathbb{Q}_p) := \tilde{H}_*(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q}_p.
\]

We have (see Theorem 1.1 of [CE12]) an exact sequence:

\[
0 \to \text{Hom}(\tilde{H}^*[1](\text{SL}, \mathbb{Z}_p)[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{H}^*(\text{SL}, \mathbb{Z}_p) \to \text{Hom}(\tilde{H}^*(\text{SL}, \mathbb{Z}_p), \mathbb{Z}_p) \to 0,
\]

where \( M[1] \) denotes the usual shift of \( M \). The modules \( \tilde{H}^*(\text{SL}, \mathbb{Z}_p) \) are finitely generated and so the first term of this sequence is torsion; tensoring with \( \mathbb{Q} \) we obtain:

\[
\tilde{H}^*(\text{SL}, \mathbb{Q}_p) \simeq \text{Hom}(\tilde{H}_*(\text{SL}, \mathbb{Z}_p), \mathbb{Q}_p).
\]

(Wagoner and Milgram considers similar completed cohomology groups when studying the continuous algebraic K-theory of local fields, see [Wag76], p.244.) There is naturally an identification of \( \tilde{H}_*(\text{SL}, \mathbb{Q}_p) \) with \( H_*^\text{cont}(Y_\infty, \mathbb{Z}_p) \otimes \mathbb{Q}_p = H_*^\text{cont}(SK(\mathcal{O}, \mathcal{O}_p), \mathbb{Z}_p) \otimes \mathbb{Q}_p \) by Theorem 1.17. We may denote the latter group by \( H_*^\text{cont}(Y_\infty, \mathbb{Q}_p) \).

### 2.4. The Eilenberg–Moore spectral sequence.

A more direct way to compute the cohomology of the fibre from the cohomology of the total space and the cohomology of the base is by using the Eilenberg–Moore spectral sequence. This is especially practical in this case since we know the cohomology of \( BSL(\mathbb{Z})^+ \) and \( BSL(\mathbb{Z})^+ \) with coefficients in \( \mathbb{Q}_p \), namely, as exterior algebras \( \Lambda_{\mathbb{Q}_p}[x_3, x_5, x_7, \ldots] \) and \( \Lambda_{\mathbb{Q}_p}[\eta_5, \eta_9, \eta_{13}, \ldots] \) respectively. (In the former case, we are taking the continuous cohomology as defined at the end of the previous section.) In particular, we take the inverse limits of the Eilenberg–Moore spectral sequences associated to equation (1)

\[
SK(\mathcal{O}, p^m) \longrightarrow SK(\mathcal{O}) \longrightarrow SK(\mathcal{O}/p^m)
\]

with coefficients in \( \mathbb{Z}/p^n\mathbb{Z} \) take the corresponding inverse limit in \( r \), and then tensor with \( \mathbb{Q} \). (Relevant here is Proposition 1 of [Wag76].) Here we use the fact that the cohomology of \( SK(\mathcal{O}, p^m) \) with coefficients in \( \mathbb{Z}/p^n\mathbb{Z} \) is uniformly bounded and that the inverse limit over \( \mathbb{Z}/p^n\mathbb{Z} \) recovers the homology groups \( \tilde{H}(\text{SL}, \mathbb{Z}_p) \) by Theorem 1.17. The uniform boundedness of these groups ensures that all inverse limits are Mittag-Leffler and so there are no issues with the derived functor \( \lim^1 \), which will always vanish. What is less obvious, however, is the structure of \( \Lambda_{\mathbb{Q}_p}[\eta_5, \eta_9, \eta_{13}, \ldots] \) as a module for \( \Lambda_{\mathbb{Q}_p}[x_3, x_5, x_7, \ldots] \). The natural supposition is (up to scaling by a non-zero constant depending on \( n \)) that \( x_{4n-1} \) acts as zero and \( x_{4n+1} \) acts as \( \eta_{4n+1} \). The latter claim, however, is equivalent to showing that the \( p \)-adic
regulator map \(K_{4n+1}(\mathbb{Z}; \mathbb{Z}_p) \otimes \mathbb{Q}_p \to K_{4n+1}(\mathbb{Z}_p; \mathbb{Z}_p) \otimes \mathbb{Q}_p\) is an isomorphism. To see this equivalence, note that \(K(\mathbb{Z})\) and \(K(\mathbb{Z}_p)\) are infinite loop spaces (and hence \(H\)-spaces), and so the Milnor–Moore theorem ([MM65], Appendix) identifies the rational classes in \(K\)-theory with the rational primitive classes in homology under the Hurewicz map. More precisely, for \(K(\mathbb{Z}_p)\), we use the \(p\)-adic Milnor–Moore theorem, which gives the corresponding relationship between the classes in \(K_*(\mathbb{Z}_p; \mathbb{Z}_p) \otimes \mathbb{Q}\) and the rational primitive classes in the continuous homology of \(G = \text{SL}(\mathbb{Z}_p)\) (see Proposition 3 of \textit{ibid} and the subsequent arguments; indeed, this is how Wagoner and Milgram computed the groups \(K_*(\mathbb{Z}_p; \mathbb{Z}_p) \otimes \mathbb{Q}\) in the first place.)

As previously noted, however, showing that maps between global and local \(K\)-groups are injective is a problem whose difficulty may be of a similar level to Leopoldt’s conjecture (see Conjecture 1.9 and the subsequent remarks).

One context in which we know this map is an isomorphism is for regular primes, by Lemma 1.11. Thus we make the following Ansatz:

\(\text{(\star)} \quad \text{Either } p \text{ is regular, or } F = \mathbb{Q} \text{ and Conjecture 1.9 holds.}\)

Under this assumption, the Eilenberg–Moore spectral sequence allows for a complete computation of \(\check{H}^*(\mathbb{Q}_p)\). Specifically, we have a spectral sequence:

\[
E_{2}^{r,s} = \text{Tor}_{s}^{\Lambda_{\mathbb{Q}_p}}(\mathbb{Q}, \Lambda_{\mathbb{Q}_p}[\eta_5, \eta_9, \eta_{13}, \ldots]) \Rightarrow \check{H}^*(\text{SL}, \mathbb{Q}_p) \\
\simeq \text{Tor}_{s}^{\Lambda_{\mathbb{Q}_p}}(\mathbb{Q}, \mathbb{Q}_p) \Rightarrow \check{H}^*(\text{SL}, \mathbb{Q}_p) \\
\simeq \bigotimes \text{Tor}_{s}^{\Lambda_{\mathbb{Q}_p}}(\mathbb{Q}, \mathbb{Q}_p) \Rightarrow \check{H}^*(\text{SL}, \mathbb{Q}_p).
\]

This sequence converges because the fundamental group of the base is abelian (since it comes from a + construction). The first isomorphism requires the assumption that the module structure arises from a surjection of exterior algebras \(\Lambda_{\mathbb{Q}_p}[x_s] \to \Lambda_{\mathbb{Q}_p}[\eta_s]\), which is exactly the content of assumption (\(\star\)). On the other hand, the final sequence degenerates and may be computed explicitly; the limit is given by a tensor product of the polynomial algebras on \(x_{4n-1}\), shifted by 1 degree. In particular, we have the following³.

