Some Systems of Multivariable Orthogonal q-Racah Polynomials

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Abstract. In 1991 Tratnik derived two systems of multivariable orthogonal Racah polynomials and considered their limit cases. q-Extensions of these systems are derived, yielding systems of multivariable orthogonal q-Racah polynomials, from which systems of multivariable orthogonal q-Hahn, dual q-Hahn, q-Krawtchouk, q-Meixner, and q-Charlier polynomials follow as special or limit cases.

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1. Introduction

The Racah polynomials [1, 2, 25], defined by

$$\begin{aligned} r_n(x) &= r_n(x;\alpha,\beta,\gamma,N) \\ &= (\alpha+1)_n(\beta+\gamma+1)_n(-N)_n \\ &\quad\times {}_4F_3\left[\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma-N \\ \alpha+1, \beta+\gamma+1, -N \end{matrix};1\right] \end{aligned} \tag{1.1}$$

for $n = 0, 1, \ldots, N$, satisfy the discrete orthogonality relation

$$\sum_{x=0}^{N} r_n(x) r_m(x) \rho(x) = \lambda_n \delta_{n,m} \tag{1.2}$$

for $n, m = 0, 1, \ldots, N$, with the weight function

$$\begin{aligned} \rho(x) &= \rho(x;\alpha,\beta,\gamma,N) \\ &= \frac{\gamma-N+2x}{\gamma-N}\ \frac{(\gamma-N)_x(\alpha+1)_x(\beta+\gamma+1)_x(-N)_x}{x!\,(\gamma-\alpha-N)_x(-\beta-N)_x(\gamma+1)_x} \end{aligned} \tag{1.3}$$

and the normalization constant

$$\begin{aligned} \lambda_n &= \lambda_n(\alpha,\beta,\gamma,N) \\ &= \frac{(\alpha+\beta+2)_N(-\gamma)_N}{(\beta+1)_N(\alpha-\gamma+1)_N}(\alpha-\gamma+1)_n(\beta+\gamma+1)_n\,(\alpha+\beta+2+N)_n \\ &\quad\times \frac{(\alpha+\beta+1)n!\,(\alpha+1)_n(\beta+1)_n(-N)_n}{(\alpha+\beta+1+2n)(\alpha+\beta+1)_n}, \end{aligned} \tag{1.4}$$

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where N is a nonnegative integer. In 1991 Tratnik [24] extended the Racah polynomials to a system of multivariable orthogonal polynomials (in a slightly different notation)

$$\begin{aligned} R_{\mathbf{n}}(\mathbf{x}) &= R_{\mathbf{n}}(\mathbf{x}; a_1, \dots, a_{s+1}, \eta, N) \\ &= \prod_{k=1}^{s} r_{n_k}(x_k - N_{k-1}; 2N_{k-1} + \eta + \alpha_k - a_1, \alpha_{k+1} - 1, N_{k-1} + \alpha_k + x_{k+1}, x_{k+1} - N_{k-1}), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} &\mathbf{x} = (x_1, \dots, x_s),\ x_{s+1} = N,\ \mathbf{n} = (n_1, \dots, n_s), N_0 = 0, \\ &N_k = \sum_{j=1}^{k} n_j,\ 1 \le k \le s, \qquad \alpha_k = \sum_{j=1}^{k} a_j,\ 1 \le k \le s+1, \end{aligned} \tag{1.6}$$

and $N_s \le N$. Clearly, $R_{\mathbf{n}}(\mathbf{x})$ is a polynomial of total degree N_s in the variables $y_1, \dots, y_s$ with $y_k = x_k(x_k + \alpha_k)$, $k = 1, \dots, s$.

Tratnik showed that these polynomials satisfy the discrete orthogonality relation

$$\sum_{\mathbf{x}} R_{\mathbf{n}}(\mathbf{x}) R_{\mathbf{m}}(\mathbf{x}) \rho(\mathbf{x}) = \lambda_n \delta_{\mathbf{n},\mathbf{m}} \tag{1.7}$$

for $N_s, M_s \le N$, where $M_k = \sum_{j=1}^{k} m_j$ and the summation is over all $\mathbf{x} = (x_1, \dots, x_s)$ with $x_k = 0, 1, \dots, N$ for $k = 1, \dots, s$, $\delta_{\mathbf{n},\mathbf{m}} = \prod_{k=1}^{s} \delta_{n_k, m_k}$, the weight function is

$$\begin{aligned} \rho(\mathbf{x}) &= \rho(\mathbf{x}; a_1, \dots, a_{s+1}, \eta, N) \\ &= \frac{N!\,\Gamma(\alpha_s + N + 1)}{\Gamma(a_{s+1} + N)\Gamma(\alpha_{s+1} + N)} \frac{(a_1)_{x_1} (\eta + 1)_{x_1}}{x_1!\,(a_1 - \eta)_{x_1}} \\ &\quad \times \prod_{k=1}^{s} \frac{\Gamma(a_{k+1} + x_{k+1} - x_k)\Gamma(\alpha_{k+1} + x_{k+1} + x_k)}{(x_{k+1} - x_k)!\,\Gamma(\alpha_k + x_{k+1} + x_k + 1)} \frac{\alpha_k + 2x_k}{\alpha_k} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \lambda_n &= \lambda_n(a_1, \dots, a_{s+1}, \eta, N) \\ &= (\alpha_s + N)_{N_s} (\eta + 1 - a_1 - N)_{N_s} (-N)_{N_s} \\ &\quad \times \frac{\Gamma(a_1 - \eta)\Gamma(\alpha_s + N + 1)\Gamma(\alpha_{s+1} - a_1 + \eta + N_s + N + 1)}{\Gamma(a_1)\Gamma(\eta + 1)\Gamma(a_{s+1} + N)\Gamma(a_1 - \eta + N)} \\ &\quad \times \Bigg[\prod_{k=1}^{s} \alpha_k^{-1} n_k!\,(\alpha_{k+1} - a_1 + \eta + N_k + N_{k-1})_{n_k} \\ &\qquad \times \frac{\Gamma(a_{k+1} + n_k)\Gamma(\alpha_k - a_1 + N_k + N_{k-1} + 1)}{\Gamma(\alpha_{k+1} - a_1 + \eta + 2N_k + 1)} \Bigg]. \end{aligned} \tag{1.9}$$

Note that $\rho(\mathbf{x}) = 0$ if $x_{k+1} < x_k$ for some $k < s$. He also pointed out the special case of the multivariable Hahn polynomials of Karlin and McGregor [14], the limit cases of the multivariable Krawtchouk, Meixner and Charlier polynomials, and used permutations of the parameters and variables to derive a second system of polynomials that are orthogonal with respect to the weight function in (1.8).

In this paper we derive q-extensions of Tratnik's $R_{\mathbf{n}}(\mathbf{x}; a_1, \ldots, a_{s+1}, \eta, N)$ polynomials and of his second system of multivariable orthogonal Racah polynomials, from which systems of multivariable orthogonal q-Hahn, dual q-Hahn, q-Krawtchouk, q-Meixner, and q-Charlier polynomials follow as special or limit cases. Some q-extensions of Tratnik's [23] multivariable orthogonal Wilson polynomials and of his [21] multivariable biorthogonal generalization of the Wilson polynomials are given in our papers [11] and [10], respectively.

