Completed Cohomology — A Survey

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This note summarizes the theory of p-adically completed cohomology. This construction was first introduced in paper [4] (although insufficient attention was given there to the integral aspects of the theory), and then further developed in the papers [2] and [6]. The papers [4] and [2] may give the impression that p-adically completed cohomology is some sort of auxiliary construction that can be used to prove theorems (of either a p-adic or classical nature) about automorphic forms. However, we believe that p-adically completed cohomology is in fact an object of fundamental importance, and that it provides the best approximation that we know of to spaces of p-adic automorphic forms. (In particular, unlike the spaces that go by this name that are sometimes constructed by arithmetico-geometric means in the theory of modular curves, or more generally Shimura varieties, p-adically completed cohomology admits a representation of the p-adic group, and thus allows the introduction of representation-theoretic methods into the study of p-adic properties of automorphic forms.)

A systematic exposition of the theory, and of its (largely conjectural, at this point) applications to the p-adic aspects of the Langlands correspondence between automorphic eigenforms and Galois representations, will be given in the paper [3]. These notes provide a summary of some of the basic points of the theory, as well as one of the main conjectures of [3] (Conjecture 6.1 below).

1 Definitions

Let G_0 be a pro-finite group, assumed to admit a countable basis of neighbourhoods of the identity, consisting of normal open subgroups, say

$$\cdots \subset G_r \subset \cdots \subset G_1 \subset G_0.$$

Suppose given a tower of topological spaces

$$\cdots \to X_r \to \cdots \to X_1 \to X_0,$$

each equipped with an action of G_0, such that:

1. The maps $X_{r+1} \to X_r$ are G_0-equivariant.
2. G_r acts trivially on X_r, and realizes X_r as a G_0/G_r-torsor over X_0. (In particular, all the maps in the tower are finite coverings.)

In this context we may define the p-adically completed homology and cohomology modules attached to the tower $X_\bullet$ (having fixed the prime p), namely:

$$\tilde{H}_\bullet := \varprojlim_r H_\bullet(X_r, \mathbb{Z}_p)$$

and

$$\tilde{H}^\bullet := \varprojlim_s \varinjlim_r H^\bullet(X_r, \mathbb{Z}/p^s\mathbb{Z}).$$

We can also consider Borel-Moore and compactly supported variants:

$$\tilde{H}_\bullet^{BM} := \varprojlim_r H_\bullet^{BM}(X_r, \mathbb{Z}_p)$$

and

$$\tilde{H}_c^\bullet := \varprojlim_s \varinjlim_r H_c^\bullet(X_r, \mathbb{Z}/p^s\mathbb{Z}).$$

There are natural maps $\tilde{H}_\bullet \to \tilde{H}_\bullet^{BM}$ and $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

From now on we suppose that each X_r is a manifold which is homotopic to a finite simplicial complex. This ensures us that all the homology spaces (usual or Borel–Moore) and cohomology spaces (usual or compactly supported) of X_r that we have written down are finitely generated over the indicated ring of coefficients, and that they are related by appropriate forms of duality.[1]

We also suppose from now on that G_0 is a p-adic analytic group. Without this hypothesis, it seems impossible to control the inverse limits involved in our constructions. On the other hand, with this hypothesis, one can control these limits in a very satisfactory way. Indeed, Lazard [7] has proved that completed group ring $\mathbb{Z}_p[[G_0]]$ is Noetherian, and Schneider and Teitelbaum [8] have explain how this result can be applied to control various analytic difficulties that arise in the study of p-adic Banach-space representations of G_0.

In particular, we have the following result (which strengthens the results stated in [4], in so far as it deals explicitly with homology as well as cohomology, and pays attention to the integral aspects of the constructions, and not just to the objects obtained after tensoring up with $\mathbb{Q}_p$).

Theorem 1.1. 1. *The natural action of $\mathbb{Z}_p[[G_0]]$ on $\tilde{H}_\bullet$ and $\tilde{H}_\bullet^{BM}$ makes each of these spaces a finitely generated left module over $\mathbb{Z}_p[[G_0]]$. Furthermore, the canonical topology on each of these modules (obtained by writing it as a quotient of a finite direct sum of copies of $\mathbb{Z}_p[[G_0]]$) coincides with projective limit topology.*

   Note that this implies in particular that each of the torsion submodules $\tilde{H}_\bullet[p^\infty]$ and $\tilde{H}_\bullet^{BM}[p^\infty]$ is finitely generated over $\mathbb{Z}_p[[G_0]]$, and so in particular, is of bounded exponent.