\textbf{Theorem 2.2.} Let \(F = \mathbb{Q}\), and assume that either \(p\) is regular or Conjecture 1.9 holds. Then there is an isomorphism \(\check{H}^*(\text{SL}, \mathbb{Q}_p) \simeq \mathbb{Q}_p[x_2, x_6, x_{10}, \ldots].\) In particular, there is an equality \(\dim \check{H}^n(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q}_p = 0\) unless \(n\) is even, and \(\dim \check{H}^n(\text{SL}, \mathbb{Z}_p) \otimes \mathbb{Q}_p\) is the coefficient of \(q^n\) in

\[
\prod_{k=1}^{\infty} \frac{1}{(1 - q^{4k-2})} = \prod_{k=1}^{\infty} (1 + q^{2k}) = 1 + q^2 + 4q^4 + 2q^6 + 2q^8 + 3q^{10} + 4q^{12} + 5q^{14} + 6q^{16} + 8q^{18} + \ldots
\]

\textbf{Remark 2.3. Where do the Borel classes go?} The maps \(H^*(\Gamma, \mathbb{Q}_p) \to H^*(\Gamma(p^r), \mathbb{Q}_p)\) are isomorphisms (in the stable range) for all \(r\), and yet, under the assumptions of Theorem 2.2, \(\check{H}^*(\text{SL}, \mathbb{Q}_p) = 0\) in all odd degrees; how do we reconcile these two statements? The explanation is that the classes must become infinitely \(p\)-divisible up the congruence tower. In

³In particular, the rational cohomology conjecturally coincides with the stable rational cohomology of the classical group \(\text{Sp}_{2n}(\mathbb{Z})\). This is most likely a coincidence; the analogous computation for a general number field \(F\) of signature \((r_1, r_2)\) yields (conditionally on the analogue of (\(\star\))):

\[
\check{H}^*(\text{SL}, \mathbb{Q}_p) = \bigotimes_{r_1} \mathbb{Q}_p[x_2, x_6, x_{10}, \ldots] \otimes \bigotimes_{r_2} \mathbb{Q}_p[x_2, x_4, x_6, \ldots],
\]

which is not the stable cohomology of \(\text{Sp}_{2n}(\mathbb{Q})\) as soon as \(F\) is not totally real, and nor is it the stable rational cohomology of any classical group.
general, the divisibility of Borel classes in the limit is equivalent to Conjecture 1.9 by Proposition 1.14; this holds for all but finitely many of the primitive classes. This is in contrast to what happens for $\text{Sp}_{2n}(\mathbb{Z})$. According to Deligne [Del78], the stable class in degree two does not become $p$-divisible for any $p$ up the congruence tower. A back of the envelope calculation suggested by the methods of this paper indicates that the same should be true for all stable classes in $\text{Sp}_{2n}(\mathbb{Z})$.

**Remark 2.4.** Since $SK(\mathcal{O}; \mathcal{O}_p)$ is an infinite loop space and thus an $H$-space, one way to interpret Theorem 2.2 is simply to note that for $H$-spaces, the homology is rationally a polynomial algebra on its homotopy. However, one must be slightly careful with such a statement, since our homotopy groups are with respect to coefficients in $\mathbb{Z}_p$, and are homology and cohomology groups with coefficients in $\mathbb{Q}_p$ are defined in terms of inverse limits (and are thus "continuous" (co-)homology groups). A similar remark applies to the cohomology ring of $\text{SL}(\mathcal{O}_p)$ and the $K$-theory of local fields with coefficients in $\mathbb{Z}_p$.

**2.5. The Hochschild–Serre Spectral Sequence III: determining the differentials.**

We have an isomorphism $H^*(G) \otimes \mathbb{Q}_p = \Lambda[x_3, x_5, x_7, \ldots]$. We assume in this section that there is an isomorphism $\tilde{H}^*(\text{SL}, \mathbb{Q}_p) \simeq \mathbb{Q}_p[x_2, x_6, x_{10}, \ldots]$ as in §2.4. However, in order to avoid confusion, we shall use different notation, and in particular we shall write:

$$\tilde{H}^*(\text{SL}, \mathbb{Q}_p) \simeq \mathbb{Q}_p[\hat{x}_2, \hat{x}_6, \hat{x}_{10}, \ldots].$$

It follows that the second page of the Hochschild–Serre spectral sequence $E_{s+t}^2$ is given by

$$\mathbb{Q}_p[\hat{x}_2, \hat{x}_6, \hat{x}_{10}, \ldots] \otimes \Lambda[x_3, x_5, x_7, \ldots]$$

We may write this out on page two as follows:

<table>
<thead>
<tr>
<th>q</th>
<th>10 \hat{x}_2^5, \hat{x}_2^2 \hat{x}<em>6, \hat{x}</em>{10}</th>
<th>9 0 0</th>
<th>8 \hat{x}_2^4, \hat{x}_2 \hat{x}_6 0 0</th>
<th>7 0 0 0 0</th>
<th>6 \hat{x}_2^3 \hat{x}_6 0 0 \hat{x}_2^3 x_3, \hat{x}_6 x_3 0</th>
<th>5 0 0 0 0 0 0</th>
<th>4 \hat{x}_2^2 0 0 \hat{x}_2^2 x_3 0 \hat{x}_2^2 x_5 0</th>
<th>3 0 0 0 0 0 0 0 0</th>
<th>2 \hat{x}_2 0 0 \hat{x}_2 x_3 0 \hat{x}_2 x_5 0 \hat{x}_2 x_7 \hat{x}_2 x_3 x_5</th>
<th>1 0 0 0 0 0 0 0 0 0</th>
<th>0 1 0 0 x_3 0 x_5 0 x_7 x_3 x_5 x_9 x_3 x_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10 11</td>
<td>p</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Koszul complex associated to a polynomial algebra induces a spectral sequence

$$\mathbb{Q}_p[\hat{x}_2, \hat{x}_6, \hat{x}_{10}, \ldots] \otimes \Lambda[x_3, x_5, x_7, \ldots] = \Lambda[x_5, x_9, x_{13}, \ldots] \otimes (\mathbb{Q}_p[\hat{x}_2, \hat{x}_6, \hat{x}_{10}, \ldots] \otimes \Lambda[x_3, x_7, x_{11}, \ldots]) \Rightarrow \Lambda[x_5, x_9, x_{13}, \ldots].$$
given explicitly by \( d^r = 0 \) unless \( r = 4n - 1 \), in which case \( d^{4n-1}(\hat{x}_{4n-2}) = x_{4n-1} \) and is zero otherwise. For example, page 4 is given as follows:

\[
\begin{array}{cccccccccccccccc}
q & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\hline
0 & \hat{x}_6^2 & 0 & 0 & 0 & 0 & \hat{x}_6 x_5 & 0 & \hat{x}_6 x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & \hat{x}_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & x_5 & 0 & x_7 & 0 & x_9 & 0 & x_{11} & x_5 x_7 & x_{13} & \ldots \\
\end{array}
\]

Under our assumptions, there is a natural map of spectral sequences from this to Hochschild–Serre provided that one knows the maps \( d^{4n-1}(\hat{x}_{4n-2}) = x_{4n-1} \) coincides with the corresponding maps in Hochschild–Serre (equivalently, the \( y_{4n-2} \) are transgressive). By induction, this reduces to showing that the maps \( K_{4n-1}(\mathbb{Z}_p; \mathbb{Z}_p) \to \tilde{K}_{4n-2}(\mathbb{Z}, \mathbb{Z}_p; \mathbb{Z}_p) \) are rational isomorphisms, which we certainly assumed in §2.4 in order to compute \( \tilde{H}^*(\mathbb{Q}_p) \) in the first place. Thus, by the Zeeman comparison theorem, these two spectral sequences coincide ([Zee58], see also Theorem 3.27 of [McC01]).