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2. Multivariable orthogonal q-Racah polynomials.

Analogous to (1.1) we define the q-Racah polynomials [2] by

$$\begin{aligned} r_n(x;q) &= r_n(x;a,b,c,N;q) \\ &= (aq, bcq, q^{-N}; q)_n (q^N/c)^{n/2} \\ &\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N} \\ aq, bcq, q^{-N} \end{matrix} ; q, q \right] \end{aligned} \tag{2.1}$$

for $n = 0, 1, \ldots, N$, where

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a_1, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n,$$

and we use the notation of our book [9]. The power $(q^N/c)^{n/2}$ is chosen in (2.1) so that certain symmetry properties of the q-Racah polynomials are satisfied; for example, by the Sears transformation formula [9, (2.10.4)] it follows from (2.1) that

$$r_n(x; a, b, c, N; q) = r_n(N - x; b, a, c^{-1}, N; q). \tag{2.2}$$

Note that $r_n(x; q)$ is a polynomials of degree n in the variable $z = q^{-x} + cq^{x-N}$.

Askey and Wilson [2] showed that the q-Racah polynomials satisfy the orthogonality relation [9, (7.2.18)]

$$\sum_{x=0}^{N} r_n(x;q) r_m(x;q) \rho(x;q) = \lambda_n(q) \delta_{n,m} \tag{2.3}$$

for $n, m = 0, 1, \ldots, N$, with

$$\begin{aligned} \rho(x;q) &= \rho(x;a,b,c,N;q) \\ &= \frac{1 - cq^{2x-N}}{1 - cq^{-N}} \frac{(cq^{-N}, aq, bcq, q^{-N}; q)_x}{(q, ca^{-1}q^{-N}, b^{-1}q^{-N}, cq; q)_x} (abq)^{-x} \end{aligned} \tag{2.4}$$

and

$$\lambda_n(q) = \lambda_n(a, b, c, N; q)$$

$$= \frac{(c^{-1}, abq^2; q)_N}{(aq/c, bq; q)_N} (q, aq, bq, aq/c, bcq, q^{-N}; q)_n \times \frac{1 - abq}{1 - abq^{2n+1}} \frac{(abq^{N+2}; q)_n}{(abq; q)_n}. \tag{2.5}$$

Analogous to (1.5) we define the multivariable q-Racah polynomials by

$$R_{\mathbf{n}}(\mathbf{x}; q) = R_{\mathbf{n}}(\mathbf{x}; a_1, \ldots, a_{s+1}, b, N; q) = \prod_{k=1}^{s} r_{n_k}(x_k - N_{k-1}; bA_k q^{2N_{k-1}}/a_1, a_{k+1}q^{-1}, A_k q^{x_{k+1}+N_{k-1}}, x_{k+1} - N_{k-1}; q), \tag{2.6}$$

where, in addition to the definitions of $\mathbf{x}$, $\mathbf{n}$, x_{s+1} and N_k given in (1.6), we let

$$A_0 = 1, \quad A_k = \prod_{j=1}^{k} a_j, \quad k = 1, \ldots, s+1. \tag{2.7}$$

Note that $R_{\mathbf{n}}(\mathbf{x}; q)$ is a polynomial of total degree N_s in the variables $z_1, \ldots, z_s$ where

$$z_k = q^{-x_k} + A_k q^{x_k}, \quad k = 1, \ldots, s.$$

In order to derive an orthogonality relation for these polynomials we start by considering the following q-extensions of the weight function $\rho(\mathbf{x})$ in (1.8):

$$\rho(\mathbf{x}; q) = \rho(\mathbf{x}; a_1, \ldots, a_{s+1}, b, N; q) = \frac{(q, qA_s; q)_N}{(a_{s+1}, A_{s+1}; q)_N} \frac{(a_1, bq; q)_{x_1}}{(q, a_1/b; q)_{x_1}} \times \prod_{k=1}^{s} \frac{(a_{k+1}; q)_{x_{k+1}-x_k} (A_{k+1}; q)_{x_{k+1}+x_k} (1 - A_k q^{2x_k})}{(q; q)_{x_{k+1}-x_k} (qA_k; q)_{x_{k+1}+x_k} (1 - A_k)} (c_{k,s})^{-x_k}, \tag{2.8}$$

where $c_{1,s}, \ldots, c_{s,s}$ are to be determined (see (2.15)) so that the orthogonality relation

$$\sum_{x_s=0}^{N} \sum_{x_{s-1}=0}^{x_s} \cdots \sum_{x_2=0}^{x_3} \sum_{x_1=0}^{x_2} R_{\mathbf{n}}(\mathbf{x}; q) R_{\mathbf{m}}(\mathbf{x}; q) \rho(\mathbf{x}; q) = \lambda_{\mathbf{n}}(q) \delta_{\mathbf{n},\mathbf{m}} \tag{2.9}$$

holds for N_s, $M_s \leq N$ and certain normalization constants $\lambda_{\mathbf{n}}(q)$, which will be given in (2.14).

The summation over x_1 in (2.9) can be evaluated via (2.3), which is derived in [9, §7.2] by means of the terminating ${}_6\phi_5$ summation [9, (2,4,2)], to obtain for $s \geq 2$ that

$$\sum_{x_1=0}^{x_2} \frac{(a_1, bq; q)_{x_1} (a_2; q)_{x_2-x_1} (A_2; q)_{x_2+x_1} (1 - a_1 q^{2x_1})}{(q, a_1/b; q)_{x_1} (q)_{x_2-x_1} (a_1 q; q)_{x_2+x_1} (1 - a_1)} (c_{1,s})^{-x_1} \times r_{n_1}(x_1; b, a_2 q^{-1}, a_1 q^{x_2}, x_2; q) r_{m_1}(x_1; b, a_2 q^{-1}, a_1 q^{x_2}, x_2; q)$$
$$= \frac{1 - a_2 b}{1 - a_2 b q^{2n_1}} \frac{(bq, a_2, q, A_2, bA_2 q/a_1; q)_{n_1}}{(a_2 b; q)_{n_1}} \left(\frac{b}{a_1}\right)^{n_1} q^{n_1^2} \times \frac{(A_2 q^{n_1}, bA_2 q^{n_1+1}/a_1; q)_{x_2}}{(q; q)_{x_2-n_1} (a_1/b; q)_{x_2-n_1}} (bq^{2n_1+1})^{-x_2} \delta_{n_1,m_1} \tag{2.10}$$

provided we choose $c_{1,s} = bq$ for $s \geq 2$. In the single variable case $s = 1$ it is clear from (2.3) and (2.4) that $c_{1,1} = bq$, and hence we conclude that $c_{1,s} = bq$ for $s \geq 1$.

