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q-Extensions of Erdélyi's Fractional Integral Representations for Hypergeometric Functions and Some Summation Formulas for Double q-Kampé de Fériet Series

George Gasper

Dedicated to Dick Askey on the occasion of his 66th birthday, and to the memory of D.B. Sears whose ${}_4\phi_3$ transformation formula plays a crucial role in this paper

Abstract. q-Analogues of Erdélyi's fractional integral representations of hypergeometric functions are derived and extended to expansion formulas for certain ${}_3\phi_2$ and ${}_4\phi_3$ basic hypergeometric series. Special cases of some of the derived formulas are used to derive new summation formulas for double hypergeometric and basic hypergeometric Kampé de Fériet series, including a summation formula for a double basic hypergeometric Kampé de Fériet series that was conjectured in work of J. Van der Jeugt, S.N. Pitre, and K. Srinivasa Rao on the evaluation of the 9-j recoupling coefficients appearing in the quantum theory of angular momentum.

1. Introduction

Let $z \neq 0$ and $|\arg(1-z)| < \pi$. In 1939, Erdélyi [**4**] used fractional integration by parts and transformation formulas for ${}_2F_1$ hypergeometric functions to show that Euler's integral

$$(1.1) \qquad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt,$$

where $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, and Bateman's [**3**] extension of (1.1)

$$(1.2) \quad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}{}_2F_1(\alpha,\beta;\lambda;tz)\,dt,$$

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where $\operatorname{Re}\gamma > \operatorname{Re}\lambda > 0$, have extensions of the forms [**4**, equations (17), (11), and (20), respectively]

$$(1.3)\qquad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)\Gamma(\gamma-\mu)}\int_0^1 t^{\mu-1}(1-t)^{\gamma-\mu-1}(1-tz)^{\lambda-\alpha-\beta} \cdot {}_2F_1(\lambda-\alpha,\lambda-\beta;\mu;tz)\,{}_2F_1(\alpha+\beta-\lambda,\lambda-\mu;\gamma-\mu;\frac{(1-t)z}{1-tz})\,dt,$$

where $\operatorname{Re}\gamma > \operatorname{Re}\mu > 0$,

$$(1.4)\qquad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}(1-tz)^{-\alpha'} \cdot {}_2F_1(\alpha-\alpha',\beta;\lambda;tz)\,{}_2F_1(\alpha',\beta-\lambda;\gamma-\lambda;\frac{(1-t)z}{1-tz})\,dt,$$

where $\operatorname{Re}\gamma > \operatorname{Re}\lambda > 0$, and

$$(1.5)\qquad {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\gamma+\mu-\lambda-\nu)}\int_0^1 t^{\nu-1}(1-t)^{\gamma+\mu-\lambda-\nu-1} \cdot {}_2F_1(\mu-\lambda,\gamma-\lambda;\gamma+\mu-\lambda-\nu;1-t)\,{}_3F_2(\alpha,\beta,\mu;\lambda,\nu;tz)\,dt,$$

where $\operatorname{Re}(\lambda,\nu,\gamma+\mu-\lambda-\nu) > 0$. Erdélyi also considered special cases and confluent limit cases of his formulas and some formulas obtained by applying transformation formulas to the hypergeometric functions in the integrands.

Let $n = 0, 1, 2, \ldots$. In [**5**] the author pointed out some important applications of Erdélyi's fractional integral (1.3) (such as to derive Dirichlet-Mehler type integral representations for Jacobi polynomials and for generalized Legendre functions, and to prove the positivity of certain sums of generalized Legendre functions) and derived the following discrete analogue of (1.3) for ${}_3F_2$ series

$$(1.6)\qquad {}_3F_2(\alpha,\beta,-n;\gamma,\delta;1) = \sum_{k=0}^{n}\binom{n}{k}\frac{(\mu)_k(\lambda+\delta-\alpha-\beta)_k(\gamma-\mu)_{n-k}}{(\gamma)_n(\delta)_k} \cdot {}_3F_2(\lambda-\alpha,\lambda-\beta,-k;\mu,\lambda+\delta-\alpha-\beta;1) \cdot {}_3F_2(\alpha+\beta-\lambda,\lambda-\mu,k-n;\gamma-\mu,\delta+k;1),$$

where, as elsewhere, we use the standard notations in the Gasper and Rahman book [**7**] for the shifted factorial $(a)_n$, generalized hypergeometric functions, etc. This formula led to a Dirichlet-Mehler type formula for the Hahn polynomials. Formula (1.3) follows from (1.6) by replacing n, k, δ in (1.6) by $Nz, Nzt, -N$, respectively, letting $Nz \to +\infty$ through integer values of Nz with z fixed and $0 < z < 1$, and then using analytic continuation with respect to z.

In this paper we will show that (1.6) has a q-analogue of the form

$$(1.7)\qquad {}_3\phi_2(\alpha,\beta,q^{-n};\gamma,\delta;q,q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\mu,\lambda\delta/\alpha\beta;q)_k(\gamma/\mu;q)_{n-k}}{(\gamma;q)_n(\delta;q)_k}\mu^n\left(\frac{\alpha\beta}{\lambda\mu}\right)^k \cdot {}_3\phi_2(\lambda/\alpha,\lambda/\beta,q^{-k};\mu,\lambda\delta/\alpha\beta;q,q) \cdot {}_3\phi_2(\alpha\beta/\lambda,\lambda/\mu,q^{k-n};\gamma/\mu,\delta q^k;q,q).$$

It can be shown that (1.6) is a limit case of (1.7) by replacing the parameters $\alpha,\beta,\gamma,\delta,\lambda,\mu$ in (1.7) by $q^\alpha, q^\beta, q^\gamma, q^\delta, q^\lambda, q^\mu$, respectively, and letting $q \to 1^-$. In addition, by letting $0 < q < 1$, which will be assumed to hold in the following q-integral (see [**7**, §1.11]) formulas, replacing $k,\alpha,\beta,\gamma,\delta,\lambda,\mu$ in (1.7) by