### 3. Classical Cohomology Groups

We turn in this section to some explicit computations. Let \( \Gamma_N = \text{SL}_N(\mathcal{O}_F) \) for some \( N \) which is sufficiently large so that \( \tilde{H}_*(\text{SL}_N, \mathbb{Z}_p) = \tilde{H}_*(\text{SL}_N, \mathbb{Z}_p) \) for \( * \) in the range of computation (this is \( * \leq 3 \) except in section 3.2). Let \( G_N = \text{SL}_N(\mathbb{Z}_p) \). Benson Farb asked (personal communication) whether one can compute the homology groups \( H_2(\Gamma(p), \mathbb{F}_p) \) — we give an complete answer below. The information about such groups is encoded in the completed cohomology groups \( \tilde{H}^* \) together with the differentials of the Hochschild–Serre spectral sequence. This exercise probably only serves to indicate why the completed homology groups \( \tilde{H}_* \) and completed \( K \)-groups \( \tilde{K}_* \) are more natural objects of study than their finite (unstable) analogues.

We make the following assumptions on \( p \):

\( (**): \ p \) does not divide \( w_F \), and \( \tilde{H}^2(\text{SL}, \mathbb{Q}_p/\mathbb{Z}_p) \simeq H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1)) \).
By Theorem 0.2 this holds if \( p \) does not divide \( w_F|K_2(\mathcal{O}_F)| \), and conjecturally always holds for \( p > 2 \). In fact, we shall state the theorems below with the stronger hypothesis \( p \nmid w_F|K_2(\mathcal{O}_F)| \), but we only use the assumption above. Let \( d = [F : \mathbb{Q}] \). Because \( p \) does not divide \( w_F \), we have \( \tilde{H}^1(\text{SL}_N, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \) (once more using [BMS67]), and thus, from the long exact sequence of completed cohomology corresponding to the short exact sequence of modules:

\[
0 \to \mathbb{F}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0,
\]

we find that

\[
\tilde{H}^2(\text{SL}_N, \mathbb{F}_p) = \tilde{H}^2(\text{SL}_N, \mathbb{Q}_p/\mathbb{Z}_p)[p] = H^1(G_S, \mathbb{Q}_p/\mathbb{Z}_p(-1))[p] \simeq H^1(G_S, \mathbb{F}_p(-1)).
\]

The Hochschild–Serre spectral sequence \( H^i(G_N(p^m), \tilde{H}^j(\text{SL}_N, \mathbb{F}_p)) \Rightarrow H^{i+j}(\Gamma_N(p^m), \mathbb{F}_p) \) gives rise in the usual way (using \( \tilde{H}^1(\text{SL}_N, \mathbb{F}_p) = 0 \)) to the exact sequence:

\[
0 \to H^2(G_N(p^m), \mathbb{F}_p) \to H^2(\Gamma_N(p^m), \mathbb{F}_p) \to H^1(G_S, \mathbb{F}_p(-1)) \to H^3(G_N(p^m), \mathbb{F}_p) \to H^3(\Gamma_N(p^m), \mathbb{F}_p).
\]

Consider the cohomology ring \( H^*(G_N(p^m), \mathbb{F}_p) \). Since \( p > 2 \), we deduce that the group \( G_N(p^m) \) is \( p \)-powerful, and (by a theorem of Lazard) that \( G_N(p^m) \) is a Poincaré homology group of dimension \( \dim(G_N) = d(N^2 - 1) \). More explicitly, by Lazard [Laz55] (Ch V, 2.2.6.3 and 2.2.7.2, p.167), we deduce that the cohomology of \( G_N(p^m) \) is an exterior algebra on the degree one classes:

\[
H^*(G_N(p^m), \mathbb{F}_p) \simeq \wedge^*(H^1(G_N(p^m), \mathbb{F}_p)).
\]

The group \( G_N/G_N(p^m) = \text{SL}_N(\mathcal{O}/p^m) \) acts naturally on this ring, the action factors through \( \text{PSL}_N(\mathcal{O}/p) \). Let us denote \( H^1(G_N(p^m), \mathbb{F}_p) \) by \( M \) as an \( \text{SL}_N(\mathcal{O}/p) \)-module. We have

\[
M \simeq \text{Hom}(H_1(G_N(p^m), \mathbb{F}_p), \mathbb{F}_p) \simeq \text{Hom}(M^0_N(\mathbb{F}_p), \mathbb{F}_p),
\]

where \( M^0_N(\mathbb{F}_p) \) denotes the matrices of trace zero. We deduce that there is an exact sequence:

(4) \[
0 \to \wedge^2 M \to H^2(\Gamma_N(p^m), \mathbb{F}_p) \to H^1(G_S, \mathbb{F}_p(-1)) \to \wedge^3 M.
\]

Since the map \( H^3(G_N(p^m), \mathbb{F}_p) \to H^3(G_N(p^{m+1}), \mathbb{F}_p) \) is zero for all \( m \geq 1 \), we also deduce that:

**Lemma 3.1.** Suppose that \( p \) does not divide \( w_F|K_2(\mathcal{O}_F)| \). For \( m \geq 1 \), there is an exact sequence:

\[
0 \to \wedge^2 M \to H^2(\Gamma_N(p^m), \mathbb{F}_p) \to H^1(G_S, \mathbb{F}_p(-1)) \to 0.
\]

For \( m \geq 2 \), this sequence splits.

**Proof.** By the computation above, we have exact sequences and commutative diagrams as follows:

\[
\begin{array}{cccccc}
0 & \to & \wedge^2 M & \to & H^2(\Gamma_N(p^m), \mathbb{F}_p) & \to & H^1(G_S, \mathbb{F}_p(-1)) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \wedge^2 M & \to & H^2(\Gamma_N(p^{m+1}), \mathbb{F}_p) & \to & H^1(G_S, \mathbb{F}_p(-1))
\end{array}
\]

where \( m > 1 \). The equality in the third column follows from the fact that the invariants of \( \tilde{H}^2(\text{SL}_N, \mathbb{F}_p) \) under \( G_N(p^m) \) do not depend on \( m \), because the entire group \( G_N \) acts trivially. On the other hand, the maps on the first and last columns are identically zero, because they are induced from the map of groups

\[
G_N(p)/G_N(p^2) \to G_N(p^m)/G_N(p^{m+1}),
\]
which is the zero map. In particular, the vanishing of the map in the last column implies by commutativity that the image of $H^1(G_S, \mathbb{F}_p(-1))$ in $\wedge^3 M$ must be trivial, which shows that the sequence of the lemma is exact. On the other hand, the vanishing of the map in the first row implies that the quotient $H^2(\Gamma_N(p), \mathbb{F}_p)/\wedge^2 M = H^1(G_S, \mathbb{F}_p(-1))$ maps into $H^2(\Gamma_N(p^m), \mathbb{F}_p)$ in a way that is compatible with the map to $H^1(G_S, \mathbb{F}_p(-1))$, which splits the sequence for $m > 1$. □

3.1. A mild improvement of Lemma 3.1

**Lemma 3.2.** Suppose that $p > 3$ splits completely in $F$ and does not divide the order of $K_2(\mathcal{O}_F)$ or $K_3(\mathcal{O}_F)$. Then, for all $m$, there is an isomorphism

$$H^2(\Gamma_N(p^m), \mathbb{F}_p) \simeq \wedge^2 M \oplus (\mathbb{F}_p)^d.$$ 

**Remark 3.3.** The condition on $p$ is never satisfied unless $F$ is totally real, since otherwise $K_3(\mathcal{O}_F)$ is infinite. If $F$ is totally real, however, then the assumptions hold for all but finitely many $p$ that split completely in $F$. For example, if $F = \mathbb{Q}$, then the assumptions hold for $p > 3$. The splitting condition should not be necessary — we assume it in order to invoke a theorem of [EF82] concerning the homology of $GL_n(\mathbb{F}_p)$ of some particular module; the methods of [EF82] should also apply to $GL_n(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$, which would be sufficient provided that $p$ is unramified in $F$.