2. *The spaces $\tilde{H}^\bullet$ and $\tilde{H}_c^\bullet$ are p-adically complete and their p-power torsion submodules have bounded exponent.*

3. *There are short exact sequences*

$$0 \to \mathrm{Hom}_{\mathrm{cont}}(\tilde{H}_{\bullet-1}, \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{H}^\bullet \to \mathrm{Hom}_{\mathrm{cont}}(\tilde{H}_\bullet, \mathbb{Z}_p) \to 0$$

[1] The paper [6] provides a sheaf-theoretic description of $\tilde{H}^\bullet$ and $\tilde{H}_c^\bullet$, which allows for more flexibility in the basic set-up of the theory than we permit here. However, we will have no need for this extra generality in the applications that we have in mind.

(*here* cont *means continuous with respect to the canonical topology on the source, and the evident topology — either discrete or p-adic — on the target*) *and*

$$0 \to \mathrm{Hom}(\tilde{H}^{\bullet+1}[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{H}_\bullet \to \mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p) \to 0$$

(*since the natural topology on $\tilde{H}^\bullet$ is the p-adic topology, and since all $\mathbb{Z}_p$-linear maps are automatically p-adically continuous, there is no need to explicitly specify any continuity conditions on the maps appearing in this exact sequence*)*, as well as similar short exact sequences relating $\tilde{H}_\bullet^{BM}$ and $\tilde{H}_c^\bullet$. These exact sequences are compatible with the maps $\tilde{H}_\bullet \to \tilde{H}_\bullet^{BM}$, and the maps $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.*

4. *For each $r \geq 0$, there are Hochschild–Serre type spectral sequences*

$$E_2^{i,j} := H^i(G_r, \tilde{H}^j) \implies H^{i+j}(X_r, \mathbb{Z}_p)$$

and

$$E_2^{i,j} := H^i(G_r, \tilde{H}_c^j) \implies H_c^{i+j}(X_r, \mathbb{Z}_p),$$

compatible with the maps $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

Remark 1.2. If M is any continuous G_0-module over $\mathbb{Z}_p$, then we may associate a family of local systems $\mathcal{M}_\bullet$ on the tower $X_\bullet$ to M, and there is an analogue of part (4) of the theorem for the cohomology of the local system $\mathcal{M}_r$.

Remark 1.3. The theorem shows the advantages of working with both homology and cohomology. It is on the homology side that the algebraic aspects of the theory are most transparent, while the cohomology side is better adapted to comparison with the situation at finite levels.

It is useful to make some additional definitions on the cohomology side, as follows:

$$\hat{H}^\bullet := \text{ the } p\text{-adic completion of } \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p) = \varprojlim_s \big(\varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)/p^s H^\bullet(X_r, \mathbb{Z}_p)\big)$$

and

$$T_p H^\bullet := \text{ the } p\text{-adic Tate module of } \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p) = \varprojlim_s \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)[p^s]$$

(the transition maps being given by multiplication by p). Note that the p-adic completion kills the p-divisible part of $\varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)$, while the p-adic Tate module knows only about the torsion subgroup of this divisible part. There is an exact sequence

$$0 \to \hat{H}^\bullet \to \tilde{H}^\bullet \to T_p H^{\bullet+1} \to 0. \tag{1}$$

Since $T_p H^{\bullet+1}$ is torsion-free and p-adically complete, we may take the $\mathbb{Z}_p$ dual of this exact sequence to obtain a short exact sequence

$$0 \to \mathrm{Hom}(T_p H^{\bullet+1}, \mathbb{Z}_p) \to \mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p) \to \mathrm{Hom}(\hat{H}^\bullet, \mathbb{Z}_p) \to 0.$$

Recall from Theorem 1.1 (3) that $\mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p)$ is the p-torsion free quotient of $\tilde{H}_\bullet$.

There are similar constructions for compactly supported cohomology, and their formation is compatible with the maps $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

2 Non-commutative Iwasawa theory

If M is any finitely generated left $\mathbb{Z}_p[[G_0]]$-module, then we define

$$E^\bullet(M) := \mathrm{Ext}^\bullet(M, \mathbb{Z}_p[[G_0]]).$$

These are again naturally $\mathbb{Z}_p[[G_0]]$-modules. (We compute the Exts via the left $\mathbb{Z}_p[[G_0]]$-module structure on $\mathbb{Z}_p[[G_0]]$, leaving a right $\mathbb{Z}_p[[G_0]]$-module structure on $E^\bullet(M)$. We convert this back to a left-module structure via the canonical anti-involution of $\mathbb{Z}_p[[G_0]]$ induced by $g \mapsto g^{-1}$.) We note that these modules are unchanged (up to a natural isomorphism, and applying the forgetful functor from $\mathbb{Z}_p[[G_0]]$-modules to $\mathbb{Z}_p[[G_0']]$-modules) if we replace G_0 by any open subgroup G_0'.