$n-k, q^\alpha, q^\beta, q^\gamma, q^{1-n}/z, q^\lambda, q^\mu$, respectively, and then setting $t=q^k$ and letting $n\to+\infty$, we obtain that Erdélyi's fractional integral (1.3) has the fractional q-integral analogue

$$(1.8)\qquad {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z)=\frac{\Gamma_q(\gamma)}{\Gamma_q(\mu)\Gamma_q(\gamma-\mu)}\int_0^1 t^{\mu-1}\frac{(tq,tzq^{\alpha+\beta-\lambda};q)_\infty}{(tq^{\gamma-\mu},tz;q)_\infty}\cdot {}_2\phi_1(q^{\lambda-\alpha},q^{\lambda-\beta};q^\mu;q,tzq^{\alpha+\beta-\lambda})\cdot {}_3\phi_2(q^{\alpha+\beta-\lambda},q^{\lambda-\mu},t^{-1};q^{\gamma-\mu},q/tz;q,q)\,d_qt,$$

where $\operatorname{Re}\gamma>\operatorname{Re}\mu>0$ and the q-integral is defined by

$$\int_0^1 f(t)\,d_qt=(1-q)\sum_{n=0}^\infty f(q^n)\,q^n.$$

For additional information about fractional q-integrals see [**7**, §1.11 and the Notes for §1.11 on p. 29]. It is clear that (1.8) tends to (1.3) as $q\to1^-$ since

$$\lim_{q\to1^-}\Gamma_q(\gamma)=\Gamma(\gamma)$$

and, by the q-binomial theorem [**7**, §1.3],

$$\lim_{q\to1^-}\frac{(tq;q)_\infty}{(tq^{\gamma-\mu};q)_\infty}\frac{(tzq^{\alpha+\beta-\lambda};q)_\infty}{(tz;q)_\infty}=(1-t)^{\gamma-\mu-1}(1-tz)^{\lambda-\alpha-\beta}.$$

Hence, (1.7) is both a discrete analogue of (1.8) and a discrete q-extension of (1.3). Setting $\lambda=\alpha+\beta$ in (1.8) and then replacing μ by λ gives

$$(1.9)\qquad {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z)=\frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)}\int_0^1 t^{\lambda-1}\frac{(tq;q)_\infty}{(tq^{\gamma-\lambda};q)_\infty}\cdot {}_2\phi_1(q^\alpha,q^\beta;q^\lambda;q,tz)\,d_qt,$$

where $\operatorname{Re}\gamma>\operatorname{Re}\lambda>0$, which is a q-analogue of Bateman's integral (1.2). Thomae's (see [**7**, (1.11.9)]) q-analogue of Euler's integral (1.1)

$$(1.10)\qquad {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z)=\frac{\Gamma_q(\gamma)}{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}\int_0^1 t^{\beta-1}\frac{(tq,tzq^\alpha;q)_\infty}{(tq^{\gamma-\beta},tz;q)_\infty}d_qt,$$

where $\operatorname{Re}\gamma>\operatorname{Re}\beta>0$, and its q-beta integral [**7**, (1.11.7)] special case are special cases of both (1.8) and (1.9).

We will also show that Erdélyi's formulas (1.4) and (1.5) have the discrete q-extensions:

$$(1.11)\qquad {}_3\phi_2(\alpha,\beta,q^{-n};\gamma,\delta;q,q)=\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\lambda,\delta/\alpha';q)_k(\gamma/\lambda;q)_{n-k}}{(\gamma;q)_n(\delta;q)_k}\lambda^n\left(\frac{\alpha'}{\lambda}\right)^k\cdot {}_3\phi_2(\alpha/\alpha',\beta,q^{-k};\lambda,\delta/\alpha';q,q)\cdot {}_3\phi_2(\alpha',\beta/\lambda,q^{k-n};\gamma/\lambda,\delta q^k;q,q)$$

and

$$(1.12)\qquad {}_3\phi_2(\alpha,\beta,q^{-n};\gamma,\delta;q,q)=\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\lambda;q)_n(\nu;q)_k(\gamma\mu/\lambda\nu;q)_{n-k}}{(\gamma,\mu;q)_n}\nu^{n-k}\cdot {}_3\phi_2(\mu/\lambda,\gamma/\lambda,q^{k-n};\gamma\mu/\lambda\nu,q^{1-n}/\lambda;q,q^{1-k}/\nu)\cdot {}_4\phi_3(\alpha,\beta,\mu,q^{-k};\lambda,\nu,\delta;q,q).$$

As in the derivation of (1.8) as a limit case of (1.7), formulas (1.11) and (1.12), respectively, have as limit cases the following q-integral analogues of Erdélyi's fractional integrals (1.4) and (1.5):

$$(1.13)\qquad {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\lambda)\Gamma_q(\gamma-\lambda)}\int_0^1 t^{\lambda-1}\frac{(tq,tzq^{\alpha'};q)_\infty}{(tq^{\gamma-\lambda},tz;q)_\infty} \cdot {}_2\phi_1(q^{\alpha-\alpha'},q^\beta;q^\lambda;q,tzq^{\alpha'})\,{}_3\phi_2(q^{\alpha'},q^{\beta-\lambda},t^{-1};q^{\gamma-\lambda},q/tz;q,q)\,d_qt,$$

where $\operatorname{Re}\gamma > \operatorname{Re}\lambda > 0$, and

$$(1.14)\qquad {}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,z) = \frac{\Gamma_q(\gamma)\Gamma_q(\mu)}{\Gamma_q(\lambda)\Gamma_q(\nu)\Gamma_q(\gamma+\mu-\lambda-\nu)}\int_0^1 t^{\nu-1}\frac{(tq;q)_\infty}{(tq^{\gamma+\mu-\lambda-\nu};q)_\infty} \cdot {}_3\phi_1(q^{\mu-\lambda},q^{\gamma-\lambda},t^{-1};q^{\gamma+\mu-\lambda-\nu};q,tq^{\lambda-\nu})\,{}_3\phi_2(q^\alpha,q^\beta,q^\mu;q^\lambda,q^\nu;q,tz)\,d_qt,$$

