On a restriction problem of de Leeuw type for Laguerre multipliers

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Dedicated to Károly Tandori on the occasion of his 70th birthday

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Abstract. In 1965 K. de Leeuw [3] proved among other things in the Fourier transform setting: *If a continuous function $m(\xi_1,\ldots,\xi_n)$ on $\mathbf{R}^n$ generates a bounded transformation on $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then its trace $\tilde{m}(\xi_1,\ldots,\xi_k) = m(\xi_1,\ldots,\xi_k,0,\ldots,0)$, $k < n$, generates a bounded transformation on $L^p(\mathbf{R}^k)$.* In this paper, the analogous problem is discussed in the setting of Laguerre expansions of different orders.

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1 Introduction

The purpose of this paper is to discuss the question: suppose $\{m_k\}_{k\in\mathbf{N}_0}$ generates a bounded transformation with respect to a Laguerre function expansion of order α on some L^p–space, does it also generate a corresponding bounded map with respect to a Laguerre function expansion of order β? To become more precise let us first introduce some notation. Consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f :\ \|f\|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma\, dx)^{1/p} < \infty\}\ , \quad 1 \le p < \infty,$$

$$L^\infty_{w(\gamma)} = \{f :\ \|f\|_{L^\infty_{w(\gamma)}} = \text{ess sup}_{x>0}|f(x)e^{-x/2}| < \infty\}, \quad p = \infty,$$

where $\gamma > -1$. Let $L^\alpha_n(x)$, $\alpha > -1$, $n \in \mathbf{N}_0$, denote the classical Laguerre polynomials (see Szegö [15, p. 100]) and set

$$R^\alpha_n(x) = L^\alpha_n(x)/L^\alpha_n(0), \qquad L^\alpha_n(0) = A^\alpha_n = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

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Associate to f its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier Laguerre coefficients of f are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x}\, dx \tag{1}$$

(if the integrals exist). A sequence $m = \{m_k\}_{k\in\mathbf{N}_0}$ is called a (bounded) multiplier from $L^p_{w(\gamma)}$ into $L^q_{w(\delta)}$, notation $m \in M^{p,q}_{\alpha;\,\gamma,\,\delta}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^q_{w(\delta)}} \le C \| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha \|_{L^p_{w(\gamma)}}$$

for all polynomials f; the smallest constant C for which this holds is called the multiplier norm $\|m\|_{M^{p,q}_{\alpha;\,\gamma,\,\delta}}$. For the sake of simplicity we write $M^{p,q}_{\alpha;\,\gamma} := M^{p,q}_{\alpha;\,\gamma,\,\gamma}$ if $\gamma = \delta$ and, if additionally $p = q$, $M^p_{\alpha;\,\gamma} := M^{p,p}_{\alpha;\,\gamma}$.

We are **mainly** interested in the question: when is $M^{p,q}_{\alpha;\,\alpha}$ continuously embedded in $M^{p,q}_{\beta;\,\beta}$:

$$M^{p,q}_{\alpha;\,\alpha} \hookrightarrow M^{p,q}_{\beta;\,\beta}\,, \quad 1 \le p \le q \le \infty\ ?$$

The Plancherel theory immediately yields

$$l^\infty = M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad \alpha, \beta > -1.$$

A combination of sufficient multiplier conditions with necessary ones indicates which results are to be expected. To this end, define the fractional difference operator Δ^δ of order δ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the series converges), the classes $wbv_{q,\delta}$, $1 \le q \le \infty$, $\delta > 0$, of weak bounded variation (see [5]) of bounded sequences which have finite norm $\|m\|_{q,\delta}$, where

$$\begin{aligned} \|m\|_{q,\delta} &:= \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} \left(\sum_{k=N}^{2N} |(k+1)^\delta \Delta^\delta m_k|^q \frac{1}{k+1} \right)^{1/q}, \qquad q < \infty, \\ \|m\|_{\infty,\delta} &:= \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} |(k+1)^\delta \Delta^\delta m_k| \qquad\qquad\qquad\qquad\quad , \qquad q = \infty. \end{aligned}$$

Observing the duality (see [14])

$$M^p_{\alpha;\,\gamma} = M^{p'}_{\alpha;\,\alpha p' - \gamma p'/p}\,, \quad -1 < \gamma < p(\alpha+1) - 1, \quad 1 < p < \infty, \tag{2}$$

where $1/p+1/p'=1$, we may restrict ourselves to the case $1<p<2$. The Corollary 1.2 b) in [14] gives the embedding

$$M^p_{\alpha;\alpha} \hookrightarrow wbv_{p',s}, \quad s=(2\alpha+2/3)(1/p-1/2), \quad \alpha>-1/3, \tag{3}$$

when $(2\alpha+2)(1/p-1/2)>1/2$. Theorem 5 in [5] gives the first embedding in

$$wbv_{p',s} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\beta;\beta},$$

whereas the last one follows from Corollaries 1.2 and 4.5 in [14] provided $s>\max\{(2\beta+2)(1/p-1/2),\ 1\}$, $\beta>-1$. Hence, choosing $\gamma=\alpha$ in (2), we obtain

