Norm inequalities for fractional integrals of Laguerre and Hermite expansions

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Abstract. Suppose the fractional integration operator I^σ is generated by the sequence $\{(k+1)^{-\sigma}\}$ in the setting of Laguerre and Hermite expansions. Then, via projection formulas, the problem of the norm boundedness of I^σ is reduced to the well-known fractional integration on the half-line. A corresponding result with respect to the modified Hankel transform is derived and its connection with the Laguerre fractional integration is indicated.

1 Introduction

The aim of this note is to present a new approach for the derivation of boundedness properties of fractional integration operators generated by multipliers with respect to Laguerre and Hermite expansions as well as with respect to modified Hankel transforms. The method of proof consists in reducing the problem to classical fractional integration via projection formulas as given, e.g., in Askey and Fitch [3] and then applying weighted norm inequalities for the resulting integral operators as given in Samko, Kilbas and Marichev [12] and in Andersen and Heinig [1], thus separating two difficulties: The orthogonal setting is handled via the projection formulas, and the behavior of classical fractional integration on the half-line in a weighted setting is essentially known.

Let us mention two different approaches in the case of Laguerre series.
(i) In Gasper, Stempak and Trebels [6] the delicate convolution structure for Laguerre expansions $\sum a_k L_k^\alpha$ is essentially used (thus restricting the parameter α to nonnegative values) to obtain an unweighted fractional integration theorem which is then extended to power weights by a method due to E. M. Stein and G. Weiss.
(ii) Kanjin and Sato [10] consider Laguerre expansions $\sum a_k \mathcal{L}_k^\alpha$ with respect to the orthonormalized Laguerre functions, apply the intricate transplantation theorem due to Kanjin (cf. [10, Theorem A]) to reduce the problem to the case $\alpha = 0$ and, by a transplantation theorem between $\sum a_k \mathcal{L}_k^0$ and one-dimensional Fourier series, can

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switch to fractional integration for Fourier series (generated via a multiplier sequence). The advantage of this method is that negative values of α are also admitted.

The benefits of the present approach are the following: (a) It is elementary; (b) it improves, e.g., the known fractional integration theorems for Laguerre expansions in two ways: more power weights are admitted and sharper norm inequalities in the following sense are derived ($\sigma > 0$)

$$\| \sum (k+1)^{-\sigma} a_k L_k^{\alpha} \|_r \lesssim \| \sum (k+1)^{-\sigma} a_k L_k^{\alpha+\sigma} \|_q \lesssim \| \sum a_k L_k^{\alpha} \|_p$$

(the notation $A \lesssim B$ is used to indicate that $A \leq c \cdot B$ with a constant c independent of significant quantities; the parameters r, p, σ, α satisfy the standard relation $1/r = 1/p - \sigma/(\alpha+1)$ in the “natural” weight case with respect to the convolution structure); (c) in principle those other than power weights are admitted; (d) the method should work for further expansions or integral transforms whenever relevant projection formulas and relevant results on classical fractional integration are available.

2 Laguerre expansions

We take over the notation from [9] and consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \Big\{ f : \|f\|_{L^p_{w(\gamma)}} = \Big(\int_0^\infty |f(x) e^{-x/2}|^p x^\gamma \, dx \Big)^{1/p} < \infty \Big\}, \quad 1 \leq p < \infty, \ \gamma > -1.$$

We denote the Laguerre polynomials by L_k^α, $\alpha > -1$, $k \in \mathbf{N}_0$ (see [16, p. 100]), and normalize them by

$$R_k^\alpha(x) = L_k^\alpha(x)/L_k^\alpha(0), \qquad L_k^\alpha(0) = A_k^\alpha = \binom{k+\alpha}{k} = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}\,.$$

If $\gamma < p(\alpha+1) - 1$, one can formally expand $f \in L^p_{w(\gamma)}$ into a Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x), \quad \hat{f}_\alpha(k) = \int_0^\infty f(x) R_k^\alpha(x) x^\alpha e^{-x} \, dx\,.$$