By doing the summations over $x_1, x_2, \ldots, x_j$ in (2.9) for small j one is led to conjecture that after summing over $x_1, \ldots, x_j$ in (2.9) and setting $c_{k,s} = a_k$ for $2 \le k \le s-1$ we have

$$\left[\prod_{k=1}^{j} \frac{(1-bA_{k+1}/a_1)}{(1-bA_{k+1}q^{2N_k}/a_1)} \frac{(q, a_{k+1}; q)_{n_k} (bA_k q/a_1; q)_{N_k+N_{k-1}}}{(bA_{k+1}/a_1; q)_{N_k+N_{k-1}}} \delta_{n_k,m_k}\right]$$
$$\times \frac{(A_{j+1}, qbA_{j+1}/a_1; q)_{N_j+x_{j+1}}}{(q, a_1/b; q)_{x_{j+1}-N_j}} \Big(\frac{b}{a_1}\Big)^{N_j} q^{N_j^2} \Big(\frac{bA_j q^{2N_j+1}}{a_1}\Big)^{-x_{j+1}} \tag{2.11}$$

as the sum for $j = 1, 2, \ldots, s-1$.

To prove (2.11) by induction on j, assume that $j \le s-2$ for $s \ge 3$, multiply (2.11) by the remaining x_{j+1}-dependent part of the weight function and polynomials, and then sum over x_{j+1} to get

$$\left[\prod_{k=1}^{j} \frac{(1-bA_{k+1}/a_1)}{(1-bA_{k+1}q^{2N_k}/a_1)} \frac{(q, a_{k+1}; q)_{n_k} (qbA_k/a_1; q)_{N_k+N_{k-1}}}{(bA_{k+1}/a_1; q)_{N_k+N_{k-1}}} \delta_{n_k,m_k}\right]$$
$$\times \Big(\frac{b}{a_1}\Big)^{N_j} q^{N_j^2} \sum_{x_{j+1}=0}^{x_{j+2}} \frac{(A_{j+1}, qbA_{j+1}/a_1; q)_{N_j+x_{j+1}}}{(q, a_1/b; q)_{x_{j+1}-N_j}} \Big(\frac{bA_j q^{2N_j+1}}{a_1}\Big)^{-x_{j+1}}$$
$$\times \frac{(a_{j+2}; q)_{x_{j+2}-x_{j+1}} (A_{j+2}; q)_{x_{j+2}+x_{j+1}} (1-A_{j+1}q^{2x_{j+1}})}{(q; q)_{x_{j+2}-x_{j+1}} (qA_{j+1}; q)_{x_{j+2}+x_{j+1}} (1-A_{j+1})} (a_{j+1})^{-x_{j+1}}$$
$$\times r_{n_{j+1}}(x_{j+1}-N_j; bA_{j+1}q^{2N_j}/a_1, a_{j+2}q^{-1}, A_{j+1}q^{N_j+x_{j+2}}, x_{j+2}-N_j; q)$$
$$\times r_{m_{j+1}}(x_{j+1}-N_j; bA_{j+1}q^{2N_j}/a_1, a_{j+2}q^{-1}, A_{j+1}q^{N_j+x_{j+2}}, x_{j+2}-N_j; q). \tag{2.12}$$

Note that the summand above vanishes for $0 \le x_{j+1} < N_j$. Hence, the sum is effectively from $x_{j+1} = N_j$ to x_{j+2}. Replacing $x_{j+1} - N_j$ by x, say, the sum in (2.12) then becomes

$$\frac{(A_{j+1}, qbA_{j+1}/a_1; q)_{2N_j} (a_{j+2}; q)_{x_{j+2}-N_j} (A_{j+2}; q)_{x_{j+2}+N_j} (1-A_{j+1}q^{2N_j})}{(q; q)_{x_{j+2}-N_j} (qA_{j+1}; q)_{x_{j+2}+N_j} (1-A_{j+1})}$$
$$\times \Big(\frac{bA_{j+1}}{a_1} q^{2N_j+1}\Big)^{-N_j} \sum_{x=0}^{x_{j+2}-N_j} \frac{(1-A_{j+1}q^{2N_j+2x})}{(1-A_{j+1}q^{2N_j})} \frac{(A_{j+1}q^{2N_j}, A_{j+2}q^{N_j+x_{j+2}}; q)_x}{(q, q^{N_j+1-x_{j+2}}/a_{j+2}; q)_x}$$
$$\times \frac{(bA_{j+1}q^{2N_j+1}/a_1, q^{N_j-x_{j+2}}; q)_x}{(a_1/b, A_{j+1}q^{N_j+x_{j+2}+1}; q)_x} \Big(\frac{bA_{j+2}}{a_1} q^{2N_j}\Big)^{-x}$$
$$\times r_{n_{j+1}}(x; bA_{j+1}q^{2N_j}/a_1, a_{j+2}q^{-1}, A_{j+1}q^{N_j+x_{j+2}}, x_{j+2}-N_j; q)$$
$$\times r_{m_{j+1}}(x; bA_{j+1}q^{2N_j}/a_1, a_{j+2}q^{-1}, A_{j+1}q^{N_j+x_{j+2}}, x_{j+2}-N_j; q)$$
$$= \frac{(qA_{j+1}, qbA_{j+1}/a_1; q)_{2N_j} (a_{j+2}; q)_{x_{j+2}-N_j} (A_{j+2}; q)_{x_{j+2}+N_j}}{(q; q)_{x_{j+2}-N_j} (qA_{j+1}; q)_{x_{j+2}+N_j}}$$
$$\times \Big(\frac{bA_{j+1}}{a_1} q^{2N_j+1}\Big)^{-N_j} (q, bA_{j+1}q^{2N_j+1}/a_1, a_{j+2}, A_{j+2}q^{N_j+x_{j+2}}, q^{N_j-x_{j+2}}; q)_{n_{j+1}}$$
$$\times \frac{(1-bA_{j+2}q^{2N_j}/a_1)(bA_{j+2}q^{N_j+x_{j+2}+1}/a_1; q)_{n_{j+1}}}{(1-bA_{j+2}q^{N_j+x_{j+2}}/a_1)(bA_{j+2}q^{2N_j}/a_1; q)_{n_{j+1}}}$$

$$\times \frac{(q^{-N_j-x_{j+2}}/A_{j+1}, bA_{j+2}q^{2N_j+1}/a_1;q)_{x_{j+2}-N_j}}{(bq^{N_j-x_{j+2}+1}/a_1, a_{j+2};q)_{x_{j+2}-N_j}}\delta_{n_{j+1},m_{j+1}}$$

by (2.3). Substituting this into (2.12) and simplifying, we find that the right-hand side of (2.12) is (2.11) with j replaced by $j+1$. This proves (2.11).