**Proof.** The assumptions on $p$ imply that $H^3(\Gamma_N, \mathbb{F}_p) = H^3(\Gamma_N, \mathbb{F}_p) = 0$, and hence that $\tilde{H}^2(SL_N, \mathbb{F}_p) \simeq H^3(G_N, \mathbb{F}_p) \simeq (\mathbb{F}_p)^d$. Consider the classical Hochschild–Serre spectral sequence

$$H^i(SL_N(\mathcal{O}/p), H^j(\Gamma_N(p), \mathbb{F}_p)) \Rightarrow H^{i+j}(\Gamma_N, \mathbb{F}_p).$$

As representations for $SL_N(\mathcal{O}/p)$, we have $H^1(\Gamma_N(p), \mathbb{F}_p) = M$ and $H^0(\Gamma_N(p), \mathbb{F}_p) = \mathbb{F}_p$. By Quillen’s computation of $K_*(\mathbb{F}_q)$ in [Qui72] (Theorem 6), if $p$ is unramified in $F$, then $K_n(\mathcal{O}/p) \otimes \mathbb{F}_p = 0$ for $n > 0$, or equivalently that $H^n(SL_N(\mathcal{O}/p), \mathbb{F}_p) = 0$ for all $n > 0$. Since $M$ has no $SL_N(\mathcal{O}/p)$-invariants, the second page of the sequence is as follows:

<table>
<thead>
<tr>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H^0(\mathbb{F}_p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^2(\Gamma_N(p), \mathbb{F}_p)$</td>
</tr>
<tr>
<td>$H^1(\mathbb{F}_p, M)$</td>
</tr>
<tr>
<td>$H^2(\mathbb{F}_p, M)$</td>
</tr>
</tbody>
</table>

| $0$ |
| $1$ |
| $2$ |
| $3$ |
| $4$ |

0 0 0 0 0

Since $H^2(\Gamma_N, \mathbb{F}_p) = 0$ and $H^3(\Gamma_N, \mathbb{F}_p) = 0$ by assumption, we deduce that there is an isomorphism:

$$H^0(\mathbb{F}_p, H^2(\Gamma_N(p), \mathbb{F}_p)) \simeq H^2(\mathbb{F}_p, M).$$

By Proposition 3.0 of [EF82], there is an isomorphism: $H_2(\mathbb{F}_p, M^*) \simeq \mathbb{Z}/p\mathbb{Z}$, which implies that $H^2(\mathbb{F}_p, M) = \mathbb{Z}/p\mathbb{Z}$, and hence that $H^2(\mathbb{F}_p, M)$ has dimension at
least one. Explicitly, since $GL_N(F_p)/SL_N(F_p) = F_p^\times$ has order prime to $p$, the Hochschild–Serre spectral sequence degenerates and one has an isomorphism:

$$H^s(GL_N(F_p), M) = H^s(SL_N(F_p), M)F^\times,$$

and so a lower bound for the former group gives a lower bound for the latter. Since $p$ is totally split in $F$, we have $SL_N(\mathcal{O}/p) = \prod_{v|p}SL_N(F_p)$, and thus $H^2(SL_N(\mathcal{O}/p), M)$ has dimension at least $[F:Q]$. Since $H^0(SL_N(\mathcal{O}/p), \wedge^2 M) = 0$, from the exact sequence 4, we deduce that there is an injection:

$$H^0(SL_N(\mathcal{O}/p), H^2(\Gamma_N(p), F_p)) \to H^1(G_S, F_p(-1)) \simeq (F_p)^[F:Q].$$

The latter equality relies on the fact that $p$ does not divide $w_F[K_2(\mathcal{O}_F)]$; however, we are assuming that $p$ does not divide the order of $K_2(\mathcal{O}_F)$, and $p > 2$ cannot divide $w_F$ because $F$ is totally real. As we just proved the left hand side has dimension at least $[F:Q]$, this map is an isomorphism, which therefore splits 4, proving the lemma.

This allows us to answer the question of Farb:

**Corollary 3.4.** Let $p > 3$, and let $\Gamma_N(p)$ denote the congruence subgroup of $SL_N(Z)$. Then for sufficiently large $N$, there are isomorphisms of $SL_N(F_p)$-modules:

$$H^2(\Gamma_N(p), F_p) \cong \wedge^2 M \oplus F_p, \quad H_2(\Gamma_N(p), F_p) \cong \wedge^2 M^* \oplus F_p,$$

where $M^*$ is isomorphic to $M^0_N(F_p)$. In particular, $\dim H^2(\Gamma_N(p), F_p) = \left(\frac{N^2 - 1}{2}\right) + 1$.

3.2. **Remarks on cohomology in higher degrees.** One may continue computations as above in higher degrees, although the analysis becomes more and more intricate. Consider, for example, the group $H^3(\Gamma_N(p^m), F_p)$ when $F = Q$. For $p > 3$, we should have $\widetilde{H}_3(F_p) = 0$. The Hochschild–Serre spectral sequence $H_i(G_N(p^m), \widetilde{H}_j(SL_N, F_p)) \Rightarrow H_{i+j}(\Gamma_N(p^m), F_p)$ would then yield:

| $q$ | 0 0 0 |
| 3  | $F_p$ M $\wedge^2 M$ |
| 2  | $F_p$ M $\wedge^2 M$ |
| 1  | 0 0 0 0 |
| 0  | $F_p$ M $\wedge^2 M$ $\wedge^3 M$ $\wedge^4 M$ |

Since $H^2(\Gamma_N(p^m), F_p) = \wedge^2 M \oplus F_p$, this would give an exact sequence:

$$0 \to \wedge^3 M \to H^3(\Gamma_N(p^m), F_p) \to M \to \wedge^4 M.$$

In this case, the map $H^3(\Gamma_N(p^m), F_p) \to H^3(\Gamma_N(p^{m+1}), F_p)$ is zero. Hence the order of $H^3(\Gamma_N(p^m), F_p)$ should be the order of $\wedge^3 M$, up to an error that is bounded by the order of $M$. More generally, we have:
Lemma 3.5. For $F = \mathbb{Q}$, all primes $p$, and all sufficiently large $N$, the natural edge map:
\[ \wedge^k M = H^k(G_N(p^m), F_p) \to H^k(\Gamma_N(p^m), F_p) \]
has kernel and cokernel whose dimensions are $O(N^{2(k-2)})$, where the implied constant does not depend on $N$. In particular,
\[ \dim H^k(\Gamma_N(p^m), F_p) = \binom{N^2 - 1}{k} + O(N^{2(k-2)}) = \frac{N^{2k}}{k!} - \binom{k+1}{2} \frac{N^{2k-2}}{k!} + O(N^{2k-4}) \]

Proof. This follows from the Hochschild–Serre spectral sequence, noting that the first row is zero (since $\check{H}^1(SL_N, F_p) = 0$).