where $\operatorname{Re}(\lambda,\nu,\gamma+\mu-\lambda-\nu) > 0$. Direct derivations of (1.8), (1.13), and (1.14) can be given by using the q-beta integral and transformation formulas. The fractional q-integral formulas (6.6) and (6.7) in Al-Salam and Verma [**1**], with the missing factor $(1-q)^{\gamma-\lambda}$ inserted in front of each q-integral, follow from (1.14) and (1.8), respectively, by applying the transformation formula [**7**, Ex. 1.15(ii)] with $q^n = t$ to the ${}_3\phi_1$ series in (1.14), and the transformation formulas [**7**, (III.13) and (III.12)] to the ${}_3\phi_2$ series in (1.8).

After obtaining the above formulas, the author realized that their derivations could be extended to give more general expansion formulas that contained terminating balanced ${}_4\phi_3$ series, and hence were applicable to the most general classical orthogonal polynomials (see [**2**] and [**7**]), the *q-Racah polynomials*

$$W_n(x;a,b,c,N;q) = {}_4\phi_3\left[\begin{matrix} q^{-n},abq^{n+1},q^{-x},cq^{x-N} \\ aq,bcq,q^{-N}\end{matrix};q,q\right]$$

and the *Askey-Wilson polynomials*

$$p_n(x;a,b,c,d\mid q) = a^{-n}(ab,ac,ad;q)_n\,{}_4\phi_3\left[\begin{matrix} q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta} \\ ab,ac,ad\end{matrix};q,q\right],$$

where $x=\cos\theta$, and to their hypergeometric and basic hypergeometric limit cases. These derivations, whose limit cases give direct derivations of formulas (1.7) – (1.14), will be presented in §§ 2–4. In §5 we derive some new summation formulas for double hypergeometric and basic hypergeometric Kampé de Fériet series, one of which contains as a limit case the sum of a double q-Kampé de Fériet series that arose in work of J. Van der Jeugt, S.N. Pitre, and K. Srinivasa Rao on the evaluation of the 9-j recoupling coefficients, and was stated without proof in [**11**, (40)], see [**9**, p. 129].

2. Derivation of a terminating balanced ${}_4\phi_3$ extension of (1.7)

Let

$$(2.1)\qquad \alpha\beta\nu q = \gamma\delta\epsilon$$

so that

$$(2.2)\qquad \Phi_n = {}_4\phi_3(\alpha,\beta,\nu q^n,q^{-n};\gamma,\delta,\epsilon;q,q)$$

is a terminating balanced ${}_4\phi_3$ series. Notice that if we use the balanced condition (2.1) to set $\epsilon = \alpha\beta\nu q/\gamma\delta$ in the above ${}_4\phi_3$ series and let $\nu \to 0$, then Φ_n tends to the ${}_3\phi_2(\alpha,\beta,q^{-n};\gamma,\delta;q,q)$ series that are on the left-hand sides of formulas (1.7), (1.11), and (1.12).

In order to give a ${}_4\phi_3$ extension of the derivation of (1.6) in [**5**], we first observe that

$$\Phi_n = A_n \sum_{j=0}^{n} \frac{(\nu\lambda/\gamma, \lambda, \nu q^{-n}; q)_j (\nu\lambda q/\gamma; q)_{2j}}{(q, \nu q/\gamma, \lambda q^{1-n}/\gamma, \nu\lambda q^{n+1}/\gamma; q)_j (\nu\lambda/\gamma; q)_{2j}} \left(\frac{q}{\gamma}\right)^j \cdot {}_4\phi_3(\alpha, \beta, \nu\lambda q^j/\gamma, q^{-j}; \lambda, \delta, \epsilon; q, q) \tag{2.3}$$

with

$$A_n = \frac{(\gamma/\lambda, \nu q/\gamma; q)_n}{(\gamma, \nu\lambda q/\gamma; q)_n} \lambda^n, \tag{2.4}$$

by changing the order of summation on the right-hand side of (2.3) and using the ${}_6\phi_5$ summation formula [**7**, (2.4.2)]. Apply the Sears' [**10**], [**7**, (2.10.4)] transformation formula for terminating balanced ${}_4\phi_3$ series to the ${}_4\phi_3$ series in (2.3) to obtain

$$\begin{aligned} &{}_4\phi_3(\alpha, \beta, \nu\lambda q^j/\gamma, q^{-j}; \lambda, \delta, \epsilon; q, q) \\ &= \frac{(\delta\gamma/\nu\lambda q^j, \epsilon\gamma/\nu\lambda q^j; q)_j}{(\delta, \epsilon; q)_j} \left(\frac{\nu\lambda q^j}{\gamma}\right)^j \\ &\cdot {}_4\phi_3(\lambda/\alpha, \lambda/\beta, \nu\lambda q^j/\gamma, q^{-j}; \lambda, \lambda\delta/\alpha\beta, \lambda\epsilon/\alpha\beta; q, q) \\ &= \frac{(\delta\gamma/\nu\lambda q^j, \epsilon\gamma/\nu\lambda q^j, \lambda/\mu, \nu q/\gamma; q)_j}{(\delta, \epsilon, \lambda, \nu\mu q/\gamma; q)_j} \left(\frac{\nu\lambda\mu q^j}{\gamma}\right)^j \\ &\cdot \sum_{k=0}^{j} \frac{(\nu\mu/\gamma, \mu, \nu\lambda q^j/\gamma, q^{-j}; q)_k (\nu\mu q/\gamma; q)_{2k}}{(q, \nu q/\gamma, \mu q^{1-j}/\lambda, \nu\mu q^{j+1}/\gamma; q)_k (\nu\mu/\gamma; q)_{2k}} \left(\frac{q}{\lambda}\right)^k \\ &\cdot {}_4\phi_3(\lambda/\alpha, \lambda/\beta, \nu\mu q^k/\gamma, q^{-k}; \mu, \lambda\delta/\alpha\beta, \lambda\epsilon/\alpha\beta; q, q). \end{aligned} \tag{2.5}$$