We define the codimension of M to the maximal value of i such that $E^i(M) \neq 0$. If G_0 is commutative, than this agrees with the usual notion of codimension of support of the coherent sheaf associated to M on $\mathrm{Spec}\mathbb{Z}_p[[G_0]]$. In general, even if G_0 is non-commutative (which is typically the case), this notion behaves entirely analogously to the usual notion of codimension of support of a sheaf in algebraic geometry [9].

We also remark that $\mathbb{Z}_p[[G_0]]$ admits a skew-field of fractions, say $\mathcal{L}$, and hence we can speak of a finitely generated $\mathbb{Z}_p[[G_0]]$-module M being torsion-free (the natural map $M \to \mathcal{L} \otimes M$ is injective) or torsion ($\mathcal{L} \otimes M = 0$), and we can speak of the rank of the (torsion-free part of) M (i.e. the rank of the $\mathcal{L}$-vector space $\mathcal{L} \otimes M$). Having positive rank is equivalent to being of codimension 0, while being torsion (or equivalently, being of rank 0) is equivalent to being of positive codimension.

3 Poincaré duality

We suppose that X_0 (and hence every X_r) is equidimensional of some dimension d. We then have two spectral sequences that express Poincaré duality in the p-adically completed situation. Note that both work with homology, rather than cohomology. (The reason for this is that the functors E^i intervene.)

Here are the spectral sequences:

$$E_2^{i,j} := E^i(\tilde{H}_j) \implies \tilde{H}^{BM}_{d-i-j}$$

and

$$E_2^{i,j} := E^i(\tilde{H}^{BM}_j) \implies \tilde{H}_{d-i-j}.$$

They are compatible with the maps $\tilde{H}_\bullet \to \tilde{H}^{BM}_\bullet$.

We point out the amplitude of the δ-functor $E^\bullet$ is given by the dimension of the group G_0. In applications, this can often be quite a bit larger than the dimension d of the spaces X_r, which can lead to interesting tension in these spectral sequences.

4 A simple example of everything so far

An illustrative example is given by taking $G_0 = \mathbb{Z}_p$, $G_r := p^r G_r$ for $r > 0$, and $X_r := \mathbb{R}/p^r\mathbb{Z}$ for each $r \geq 0$. We let $G_0/G_r := \mathbb{Z}_p/p^r\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}/p^r\mathbb{Z}$ act on X_r in the obvious manner by translations. Thus each X_r is a circle, and each map $X_{r+1} \to X_r$ is a degree p covering map. Note that $\mathbb{Z}_p[[G_0]] = \mathbb{Z}_p[[T]]$.

Clearly $\tilde{H}_0 = \mathbb{Z}_p$, with trivial G_0-action, while $\tilde{H}_1 = 0$ (the inverse limit of a sequence of copies of $\mathbb{Z}_p$ under the multiplication by p map obviously vanishes). Similarly $\tilde{H}^0 = \mathbb{Z}_p$, while $\tilde{H}^1 = 0$. (Since the X_r are compact, Borel-Moore homology agrees with usual homology, and compactly supported cohomology agrees with usual cohomology.)

Since $G_r := p^r\mathbb{Z}_p$ is procyclic, we see that $H^0(G_r, \mathbb{Z}_p) = H^1(G_r, \mathbb{Z}_p) = \mathbb{Z}_p$, and thus we do indeed recover the cohomology of the circle X_r from the completed cohomology via the Hochschild–Serre type spectral sequence.

Similarly, using the resolution

$$0 \to \mathbb{Z}_p[[T]] \xrightarrow{T\cdot} \mathbb{Z}_p[[T]] \to \mathbb{Z}_p \to 0,$$

we compute that $E^0(\mathbb{Z}_p) = 0$, while $E^1(\mathbb{Z}_p) = \mathbb{Z}_p$. This is immediately seen to be consistent with the Poincaré duality spectral sequence.