Now set $j = s-1$ in (2.11) and substitute into the left-hand side of (2.9) to find that if we set $c_{s,s} = a_s$ for $s \geq 2$, then

$$\begin{aligned}
&\sum_{\mathbf{x}} R_{\mathbf{n}}(\mathbf{x};q)R_{\mathbf{m}}(\mathbf{x};q)\rho(\mathbf{x};q) \\
&= \sum_{x_s=0}^{N}\sum_{x_{s-1}=0}^{x_s}\cdots\sum_{x_2=0}^{x_3}\sum_{x_1=0}^{x_2} R_{\mathbf{n}}(\mathbf{x};q)R_{\mathbf{m}}(\mathbf{x};q)\rho(\mathbf{x};q) \\
&= \left[\prod_{k=1}^{s-1}\frac{(1-bA_{k+1}/a_1)(q,a_{k+1};q)_{n_k}(qbA_k/a_1;q)_{N_k+N_{k-1}}}{(1-bA_{k+1}q^{2N_k}/a_1)(bA_{k+1}/a_1;q)_{N_k+N_{k-1}}}\delta_{n_k,m_k}\right] \\
&\times \frac{(q,A_s,qbA_s/a_1;q)_{2N_{s-1}}(A_{s+1}q^N,q^{-N};q)_{N_{s-1}}}{(A_sq^{N+1},q^{1-N}/a_{s+1};q)_{N_{s-1}}}(A_{s+1}q^{N_{s-1}})^{-N_{s-1}} \\
&\times \sum_{x=0}^{N-N_{s-1}}\frac{(1-A_sq^{2N_{s-1}+2x})}{(1-A_sq^{2N_{s-1}})}\frac{(A_sq^{2N_{s-1}},\frac{bA_s}{a_1}q^{2N_{s-1}+1},A_{s+1}q^{N+N_{s-1}},q^{N_{s-1}-N};q)_x}{(q,a_1/b,q^{1-N+N_{s-1}}/a_{s+1},A_sq^{N+N_{s-1}+1};q)_x} \\
&\times \left(\frac{bA_{s+1}}{a_1}q^{2N_{s-1}}\right)^{-x} r_{n_s}(x;bA_sq^{2N_{s-1}}/a_1,a_{s+1}q^{-1},A_sq^{N+N_{s-1}},N-N_{s-1};q) \\
&\times r_{m_s}(x;bA_sq^{2N_{s-1}}/a_1,a_{s+1}q^{-1},A_sq^{N+N_{s-1}},N-N_{s-1};q) \\
&= \lambda_{\mathbf{n}}(q)\delta_{\mathbf{n},\mathbf{m}}
\end{aligned} \tag{2.13}$$

with

$$\begin{aligned}
\lambda_{\mathbf{n}}(q) &= \lambda_{\mathbf{n}}(a_1,\ldots,a_{s+1},b,N;q) \\
&= \frac{(qA_s,qbA_{s+1}/a_1;q)_N}{(a_{s+1},a_1/b;q)_N}\left(\frac{a_1}{qbA_s}\right)^N \\
&\times (bA_{s+1}q^{N+1}/a_1,A_{s+1}q^N,bq^{1-N}/a_1,q^{-N};q)_{N_s} \\
&\times \prod_{k=1}^{s}\frac{(q,a_{k+1};q)_{n_k}(qbA_k/a_1;q)_{N_k+N_{k-1}}(1-bA_{k+1}/a_1)}{(bA_{k+1}/a_1;q)_{N_k+N_{k-1}}(1-bA_{k+1}q^{2N_k}/a_1)},
\end{aligned} \tag{2.14}$$

where, in the weight function $\rho(\mathbf{x};q)$ in (2.8) we set

$$c_{1,s} = bq \quad \text{for } s \geq 1, \quad c_{k,s} = a_k \quad \text{for } 2 \leq k \leq s, \tag{2.15}$$

which completes the proof of (2.9) with

$$\begin{aligned}
\rho(\mathbf{x};q) &= \frac{(q,qA_s;q)_N}{(a_{s+1},A_{s+1};q)_N}\frac{(a_1,bq;q)_{x_1}}{(q,a_1/b;q)_{x_1}}\left(\frac{a_1}{bq}\right)^{x_1} \\
&\times \prod_{k=1}^{s}\frac{(a_{k+1};q)_{x_{k+1}-x_k}(A_{k+1};q)_{x_{k+1}+x_k}(1-A_kq^{2x_k})}{(q;q)_{x_{k+1}-x_k}(qA_k;q)_{x_{k+1}+x_k}(1-A_k)}a_k^{-x_k}.
\end{aligned} \tag{2.16}$$

Analogous to Tratnik's [24] permutations of the parameters and variables, we consider the following permutations

$$
\begin{aligned}
&a_1 \longleftrightarrow (A_s q^{2N})^{-1}, \quad a_{k+1} \longleftrightarrow a_{s-k+1}, \quad k=1,\dots,s-1, \\
&a_{s+1} \longleftrightarrow bq, \quad x_k \longleftrightarrow N - x_{s-k+1}, \quad k=1,\dots,s.
\end{aligned} \tag{2.17}
$$

Clearly the summation region in (2.9) remains unchanged under these permutations, and the weight function $\rho(\mathbf{x};q)$ in (2.16) changes to a multiple of itself, namely,

$$
\frac{(a_{s+1}, a_1/b, A_{s+1};q)_N}{(a_1, bq, qA_s;q)_N} \Big(\frac{qbA_s}{a_1}\Big)^N \rho(\mathbf{x}; a_1,\dots,a_{s+1}, b, N; q). \tag{2.18}
$$

On the other hand the polynomial $R_{\mathbf{n}}(\mathbf{x};q)$ transforms to

$$
R_{\mathbf{n}}((N-x_s, N-x_{s-1},\dots,N-x_1); (A_s q^{2N})^{-1}, a_s, a_{s-1},\dots,a_2, bq, a_{s+1}q^{-1}, N; q), \tag{2.19}
$$

which, since (2.6) is not invariant under (2.17), gives a second family of multivariable orthogonal q-Racah polynomials. By replacing n_k by n_{s-k+1} for $k=1,\dots,s$, setting $N_k^* = \prod_{j=k}^{s} n_j$ and using (2.9) we obtain that the polynomials

$$
\begin{aligned}
&\tilde{R}_{\mathbf{n}}(\mathbf{x};q) = \tilde{R}_{\mathbf{n}}(\mathbf{x}; a_1,\dots,a_{s+1}, b, N; q) \\
&= r_{n_1}(N - N_2^* - x_1; A_{s+1}q^{2N_2^*-1}/a_1, b, q^{N_2^*-N}/a_1, N-N_2^*; q) \\
&\times \prod_{k=2}^{s} r_{n_k}(N - N_{k+1}^* - x_{kj}; A_{s+1}q^{2N_{k+1}^*-1}/A_k, a_k q^{-1}, q^{N_{k+1}^*-N-x_{k-1}}/A_k, N - N_{k+1}^* - x_{k-1}; q)
\end{aligned} \tag{2.20}
$$

are a q-extension of Tratnik's [24, (2.12)] second multivariable Racah polynomials and that they satisfy the orthogonality relation