Now substitute (2.5) into (2.3), change the order of summation, and then apply some of the identities in [**7**, Appendix I] to simplify the products of q-shifted factorials and get

$$\begin{aligned} \Phi_n = A_n &\sum_{k=0}^{n} \frac{(\lambda\epsilon/\alpha\beta, \lambda\delta/\alpha\beta, \nu\mu/\gamma, \mu, \nu q^n, q^{-n}; q)_k (\nu\lambda q/\gamma; q)_{2k}}{(q, \delta, \epsilon, \nu q/\gamma, \nu\lambda q^{n+1}/\gamma, \lambda q^{1-n}/\gamma; q)_k (\nu\mu/\gamma; q)_{2k}} \left(\frac{\epsilon\delta}{\lambda\nu}\right)^k \\ &\cdot {}_8W_7(\nu\lambda q^{2k}/\gamma; \lambda/\mu, \nu\lambda q^{k+1}/\gamma\delta, \nu\lambda q^{k+1}/\gamma\epsilon, \nu q^{n+k}, q^{k-n}; q, \epsilon\delta\mu/\nu\lambda) \\ &\cdot {}_4\phi_3(\lambda/\alpha, \lambda/\beta, \nu\mu q^k/\gamma, q^{-k}; \mu, \lambda\delta/\alpha\beta, \lambda\epsilon/\alpha\beta; q, q). \end{aligned} \tag{2.6}$$

Since, by the transformation formulas [**7**, (III.18) and then (III.15)],

$$\begin{aligned}(2.7)\qquad & {}_8W_7(\nu\lambda q^{2k}/\gamma;\lambda/\mu,\nu\lambda q^{k+1}/\gamma\delta,\nu\lambda q^{k+1}/\gamma\epsilon,\nu q^{n+k},q^{k-n};q,\epsilon\delta\mu/\nu\lambda)\\ &= \frac{(\nu\lambda q^{2k+1}/\gamma,\mu\epsilon q^k/\lambda;q)_{n-k}}{(\nu\mu q^{2k+1}/\gamma,\epsilon q^k;q)_{n-k}}\\ &\cdot {}_4\phi_3(\lambda/\mu,\delta/\nu q^n,\nu\lambda q^{k+1}/\gamma\epsilon,q^{k-n};\delta q^k,\lambda q^{1+k-n}/\gamma,\lambda q^{1-n}/\mu\epsilon;q,q)\\ &= \frac{(\nu\lambda q^{2k+1}/\gamma,\mu\epsilon q^k/\lambda,q^{1-n}/\epsilon;q)_{n-k}}{(\nu\mu q^{2k+1}/\gamma,\epsilon q^k,\lambda q^{1-n}/\mu\epsilon;q)_{n-k}}\left(\frac{\lambda}{\mu}\right)^{n-k}\\ &\cdot {}_4\phi_3(\alpha\beta/\lambda,\lambda/\mu,\nu q^{k+n},q^{k-n};\gamma/\mu,\delta q^k,\epsilon q^k;q,q),\end{aligned}$$

it follows from (2.1), (2.2), (2.4), (2.6), and (2.7) that we have the expansion

$$\begin{aligned}(2.8)\qquad & {}_4\phi_3(\alpha,\beta,\nu q^n,q^{-n};\gamma,\delta,\epsilon;q,q)\\ &= \frac{(\gamma/\mu,\nu q/\gamma;q)_n}{(\gamma,\nu\mu q/\gamma;q)_n}\mu^n\\ &\cdot \sum_{k=0}^{n}\frac{(\nu\mu/\gamma,\mu,\lambda\delta/\alpha\beta,\lambda\epsilon/\alpha\beta,\nu q^n,q^{-n};q)_k(\nu\mu q/\gamma;q)_{2k}}{(q,\nu q/\gamma,\epsilon,\delta,\mu q^{1-n}/\gamma,\nu\mu q^{n+1}/\gamma;q)_k(\nu\mu/\gamma;q)_{2k}}\left(\frac{\epsilon\delta}{\lambda\nu}\right)^k\\ &\cdot {}_4\phi_3(\lambda/\alpha,\lambda/\beta,\nu\mu q^k/\gamma,q^{-k};\mu,\lambda\delta/\alpha\beta,\lambda\epsilon/\alpha\beta;q,q)\\ &\cdot {}_4\phi_3(\alpha\beta/\lambda,\lambda/\mu,\nu q^{k+n},q^{k-n};\gamma/\mu,\delta q^k,\epsilon q^k;q,q)\end{aligned}$$

with $\alpha\beta\nu q=\gamma\delta\epsilon$, which is the desired ${}_4\phi_3$ extension of (1.7). Notice that the three ${}_4\phi_3$ series in (2.8) are terminating balanced series, and that (1.7) follows from (2.8) by setting $\epsilon=\alpha\beta\nu q/\gamma\delta$ in (2.8) and then letting $\nu\to 0$.