Since the polynomials are dense in these $L^p_{w(\gamma)}$-spaces, we restrict ourselves in the following to considering polynomials $f = \sum a_k L_k^\alpha(x)$ with only finitely many non-zero coefficients a_k. Motivated by the definition of fractional integration for Fourier series discussed by Hardy and Littlewood, we define a fractional integral operator $I^\sigma,\ \sigma > 0$, for Laguerre expansions by

$$I^\sigma(\sum_{k=0}^{\infty} a_k L_k^\alpha(x)) = \sum_{k=0}^{\infty}(k+1)^{-\sigma} a_k L_k^\alpha(x)\,.$$

In order to use also related fractional integrals, equivalent to I^σ (with respect to the mapping behavior), we need the idea of multipliers. A sequence $m = \{m_k\}$ is called a (bounded) multiplier from $L^p_{w(\gamma)}$ into $L^r_{w(\gamma)}$, denoted by $m \in M^{p,r}_{w(\gamma)}$, if

$$\Big\|T_m(\sum_{k=0}^{\infty} a_k L_k^\alpha)\Big\|_{L^r_{w(\gamma)}} := \Big\|\sum_{k=0}^{\infty} m_k a_k L_k^\alpha\Big\|_{L^r_{w(\gamma)}} \le C\,\Big\|\sum_{k=0}^{\infty} a_k L_k^\alpha\Big\|_{L^p_{w(\gamma)}}$$

for all polynomials; the smallest constant C for which this holds is called the multiplier norm $\|m\|_{M^{p,r}_{w(\gamma)}}$ which coincides of course with the operator norm of T_m. We mention that I^σ extends to a bounded operator from $L^p_{w(\gamma)}$ to $L^p_{w(\gamma)}$ when $0 \le \alpha p/2 \le \gamma \le \alpha$, $1 \le p \le \infty$, (see [6]) or when $-1 < \gamma < (\alpha+1)p - 1$, $1 \le p \le \infty$, and $-1 < \alpha \le \min\{1 - 2/(3p), 1/3 + 2/(3p)\}$; this follows from a Corollary due to Poiani [11, p. 11] and [17, Theorem 3.3]. By [4, 1.18], [9, II, Theorem 3.1] and [17, Theorem 3.3] it is then clear that, e.g., the multiplier sequences

$$\Big\{\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma+1)}\Big\}_{k\in\mathbf{N}_0}, \qquad \{(k^\delta + c + 1)^{-\sigma/\delta}\}_{k\in\mathbf{N}_0}, \quad c > -1\,,\ \delta > 0,$$

generate fractional integral operators on $L^p_{w(\gamma)}, 1 \le p \le \infty$, which are equivalent to the above I^σ. As a consequence, without loss of generality we can work with any of the fractional integrals generated by the preceding sequences.

THEOREM 1. *Let $\alpha > -1$ and $1 < p \le r < \infty$. Assume further that $0 < \sigma < \alpha + 1$, $a < (\alpha+1)/p'$, $b < (\alpha+1)/r$, $a + b \ge -\sigma(2\alpha+1)$. Then*

$$\|I^\sigma f\|_{L^r_{w(\alpha-br)}} \lesssim \|f\|_{L^p_{w(\alpha+ap)}}, \qquad \frac{1}{r} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha+1}\,.$$

REMARK 1. Not only is Theorem 1 proved by a new method, but this method also extends the parameter range $\alpha \ge 0$, $a + b \ge 0$ in [6, Theorem 1.1] to $\alpha > -1$, $a + b \ge -\sigma(2\alpha+1)$. Moreover, the result of Kanjin and Sato [10, Theorem] is completely recovered without the use of transplantation theorems. We only have to choose $a = \alpha(1/2 - 1/p)$, $b = \alpha(1/r - 1/2)$; then $1/r = 1/p - \sigma$; the condition $a + b \ge -\sigma(2\alpha+1)$ causes no new restriction, and the assumptions $a < (\alpha+1)/p'$, $b < (\alpha+1)/r$, lead to the restriction of the $(p,\, r)$-range as given in [10].

Remark 2. The proof of Theorem 1 also gives the following result which makes precise the norm estimates indicated in the Introduction.