$$
\sum_{\mathbf{x}} \tilde{R}_{\mathbf{n}}(\mathbf{x};q)\tilde{R}_{\mathbf{m}}(\mathbf{x};q)\rho(\mathbf{x};q) = \tilde{\lambda}_{\mathbf{n}}(q)\delta_{\mathbf{n},\mathbf{m}} \tag{2.21}
$$

for N_s, $M_s \le N$, where

$$
\begin{aligned}
&\tilde{\lambda}_{\mathbf{n}}(q) = \tilde{\lambda}_{\mathbf{n}}(a_1,\dots,a_{s+1}, b, N; q) \\
&= \frac{(qA_s, qbA_{s+1}/a_1;q)_N}{(a_1/b, a_{s+1};q)_N}\Big(\frac{a_1}{qbA_s}\Big)^N \\
&\times (bA_{s+1}q^{N+1}/a_1, bq^{1-N}/a_1, A_{s+1}q^N, q^{-N};q)_{N_1^*} \\
&\times \frac{(q,qb;q)_{n_1}(A_{s+1}/a_1;q)_{N_1^*+N_2^*}(1-bA_{s+1}/a_1)}{(bA_{s+1}/a_1;q)_{N_1^*+N_2^*}(1-bA_{s+1}q^{2N_1^*}/a_1)} \\
&\times \prod_{k=1}^{s-1} \frac{(q, a_{k+1};q)_{n_{k+1}}(A_{s+1}/A_{k+1};q)_{N_{k+1}^*+N_{k+2}^*}(1-A_{s+1}/qA_k)}{(A_{s+1}/qA_k;q)_{N_{k+1}^*+N_{k+2}^*}(1-A_{s+1}q^{2N_{k+1}^*}/qA_k)}.
\end{aligned} \tag{2.22}
$$

3. Some limit cases of (2.9) and (2.21)

The multivariable q-Racah polynomials in (2.6) and (2.20) contain as limit cases Tratnik's [24] systems of multivariable Racah, Hahn, dual Hahn, Krawtchouk, Meixner, and Charlier polynomials.

Here we will derive limit cases of the orthogonality relations (2.9) and (2.21) containing multivariable q-Hahn, dual q-Hahn, q-Krawtchouk, q-Meixner, and q-Charlier polynomials. We start with the q-analogue of the dual Hahn polynomials defined by

$$\begin{aligned} d_n(x;b,c,N;q) &= \lim_{a\to 0} r_n(x;a,b,c,N;q) \\ &= (bcq,q^{-N};q)_n \left(q^N/c\right)^{n/2} {}_3\phi_2 \left[\begin{matrix} q^{-n}, q^{-x}, cq^{x-N} \\ bcq, q^{-N} \end{matrix} ; q, q \right] \end{aligned} \tag{3.1}$$

for $n = 0, 1, \ldots, N$, and let

$$\begin{aligned} D_{\mathbf{n}}(\mathbf{x};q) &= D_{\mathbf{n}}(\mathbf{x};a_1,\ldots,a_{s+1},N;q) \\ &= \lim_{b\to 0} R_{\mathbf{n}}(\mathbf{x};a_1,\ldots,a_{s+1},b,N;q) \\ &= \prod_{k=1}^{s} d_{n_k}(x_k - N_{k-1}; a_{k+1}q^{-1}, A_k q^{x_{k+1}+N_{k-1}}, x_{k+1} - N_{k-1}; q) \end{aligned} \tag{3.2}$$

where $N_s \le N$ and $x_{s+1} = N$. Then, by taking the $b \to 0$ limit of (2.9) we get the orthogonality relation

$$\sum_{\mathbf{x}} D_{\mathbf{n}}(\mathbf{x};q) D_{\mathbf{m}}(\mathbf{x};q) \rho_D(\mathbf{x};q) = \lambda_D(\mathbf{n};q) \delta_{\mathbf{n},\mathbf{m}} \tag{3.3}$$

for N_s, $M_s \le N$, were

$$\begin{aligned} \rho_D(\mathbf{x};q) &= \rho_D(\mathbf{x};a_1,\ldots,a_{s+1},N;q) \\ &= \frac{(q,qA_s;q)_N}{(a_{s+1},A_{s+1};q)_N} \frac{(a_1;q)_{x_1}}{(q;q)_{x_1}} (-q)^{-x_1} q^{-\binom{x_1}{2}} \\ &\times \prod_{k=1}^{s} \frac{(a_{k+1};q)_{x_{k+1}-x_k} (A_{k+1};q)_{x_{k+1}+x_k} (1 - A_k q^{2x_k})}{(q;q)_{x_{k+1}-x_k} (qA_k;q)_{x_{k+1}+x_k} (1 - A_k)} a_k^{-x_k}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \lambda_D(\mathbf{n};q) &= \lambda_D(\mathbf{n};a_1,\ldots,a_{s+1},N;q) \\ &= \frac{(qA_s;q)_N (A_{s+1}q^N, q^{-N};q)_{N_s}}{(a_{s+1};q)_N} (-qA_s)^{-N} q^{-\binom{N}{2}} \\ &\times \prod_{k=1}^{s} (q, a_{k+1};q)_{n_k}, \end{aligned} \tag{3.5}$$

and the summation in (3.3) is over the same region as in (2.13).

On the other hand, as $b \to 0$ the orthogonality relation (2.21) approaches the limit

$$\sum_{\mathbf{x}} T_{\mathbf{n}}(\mathbf{x};q) T_{\mathbf{m}}(\mathbf{x};q) \rho_T(\mathbf{x};q) = \lambda_T(\mathbf{n};q) \delta_{\mathbf{n},\mathbf{m}} \tag{3.6}$$

for N_s, $M_s \le N$, where

$$\begin{aligned} T_{\mathbf{n}}(\mathbf{x};q) &= T_{\mathbf{n}}(\mathbf{x};a_1,\ldots,a_{s+1},N;q) \\ &= d_{n_1}(N - N_2^* - x_1; A_{s+1}q^{N+N_2^*-1}, q^{N_2^*-N}/a_1, N - N_2^*; q) \\ &\times \prod_{k=2}^{s} r_{n_k}\left(N - N_{k+1}^* - x_k; \frac{A_{s+1}}{A_k} q^{2N_{k+1}^*-1}, \frac{a_k}{q}, \frac{q^{N_{k+1}^*-N-x_{k-1}}}{A_k}, N - N_{k+1}^* - x_{k-1}; q\right), \end{aligned} \tag{3.7}$$

$\rho_T(\mathbf{x};q) = \rho_D(\mathbf{x};q)$ is the same weight function as in (3.4), and

$$\begin{aligned}\lambda_T(\mathbf{n};q) &= \lambda_T(\mathbf{n};a_1,\ldots,a_{s+1},N;q)\\ &= \frac{(qA_s;q)_N}{(a_{s+1};q)_N}(-qA_s)^{-N}q^{-\binom{N}{2}}\\ &\times (A_{s+1}q^N,q^{-N};q)_{N_1^*}(q;q)_{n_1}(A_{s+1}/a_1;q)_{N_1^*+N_2^*}\\ &\times \prod_{k=1}^{s-1}\frac{(q,a_{k+1};q)_{n_{k+1}}(A_{s+1}/A_{k+1};q)_{N_{k+1}^*+N_{k+2}^*}(1-A_{s+1}/qA_k)}{(A_{s+1}/qA_k;q)_{N_{k+1}^*+N_{k+2}^*}(1-A_{s+1}q^{2N_{k+1}^*}/qA_k)}.\end{aligned} \tag{3.8}$$