Remark 3.6. It follows from Theorem 1.5 of [CEFN14] that, for all $F$ and $p$, the quantity $\dim H^k(\Gamma_N(p^m), F_p)$ (in the stable range) is actually a polynomial in $N$.

Remark 3.7. One obtains a corresponding result for number fields, except $\check{H}^1(SL_N, F_p)$ does not vanish in general if $F$ contains $p$th roots of unity; and thus one only obtains the estimate (without any assumption on $p$ or $F$):
\[ \dim H^k(\Gamma_N(p^m), F_p) = \frac{N^{2kd}}{k!} + O(N^{2kd-2}). \]

Remark 3.8. One may sensibly define the stable homology groups $H_*^{\text{stab}}(\Gamma_N(p^m), \mathbb{Z}_p)$ and $H_*^{\text{stab}}(\Gamma_N(p^m), F_p)$ for sufficiently large $N$ to be the images of $\check{H}_*(SL_N, \mathbb{Z}_p)$ and $\check{H}_*(SL_N, F_p)$ in $H_*(\Gamma_N(p^m), \mathbb{Z}_p)$ and $H_*(\Gamma_N(p^m), F_p)$ respectively. Our results certainly imply that the action of $G_N/G_N(p^m)$ on $H_*^{\text{stab}}(\Gamma_N(p^m), \mathbb{Z}_p)$ is trivial in the stable range. Moreover, one expects that these groups should be directly related (perhaps even equal) to the continuous homology groups $H_*^{\text{cont}}(\mathcal{S}_m, \mathbb{Z}_p)$, which then relate to the $K$-theory (with coefficients) of the rings $\mathbb{Z}/p^m \mathbb{Z}$. Indeed, one can see a reflection of the calculation of these groups (in the simplest situations) in the computations above.

4. Partially Completed $K$-groups

Suppose that $F$ is a number field, and that $p | j$ is a prime in $\mathcal{O}_F$. Then one can apply the analysis above to the partially completed homology groups
\[ \check{H}_s(SL_N, p, \mathbb{Z}_p) := \lim_{\leftarrow} H_s(\Gamma_N(p^m), \mathbb{Z}_p). \]

(More generally, one can do this for any set of primes dividing $p$.) The arguments of [CELI] show that these groups do not depend on $N$ for sufficiently large $N$ (and have a trivial action of $SL_N(\mathcal{O}_p)$) and for such $N$ we write $\check{H}_s(SL, p, \mathbb{Z}_p)$. Although we no longer have recourse to the Poitou–Tate sequence (which requires that $S$ contain all places above $p$), it turns out that these groups are even simpler to understand than the standard completed cohomology groups under the following favorable circumstances. In particular, we shall find contexts in which all the partially completed homology groups (beyond $\check{H}_0$) vanish identically.

Let $F/\mathbb{Q}$ be an imaginary quadratic field. Suppose that $p$ splits in $F$, and let $v | p$ be a place above $p$. Let $\Phi$ be the Galois group of the maximal pro-$p$ extension of $F(\zeta_p)$ over $F$ unramified outside the places $S$ above $p$. Let $D_v$ be the decomposition group at $v$. 

Definition 4.1. Let $p$ be a prime that splits in an imaginary quadratic field $F/\mathbb{Q}$. The prime $p$ is \textbf{very regular} if the map:

$$\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to D_v \subseteq \Phi$$

is surjective for either $v|p$.

If the result is true for one $v|p$, then it is also true for the other (the images of $D_v$ for $v|p$ are permuted by the action of $\text{Gal}(F/\mathbb{Q})$.)

Lemma 4.2. Suppose that $p$ is very regular. Then the map $H^i(G_S, M) \to H^i(D_v, M)$ is an isomorphism for all $i$ and all $G_S$-modules $M$ that are sub-quotients of the Tate twists $\mathbb{Z}_p(n)$ for any $n \in \mathbb{Z}$.

Before proving this lemma, we note the following consequence, which is the reason for considering very regular primes.

Corollary 4.3. If $p$ is a very regular prime, then the map $K_n(\mathcal{O}_F) \otimes \mathbb{Z}_p \to K_n(\mathcal{O}_p; \mathbb{Z}_p)$ is an isomorphism for $n > 1$.

This follows from the description of these groups in terms of Galois cohomology (Theorem 1.3) and the fact that the corresponding map on Galois cohomology is an isomorphism.

Proof of Lemma 4.2. For such modules $M$ as in the statement of the lemma, there is a canonical isomorphism $H^1(G_S, M) \simeq H^1(\Phi, M)$, because the cohomology groups will only depend on the pro-$p$ completion of the fixed field of any relevant module. There is a natural isomorphism $H^0(G_S, M) = H^0(D_v, M_v)$, and the map:

$$H^1(G_S, M) \to H^1(D_v, M_v),$$

is injective by inflation–restriction. Let $\Sigma$ denote the set of Selmer conditions where no condition is imposed at $v|p$, and the dual Selmer condition $\Sigma^\ast$ consists of classes that are totally trivial locally at $v|p$, and the classes are unramified outside $p$. We have $H^1_\Sigma(F, M) = H^1(G_S, M)$ by definition, where $H^1_\Sigma(F, M)$ as usual denotes classes in $H^1(F, M)$ which satisfy the local conditions corresponding to $\Sigma$. Since $p$ is very regular, the group $H^1_\Sigma(F, M^\ast) = 0$ because any such class is trivial in $H^1(D_v, M^\ast)$. To this point, we have not used the fact that $F$ is an imaginary quadratic field. This condition arises in the numerical computation of Selmer groups via the formula of Greenberg–Wiles [Gre89, Wil95, DDT97] and the global Euler characteristic formula [Mil06, Tat63]. From the Greenberg–Wiles formula, we have

$$|H^1(G_S, M)| = \frac{|H^0(F, M)|}{|H^0(F, M^\ast)|} \cdot \frac{|H^1(D_v, M)|^2}{|H^0(D_v, M)|^2} \cdot \frac{1}{|M|}.$$ 

From the local Euler characteristic formula,

$$|H^1(D_v, M)| = |H^0(D_v, M)| \cdot |H^2(D_v, M)| \cdot |M| = |H^0(D_v, M)| \cdot |H^0(D_v, M^\ast)| \cdot |M|,$$
and thus

\[ |H^1(G_S, M)| = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|^2}{|H^0(D_v, M)|^2} \cdot \frac{1}{|M|} = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|}{|H^0(D_v, M)|} \cdot \frac{1}{|M|} = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|}{|H^0(D_v, M^*)|} \cdot \frac{1}{|M|} = |H^1(D_v, M)|.

It follows that \( H^1(G_S, M) \to H^1(D_v, M) \), which we already showed was an injection, is actually an isomorphism. On the other hand, by the global and local Euler characteristic formulae,

\[ |H^2(G_S, M)| = \frac{|H^1(G_S, M)|}{|H^0(G_S, M)|} \cdot \frac{H^0(G_C, M)}{|M|^2} = \frac{|H^1(D_v, M)|}{|H^0(D_v, M)|} \cdot \frac{1}{|M|} = |H^2(D_v, M)|.