Corollary 2. *Set $2/q = 1/p + 1/r$. Then, under the hypotheses of Theorem* 1, *there holds*

$$\| \sum (k+1)^{-\sigma} a_k L_k^\alpha \|_{L^r_{w(\alpha-br)}} \lesssim \| \sum (k+1)^{-\sigma} a_k L_k^{\alpha+\sigma} \|_{L^q_{w(\alpha+\sigma)}} \lesssim \| \sum a_k L_k^\alpha \|_{L^p_{w(\alpha+ap)}} .$$

Proof of Theorem 1. From [3, (3.30) and (3.31)] we have that if $\alpha > -1$, $\sigma > 0$, then the following projection formulas for Laguerre polynomials hold

$$\frac{x^{\alpha+\sigma} L_k^{\alpha+\sigma}(x)}{\Gamma(k+\alpha+\sigma+1)} = \frac{1}{\Gamma(\sigma)} \int_0^x \frac{(x-y)^{\sigma-1} y^\alpha L_k^\alpha(y)\, dy}{\Gamma(k+\alpha+1)} , \tag{1}$$

$$e^{-x} L_k^\alpha(x) = \frac{1}{\Gamma(\sigma)} \int_x^\infty (y-x)^{\sigma-1} e^{-y} L_k^{\alpha+\sigma}(y)\, dy . \tag{2}$$

On the one hand, we have by (1)

$$\left(\int_0^\infty \left| \sum_{k=0}^\infty \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma+1)} a_k L_k^{\alpha+\sigma}(x) e^{-x/2} \right|^q x^{\alpha+\sigma} dx \right)^{1/q}$$

$$\lesssim \left(\int_0^\infty \left| \int_0^x (x-y)^{\sigma-1} e^{-(x-y)/2} (\sum a_k L_k^\alpha(y) y^\alpha e^{-y/2})\, dy \right|^q x^{-(\alpha+\sigma)q/q'} dx \right)^{1/q} =: J_1 .$$

Also, by (2),

$$\left(\int_0^\infty \left| \sum_{k=0}^\infty b_k L_k^\alpha(x) e^{-x/2} \right|^r x^{\alpha-br} dx \right)^{1/r}$$

$$\lesssim \left(\int_0^\infty \left| \int_x^\infty (y-x)^{\sigma-1} e^{-(y-x)/2} (\sum b_k L_k^{\alpha+\sigma}(y) e^{-y/2})\, dy \right|^r x^{\alpha-br} dx \right)^{1/r} =: J_2 .$$

Observing that $e^{-|x-y|/2} \le 1$, we see that the problem is reduced to finding weighted L^p-L^q estimates for the Riemann- Liouville fractional integral

$$I_+^\sigma h(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} h(s)\, ds$$

and for the Weyl fractional integral

$$I_-^\sigma h(t) = \frac{1}{\Gamma(\sigma)} \int_t^\infty (s-t)^{\sigma-1} h(s)\, ds .$$

Convenient norm estimates can be found in the book of Samko, Kilbas and Marichev [12] from which we now quote Theorem 5.4.

THEOREM A. *Let $\sigma > 0$, $p \geq 1$, and $p \leq q \leq p/(1-p\sigma)$ when $1 \leq p < 1/\sigma$ (in the case $p = 1$ the right endpoint $p/(1-p\sigma)$ is excluded), or $p \leq q < \infty$ when $p \geq 1/\sigma$. Suppose also that $-\infty < N < \infty$ and $M < p-1$ when we consider I_+^σ, or $M > \sigma p - 1$ in the case I_-^σ, and*

$$\frac{N+1}{q} = \frac{M+1}{p} - \sigma \,. \tag{3}$$

Then

$$\left(\int_0^\infty |I_\pm^\sigma h(t)|^q t^N dt\right)^{1/q} \lesssim \left(\int_0^\infty |h(t)|^p \, t^M dt\right)^{1/p}.$$

In order to estimate J_1 choose the parameters $N = -(\alpha+\sigma)q/q'$ and $M = \alpha(1-p) + ap$ in Theorem A. Then the condition $M < p-1$ is equivalent to $a < (\alpha+1)/p'$; in the case $1 < p < 1/\sigma$ the condition $q \leq p/(1-p\sigma)$ is equivalent to $\sigma \geq 1/p - 1/q$. Furthermore (3) leads to

$$\frac{\alpha+\sigma+1}{q} = \frac{\alpha+1}{p} + a \,. \tag{4}$$

Thus, if $p \leq q$, we have under the above conditions

$$\| \sum (k+1)^{-\sigma} a_k L_k^{\alpha+\sigma} \|_{L^q_{w(\alpha+\sigma)}} \lesssim J_1 \lesssim \| \sum a_k L_k^\alpha \|_{L^p_{w(\alpha+ap)}} \,. \tag{5}$$