If we multiply (2.9) by

$$\prod_{k=1}^{s}\Big(\frac{bA_k}{a_1}q^{N_{k-1}+1}\Big)^{-n_k}\Big(\frac{bA_k}{a_1}q^{M_{k-1}+1}\Big)^{-m_k}$$

and take the limit $b\to\infty$, we obtain the limiting relation

$$\sum_{\mathbf{x}} D_{\mathbf{n}}^*(\mathbf{x};q)D_{\mathbf{m}}^*(\mathbf{x};q)\rho_{D^*}(\mathbf{x},q) = \lambda_{D^*}(\mathbf{n};q)\delta_{\mathbf{n},\mathbf{m}} \tag{3.9}$$

for $N_s,\ M_s \le N$, where

$$\begin{aligned}D_{\mathbf{n}}^*(\mathbf{x};q) &= D_{\mathbf{n}}^*(\mathbf{x};a_1,\ldots,a_{s+1},N;q)\\ &= \prod_{k=1}^{s} d_{n_k}^*(x_k-N_{k-1};a_{k+1}q^{-1},A_kq^{x_{k+1}+N_{k-1}},x_{k+1}-N_{k-1};q)\end{aligned} \tag{3.10}$$

with

$$\begin{aligned}d_n^*(x;b,c,N;q) &= \lim_{a\to\infty}(aq)^{-n}r_n(x;a,b,c,N;q)\\ &= (bcq,q^{-N};q)_n(-1)^nq^{\binom{n}{2}}(q^N/c)^{n/2}\,{}_3\phi_2\left[\begin{matrix}q^{-n},q^{-x},cq^{x-N}\\ bcq,q^{-N}\end{matrix};q,bq^{n+1}\right]\end{aligned} \tag{3.11}$$

for $n=0,1,\ldots,N$,

$$\begin{aligned}\rho_{D^*}(\mathbf{x};q) &= \rho_{D^*}(\mathbf{x};a_1,\ldots,a_{s+1},N;q)\\ &= \frac{(q,qA_s;q)_N}{(a_{s+1},A_{s+1};q)_N}\frac{(a_1;q)_{x_1}}{(q;q)_{x_1}}(-a_1)^{x_1}q^{-\binom{x_1}{2}}\\ &\times\prod_{k=1}^{s}\frac{(a_{k+1};q)_{x_{k+1}-x_k}(A_{k+1};q)_{x_{k+1}+x_k}(1-A_kq^{2x_k})}{(q;q)_{x_{k+1}-x_k}(qA_k;q)_{x_{k+1}+x_k}(1-A_k)}a_k^{-x_k},\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\lambda_{D^*}(\mathbf{n};q) &= \lambda_{D^*}(\mathbf{n};a_1,\ldots,a_{s+1},N;q)\\ &= \frac{(qA_s;q)_N}{(a_{s+1};q)_N}(-a_{s+1})^Nq^{\binom{N}{2}}A_{s+1}^{N_s}q^{N_s(N_s+1)}\\ &\times\prod_{k=1}^{s}(q,a_{k+1};q)_{n_k}(A_kq^{N_{k-1}+1})^{-2n_k}q^{N_{k-1}-N_k}a_{k+1}^{-N_k-N_{k-1}}.\end{aligned} \tag{3.13}$$

Note that $\rho_{D^*}(\mathbf{x};q) = (a_1/q)^{x_1}\ q^{\binom{x_1}{2}}\ \rho_D(\mathbf{x};q)$ and that $D_{\mathbf{n}}^*(\mathbf{x};q)$ is not a constant (independent of $\mathbf{x}$) multiple of $D_{\mathbf{n}}(\mathbf{x};q)$. Similarly, by inverting the base q in the orthogonality relation (3.9) it follows that (3.9) is equivalent to (3.3).

To obtain q-analogues of the multivariable Hahn polynomials in Karlin and McGregor [14] and in Tratnik [24], multiply (2.9) by $a_1^{N_s}$, take the limit $a_1 \to 0$, replace b by a_1 and a_k by qa_k for $k = 2, 3, \ldots, s+1$, and make the change of variables $y_1 = x_1$, $y_k = x_k - x_{k-1}$ for $k = 2, 3, \ldots, s$. This yields the multivariable q-Hahn polynomial orthogonality relation

$$\sum_{\mathbf{y}} H_{\mathbf{n}}(\mathbf{y}; q) H_{\mathbf{m}}(\mathbf{y}; q) \rho_H(\mathbf{y}; q) = \lambda_H(\mathbf{n}; q) \delta_{\mathbf{n},\mathbf{m}} \tag{3.14}$$

for N_s, $M_s \le N$, where

$$\begin{aligned} H_{\mathbf{n}}(\mathbf{x}; q) &= H_{\mathbf{n}}(\mathbf{x}; a_1, \ldots, a_{s+1}, N; q) \\ &= \prod_{k=1}^{s} h_{n_k}(Y_k - N_{k-1}; A_k q^{2N_k+k-1}, a_{k+1}, Y_{k+1} - N_{k-1}; q) \end{aligned} \tag{3.15}$$

with the single-variable q-Hahn polynomials defined by

$$\begin{aligned} h_n(x; a, b, N; q) &= \lim_{c \to 0} (cq^{-N})^{n/2} r_n(x; a, b, c, N; q) \\ &= (aq, q^{-N}; q)_n \; {}_3\phi_2 \left[\begin{matrix} q^{-n}, abq^{n+1}, q^{-x} \\ aq, q^{-N} \end{matrix} ; q, q \right] \end{aligned} \tag{3.16}$$

for $n = 0, 1, \ldots, N$,

$$\begin{aligned} \rho_H(\mathbf{y}; q) &= \rho_H(\mathbf{y}; a_1, \ldots, a_{s+1}, N; q) \\ &= \frac{(q^{-N}; q)_{Y_s}}{(q^{-N}/a_{s+1}; q)_{Y_s}} a_{s+1}^{-Y_s} \prod_{k=1}^{s} \frac{(qa_k; q)_{y_k}}{(q; q)_{y_k}} (qa_k)^{-Y_k}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \lambda_H(\mathbf{n}; q) &= \lambda_H(\mathbf{n}; a_1, \ldots, a_{s+1}, N; q) \\ &= \frac{(A_{s+1} q^{s+1}; q)_{N+N_s} (q^{-N}; q)_{N_s}}{(qa_{s+1}; q)_N} (A_s q^{N_s+s})^{-N} (-1)^{N_s} q^{\binom{N_s}{2}} \\ &\times \prod_{k=1}^{s} \frac{(q, qa_{k+1}; q)_{n_k} (A_k q^k; q)_{N_k+N_{k-1}} (1 - A_{k+1} q^k)}{(A_{k+1} q^k; q)_{N_k+N_{k-1}} (1 - A_{k+1} q^{k+2N_k})} \left(A_k q^{k+2N_{k-1}} \right)^{n_k}, \end{aligned} \tag{3.18}$$

$Y_k = \sum_{j=1}^{k} y_j$ for $1 \le k \le s$ so that $Y_1 = y_1 = x_1$ and $x_k = Y_k$ for $1 \le k \le s$, $Y_{s+1} = x_{s+1} = N$, and the summation is over all $\mathbf{y}$ with $y_k = 0, 1, \ldots$, and $Y_s \le N$. Also see Dunkl [4] and Rosengren [18].