5 Congruence quotients of symmetric spaces

We now suppose that $\mathbb{G}$ is a connected reductive linear algebraic group over $\mathbb{Q}$. We let $\mathbb{A}$ denote the adèle ring over $\mathbb{Q}$, let $\mathbb{A}^\infty$ denote the ring of finite adèles, and let $\mathbb{A}^{\infty,p}$ denote the ring of prime-to-p finite adèles. We fix a compact open subgroup $K^{\infty,p}$ of $\mathbb{G}(\mathbb{A}^{\infty,p})$ (the tame level), take G_0 to be a sufficiently small compact open subgroup of $\mathbb{G}(\mathbb{Q}_p)$, and let $(G_r)_{r\geq 1}$ be a sequence of normal open subgroups of G_0 that form a neighbourhood basis of the identity in G_0.

We let K_∞° denote the connected component of the identity of a maximal compact subgroup K_∞ of $\mathbb{G}(\mathbb{R})$, and let A_∞° denote the connected component of the identity of the group of real points A_∞ of a maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$.

We define

$$X_r := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ G_r K^{\infty,p}.$$

The X_r form a tower with an action of G_0 satisfying the axioms of Section 1, and so the preceding theory applies. Furthermore, the usual argument about limits of (co)homology of arithmetic quotients in the adèlic setting shows that $\tilde{H}_\bullet$, $\tilde{H}^\bullet$, etc. all inherit not just an action of G_0, but of the entire p-adic group $\mathbb{G}(\mathbb{Q}_p)$. They also inherit an action of a suitably completed Hecke algebra $\mathbb{T}$ (built up out of spherical Hecke operators at primes $\ell \neq p$ over which $\mathbb{G}$ and the tame level $K^{\infty,p}$ are unramified).

The following theorem is the main result of [2]. (The result about $T_pH^{\bullet+1}$ being torsion, even in the middle dimension in the discrete series case, is not stated there, but follows from the proof.)

Theorem 5.1. *If, in the preceding setting, the group $\mathbb{G}$ is semi-simple, then the modules $\tilde{H}_n$ are torsion $\mathbb{Z}_p[[G_0]]$-modules, unless the group $\mathbb{G}(\mathbb{R})$ admits discrete series and n is the "middle dimension". In this latter case,* $\mathrm{Hom}(\hat{H}^n, \mathbb{Z}_p)$ *(and hence $\tilde{H}_n$) has positive rank, while* $\mathrm{Hom}(T_pH^{n+1}, \mathbb{Z}_p)$ *is torsion over $\mathbb{Z}_p[[G_0]]$.*

We believe that this is actually only the beginning of the story with regard to the codimensions of the various $\tilde{H}_\bullet$. In the following section, we will go on to describe how we expect the rest of the story to play out. But first, we will close this section by discussing the boundary long exact sequence.

Each X_r embeds as an open subset of its Borel-Serre compactification $\overline{X}_r$. We let ∂X_r denote $\overline{X}_r \setminus X_r$; it is the boundary of $\overline{X}_r$. As r varies the spaces $\overline{X}_r$ form a tower to which the p-adically completed cohomology machinery applies, and hence we may define completed (co)homology spaces $\tilde{H}_\bullet(\partial)$ and $\tilde{H}^\bullet(\partial)$. There are long exact sequences

$$\cdots \to \tilde{H}_\bullet(\partial) \to \tilde{H}_\bullet \to \tilde{H}_\bullet^{BM} \to \tilde{H}_{\bullet-1}(\partial) \to \cdots \tag{2}$$

and

$$\cdots \to \tilde{H}^{\bullet-1}(\partial) \to \tilde{H}_c^\bullet \to \tilde{H}^\bullet \to \tilde{H}^\bullet(\partial) \to \cdots . \tag{3}$$

The complement ∂X_r is a union of strata $\partial X_{\mathbb{P},r}$ indexed by the (equivalence classes of) proper parabolic subgroups $\mathbb{P}$ of $\mathbb{G}$. For fixed $\mathbb{P}$ and varying r the strata $\partial X_{\mathbb{P},r}$ again form a tower to which the p-adically completed cohomology machine applies, allowing us to define $\tilde{H}_\bullet(\partial_\mathbb{P})$ and $\tilde{H}^\bullet(\partial_\mathbb{P})$. There is a Mayer-Vietoris spectral sequence relating the various $\tilde{H}_\bullet(\partial_\mathbb{P})$ (resp. $\tilde{H}^\bullet(\partial_\mathbb{P})$) to $\tilde{H}_\bullet(\partial)$ (resp. $\tilde{H}^\bullet(\partial)$).