Moreover, if we replace the parameters $\alpha,\beta,\gamma,\delta,\epsilon,\lambda,\mu,\nu$ in (1.7) by $q^\alpha,q^\beta,q^\gamma,q^\delta,q^\epsilon,q^\lambda,q^\mu,q^\nu$, respectively, and then let $q\to 1^-$, we obtain the expansion

$$\begin{aligned}(2.9)\quad & {}_4F_3(\alpha,\beta,n+\nu,-n;\gamma,\delta,\epsilon;1)=\frac{(\gamma-\mu)_n(1+\nu-\gamma)_n}{(\gamma)_n(1+\nu+\mu-\gamma)_n}\sum_{k=0}^{n}\frac{(\nu+\mu-\gamma)_k}{k!}\\ &\cdot\frac{(\mu)_k(\lambda+\delta-\alpha-\beta)_k(\lambda+\epsilon-\alpha-\beta)_k(n+\nu)_k(-n)_k(1+\nu+\mu-\gamma)_{2k}}{(1+\nu-\gamma)_k(\epsilon)_k(\delta)_k(1+\mu-\gamma-n)_k(n+1+\nu+\mu-\gamma)_k(\nu+\mu-\gamma)_{2k}}\\ &\cdot {}_4F_3(\lambda-\alpha,\lambda-\beta,k+\nu+\mu-\gamma,-k;\mu,\lambda+\delta-\alpha-\beta,\lambda+\epsilon-\alpha-\beta;1)\\ &\cdot {}_4F_3(\alpha+\beta-\lambda,\lambda-\mu,n+k+\nu,k-n;\gamma-\mu,k+\delta,k+\epsilon;1),\end{aligned}$$

with $\alpha+\beta+\nu+1=\gamma+\delta+\epsilon$, which is an extension of (1.6) that contains three terminating balanced ${}_4F_3$ series. As in [**5**], the cases when $\mu=q^{\frac12}$ in (2.8) and $\mu=\frac12$ in (2.9), respectively, give Dirichlet-Mehler type formulas for the ${}_4\phi_3$ q-Racah and the ${}_4F_3$ Racah polynomials.

3. Derivation of a terminating balanced ${}_4\phi_3$ extension of (1.11)

Since Erdélyi's fractional integral (1.4) cannot be derived directly from his fractional integral (1.3) by applying transformation formulas to the ${}_2F_1$ functions in the integrands, in order to derive a ${}_4\phi_3$ extension of (1.11) we have to proceed substantially differently than in our derivation of (2.8).

First observe that, by the q-Saalschütz formula [**7**, (1.7.2)],

$${}_3\phi_2(\alpha/\alpha',\epsilon/\alpha,q^{-k};\epsilon,q^{1-k}/\alpha';q,q)=\frac{(\epsilon\alpha'/\alpha,\alpha;q)_k}{(\epsilon,\alpha';q)_k},$$

which gives

$$
\begin{aligned}
(3.1)\qquad & {}_4\phi_3(\alpha,\beta,\nu q^n,q^{-n};\gamma,\delta,\epsilon;q,q)\\
&=\sum_{k=0}^{n}\frac{(\alpha',\beta,\nu q^n,q^{-n};q)_k}{(q,\gamma,\delta,\epsilon\alpha'/\alpha;q)_k}q^k\sum_{j=0}^{k}\frac{(\alpha/\alpha',\epsilon/\alpha,q^{-k};q)_j}{(q,\epsilon,q^{1-k}/\alpha';q)_j}q^j\\
&=\sum_{j=0}^{n}\frac{(\beta,\alpha/\alpha',\epsilon/\alpha,\nu q^n,q^{-n};q)_j}{(q,\gamma,\delta,\epsilon,\epsilon\alpha'/\alpha;q)_j}(q\alpha')^j\\
&\cdot\ {}_4\phi_3(\alpha',\beta q^j,\nu q^{n+j},q^{j-n};\gamma q^j,\delta q^j,\epsilon\alpha' q^j/\alpha;q,q)
\end{aligned}
$$

by setting $k=j+m$ and changing the order of summation. Next, from the $\lambda=\beta$ case of (2.8) we obtain that when the balanced condition (2.1) holds

$$
\begin{aligned}
(3.2)\qquad & {}_4\phi_3(\alpha',\beta q^j,\nu q^{n+j},q^{j-n};\gamma q^j,\delta q^j,\epsilon\alpha' q^j/\alpha;q,q)\\
&=\frac{(\gamma/\lambda,\nu q^{j+1}/\gamma;q)_{n-j}}{(\gamma q^j,\nu\lambda q^{2j+1}/\gamma;q)_{n-j}}(\lambda q^j)^{n-j}\sum_{m=0}^{n-j}\frac{(\nu\lambda q^{2j+1}/\gamma;q)_{2m}}{(\nu\lambda q^{2j}/\gamma;q)_{2m}}\\
&\cdot\frac{(\nu\lambda q^{2j}/\gamma,\lambda q^j,\epsilon q^j/\alpha,\delta q^j/\alpha',\nu q^{n+j},q^{j-n};q)_m}{(q,\nu q^{j+1}/\gamma,\delta q^j,\alpha'\epsilon q^j/\alpha,\lambda q^{1+j-n}/\gamma,\nu\lambda q^{n+j+1}/\gamma;q)_m}\left(\frac{\alpha' q^{1-j}}{\gamma}\right)^m\\
&\cdot\ {}_4\phi_3(\alpha',\beta/\lambda,\nu q^{n+j+m},q^{j+m-n};\gamma/\lambda,\delta q^{j+m},\alpha'\epsilon q^{j+m}/\alpha;q,q).
\end{aligned}
$$

Now substitute (3.2) into (3.1), set $m=k-j$, and change the order of summation to get the expansion

$$
\begin{aligned}
(3.3)\qquad & {}_4\phi_3(\alpha,\beta,\nu q^n,q^{-n};\gamma,\delta,\epsilon;q,q)\\
&=\frac{(\gamma/\lambda,\nu q/\gamma;q)_n}{(\gamma,\nu\lambda q/\gamma;q)_n}\lambda^n\\
&\cdot\sum_{k=0}^{n}\frac{(\nu\lambda/\gamma,\lambda,\epsilon/\alpha,\delta/\alpha',\nu q^n,q^{-n};q)_k(\nu\lambda q/\gamma;q)_{2k}}{(q,\nu q/\gamma,\delta,\alpha'\epsilon/\alpha,\lambda q^{1-n}/\gamma,\nu\lambda q^{n+1}/\gamma;q)_k(\nu\lambda/\gamma;q)_{2k}}\left(\frac{\alpha' q}{\gamma}\right)^k\\
&\cdot\ {}_4\phi_3(\alpha/\alpha',\beta,\nu\lambda q^k/\gamma,q^{-k};\lambda,\delta/\alpha',\epsilon;q,q)\\
&\cdot\ {}_4\phi_3(\alpha',\beta/\lambda,\nu q^{n+k},q^{k-n};\gamma/\lambda,\delta q^k,\alpha'\epsilon q^k/\alpha;q,q)
\end{aligned}
$$

with $\alpha\beta\nu q=\gamma\delta\epsilon$, which is a ${}_4\phi_3$ extension of (1.11) that contains three terminating balanced ${}_4\phi_3$ series. Setting $\epsilon=\alpha\beta\nu q/\gamma\delta$ in (3.3) and letting $\nu\to 0$ gives (1.11).