Proposition 1.1 *Let $1<p<\infty$ and α be such that $(2\alpha+2/3)|1/p-1/2|>1$. Then*

$$M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta}, \quad -1<\beta<\alpha-2/3.$$

In the same way we can derive a result for $M^{p,q}$–multipliers. The necessary condition in [6, Cor. 1.3] can easily be extended in the sense of [6, Cor. 2.5 b)] to

$$\sup_k |(k+1)^\sigma m_k| + \sup_n \Big(\sum_{k=n}^{2n} |(k+1)^{\sigma+s}\Delta^s m_k|^{q'}/k\Big)^{1/q'} \le C\|m\|_{M^{p,q}_{\alpha;\alpha}},$$

where $\alpha>-1/3$, $1/q=1/p-\sigma/(\alpha+1)$, $1<p<q<2$, $(\alpha+1)(1/q-1/2)>1/4$, and $s=(2\alpha+2/3)(1/q-1/2)>0$. Using this and the sufficient condition for $M^{p,q}_{\beta;\beta}$–multipliers given in [4, Cor. 1.2], which is proved only for $\beta\ge 0$, we obtain

$$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}, \quad 0\le\beta<\alpha-2/3,\ (2\alpha+2/3)(1/q-1/2)>1,\ 1<p<q<2.$$

In this context let us mention that the same technique yields for $1<p,q<2$

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}, \quad (2\alpha+2/3)(1/p-1/2)>\max\{(2\beta+2)(1/q-1/2),\ 1\}. \tag{4}$$

This embedding is in so far interesting as it allows to go from p, $1<p<2$, to $q\ne p$, $1<q<2$, connected with a loss in the size of β if $q<p$ or a gain in β if $1<p<q<2$; e.g.

$$M^{4/3}_{10;10} \hookrightarrow M^q_{5;5}, \quad 1.08\le q\le 2, \quad \text{or} \quad M^{8/7}_{2;2} \hookrightarrow M^q_{4;4}, \quad 3/2\le q\le 2.$$

Improvements of (4) can be expected by better necessary conditions and/or better sufficient conditions; but this technique **cannot** give something like

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}, \quad (\alpha+1)(1/p-1/2)>(\beta+1)(1/q-1/2),\ 1<p,\ q<2,$$

which is suggested by (4) when choosing "large" α with p near 2 since then the number $4(1/p-1/2)/3$, which describes the smoothness gap between the necessary conditions and the sufficient conditions in [14, Cor. 1.2], is "negligible".

Concerning the general problem "When does $M^{p,q}_{\alpha;\gamma_1,\delta_1} \hookrightarrow M^{p,q}_{\beta;\gamma_2,\delta_2}$ hold?", we mention results in Stempak and Trebels [14, Cor. 4.3]: For $1 < p < \infty$ there holds

$$M^p_{\beta;\beta p/2+\delta} = M^p_{0;\delta} \quad \text{if} \begin{cases} -1-\beta p/2 < \delta < p-1+\beta p/2\,, & -1 < \beta < 0\,, \\ -1 < \delta < p-1\,, & 0 \le \beta, \end{cases}$$

which for $\delta = 0$ contains half of Kanjin's [9] result and for $\delta = p/4 - 1/2$ Thangavelu's [16]. In particular, there holds for $-1 < \beta < \alpha,\ 1 < p < \infty$,

$$M^p_{\beta;\beta} = M^p_{\beta;\,\beta p/2+\beta p(1/p-1/2)} = M^p_{\alpha;\,\alpha p/2+\beta p(1/p-1/2)}\,, \quad (2\beta+2)|1/p-1/2| < 1. \qquad (5)$$

These results are based on Kanjin's [9] transplantation theorem and its weighted version in [14]. The latter gives further insight into our problem in so far as it implies that the restriction $\beta < \alpha - 2/3$ in Proposition 1.1 is not sharp.
To this end we first note that the following extension of Corollary 4.4 in [14] holds

$$wbv_{2,s} \hookrightarrow M^p_{\alpha;\,\alpha p/2+\eta(p/2-1)}, \quad 0 \le \eta \le 1, \quad 1 < p \le 2, \quad s > 1/p.$$