Clearly the summation region in (3.14) and the weight function in (3.17) are invariant under any permutation of the labels $(1, 2, \ldots, s)$. If we set $y_{s+1} = N - Y_s$, then they are also invariant (apart from the renormalization of the weight function) under any permutation of the labels $(1, 2, \ldots, s+1)$; i.e., under any simultaneous permutation of $(a_1, a_2, \ldots, a_{s+1})$ and $(y_1, y_2, \ldots, y_{s+1})$. Since the polynomials in (3.15) are generally not invariant under these permutations, this generates distinct systems satisfying the orthogonality relation (3.14) via these permutations, which are q-extensions of those mentioned at the top of page 2341 in Tratnik [24].

Next, consider the q-Krawtchouk polynomials defined by

$$k_n(x;b,N;q) = \lim_{a\to\infty}(aq)^{-n}h_n(x;a,b,N;q)$$
$$= (q^{-N};q)_n(-1)^n q^{\binom{n}{2}}\ {}_2\phi_1\Big(q^{-n},q^{-x};q^{-N};q,bq^{n+1}\Big) \tag{3.19}$$

for $n = 0,1,\ldots,N$. In view of (3.19), if we multiply (3.14) by $a_1^{-2N_s}$, let $a_1 \to \infty$, and then replace a_k by a_{k-1} for $2 \le k \le s+1$, we obtain the multivariable q-Krawtchouk orthogonality relation

$$\sum_{\mathbf{y}} K_{\mathbf{n}}(\mathbf{y};q)K_{\mathbf{m}}(\mathbf{y};q)\rho_K(\mathbf{y};q) = \lambda_K(\mathbf{n};q)\delta_{\mathbf{n},\mathbf{m}} \tag{3.20}$$

for N_s, $M_s \le N$, where

$$K_{\mathbf{n}}(\mathbf{y};q) = K_{\mathbf{n}}(\mathbf{y};a_1,\ldots,a_s,N;q)$$
$$= \prod_{j=1}^{s} k_{n_j}(Y_j - N_{j-1};a_j,Y_{j+1}-N_{j-1};q), \tag{3.21}$$

$$\rho_K(\mathbf{y};q) = \rho_K(\mathbf{y};a_1,\ldots,a_s,N;q)$$
$$= \frac{(q^{-N};q)_{Y_s}}{(q;q)_{y_1}(q^{-N}/a_s;q)_{Y_s}}(-1)^{y_1}q^{-\binom{y_1}{2}}a_s^{-Y_s}$$
$$\times \prod_{j=2}^{s}\frac{(qa_{j-1};q)_{y_j}}{(q;q)_{y_j}}(qa_{j-1})^{-Y_j}, \tag{3.22}$$

$$\lambda_K(\mathbf{n};q) = \lambda_K(\mathbf{n};a_1,\ldots,a_s,N;q)$$
$$= \frac{(q^{-N};q)_{N_s}}{(qa_s;q)_{N_s}}(A_{s-1}q^{s+N_s})^{-N}(A_sq^{s+1})^{N+N_s}(-1)^N q^{\binom{N_s}{2}+\binom{N+N_s}{2}}$$
$$\times \prod_{j=1}^{p}(q,qa_j;q)_{n_j}(A_{j-1}q^{j+2N_{j-1}})^{-n_j}a_j^{-N_j-N_{j-1}}q^{-2N_j} \tag{3.23}$$

and the summation is over the same region as in (3.14). This orthogonality relation can also be derived by starting with the $a_1 \to 0$ limit case of (3.14).

Now consider the q-Meixner polynomials defined by

$$M_n(q^{-x};a,c;q) = {}_2\phi_1\Big(q^{-n},q^{-x};aq;q,-q^{n+1}/c\Big) \tag{3.24}$$

for $n = 0,1,\ldots$, which satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} M_n(q^{-x};a,c;q)M_m(q^{-x};a,c;q)\frac{(aq;q)_x}{(q,-acq;q)_x}c^x q^{\binom{x}{2}}$$
$$= \frac{(-c;q)_\infty(q,-q/c;q)_n}{(-acq;q)_\infty(aq;q)_n}q^{-n}\delta_{n,m}. \tag{3.25}$$

See [9, Ex. 7.12] and [15, §3.13].

To find multivariable orthogonal q-Meixner polynomials we, formally, replace the upper limit of summation N for the sum over x_s in (2.9) by ∞ and replace q^{-N} by an arbitrary number β, say. Then, from (2.16), the factor

$$\begin{aligned}&\frac{(q,qA_s;q)_N}{(a_{s+1},A_{s+1};q)_N}\frac{(a_{s+1};q)_{N-x_s}(A_{s+1};q)_{N+x_s}}{(q;q)_{N-x_s}(qA_s;q)_{N+x_s}}\\ &=\frac{(q^{-N},A_{s+1}q^N;q)_{x_s}}{(q^{1-N}/a_{s+1},A_sq^{N+1};q)_{x_s}}(q/a_{s+1})^{x_s},\end{aligned}$$

gets replaced by

$$\frac{(\beta,A_{s+1}/\beta;q)_{x_s}}{(\beta q/a_{s+1},qA_s/\beta;q)_{x_s}}(q/a_{s+1})^{x_s},$$

and, from (2.14), the factor

$$\begin{aligned}&\frac{(qA_s,qbA_{s+1}/a_1;q)_N}{(a_{s+1},a_1/b;q)_N}\Big(\frac{a_1}{qbA_s}\Big)^N\\ &=\frac{(qA_s,a_1q^{-N}/bA_{s+1};q)_N}{(q^{1-N}/a_{s+1},a_1/b;q)_N}\end{aligned}$$

gets replaced by

$$\frac{(qA_s,a_1\beta/bA_{s+1},q/a_{s+1},a_1/b\beta;q)_\infty}{(\beta q/a_{s+1},a_1/b,qA_s/\beta,a_1/bA_{s+1};q)_\infty}$$

via the identity $(a;q)_N=(a;q)_\infty/(aq^N;q)_\infty$ in [9, (I.5)]. Hence, by multiplying both sides of (2.9) by $(bq/a_1^{1/2})^{-N_s-M_s}$, taking the limits $b\to\infty$ and $a_1\to 0$, and replacing a_k by a_{k-1} for $2\le k\le s+1$, we are led to the limit case