Proceeding as in the derivation of (2.9) from (2.8), it follows from (3.3) that

$$
\begin{aligned}
(3.4)\qquad & {}_4F_3(\alpha,\beta,n+\nu,-n;\gamma,\delta,\epsilon;1)\\
&=\frac{(\gamma-\lambda)_n(1+\nu-\gamma)_n}{(\gamma)_n(1+\nu+\lambda-\gamma)_n}\sum_{k=0}^{n}\frac{(\nu+\lambda-\gamma)_k(\lambda)_k(\epsilon-\alpha)_k}{k!\,(1+\nu-\gamma)_k(\delta)_k}\\
&\cdot\frac{(\delta-\alpha')_k(n+\nu)_k(-n)_k(1+\nu+\lambda-\gamma)_{2k}}{(\alpha'+\epsilon-\alpha)_k(1+\lambda-\gamma-n)_k(n+1+\nu+\lambda-\gamma)_k(\nu+\lambda-\gamma)_{2k}}\\
&\cdot\ {}_4F_3(\alpha-\alpha',\beta,k+\nu+\lambda-\gamma,-k;\lambda,\delta-\alpha',\epsilon;1)\\
&\cdot\ {}_4F_3(\alpha',\beta-\lambda,n+k+\nu,k-n;\gamma-\lambda,k+\delta,k+\alpha'+\epsilon-\alpha;1),
\end{aligned}
$$

with $\alpha+\beta+\nu+1=\gamma+\delta+\epsilon$, which is a discrete extension of (1.4) that contains three terminating balanced ${}_4F_3$ series. The $\nu\to+\infty$ limit case of (3.4), which contains three terminating ${}_3F_2$ series, and the expansion formulas that follow by applying transformation formulas to the ${}_4\phi_3$ and ${}_4F_3$ series in (2.8), (3.3) and (2.9), (3.4), respectively, will be omitted.

4. Derivation of (1.12) and extensions

We start by observing that if $\{A_k\}$ is a sequence of complex numbers, then

$$\sum_{k=0}^{n} \frac{(q^{-n},\lambda;q)_k}{(q,\gamma;q)_k} A_k = \frac{(\lambda;q)_n}{(\gamma;q)_n} \sum_{j=0}^{n} \frac{(q^{-n},\gamma/\lambda;q)_j}{(q,q^{1-n}/\lambda;q)_j} q^j \sum_{k=0}^{n-j} \frac{(q^{j-n};q)_k}{(q;q)_k} A_k \tag{4.1}$$

by changing the order of summation and using the q-Vandermonde summation formula [**7**, (1.5.3)] in the form

$${}_2\phi_1(q^{k-n},\gamma/\lambda;q^{1-n}/\lambda;q,q) = \frac{(\gamma;q)_n(\lambda;q)_k}{(\lambda;q)_n(\gamma;q)_k}.$$

Application of formula (4.1) to the sum over k on the right-hand side (4.1) yields

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(q^{-n},\lambda;q)_k}{(q,\gamma;q)_k} A_k \\ &= \frac{(\lambda;q)_n}{(\gamma;q)_n} \sum_{j=0}^{n} \frac{(q^{-n},\gamma/\lambda;q)_j}{(q,q^{1-n}/\lambda;q)_j} q^j \frac{(\nu;q)_{n-j}}{(\mu;q)_{n-j}} \\ &\cdot \sum_{m=0}^{n-j} \frac{(q^{j-n},\mu/\nu;q)_m}{(q,q^{1+j-n}/\nu;q)_m} q^m \sum_{r=0}^{n-j-m} \frac{(q^{j+m-n},\mu;q)_r}{(q,\nu;q)_r} A_r \\ &= \frac{(\lambda,\nu;q)_n}{(\gamma,\mu;q)_n} \sum_{k=0}^{n} \left(\sum_{r=0}^{k} \frac{(q^{-k},\mu;q)_r}{(q,\nu;q)_r} A_r \right) \frac{(q^{-n},\mu/\nu;q)_{n-k}}{(q,q^{1-n}/\nu;q)_{n-k}} q^{n-k} \\ &\cdot {}_3\phi_2(q^{k-n},q^{1-n}/\mu,\gamma/\lambda;\nu q^{1+k-n}/\mu,q^{1-n}/\lambda;q,q) \end{aligned} \tag{4.2}$$

after setting $m = n-j-k$.