If we let $\mathbb{P} = \mathbb{M}\mathbb{N}$ be a Levi decomposition of the proper parabolic $\mathbb{P}$, then each $\partial_{\mathbb{P},r}$ is a bundle over a union of congruence quotients associated to $\mathbb{M}$, whose fibre is a product of circles (of dimension equal to $\dim \mathbb{N}$). A generalization to a product of circles of the example of Section 4 shows that these products of circles contribute to completed (co)homology only in degree 0, and so $\tilde{H}_\bullet(\partial_\mathbb{P})$ and $\tilde{H}^\bullet(\partial_\mathbb{P})$ are supported in the same degrees as the p-adically completed (co)homology associated to $\mathbb{M}$. To describe the precise relationship between the p-adically completed cohomology for $\partial_\mathbb{P}$ and for $\mathbb{M}$, it is convenient to take a direct limit over all tame levels. We then find that:

$$\varinjlim_{\text{tame levels}} \tilde{H}^\bullet(\partial_\mathbb{P}) \xrightarrow{\sim} \operatorname{Ind}_{\mathbb{P}(\mathbb{A}_f)}^{\mathbb{G}(\mathbb{A}_f)}\Big(\varinjlim_{\text{tame levels}} \tilde{H}^\bullet \text{ associated to } \mathbb{M}\Big), \tag{4}$$

where the induction is p-adically completed at p, and smooth at primes away from p.

We note that much of this general set-up for congruence quotients has been described by Richard Hill in [6, §4].

6 Conjectures on codimensions

We maintain the set-up of the preceding section. We write $G_\infty := \mathbb{G}(\mathbb{R})$, and we begin by defining two quantities associated to $\mathbb{G}$:

$$l_0 := \text{ rank of } G_\infty - \text{ rank of } A_\infty K_\infty,$$

and

$$q_0 = (\text{ dimension of } G_\infty - \text{ dimension of } A_\infty K_\infty - l_0\,)/2 = (d - l_0)/2,$$

where d denotes the dimension of the quotients X_r. Note that if $\mathbb{G}$ is semi-simple, then these quantities coincide with the quantities denoted by the same symbols in Borel–Wallach. Namely, l_0 denotes the "defect" of G_∞ with regard to possessing discrete series, while q_0 denotes the first "interesting" dimension for the (co)homology of X_r.

We now state our basic conjecture on codimensions.

Conjecture 6.1. *1. If $n < q_0$, then the codimension of $\tilde{H}_n$ is greater than $l_0 + q_0 - n$.*

2. The codimension of $\tilde{H}_{q_0}$ equals l_0.

3. If $n < q_0$, then the codimension of $\tilde{H}_n^{BM}$ is greater than $l_0 + q_0 - n$.

4. *The codimension of* $\tilde{H}^{BM}_{q_0}$ *equals* l_0.

5. $\tilde{H}_{q_0}$ *is p-torsion-free.*

6. $\tilde{H}^{BM}_{q_0}$ *is p-torsion-free.*

7. $\tilde{H}_n = 0$ *if* $n > q_0$.

8. $\tilde{H}^{BM}_n = 0$ *if* $n > q_0$.

In fact, these conjectures are not independent, as the following theorem makes clear.

Theorem 6.2. 1. *Suppose that parts* (1) *and* (2) (*resp. parts* (3) *and* (4)) *of Conjecture* 6.1 *holds for* G*, and for all the proper Levi subgroups of* G*. Then parts* (3) *and* (4) (*resp. parts* (1) *and* (2)) *of the conjecture also holds for* G (*and also for all its Levi subgroups*)*, as do parts* (7) *and* (8).

2. *If part* (7) (*resp. part* (8)) *of Conjecture* 6.1 *holds for* $G \times G$*, then part* (5) (*resp. part* (6)) *of the conjecture holds for* G.

Proof. Suppose that parts (1) and (2), or parts (3) and (4), of the conjecture hold for all the proper Levi subgroups of G. Applying the theorem inductively, we conclude that parts (1), (2), (3), (4), (5), and (6) of the conjecture hold for these Levi subgroups. Formula (4) then allows us to bound from below the codimensions of the p-adically completed homology spaces associated to the various boundary strata. A comparison of the invariants l_0 and q_0 for G and for its Levi subgroups, along with a consideration of the Mayer-Vietoris spectral sequence that computes the p-adically completed homology of the boundary in terms of the p-adically completed cohomology of its various strata, as well as of the long exact sequence (2), then shows that parts (1) and (2) of the conjecture for G are equivalent to parts (3) and (4) for G. Looking at the Poincaré duality spectral sequence then shows that parts (7) and (8) of the conjecture also hold.