$$\begin{aligned}&\sum_{x_s=0}^{\infty}\sum_{x_{s-1}=0}^{x_s}\cdots\sum_{x_1=0}^{x_2}\mathcal{M}_{\mathbf{n}}(\mathbf{x};q)\mathcal{M}_{\mathbf{m}}(\mathbf{x};q)\rho_{\mathcal{M}}(\mathbf{x};q)\\ &=\lambda_{\mathcal{M}}(\mathbf{n};q)\delta_{\mathbf{n},\mathbf{m}}\,,\end{aligned}\tag{3.26}$$

where

$$\begin{aligned}\mathcal{M}_{\mathbf{n}}(\mathbf{x};q)&=\mathcal{M}_{\mathbf{n}}(\mathbf{x};a_1,\ldots,a_s,\beta;q)\\ &=\Bigg[\prod_{k=1}^{s-1}(q^{N_{k-1}-x_{k+1}};q)_{n_k}(-1)^{n_k}q^{\binom{n_k}{2}+n_kN_{k-1}}A_{k-1}^{n_k/2}\\ &\qquad\times M_{n_k}(q^{N_{k-1}-x_k};q^{N_{k-1}-x_{k+1}-1},-q/a_k;q)\Bigg]\\ &\quad\times(\beta q^{N_{s-1}};q)_{n_s}(-1)^{n_s}q^{\binom{n_s}{2}+n_sN_{s-1}}A_{s-1}^{n_s/2}\\ &\quad\times M_{n_s}(q^{N_{s-1}-x_s};\beta q^{N_{s-1}-1},-q/a_s;q),\end{aligned}\tag{3.27}$$

$$\begin{aligned}\rho_{\mathcal{M}}(\mathbf{x};q)&=\rho_{\mathcal{M}}(\mathbf{x};a_1,\ldots,a_s,\beta;q)\\ &=\frac{(\beta;q)_{x_s}}{(q\beta/a_s;q)_{x_s}}\Big(\frac{q}{a_{s-1}a_s}\Big)^{x_s}\frac{(-1)^{x_1}}{(q;q)_{x_1}}q^{\binom{x_1}{2}}\\ &\quad\times\prod_{k=1}^{s-1}\frac{(a_k;q)_{x_{k+1}-x_k}}{(q;q)_{x_{k+1}-x_k}}a_{k-1}^{-x_k}\end{aligned}\tag{3.28}$$

with $a_0 = 1$, and

$$\begin{aligned}\lambda_{\mathcal{M}}(\mathbf{n};q) &= \lambda_{\mathcal{M}}(\mathbf{n};a_1,\ldots,a_s,\beta;q)\\ &= \frac{(q/a_s;q)_\infty}{(q\beta/a_s;q)_\infty}(\beta;q)_{N_s}q^{N_s(N_s-1)}A_s^{N_s}\\ &\times \prod_{k=1}^{s}(q,a_k;q)_{n_k}a_k^{-N_k-N_{k-1}}q^{N_{k-1}-N_k}. \end{aligned}\tag{3.29}$$

This orthogonality relation can be verified by starting with the q-Meixner orthogonality relation (3.25) and proceeding as in the derivation of (2.9).

The $\beta \to 0$ limit of (3.26) gives the following multivariable extension of the q-Charlier polynomial orthogonality

$$\begin{aligned}&\sum_{x_s=0}^{\infty}\sum_{x_{s-1}=0}^{x_s}\cdots\sum_{x_1=0}^{x_2}\mathcal{C}_{\mathbf{n}}(\mathbf{x};q)\mathcal{C}_{\mathbf{m}}(\mathbf{x};q)\rho_{\mathcal{C}}(\mathbf{x};q)\\ &= \lambda_{\mathcal{C}}(\mathbf{n};q)\delta_{\mathbf{n},\mathbf{m}}\,, \end{aligned}\tag{3.30}$$

where

$$\begin{aligned}\mathcal{C}_{\mathbf{n}}(\mathbf{x};q) &= \mathcal{C}_{\mathbf{n}}(\mathbf{x};a_1,\ldots,a_s;q)\\ &= \Bigg[\prod_{k=1}^{s-1}(q^{N_{k-1}-x_{k+1}};q)_{n_k}(-1)^{n_k}q^{\binom{n_k}{2}+n_kN_{k-1}}A_{k-1}^{n_k/2}\\ &\quad\times M_{n_k}(q^{N_{k-1}-x_k};q^{N_{k-1}-x_{k+1}-1},-q/a_k;q)\Bigg]\\ &\times(-1)^{n_s}q^{\binom{n_s}{2}+n_sN_{s-1}}A_{s-1}^{n_s/2}\\ &\times c_{n_s}(q^{N_{s-1}-x_s};-q/a_s;q), \end{aligned}\tag{3.31}$$

$$\begin{aligned}\rho_{\mathcal{C}}(\mathbf{x};q) &= \rho_{\mathcal{C}}(\mathbf{x};a_1,\ldots,a_s;q)\\ &= \Big(\frac{q}{a_{s-1}a_s}\Big)^{x_s}\frac{(-1)^{x_1}}{(q;q)_{x_1}}q^{\binom{x_1}{2}}\prod_{k=1}^{s-1}\frac{(a_k;q)_{x_{k+1}-x_k}}{(q;q)_{x_{k+1}-x_k}}a_{k-1}^{-x_k}, \end{aligned}\tag{3.32}$$

$$\begin{aligned}\lambda_{\mathcal{C}}(\mathbf{n};q) &= \lambda_{\mathcal{C}}(\mathbf{n};a_1,\ldots,a_s;q)\\ &= (q/a_s;q)_\infty q^{N_s(N_s-1)}A_s^{N_s}\prod_{k=1}^{s}(q,a_k;q)_{n_k}a_k^{-N_k-N_{k-1}}q^{N_{k-1}-N_k}, \end{aligned}\tag{3.33}$$

with $a_0 = 1$ and the q-Charlier polynomial defined by

$$\begin{aligned}c_n(q^{-x};a;q) &= \lim_{\beta\to 0}M_n(q^{-x};\beta,a;q)\\ &= {}_2\phi_1(q^{-n},q^{-x};0;q,-q^{n+1}/a). \end{aligned}\tag{3.34}$$

See [9, Ex. 7.13] and [15, §3.23] for the orthogonality relation and other properties of the q-Charlier polynomials. Of course (3.3), (3.6), (3.9), (3.14), (3.20), and (3.30) can also be derived directly from their single variable special cases by proceeding as in the derivation of (2.9) and (2.21).

Other discrete multivariable extensions of the Racah and Hahn polynomials are considered in [3], [5], [13], [17], [18], [20], [22] and [26]. For some related results see, e.g., [6], [7], [8], [9], [12], [16] and [19]. In view of the nonnegativity results for the linearization coefficients and for kernels containing products of q-Racah polynomials in [7] and [8] and the resulting convolution structure and positive summability methods, it would be of interest to see if any of these nonnegativity results can be extended to some of the multivariable orthogonal polynomials considered in this paper.

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