Now apply the transformation formula [**7**, (3.2.2)] to the ${}_3\phi_2$ series in (4.2) to obtain the expansion formula

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(q^{-n},\lambda;q)_k}{(q,\gamma;q)_k} A_k \\ &= \frac{(\lambda,\nu;q)_n}{(\gamma,\mu;q)_n} \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \frac{(q^{-k},\mu;q)_j}{(q,\nu;q)_j} A_j \right) \frac{(q^{-n},\gamma\mu/\lambda\nu;q)_{n-k}}{(q,q^{1-n}/\nu;q)_{n-k}} q^{n-k} \\ &\cdot {}_3\phi_2(\mu/\lambda,\gamma/\lambda,q^{k-n};\gamma\mu/\lambda\nu,q^{1-n}/\lambda;q,q^{1-k}/\nu), \end{aligned} \tag{4.3}$$

which is an extension of (1.12) since it gives (1.12) when

$$A_k = \frac{(\alpha,\beta;q)_k}{(\lambda,\delta;q)_k} q^k.$$

Similarly, (4.3) also gives the following ${}_{r+2}\phi_{s+1}$ extension of (1.12)

$$
\begin{aligned}
(4.4)\qquad & {}_{r+2}\phi_{s+1}\left[\begin{matrix} q^{-n},\lambda,a_1,a_2,\dots,a_r \\ \gamma,b_1,\dots,b_s\end{matrix}; q,z\right] \\
&= \frac{(\lambda,\nu;q)_n}{(\gamma,\mu;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},\gamma\mu/\lambda\nu;q)_{n-k}}{(q,q^{1-n}/\nu;q)_{n-k}}q^{n-k} \\
&\cdot {}_3\phi_2\left[\begin{matrix}\mu/\lambda,\gamma/\lambda,q^{k-n} \\ \gamma\mu/\lambda\nu,q^{1-n}/\lambda\end{matrix}; q,q^{1-k}/\nu\right] \\
&\cdot {}_{r+2}\phi_{s+1}\left[\begin{matrix} q^{-k},\mu,a_1,a_2,\dots,a_r \\ \nu,b_1,\dots,b_s\end{matrix}; q,z\right].
\end{aligned}
$$

and, as a limit case, a corresponding extension of (1.14), which we omit. The above extensions of (1.12) can also be derived by applying transformation formulas to special cases of the rather general expansion formulas in [**6**, (4.2) and (4.5)] and [**7**, (3.7.3) and (3.7.6)].

5. Summation formulas for some basic Kampé de Fériet series

In a June 1, 1994, email message to the author, Joris Van der Jeugt conjectured that if $|a|,|b|<1$ and $|q|>1$, then

$$
(5.1)\qquad \Phi^{0:3}_{1:1}\left[\begin{matrix} - \\ de/bc\end{matrix} : \begin{matrix} c/a,d/c,e/c \\ 1/a\end{matrix} ; \begin{matrix} a/c,d/b,e/b \\ 1/b\end{matrix} ; q,q;q\right] = \frac{(c,ab/c;q^{-1})_\infty}{(a,b;q^{-1})_\infty},
$$

where $\Phi^{0:3}_{1:1}$ is the double basic hypergeometric Kampé de Fériet series defined by

$$
\begin{aligned}
(5.2)\qquad & \Phi^{0:3}_{1:1}\left[\begin{matrix} - \\ c\end{matrix} : \begin{matrix} a_1,a_2,a_3 \\ a_4\end{matrix} ; \begin{matrix} b_1,b_2,b_3 \\ b_4\end{matrix} ; x,y;q\right] \\
&= \sum_{j,k=0}^{\infty}\frac{(a_1,a_2,a_3;q)_j(b_1,b_2,b_3;q)_k\ x^jy^k}{(c;q)_{j+k}(q,a_4;q)_j(q,b_4;q)_k},
\end{aligned}
$$

which is also called a double q-Kampé de Fériet series. Van der Jeugt also observed that by inverting the base (i.e., replacing q by q^{-1}) and using the identity

$$
(a;q^{-1})_n = (a^{-1};q)_n(-a)^nq^{-\binom{n}{2}},
$$

formula (5.1) can be written in the equivalent form

$$
(5.3)\qquad \sum_{j,k=0}^{\infty}\frac{(a/c,c/d,c/e;q)_j(c/a,b/d,b/e;q)_kb^ja^kq^{jk}}{(bc/de;q)_{j+k}(q,a;q)_j(q,b;q)_k} = \frac{(c,ab/c;q)_\infty}{(a,b;q)_\infty}
$$

with $|a|,|b|,|q|<1$, which, due to the q^{jk} power in the argument of the double series in (5.3), cannot be written in terms of the $\Phi^{0:3}_{1:1}$ function defined in (5.2). This conjecture arose in trying to find a q-analogue of the summation formula (34) for a Kampé de Fériet $F^{0:3}_{1:1}$ series in the work of J. Van der Jeugt, S.N. Pitre, and K. Srinivasa Rao [**11**] on the evaluation of the 9-j recoupling coefficients appearing in the quantum theory of angular momentum.

On June 6, 1994, the author sent Van der Jeugt a proof of (5.3) and the derivation of a q-analogue of Erdélyi's fractional integral (1.5) by using the q-binomial integral [**7**, (1.11.7)], the transformation formula [**7**, III.3], the q-Gauss sum [**7**, II.8] and changes in order of summation. This enabled Van der Jeugt, Pitre, and Srinivasa Rao to state formula (5.1) in [**11**, (40) with a change in parameters], and inspired them to use similar methods to derive the transformation and summation formulas contained in [**11**], [**12**], [**13**], also see [**9**]. It also started the author's work on the derivation of the formulas contained in this paper.

To save space, rather than presenting the author's original proof of (5.3), we will first derive an extension of (5.3) to $\Phi^{1:3}_{1:2}$ series and then obtain (5.3) as a limit case of it. A limit case of the following argument gives a direct proof of (5.3). Let $\nu = \alpha$ and $\gamma\delta = \alpha\beta q^{1-n}$. Then the ${}_3\phi_2$ series on the left-hand side of (1.12) can be summed by the q-Saalschütz formula [**7**, (II.12)], and the ${}_4\phi_3$ series on the right-hand side (1.12) reduces to a ${}_3\phi_2$ series that can be transformed via [**7**, (III.11)] to get

$$(5.4)\qquad \frac{(\gamma/\alpha,\gamma/\beta;q)_n}{(\gamma,\gamma/\alpha\beta;q)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\alpha^n(\lambda;q)_n(\alpha,\delta\lambda/\beta\mu;q)_k(\gamma\mu/\alpha\lambda;q)_{n-k}}{(\gamma,\mu;q)_n(\delta;q)_k}(\frac{\beta\mu}{\alpha\lambda})^k \cdot {}_3\phi_2(\mu/\lambda,\gamma/\lambda,q^{k-n};\gamma\mu/\alpha\lambda,q^{1-n}/\lambda;q,q^{1-k}/\alpha) \cdot {}_3\phi_2(\lambda/\beta,\lambda/\mu,q^{-k};\lambda,\delta\lambda/\beta\mu;q,q).$$