Similarly, we handle J_2 by replacing p, q in Theorem A by q, r, respectively, and choosing $N = \alpha - br$, $M = \alpha + \sigma$. Then (3) leads to

$$\frac{\alpha+1}{r} - b = \frac{\alpha+\sigma+1}{q} - \sigma \,, \tag{6}$$

which in combination with $M > \sigma q - 1$ gives $b < (\alpha+1)/r$. In the case $1 < q < 1/\sigma$ the condition $r \leq q/(1-q\sigma)$ is now equivalent to $\sigma \geq 1/q - 1/r$. Thus, if $q \leq r$, we have

$$\| \sum b_k L_k^\alpha \|_{L^r_{w(\alpha-br)}} \lesssim J_2 \lesssim \| \sum b_k L_k^{\alpha+\sigma} \|_{L^q_{w(\alpha+\sigma)}} \,. \tag{7}$$

We can combine (5) and (7) if $\sigma \geq \max\{1/p - 1/q,\, 1/q - 1/r\}$; the maximum is assumed with $1/q = (1/p + 1/r)/2$, thus, for this choice of q, we have $p \leq q \leq r$ since by hypothesis $p \leq r$. Finally, since (4) and (6) lead to the relation between r, p, α, σ, a, b, as asserted in Theorem 1, we arrive at the condition $\sigma \geq (1/p - 1/r)/2 = (\sigma - a - b)/(2\alpha+2)$ which implies $a + b \geq -\sigma(2\alpha+1)$. This finishes the proof of Theorem 1 and its Corollary.

REMARK 3. In [1] and [2] there are admitted more general weights than those of power type as in Theorem A. We note that the conditions in [1] and [2] lead to more restrictions (e.g., the order of fractional integration is assumed to be ≤ 1). In the last years, a number of papers have been published which deal with the precise description of norm inequalities with two weights for fractional integrals or fractional maximal functions; the derived characterizations seem to be hard to apply in our situation.

3 Hermite expansions

We follow again [9] and consider for $1 \leq p < \infty$ the Lebesgue space

$$L^p_{w(H)} = \Big\{ f \ : \ \|f\|_{L^p_{w(H)}} = \Big(\int_{-\infty}^{\infty} |f(x) e^{-x^2/2}|^p dx \Big)^{1/p} < \infty \Big\} .$$

The Hermite polynomials (see [16, p. 106])

$$H_n(x) = (-1)^n e^{x^2} \Big(\frac{d}{dx}\Big)^n e^{-x^2} , \quad x \in \mathbf{R} , \ n \in \mathbf{N}_0 ,$$

belong to $L^p_{w(H)}$; we associate to $f \in L^p_{w(H)}$ its Hermite expansion

$$f(x) \sim \sum_{k=0}^{\infty} \hat{f}_H(k) H_k(x) , \qquad \hat{f}_H(k) = h_k^H \int_{-\infty}^{\infty} f(t) H_k(t) e^{-t^2} dt ,$$

where $h_k^H = \|H_k\|^{-2}_{L^2_{w(H)}} = (\sqrt{\pi} 2^k k!)^{-1}$. Without loss of generality we can work with polynomials f in the following, since the polynomials are dense in $L^p_{w(H)}$. As in the Laguerre series case we define a fractional integral operator I^σ_H, $\sigma > 0$, for Hermite expansions by ($f = \sum a_k H_k$ being a polynomial)

$$I^\sigma_H \Big(\sum_{k=0}^{\infty} a_k H_k(x) \Big) = \sum_{k=0}^{\infty} (k+1)^{-\sigma} a_k H_k(x) .$$

This time we derive results on the mapping behavior of I^σ_H by reducing the problem to the corresponding one for Laguerre expansions. Observe that H_{2n} and H_{2n+1} are even and odd polynomials, respectively, and that one can uniquely decompose $f \in L^p_{w(H)}$ into its even and odd parts:

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)) , \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)) .$$

Then

$$\|f\|_{L^p_{w(H)}} \leq \|f_e\|_{L^p_{w(H)}} + \|f_o\|_{L^p_{w(H)}} \leq 2 \|f\|_{L^p_{w(H)}} , \tag{8}$$

and for their Hermite coefficients we obtain

$$(f_e)\hat{}_H(k) = \begin{cases} \hat{f}_H(k) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} , \quad (f_o)\hat{}_H(k) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \hat{f}_H(k) & \text{if } k \text{ is odd} \end{cases} .$$

Quadratic transformations [16, (5.6.1)] relate the Hermite and the Laguerre polynomials in the following way

$$H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{-1/2}(x^2)\,, \quad H_{2k+1}(x) = (-1)^k 2^{2k+1} k! x L_k^{1/2}(x^2)\,.$$

As a consequence we have

$$\sum_{k=0}^{\infty} a_{2k} H_{2k}(x) = \sum_{k=0}^{\infty} b_k L_k^{-1/2}(x^2)\,, \quad b_k = (-1)^k 2^{2k} k! a_{2k}\,,$$

$$\sum_{k=0}^{\infty} a_{2k+1} H_{2k+1}(x) = \sum_{k=0}^{\infty} c_k x\, L_k^{1/2}(x^2)\,, \quad c_k = (-1)^k 2^{2k+1} k! a_{2k+1}\,.$$

Therefore,

$$\|(I_H^\sigma f)_e\|_{L^r_{w(H)}} \lesssim \|\sum_{k=0}^{\infty}(2k+1)^{-\sigma} b_k L_k^{-1/2}\|_{L^r_{w(-1/2)}} \lesssim \|\sum_{k=0}^{\infty} b_k L_k^{-1/2}\|_{L^p_{w(-1/2)}}$$
$$\lesssim \|f_e\|_{L^p_{w(H)}}\,, \qquad \frac{1}{r} = \frac{1}{p} - 2\sigma\,,$$

by Theorem 1. Similarly,

$$\|(I_H^\sigma f)_o\|_{L^r_{w(H)}} \lesssim \|\sum_{k=0}^{\infty}(2k+2)^{-\sigma} c_k\, L_k^{1/2}\|_{L^r_{w((r-1)/2)}}$$
$$\lesssim \|\sum_{k=0}^{\infty} c_k\, L_k^{1/2}\|_{L^p_{w((p-1)/2)}} \lesssim \|f_o\|_{L^p_{w(H)}}\,, \qquad \frac{1}{r} = \frac{1}{p} - 2\sigma\,,$$

by Theorem 1 with $a = 1/2 - 1/p$, $b = 1/r - 1/2$ and $a + b = 1/r - 1/p \geq -2\sigma$ by hypothesis. On account of (8) we can combine these two estimates to obtain the following result.

THEOREM 3. *Let* $1 < p < r < \infty$ *and* $0 < \sigma < 1/2$. *Then*

$$\|\sum_{k=0}^{\infty}(k+1)^{-\sigma} a_k H_k\|_{L^r_{w(H)}} \lesssim \|\sum_{k=0}^{\infty} a_k H_k\|_{L^p_{w(H)}}\,, \qquad \frac{1}{r} = \frac{1}{p} - 2\sigma\,.$$

4 Hankel transforms

Consider the weighted Lebesgue spaces $L^p_{v(\gamma)}$, $1 \le p < \infty$, $\gamma > -1$, consisting of those measurable functions on $(0,\infty)$ which satisfy

$$\|f\|_{L^p_{v(\gamma)}} = \Big(\int_0^\infty |f(t)|^p t^{2\gamma+1} dt\Big)^{1/p} < \infty\,.$$

For fixed $\alpha \ge -1/2$ the (modified) Hankel transform of $f \in L^1_{v(\alpha)}$ is defined by

$$H_\alpha f(\tau) = \int_0^\infty \frac{J_\alpha(\tau t)}{(\tau t)^\alpha} f(t)\, t^{2\alpha+1} dt, \qquad \tau > 0\,.$$