By the 5-lemma and dévissage, to prove the lemma it suffices to show that the maps

\[ H^2(G_S, F_p(n)) \to H^2(D_v, F_p(n)) \]

are injective (equivalently, isomorphisms) for any \( n \in \mathbb{Z} \) (we use here the fact that the local and global cohomology groups have cohomological dimension 2 in this context). The only non-trivial case (i.e., when both groups are not trivial) is \( n = 1 \). In this case, there is a commutative diagram:

\[
\begin{CD}
H^1(G_S, V) @>>> H^1(G_S, F_p) @>>> H^1(G_S, F_p(1)) \\
| @| @| @| \\
H^1(D_v, V) @>>> H^1(D_v, F_p) @>>> H^1(D_v, F_p(1))
\end{CD}
\]

where \( V \) is any non-split extension of \( F_p \) by \( F_p(1) \). Since \( V \) is non-split, \( H^0(V^*) = 0 \), and hence \( H^2(V) = 0 \). Thus the maps \( H^1(F_p) \to H^1(F_p(1)) \) are surjective, and it follows (by commutativity) that map \( H^1(G_S, F_p(1)) \to H^1(D_v, F_p(1)) \) must be surjective, hence an isomorphism. \(\square\)

We now give a numerical characterization of very regular primes.

**Lemma 4.4.** Let \( F \) be an imaginary quadratic field such that \( p > 2 \) splits completely, and let \( v \mid p \) be a place above \( p \). Let \( w \) be the unique place above \( v \) in \( F(\zeta_p) \). Suppose that:

1. The maximal exponent \( p \) abelian extension of \( F(\zeta_p) \) that is unramified outside \( w \) is cyclic.
2. If \( p = pp \in \mathcal{O}_F \) and \( h = |\text{Cl}(\mathcal{O}_F)| \), then the projection of \( p^h \) generates \((1+p)/(1+p^2)\).

Then \( p \) is very regular. Conversely, if either of these conditions fail, then \( p \) is not very regular.
Proof. To show the map of pro-p groups $D_w \to \Phi$ is surjective, it suffices to prove that $D_w$ surjects onto the Frattini quotient of $\Phi$. The cokernel of this map corresponds to an abelian exponent $p$ extension of $F(\zeta_p)$ that is unramified at $w$ and at all primes away from $p$. If the cokernel is not cyclic, then there exists a quotient where $w$ splits completely, and $p$ is not very regular. Let us assume it is cyclic. By class field theory, $F$ itself admits an abelian $p$-extension $H/F$ unramified at $v$ and all primes away from $p$, and hence the extension over $F(\zeta_p)$ descends to $F$. It suffices to show that this extension is inert at $v$ if and only if the second condition holds. Equivalently, it suffices to show that the maximal abelian $p$-extension of $F$ completely split at $v|p$ and unramified outside $p$ is trivial. Let $p$ and $\overline{p}$ be the primes in $\mathfrak{O}_F$ corresponding to the place $v|p$ and its conjugate. Recall that the $p$-part of the ray class group of conductor $\overline{p}^2$ lives in an exact sequence:

$$\mathfrak{O}_F^\times \cap (1 + \overline{p}) \to (1 + \overline{p})/(1 + \overline{p}^2) \to \text{RCl}(\overline{p}^2) \otimes \mathbb{Z}_p \to \text{Cl}(\mathfrak{O}_F) \otimes \mathbb{Z}_p \to 0.$$ 

Since $F$ is an imaginary quadratic field, $\mathfrak{O}_F^\times$ consists entirely of roots of unity, and so the first term $\mathfrak{O}_F^\times \cap (1 + \overline{p})$ vanishes because we are assuming that $p$ is prime to $w_F$ (this follows because $p > 2$ is unramified in $F$). It suffices to show that $\text{RCl}(\overline{p}^2) \otimes \mathbb{Z}_p$ is generated by $p$, since, by Nakayama, any generator of $\text{RCl}(\overline{p}^2) \otimes \mathbb{Z}_p$ lifts to a generator of $\text{RCl}(\overline{p}^m) \otimes \mathbb{Z}_p$ for any $m$. Since $\text{RCl}(\overline{p}^2) \otimes \mathbb{Z}_p$ is an abelian $p$-group of order $p \cdot h_p$ (where $h_p$ is the $p$-part of $h$), it is cyclic with generator $p$ if and only if $p^h$ has order $p$. Yet $p^h$ is non-trivial in $\text{RCl}(\overline{p}^2) \otimes \mathbb{Z}_p$ if and only if its projection to $(1 + \overline{p})/(1 + \overline{p}^2)$ is non-trivial. \(\square\)

Note that the projection of any principal ideal $(\alpha)$ onto $(1 + \overline{p})/(1 + \overline{p}^2)$ is given by the image of $\alpha^{p-1}$.

We give an alternate numerical formulation of the previous lemma:

**Lemma 4.5.** Let $F$ be an imaginary quadratic field such that $p > 2$ splits completely, and let $v|p$ be a place above $p$. Let $\chi$ be the odd quadratic character corresponding to $F$. Then $p$ is very regular if and only if the following hold:

1. $p$ is a regular prime, that is, $p > 3$ does not divide $\zeta(-1), \zeta(-3), \zeta(-5), \ldots, \zeta(4 - p)$.
2. $p$ does not divide any of the $L$-values $L(\chi, -2), L(\chi, -4), L(\chi, -6), \ldots, L(\chi, 3 - p)$.
3. If $p = p\overline{p}$ in $\mathfrak{O}_F$ and $h = |\text{Cl}(\mathfrak{O}_F)|$, then the projection of $p^h$ generates $(1 + \overline{p})/(1 + \overline{p}^2)$.

**Proof.** It suffices to show that the conditions are equivalent to those of Lemma 4.4. If either $p$ is not regular or divides one of the $L$-values, then, by the main conjecture of Iwasawa theory [Wil90], there exists an unramified extension of $F_p$ by either $F_p(n)$ or $F_p(n) \otimes \chi$ for some $n \neq 0, 1 \mod (p - 1)$. Since $H^1(Q_p, F_p(n)) = 0$ for such $n$, these extensions necessarily split completely at primes above $p$, implying that $p$ is not very regular. By Lemma 4.4 and its proof the same is true of the final condition.

Now suppose that $p$ is not very regular. Equivalently, the subspace of $H^1(F(\zeta_p), F_p)$ consisting of classes that are trivial away from $p$ and for some $w$ with $w|p$ is non-zero. The group $\text{Gal}(F(\zeta_p)/F)$ acts on this space, and the third condition above implies that the projection of this space onto $H^1(F, F_p)$ is trivial. This implies the same for $H^1(F, F_p(1))$ by the Greenberg–Wiles formula. Hence there exists a non-trivial $n \neq 0, 1 \mod (p - 1)$ such that

$$H^1(G_{F,S}, F_p(n)) \simeq H^1(G_{Q,S}, F_p(n)) \oplus H^1(G_{Q,S}, F_p(n) \otimes \chi)$$

contains a class $\eta$ that is unramified at $v|p$. Suppose that $\dim H^1(G_{F,S}, F_p(n)) = 1$. Then $\eta \in H^1(Q, V)$ where $V$ is either $F_p(n)$ or $F_p(n) \otimes \chi$. As these spaces are $\text{Gal}(F/Q)$-invariant, it follows that $\eta$ is also unramified at the other prime dividing $p$. Now either:
(1) Complex conjugation acts on $V$ by $-1$, in which case by the main conjecture one obtains a divisibility of $L$-values as above, or

(2) Complex conjugation acts by $+1$ on $V$, in which case, by the Greenberg–Wiles formula, there exists an everywhere unramified class in $H^1(Q, V^*)$, that also implies a divisibly of $L$-values (again by the main conjecture).