By denoting the indexes of summation of the first and second ${}_3\phi_2$ series on the right-hand side (5.4) be i and j, respectively, setting $k = j + m$, and changing the order of summation, we find that sum over m is a ${}_2\phi_1$ series that can be summed via the q-Gauss sum [**7**, II.8]. This yields the summation formula

$$(5.5)\qquad \sum_{j,k=0}^{\infty} \frac{(q^{-n};q)_{j+k}(\mu/\lambda,\gamma/\lambda,\gamma/\alpha\beta;q)_j(\lambda/\beta,\lambda/\mu,\alpha;q)_k q^j q^k}{(\gamma/\beta;q)_{j+k}(\gamma\mu/\alpha\lambda,q^{1-n}/\lambda,q;q)_j(\lambda,\alpha\lambda q^{1-n}/\gamma\mu,q;q)_k} = \frac{(\gamma/\alpha,\mu;q)_n}{(\lambda,\gamma\mu/\alpha\lambda;q)_n}$$

with $n = 0, 1, 2, \ldots$. The double series in (5.5) terminates since $(q^{-n};q)_{j+k} = 0$ when $j + k > n$.

Setting

$$\alpha = b/d, \quad \beta = e, \quad \gamma = bc/d, \quad \lambda = b, \quad \mu = ab/c$$

converts (5.5) to the formula

$$(5.6)\qquad \Phi^{1:3}_{1:2}\left[\begin{matrix} q^{-n} \\ bc/de \end{matrix} : \begin{matrix} a/c,c/d,c/e \\ a,q^{1-n}/b \end{matrix} ; \begin{matrix} c/a,b/d,b/e \\ b,q^{1-n}/a \end{matrix} ;q,q;q\right] = \frac{(c,ab/c;q)_n}{(a,b;q)_n}$$

with $n = 0, 1, 2, \ldots$, where

$$(5.7)\qquad \Phi^{1:3}_{1:2}\left[\begin{matrix} c_1 \\ c_2 \end{matrix} : \begin{matrix} a_1,a_2,a_3 \\ a_4,a_5 \end{matrix} ; \begin{matrix} b_1,b_2,b_3 \\ b_4,b_5 \end{matrix} ;x,y;q\right] = \sum_{j,k=0}^{\infty} \frac{(c_1;q)_{j+k}(a_1,a_2,a_3;q)_j(b_1,b_2,b_3;q)_k\ x^j y^k}{(c_2;q)_{j+k}(q,a_4,a_5;q)_j(q,b_4,b_5;q)_k}.$$

Since formula (5.3) is the $n \to \infty$ limit case of (5.6), this completes our proof of (5.3).

Notice that if we replace the parameters a, b, c, d, e in (5.6) by q^a, q^b, q^c, q^d, q^e, respectively, and then let $q \to 1$, we obtain the summation formula

$$(5.8)\qquad F^{1:3}_{1:2}\left[\begin{matrix} -n \\ b+c-d-e \end{matrix} : \begin{matrix} a-c,c-d,c-e \\ a,1-n-b \end{matrix} ; \begin{matrix} c-a,b-d,b-e \\ b,1-n-a \end{matrix} ;1,1\right] = \frac{(c)_n(a+b-c)_n}{(a)_n(b)_n}$$

with $n = 0, 1, 2, \dots$, where $F^{1:3}_{1:2}$ is the double hypergeometric Kampé de Fériet series defined by

$$F^{1:3}_{1:2}\left[\begin{matrix} c_1 \\ c_2 \end{matrix} : \begin{matrix} a_1, a_2, a_3 \\ a_4, a_5 \end{matrix} ; \begin{matrix} b_1, b_2, b_3 \\ b_4, b_5 \end{matrix} ; x, y\right] = \sum_{j,k=0}^{\infty} \frac{(c_1)_{j+k}(a_1)_j(a_2)_j(a_3)_j(b_1)_k(b_2)_k(b_3)_k \; x^j y^k}{(c_2)_{j+k}(a_4)_j(a_5)_j(b_4)_k(b_5)_k \; j! \, k!}. \tag{5.9}$$

The $n \to \infty$ limit case of (5.8)

$$F^{0:3}_{1:1}\left[\begin{matrix} - \\ b+c-d-e \end{matrix} : \begin{matrix} a-c, c-d, c-e \\ a \end{matrix} ; \begin{matrix} c-a, b-d, b-e \\ b \end{matrix} ; 1, 1\right] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)}, \qquad \mathrm{Re}\, a > 0, \quad \mathrm{Re}\, b > 0. \tag{5.10}$$

which is equivalent to [**11**, (34)], was proven in Per W. Karlsson [**8**] by applying a reduction formula to a double Eulerian integral representation for $F^{0:3}_{1:1}$ series and using Gauss' summation formula

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\, c > \mathrm{Re}(a+b), \; c \neq 0, -1, -2, \dots,$$

and analytic continuation. Our derivation of (5.5) shows that one can also derive formula (5.10) from Erdélyi's fractional integral (1.5) by applying the transformation formula $_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z)$ to the $\nu = \alpha$ case of the hypergeometric series on the right-hand side of the integrand in (1.5), integrating termwise via the beta integral, letting $z \to 1^-$, summing the $_2F_1(\alpha, \beta; \delta; 1)$ series on the left-hand side of the equation with Gauss' summation formula, and then using analytic continuation. Additional summation formulas for hypergeometric and basic hypergeometric Kampé de Fériet series will be considered elsewhere.

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Department of Mathematics, Northwestern University , Evanston, IL 60208-2730
E-mail address: `george@math.nwu.edu`