(For the proof observe that for $\alpha = 0$ the parameter $\gamma = \eta(p/2-1),\ 0 \le \eta \le 1$, is admissible in [14, Theorem 1.1] and then follow the argumentation of [14, Cor. 4.4].) This combined with (3) yields for $s = (2\alpha + 2/3)(1/p - 1/2) > 1/p$

$$M^p_{\alpha;\,\alpha} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\alpha;\,\alpha p/2+p/2-1}\,, \quad 1 < p \le 2, \quad \alpha > (p+1)/(6-3p).$$

Thus, by interpolation with change of measure,

$$M^p_{\alpha;\,\alpha} \hookrightarrow M^p_{\alpha;\,\alpha p/2+\delta}\,, \quad p/2-1 \le \delta \le \alpha - \alpha p/2, \quad \alpha > (p+1)/(6-3p).$$

Since (5) gives

$$M^p_{\alpha;\,\alpha p/2+\beta p(1/p-1/2)} = M^p_{\beta;\,\beta}$$

we arrive at

Proposition 1.2 *Let $1 < p \le 2$ and $\alpha > (p+1)/(6-3p)$. Then*

$$M^p_{\alpha;\,\alpha} \hookrightarrow M^p_{\beta;\,\beta}\,, \quad (2\beta+2)(1/p-1/2) < 1, \quad -1 < \beta < \alpha.$$

The first restriction on β is equivalent to $\beta < (2p-2)/(2-p)$. This combined with the restriction on α gives $\alpha - \beta > (7-5p)/(6-3p)$, the latter being decreasing in p and taking the value $2/3$ at $p = 1$. Hence Proposition 1.2 is an improvement of the previous one for all $1 < p < 2$ provided $(p+1)/(6-3p) < \alpha \le (2p-2)/(2-p)$.

For big α's, Proposition 1.1 is certainly better. If in the transplantation theorem in [14] higher exponents could be allowed in the power weight – which is possible in the Jacobi expansion case as shown by Muckenhoupt [12] – the technique just used would give the embedding when $-1 < \beta < \alpha,\ 1 < p < 2$, and $\alpha > (p+1)/(6-3p)$. Summarizing, it is reasonable to

conjecture $$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}, \quad -1 < \beta < \alpha, \quad 1 \le p \le q \le \infty.$$

Apart from the above fragmentary results, so far we can only prove the conjecture in the extreme case when $q = \infty$ and $\beta \ge 0$; the latter restriction arises from the fact that we have to make use of the twisted Laguerre convolution (see [7]) which is proved till now only for Laguerre polynomials $L^\alpha_n(x)$ with $\alpha \ge 0$. Our main result is

Theorem 1.3 *If* $1 \le p \le \infty$, *then*

$$M^{p,\infty}_{\alpha;\alpha} \hookrightarrow M^{p,\infty}_{\beta;\beta}, \quad 0 \le \beta < \alpha.$$

Remarks. 1) One could speculate that an interpolation argument applied to

$$M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad M^\infty_{\alpha;\alpha} = M^1_{\alpha;\alpha} \hookrightarrow M^1_{\beta;\beta} = M^\infty_{\beta;\beta}, \quad \beta < \alpha,$$

could give the open case $M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta},\ 1 < p < 2$. In this respect we mention a result of Zafran [17, p. 1412] for the Fourier transform pointed out to us by A. Seeger:

> *Denote by* $M^p(\mathbf{R})$ *the set of bounded Fourier multipliers on* $L^p(\mathbf{R})$ *and by* $M^\wedge(\mathbf{R})$ *the set of Fourier transforms of bounded measures on* $\mathbf{R}$. *Then* $M^p(\mathbf{R}),\ 1 < p < 2$, *is* **not** *an interpolation space with respect to the pair* $(M^\wedge(\mathbf{R}), L^\infty(\mathbf{R}))$.

Thus de Leeuw's result mentioned at the beginning cannot be proved by interpolation.

2) It is perhaps amazing to note that the *wbv*–classes do not play only an auxiliary role in dealing with the above formulated general problem. In the framework of one-dimensional Fourier transforms/series this was shown by Muckenhoupt, Wheeden, and Wo-Sang Young [13]. That this phenomenon also occurs in the framework of Laguerre expansions can be seen from the following two theorems.

Theorem 1.4 *If* $\alpha > -1,\ \alpha \ne 0$, *then*

$$wbv_{2,1} \hookrightarrow M^2_{\alpha;\alpha+1}.$$

In the case $-1 < \alpha < 0$ *the multiplier operator is defined only on the subspace* $\{f \in L^2_{w(\alpha+1)} : \hat{f}_\alpha(0) = 0\}$.