Here J_α denotes the Bessel function of the first kind of order α (see [16, (1.71.1)]). We mention the connection with the multidimensional Fourier transform: If $\alpha = (n-2)/2$, $n \in \mathbf{N}$, the Hankel transform $H_\alpha f(\tau)$ coincides with the Fourier transform of the radial extension $f(|x|)$ to $\mathbf{R}^n$ of $f(t)$. One convenient substitute of the set of polynomials in the Laguerre case is now described by $H_\alpha(C_0^\infty)$, the set of Hankel transforms of C^∞-functions with compact support in $(0,\infty)$, since by [15, Theorem 4.7] we have that $H_\alpha(C_0^\infty)$ is dense in $L^p_{v(\alpha+\gamma)}$ for $\gamma > -1/2$, $1 < p < \infty$.

We note that the Hankel transform is self inverse on $H_\alpha(C_0^\infty)$, i.e.,

$$H_\alpha[H_\alpha f] = f\,, \qquad f \in H_\alpha(C_0^\infty)\,.$$

Motivated by the definition of the Riesz fractional integral on $L^p(\mathbf{R}^n)$ via its Fourier symbol, we define a fractional integral operator on $H_\alpha(C_0^\infty)$ by

$$I^\sigma f = H_\alpha(\tau^{-\sigma} H_\alpha(g)(\tau))\,.$$

To obtain a norm estimate for I^σ we proceed as in the Laguerre series case and work with the parameters of the Hankel transform. We slightly extend Theorems 3.1 and 3.2 of [15] and interpret them in a new way. In the "natural" weight case $\gamma = \alpha$, we can at once conclude from [15, Theorem 3.1/2] that

$$\|I^\sigma f\|_{L^r_{v(\alpha)}} \lesssim \|f\|_{L^p_{v(\alpha)}}\,, \qquad \frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2(\alpha+1)}\,, \quad f \in H_\alpha(C_0^\infty)\,.$$

For $\alpha = (n-2)/2$ this reduces to the well-known result for Riesz-potentials restricted to the radial functions of $L^p(\mathbf{R}^n)$. More generally there holds the following.

Theorem 4. *Let $\alpha \ge -1/2$ and $\sigma > 0$, $1 < p \le q \le r < \infty$, $2/q = 1/p + 1/r$. Assume further that $a < 2(\alpha+1)/p'$, $b < 2(\alpha+1)/r$ and $a + b \ge -\sigma(2\alpha+1)$. Then*

$$\|I^\sigma f\|_{L^r_{v(\alpha-br/2)}} \lesssim \|H_{\alpha+\sigma/2}(H_\alpha(I^\sigma f))\|_{L^q_{v(\alpha+\sigma/2)}} \lesssim \|f\|_{L^p_{(\alpha+ap/2)}}\,, \quad \frac{1}{r} = \frac{1}{p} - \frac{\sigma-a-b}{2(\alpha+1)}\,.$$

For the **proof of Theorem 4** we observe that the relevant projection formulas for the Bessel functions are given by [5, 8.5(32) and 8.5(33)]. Formula [5, 8.5(32)] leads to [15, (3.9)] which, when taking $g = H_{\alpha+\sigma}(H_\alpha(I^{2\sigma}f))$, may be rewritten in the form

$$I^{2\sigma}f(t) = c_{\alpha,\sigma}\int_t^\infty s(s^2-t^2)^{\sigma-1}H_{\alpha+\sigma}(H_\alpha(I^{2\sigma}f))(s)\,ds\,, \quad f\in H_\alpha(C_0^\infty),$$

provided $\alpha \geq -1/2$, $\sigma > 0$. Also for these parameters, formula [5, 8.5(33)] yields (all double integrals are absolutely convergent – see [15, p. 57])

$$H_{\alpha+\sigma}(H_\alpha(I^{2\sigma}f))(t) = \frac{c^*_{\alpha,\sigma}}{t^{2(\alpha+\sigma)}}\int_0^t (t^2-s^2)^{\sigma-1}f(s)s^{2\alpha+1}\,ds\,, \quad \sigma>0,\ f\in H_\alpha(C_0^\infty)\,.$$