Suppose now that $\dim H^1(G_{F,S}, F_p(n)) > 1$. If either $H^1(G_{Q,S}, F_p(n))$ or $H^1(G_{Q,S}, F_p(n) \otimes \chi)$ is zero, the same argument as above applies. Hence we may assume that $H^1(G_{Q,S}, V) \neq 0$ where complex conjugation acts by 1 on $V$, and the result follows as in the previous case. □

The following result shows that the partially completed homology groups at very regular primes are particularly simple:

**Lemma 4.6.** Suppose that $p$ is very regular in an imaginary quadratic field $F/\mathbb{Q}$. Let $p$ denote a prime above $p$. Then the stable $p$-completed homology groups:

$$\tilde{H}_n(SL, p, \mathbb{Z}_p) = \lim_{\to} H_n(SL, \Gamma(p^r), \mathbb{Z}_p)$$

are trivial for all $n > 0$, and equal to $\mathbb{Z}_p$ for $n = 0$.

**Proof.** The analogues of the completed $K$-groups in these contexts are the homotopy groups with coefficients in $\mathbb{Z}_p$ of the homotopy fibres $K(\theta, \theta_p; \mathbb{Z}_p)$ and $SK(\theta, \theta_p; \mathbb{Z}_p)$. Let

$$\tilde{K}_n(\theta, p) := \tilde{K}_n(\theta, \theta_p; \mathbb{Z}_p) := \pi_n(K(\theta, \theta_p; \mathbb{Z}_p))$$

$$\tilde{SK}_n(\theta, p) := \tilde{SK}_n(\theta, \theta_p; \mathbb{Z}_p) := \pi_n(SK(\theta, \theta_p; \mathbb{Z}_p))$$

for $n > 0$. By Corollary 4.3, the natural map $SK_n(\theta_F; \mathbb{Z}_p) \to SK_n(\theta_p; \mathbb{Z}_p) = SK_n(Z_p; \mathbb{Z}_p)$ is an isomorphism for all $n \geq 2$, and (because both vanish) the map is also an isomorphism for $n = 1$. Hence, from the Serre exact sequence, the completed homotopy groups $\tilde{SK}_n(\theta, p)$ vanish for all $n$. The result follows by the Hurewicz theorem. □

This lemma allows us to explicitly compute all the classical congruence homology groups at powers of a very regular prime in the stable range.

**Theorem 4.7.** Suppose that $p$ is very regular in an imaginary quadratic field $F/\mathbb{Q}$, and fix an integer $n$. Let $p$ denote a prime above $p$, let $\Gamma_N = SL_N(\theta_F)$, let $G_N = SL_N(\theta_p) = SL_N(\mathbb{Z}_p)$. Then, for all sufficiently large $N$, there are, for all positive integers $m$, isomorphisms

$$H_n(\Gamma_N(p^m), \mathbb{Z}_p) \simeq H_n(G_N(p^m), \mathbb{Z}_p),$$

$$H_n(\Gamma_N(p^m), F_p) \simeq H_n(G_N(p^m), F_p) \simeq \wedge^m M_N^0(F_p),$$

where $M_N^0(F_p)$ denotes the trace zero $N \times N$ matrices with coefficients in $F_p$. Moreover, these isomorphisms respect the $G_N/G_N(p^m)$ and $G_N/G_N(p)$ = $SL_N(F_p)$-module structures respectively.

**Proof.** This follows immediately from the Hochschild–Serre spectral sequence, since $E_{ij}^2 = 0$ unless $i = 0$, and hence the sequence degenerates immediately. Note that the vanishing of $\tilde{H}_*(SL_N, p, \mathbb{Z}_p)$ implies the vanishing of $H_*(SL_N, p, F_p)$. □

**Remark 4.8.** Algebraic local systems of $GL$ for an imaginary quadratic field are algebraic representations of $GL_N(\mathbb{C})$ as a real group, in particular, they are direct sums of representations of the form $V_\mu \otimes \overline{V}_\lambda$, where $V_\mu$ and $V_\lambda$ are algebraic representations of $GL_N$ of highest weight $\mu$ and $\lambda$ respectively, and where the bar indicates the action of $GL_N(\mathbb{C})$ is composed with complex conjugation. If $\mathcal{L}$ corresponds to such a representation with $\lambda = 0$, then
\( \mathcal{L} \) becomes locally trivial up the \( p \)-adic tower, and hence the coefficients can be pulled out as in Remark 1.2. In particular, for such local systems, we have, under the conditions of Theorem 4.7 isomorphisms

\[
H_n(\Gamma_N(\mathfrak{p}^m), \mathcal{L}) \simeq H_n(G_N(p^m), \mathcal{L} \otimes_{\mathcal{O}_F} \mathbb{Z}_p).
\]

We note that (as it must) the algebra \( H^*(G_N, \mathbb{Q}_p) \) as \( N \to \infty \) coincides with the stable cohomology groups as computed by Borel. The groups \( H^*(G_N, \mathcal{L} \otimes \mathbb{Q}_p) \) vanish for non-trivial \( \mathcal{L} \) as can be seen by considering the action of the infinitesimal character. This is consistent with the vanishing of \( H^*(\Gamma_N, \mathcal{L} \otimes \mathbb{Q}_p) \) in the stable range, as follows (essentially) from a similar computation.

We observe that very regular primes do exist. For any particular fixed field \( F \), we can give a relatively fast and explicit algorithm for computing regular primes using generalized Bernoulli numbers. We give some details in the case \( F = \mathbb{Q}(\sqrt{-1}) \). Recall that the Bernoulli numbers \( B_{2n} \) and Euler numbers \( E_{2n} \) are defined by the following Taylor series:

\[
\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} \cdot t^n, \quad \frac{2}{e^t + e^{-t}} = \sum_{n \geq 0} \frac{E_n}{n!} \cdot t^n.
\]

**Lemma 4.9.** If \( F = \mathbb{Q}(\sqrt{-1}) \), then \( p \equiv 1 \mod 4 \) is very regular if and only if the following conditions are satisfied:

1. The prime \( p \) divides neither \( B_{2n} \) nor \( E_{2n} \) for \( 2n \) less than \( p - 1 \).
2. If \( p = a^2 + b^2 \), then \((4ab)^{p-1} \not\equiv 1 \mod p^2\).