Part (2) of the theorem follows by applying a Kunnëth-type theorem to compare the completed cohomology for G to that for $G \times G$. □

Remark 6.3. In particular, the preceding theorem shows that if either parts (1) and (2), or part (3) and (4), of Conjecture 6.1 hold for every group G, then all parts of the conjecture hold for all groups G.

Remark 6.4. Richard Hill has also made a conjecture about the vanishing of completed cohomology of arithmetic quotients, namely [6, Conj. 3]. Conjecture 6.1 implies Hill's conjecture, but is quite a bit stronger in general. The various vanishing results proved in [6, §5], being consistent with [6, Conj. 3], are thus also consistent with our general conjecture.

There are several different heuristics and motivations behind this conjecture (as well as a small amount of actual evidence). For now, let us remark that it is consistent with Künneth, and it is consistent with the Poincaré duality spectral sequence. Of course, in the case when $\mathbb{G}$ is semi-simple, it is also consistent with Theorem 5.1, which shows at least that the codimension of $\tilde{H}_n$ is zero (respectively, positive) exactly when it is predicted to be so by the conjecture.

We now discuss some illustrative examples.

Example 6.5. If $\mathbb{G}$ is a torus, then $q_0 = 0$, and a generalization of the analysis of the example in Section 4 shows that in this case, Conjecture 6.1 is equivalent to Leopoldt's conjecture. (See [6, Cor. 5].)

Example 6.6. If $G = \mathrm{SL}_2(\mathbb{Q})$, then $l_0 = 0$ and $q_0 = 1$, and one finds that $\tilde{H}_0 = \mathbb{Z}_p$ has codimension 3, that while $\tilde{H}_1$ has codimension zero. (Apply Theorem 5.1.)

Example 6.7. If K is a quadratic imaginary field, and $G = \mathrm{SL}_2(K)$ (regarded as a group over $\mathbb{Q}$ by restriction of scalars, as usual), then $l_0 = 1$ and $q_0 = 1$, and one finds that $\tilde{H}_0 = \mathbb{Z}_p$, and so has codimension 6, that $\tilde{H}_1$ has codimension 1, and that $\tilde{H}_2 = 0$. (Apply Theorem 5.1 together with the Poincaré duality spectral sequence. Note that the term $E^6(\tilde{H}_0) = E^6(\mathbb{Z}_p) = \mathbb{Z}_p$ plays a crucial role in showing that $\tilde{H}_1 \neq 0$.)

Example 6.8. If G is semi-simple and simply connected and satisfies the congruence subgroup property, then $\tilde{H}_1$ is finite.

Example 6.9. If $G = \mathrm{Sp}_4(\mathbb{Q})$, then $l_0 = 0$ and $q_0 = 3$. One finds that $\tilde{H}_0 = \mathbb{Z}_p$, $\tilde{H}_1$ is finite (since G satisfies the congruence subgroup property), $\tilde{H}_2$ has codimension at least 1 (by Theorem 5.1), and $\tilde{H}_3$ has codimension 0 (again by Theorem 5.1). The Poincaré duality spectral sequence then shows that $\tilde{H}_n = 0$ for $n \geq 4$.

Remark 6.10. It is not clear in general how one might go about computing the codimension of support of (or any other information about) the completed cohomology. However, suppose that one knew that the $\mathbb{Z}_p$-cohomology of each X_r was p-torsion free (or, more generally, of bounded exponent). The Tate module term in the exact sequence (1) would then vanish, and hence there would be an isomorphism $\hat{H}^\bullet \xrightarrow{\sim} \tilde{H}^\bullet$. Now $\hat{H}^\bullet$ is the p-adic completion of the classical cohomology, and this classical cohomology may be described in terms of classical automorphic forms (at least after tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$). Thus it seems not totally inconceivable that one might be able to prove results about (e.g. the codimension of support of) $\hat{H}^\bullet$. (This is done for modular curves in [5], although admittedly the arguments there rely on the full strength of the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$.)

Thus it seems worth investigating and attempting to establish torsion freeness results for the cohomology of congruence quotients, say in the Shimura variety context. The most notable result in this direction that we are aware of is due to Boyer [1], who has established torsion-freeness for the cohomology of the so-called simple Shimura varieties attached to forms of $U(n, 1)$ that have been studied by Kottwitz, Harris, and Taylor.

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