Variable changes lead to

$$\|H_{\alpha+\sigma}(H_\alpha(I^{2\sigma}f))\|_{L^q_{v(\alpha+\sigma)}} \lesssim \left(\int_0^\infty \Big|\int_0^t (t-s)^{\sigma-1}f(s^{\frac12})\,s^\alpha ds\Big|^q t^{-(\alpha+\sigma)q/q'}dt\right)^{1/q} =: I_1\,,$$

$$\|I^{2\sigma}f\|_{L^r_{v(\alpha-br/2)}} \lesssim \left(\int_0^\infty \Big|\int_t^\infty (s-t)^{\sigma-1}H_{\alpha+\sigma}(H_\alpha(I^{2\sigma}f))(s^{\frac12})\,ds\Big|^r t^{-br/2+\alpha}dt\right)^{1/r} =: I_2\,.$$

We are now in a situation to apply Theorem A and follow the pattern of the proof to Corollary 2. Concerning I_1 choose $N = -(\alpha+\sigma)q/q'$, $M = (1-p)\alpha + ap/2$; the restrictions $\sigma \geq 1/p - 1/q$, $a/2 < (\alpha+1)/p'$ and $(\alpha+\sigma+1)/q = (\alpha+1)/p + a/2$ then result from Theorem A. Analogously, when dealing with I_2, we obtain the restrictions $\sigma \geq 1/q - 1/r$, $b/2 < (\alpha+1)/r$ and $(\alpha+\sigma+1)/q = (\alpha+1)/r + \sigma - b/2$. Taking again the maximum over the lower bounds of σ leads to $a+b \geq -2\sigma(2\alpha+1)$. Finally, since in the proof we worked with $I^{2\sigma}$ instead of I^σ, we only have to replace σ by $\sigma/2$ to establish Theorem 4.

Remark 4. Let us apply Theorem 4 to fractional integration of order $\sigma > 0$ with respect to the classical Hankel transform given by

$$\mathcal{I}^\sigma f = \mathcal{H}_\alpha(\tau^{-\sigma}\mathcal{H}_\alpha(f)(\tau))\,, \qquad \mathcal{H}_\alpha f(\tau) = \int_0^\infty (s\tau)^{\frac12}J_\alpha(s\tau)f(s)\,ds\,.$$

Here we may restrict ourselves to elements of the (self-explaining) set $\mathcal{H}_\alpha(C_0^\infty)$ which is dense in $L^p(0,\infty)$, $1<p<\infty$ (see [15, Cor. 4.8]). If we choose in Theorem 4 the parameters $a = (2\alpha+1)(1/2-1/p)$, $b = (2\alpha+1)(1/r-1/2)$, then we arrive at the following result ($\|\cdot\|_p$ denotes the standard $L^p(0,\infty)$-norm).

Corollary 5. *Let $\alpha \geq -1/2$ and $1 < p < r < \infty$. Then, for all $f \in L^p(0,\infty)$ there holds*

$$\|\mathcal{I}^\sigma f\|_r \lesssim \|f\|_p \,, \qquad \frac{1}{r} = \frac{1}{p} - \sigma \,.$$

An alternative proof of Corollary 5 in the spirit of Kanjin and Sato would be to prove the assertion for the cosine transform (thus for $\alpha = -1/2$ which is classical) and then use Guy's transplantation theorem for the Hankel transform (cf. [15, p. 53]).

Remark 5. Obviously, Theorem 4 and its proof are quite analogous to the Laguerre expansion case. That this is not accidental may be seen by a result due to Stempak [13, Theorem 1.1]. Though Stempak's theorem is only formulated for the case $r = p$ its proof actually shows the following.

Theorem B. *Let $1 \leq p \leq r < \infty$ and $\alpha > -1$. Assume that m is a function on $\mathbf{R}_+$, which is continuous except on a set of Lebesgue measure zero. Then*

$$\|m\|_{M^{p,r}(H_\alpha)} \lesssim \liminf_{\varepsilon \to 0+} \|\varepsilon^\sigma \{m(\varepsilon k^{1/2})\}_{k\in\mathbf{N}_0}\|_{M^{p,r}_{w(\alpha)}} \,, \qquad \frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2(\alpha+1)} \,, \tag{9}$$

whenever the right hand side is finite. Here the (modified) Hankel multiplier norm is given by

$$\|m\|_{M^{p,r}(H_\alpha)} = \inf\{C \,:\, \|H_\alpha(mH_\alpha f)\|_{L^r_{v(\alpha)}} \leq C\,\|f\|_{L^p_{v(\alpha)}} \,,\; f \in H_\alpha(C_0^\infty)\} \,.$$