**Proof.** The first condition follows from the formulæ

\[
\zeta(1 - 2n) = \frac{B_{2n}}{2n}, \quad L(\chi_4, -2n) = \frac{E_{2n}}{2n},
\]

the second is equivalent to the condition \( \alpha^{p-1} \not\equiv 1 \mod \alpha^2 \) where \( \alpha = a + bi \). \( \square \)

The non-regular primes \( p \) less than 100 that split in \( \mathbb{Q}(\sqrt{-1}) \) are \( p = 37 \) (which divides \( B_{92} \)) and \( p = 61 \), which divides:

\[
\frac{-\pi^7}{6! \cdot 27} \cdot L(\chi_4, -6) = L(\chi_4, 7) = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \ldots = 61 \cdot \frac{\pi^7}{6! \cdot 27}.
\]

Using Lemmas 4.4, 4.5 and 4.9 we compute the following examples for \( p = 3 \) and \( |d_F| < 200 \) as well as primes less than 100 for the ten smallest imaginary quadratic fields.

**Example 4.10.** If \( F = \mathbb{Q}(\sqrt{-|d_F|}) \), then \( p = 3 \) is very regular for a fundamental discriminant \( |d_F| < 200 \) if and only if \(-d_F \) is one of the following integers:

\[
\]

**Remark 4.11.** Following [CLS1], for an imaginary quadratic field \( F \), one may consider a heuristic model of the \( p \)-class group \( \text{Cl}(\mathcal{O}_F) \otimes \mathbb{Z}_p \) as the quotient of a free \( \mathbb{Z}_p \)-module with \( N \) relations, where \( N \) goes to infinity. Correspondingly, one might model the ray class group of conductor \( \mathfrak{p} \) for \( p \) which is completely split by allowing an extra generator for ramification at \( \mathfrak{p} \), and imposing a further relation by demanding local triviality at \( p \) (via the image of the Artin map). In this case, one would predict that the probability
that \( p \) fails to satisfy the given condition is exactly the same as the probability that the \( p \)-part of the class group is trivial, namely:

\[
\prod_{n=0}^{\infty} \left(1 - \frac{1}{p^n}\right).
\]

For example, if \( p = 3 \), then \( p \) is very regular if and only if it satisfies the final condition of Lemma 4.5, and this calculation suggests that the density of such quadratic fields (amongst those in which 3 splits) is

\[
\prod_{n=0}^{\infty} \left(1 - \frac{1}{3^n}\right) \sim 0.560126 \ldots
\]

For such fields of fundamental discriminant \(|d_F| < 1,000, 10,000, 100,000, \) and 300,000, the corresponding percentage of fields in which 3 is very regular is 68.1%, 64.6%, 61.2%, and 59.8% respectively. (This is consistent with the slow monotone rate of convergence often observed in Cohen–Lenstra phenomena, and also with the fact that the apparent error term is positive.) There are natural maps:

\[
\text{RCl}(\mathcal{O}_F^\infty)/p \otimes \mathbb{Z}_p \to \text{Cl}(\mathcal{O}_F)/p \otimes \mathbb{Z}_p \leftarrow \text{Cl}(\mathcal{O}_F) \otimes \mathbb{Z}_p.
\]

From our discussion, the two outer groups should have a similar distribution. The group in the middle, however, having one further relation, should behave like the \( p \)-part of the class group of a real quadratic field (this exact remark is made in the last paragraph of \S 8 of [CLS84]).

**Example 4.12.** For the 10 smallest imaginary quadratic fields, we have the following table:

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Here a tick indicates that $p$ is very regular for $F$, a blank square indicates that $p$ is either inert or ramified in $F$, an even integer $2n$ means that $p$ fails to be regular because $p$ divides $L(\chi, -2n)$, an odd integer $2n - 1$ means that $p$ divides $\zeta(1 - 2n)$, and the crosses indicate the failure of condition 3 of Lemma 4.5 explicitly, $\alpha^2 - 1 \equiv 1 \mod \pi^2$, where

$$\alpha = \frac{7 + \sqrt{-3}}{2}, \quad \frac{3 + \sqrt{-11}}{2}, \quad \frac{5 + \sqrt{-19}}{2}$$

respectively — note that $h_F = 1$ in each case.

It is interesting to compare the case $F = Q(\sqrt{-2})$ and $3 = \mathfrak{p}\mathfrak{p}$ to computations of non-stable completed cohomology groups in the work of the author and Dunfield [CD06] (see also [BE06]). In particular, for $N = 2$, one may show in this case that $\tilde{H}_n(p, \mathbb{Z}_p) = 0$ for all $n > 0$. Moreover, in the context of [CE11], it is theoretically possible that $\tilde{H}_n(p, \mathbb{Z}_p) = 0$ for very regular primes for the same $F$ and $p$ for other values of $N$, since one has the necessary “numerical coincidence” between dimensions of locally symmetric spaces and $p$-adic lie groups:

$$\dim \text{SL}_N(O_F) / \text{SL}_N(C) / \text{SU}_N(C) = N^2 - 1 = \dim \text{SL}_N(\mathbb{Z}_p).$$

However, under the assumption that for sufficiently large $N$ there will exist weight 0 regular cuspidal automorphic representations of level 1 for $F = Q$, one would anticipate the existence of non-stable characteristic zero classes, which would imply that $\tilde{H}_n(p, \mathbb{Z}_p)$ can only vanish over the entire non-stable range only for finitely many $N$ (for all $F$ and $p$).

**Remark 4.13.** Natural heuristics suggest that for any imaginary quadratic field $F$, there are infinitely many very regular primes (possibly with density $e^{-1} = 0.367879\ldots$). This seems hard to prove, however, since being very regular implies that $p$ is regular in $Q$, and the infinitude of regular primes is a well known open question.

**Remark 4.14.** There is a natural generalization of very regular to a prime $p$ that splits completely in a totally imaginary CM field $F/F^+$. Here one replaces a single place $v | p$ by a collection $T$ of $[F^+ : Q]$ places $v | p$, and asks that the map:

$$\prod_{v | T} \text{Gal}(Q_p/O_p) \rightarrow \prod_{v | T} D_v \subseteq \Gamma$$

is surjective, where $\Gamma$ is the Galois group of the pro-$p$ extension of $(\zeta_p)$ unramified outside $p$.

**Remark 4.15.** In contrast to the case of regular primes, it seems difficult to control the module $\tilde{H}_d(\text{SL}_n, \mathbb{Z}_p)$ for a prime $p$ that splits in $F$, even for $d = 2$. An elementary computation shows that $\tilde{H}_2(\text{SL}_n, \mathbb{Z}_p)$ is infinite if and only if $L_p(\chi, 2) \neq 0$. (Wagoner computed that the associated regulator map was non-zero for $F = Q(\sqrt{-3})$ and primes $p$ of prime norm $\leq 73$ in [Wag84].) As with $(\zeta_p(3) \in Q_p$, it seems difficult to prove that $L_p(\chi, 2) \neq 0$ in general, even for a fixed $p$ (and varying imaginary quadratic character $\chi$).

**References**


Thomas Church and Benson Farb, \textit{Hecke operators on stable cohomology}, preprint.

Thomas Church, Jordan S. Ellenberg, and Benson Farb, \textit{Hecke operators on stable cohomology}, preprint.

Thomas Church, Jordan S. Ellenberg, and Benson Farb, \textit{Hecke operators on stable cohomology}, preprint.

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Thomas Church and Benson Farb, \textit{Hecke operators on stable cohomology}, preprint.


LS76 ———, \textit{The group $K_2(\mathbb{Z})$ is cyclic of order forty-eight}, Ann. of Math. (2) \textbf{104} (1976), no. 1, 31–60. MR 0442934 (56 #1309)