Observe that the relation $1/r = 1/p - \sigma/2(\alpha+1)$ reflects the mapping behavior of a fractional integral operator of order $\sigma/2$ in the Laguerre setting. In order to apply Theorem B we face the problem to ensure that the right hand side of (9) is bounded. Here the case $1 < p < r < 2$ is the essential one. For, by duality, it is equivalent to the problem for $2 < p < r < \infty$; if $1 < p \leq 2 \leq q < \infty$ we observe that on account of $M^{2,2}_{w(\alpha)} = \ell^\infty$ we have (see [6, p. 69]) that

$$\|\varepsilon^\sigma \{m(\varepsilon k^{1/2})\}_{k\in\mathbf{N}_0}\|_{M^{p,r}_{w(\alpha)}} \lesssim \|\{\varepsilon^\sigma k^{\sigma/2} m(\varepsilon k^{1/2})\}_{k\in\mathbf{N}_0}\|_{\ell^\infty} \,, \qquad \frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2(\alpha+1)} \,,$$

which is an easy to verify condition.

Now, in the case $1 < p < r < 2$, the extension of [6, Cor. 1.2] from $\alpha \geq 0$ to $\alpha \geq -1/2$ is obvious by Theorem 1 for $a = b = 0$; note that Corollaries 1.2 and 4.5 in [14] give sufficient $M^{p,p}_{w(\alpha)}$-multiplier criteria for all $\alpha > -1$ and that the assumption $a + b = 0 \geq -\sigma(2\alpha+1)$ leads to the restriction $\alpha \geq -1/2$. Thus, using the notation $s = [\alpha+1]+2$, $\Delta m_k = m_k - m_{k+1}$, $\Delta^s = \Delta\,\Delta^{s-1}$, and choosing $m(\tau) = (1+\tau^2)^{-\sigma/2}$ in Theorem B we can estimate as follows:

$$\sup_{\varepsilon>0} \|\varepsilon^\sigma \{m(\varepsilon k^{1/2})\}_{k\in\mathbf{N}_0}\|^2_{M^{p,r}_{w(\alpha)}}$$

$$\lesssim \sup_{\varepsilon>0} \Big(\|\varepsilon^\sigma k^{\sigma/2}\{m(\varepsilon k^{1/2})\}_{k\in\mathbf{N}_0}\|_\infty^2$$

$$+ \sup_{N\in\mathbf{N}} \sum_{k=N}^{2N} |(k+1)^{s+\sigma/2}\Delta^s \varepsilon^\sigma m(\varepsilon k^{1/2})|^2 (k+1)^{-1} \Big)$$

$$\lesssim \sup_{\varepsilon>0} \Big(O(1) + \sup_{N>0} \int_N^{2N} \Big| \tau^{s+\sigma/2} \Big(\frac{d}{d\tau}\Big)^s \frac{\varepsilon^\sigma}{(1+\varepsilon^2 t)^{\sigma/2}} \Big|^2 dt \Big),$$

the latter inequality being valid on account of a slight variation of [7, (5.1)]. A straightforward computation at once yields that the last integral is uniformly bounded in $\varepsilon > 0$. Therefore, by Theorem B, the mapping behavior of the fractional integration, generated by the (modified) Hankel symbol $(1+\tau)^{-\sigma}$, can be deduced from the corresponding one for fractional integration of order $\sigma/2$ in the Laguerre series setting, i.e., from Theorem 1 in the case $a = b = 0$.

Of course, when $1 < p < r < \infty,\ \alpha \geq -1/2$,

$$\|H_\alpha((1+\tau)^{-\sigma} H_\alpha(g)(\tau))\|_{L^r_{v(\alpha)}} \lesssim \|f\|_{L^p_{v(\alpha)}}, \qquad \frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2(\alpha+1)},$$

also follows from Theorem 4, since $\tau^\sigma/(1+\tau)^\sigma$ generates a bounded operator on all $L^p_{v(\alpha)}$-spaces, $1 \leq p \leq \infty$.

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