Theorem 1.5 *If $\alpha > -1$, then*

$$M^2_{\alpha;\,\alpha+1} \hookrightarrow wbv_{2,1}\,.$$

A combination of these two results leads to

$$M^2_{\alpha;\,\alpha+1} = M^2_{\beta;\,\beta+1} = wbv_{2,1}\,, \quad \alpha,\beta > -1, \quad \alpha,\beta \neq 0, \tag{6}$$

and a combination with [14, (19)] gives

$$M^2_{\alpha;\,\alpha+1} \hookrightarrow M^p_{\alpha;\,\alpha}\,, \quad \alpha \geq 0, \quad (2\alpha+2)/(\alpha+1) < p \leq 2.$$

2 Proof of Theorem 1.3

Theorem 1.3 is an immediate consequence of the combination of the following two theorems.

Theorem 2.1 *Let $f \in L^p_{w(\alpha)}$ with $\alpha > -1$ when $1 \leq p < \infty$ and $\alpha \geq 0$ when $p = \infty$. Then there exists a function $g \in L^p_{w(\beta)}$, $-1 < \beta < \alpha$, with*

$$g(x) \sim (\Gamma(\beta+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\beta_k(x), \quad \|g\|_{L^p_{w(\beta)}} \leq C\|f\|_{L^p_{w(\alpha)}}\,.$$

Proof

First let $1 \leq p < \infty$ and, without loss of generality, let f be a polynomial (these are dense in $L^p_{w(\alpha)}$). We recall the projection formula (3.31) in Askey and Fitch [2]

$$e^{-x} L^\beta_n(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} L^\alpha_n(y)\, dy, \quad -1 < \beta < \alpha.$$

Then the following computations are justified.

$$\begin{aligned}
\|g\|_{L^p_{w(\beta)}} &= C \left(\int_0^\infty |\sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\beta_k(x) e^{-x/2}|^p x^\beta dx \right)^{1/p} \\
&= C \left(\int_0^\infty \left| \int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\alpha_k(y)\, dy \right|^p x^\beta e^{xp/2} dx \right)^{1/p} \\
&\leq C \int_1^\infty (t-1)^{\alpha-\beta-1} \left(\int_0^\infty |\sum_k \hat{f}_\alpha(k) L^\alpha_k(xt) x^{\alpha-\beta+\beta/p} e^{-x(t-1/2)}|^p dx \right)^{1/p} dt
\end{aligned}$$

after a substitution and application of the integral Minkowski inequality. Additional substitutions lead to

$$\begin{aligned}\|g\|_{L^p_{w(\beta)}} &\le C\int_0^\infty s^{\alpha-\beta-1}(s+1)^{\beta/p'-\alpha-1/p}\times \\ &\qquad \left(\int_0^\infty |\sum_k \hat{f}_\alpha(k)L_k^\alpha(y)e^{-y/2}y^{(\alpha-\beta)/p'}e^{-ys/2(s+1)}|^p y^\alpha dy\right)^{1/p} ds \\ &\le C\int_0^\infty s^{(\alpha-\beta)/p-1}(s+1)^{-(\alpha+1)/p}\left(\int_0^\infty |\sum_k \hat{f}_\alpha(k)L_k^\alpha(y)e^{-y/2}|^p y^\alpha dy\right)^{1/p} ds,\end{aligned}$$

where we used the inequality $y^{(\alpha-\beta)/p'}e^{-ys/2(s+1)} \le C((s+1)/s)^{(\alpha-\beta)/p'}$. Since $-1 < \beta < \alpha$ it is easily seen that the outer integration only gives a bounded contribution. If $f \in L^\infty_{w(\alpha)}$ then $|(k+1)^{-1/2}\hat{f}_\alpha(k)| \le C\|f\|_{L^\infty_{w(\alpha)}}$ by [10, Lemma 1] and, therefore, the Abel-Poisson means of an arbitrary $f \in L^\infty_{w(\alpha)}$ can be represented by

$$P_r f(x) = (\Gamma(\alpha+1))^{-1}\sum_k r^k \hat{f}_\alpha(k)L_k^\alpha(x), \quad 0 \le r < 1, \quad x \ge 0,$$

and, by the convolution theorem in Görlich and Markett [7, p. 169],

$$\|P_r f\|_{L^\infty_{w(\alpha)}} \le C\|f\|_{L^\infty_{w(\alpha)}}, \quad 0 \le r < 1, \quad \alpha \ge 0.$$

A slight modification of the argument in the case $1 \le p < \infty$ shows that

$$\|g_r\|_{L^\infty_{w(\beta)}} := \|(\Gamma(\beta+1))^{-1}\sum_k r^k \hat{f}_\alpha(k)L_k^\beta\|_{L^\infty_{w(\beta)}} \le C\|P_r f\|_{L^\infty_{w(\alpha)}} \le C\|f\|_{L^\infty_{w(\alpha)}}.$$

By the weak* compactness there exists a function $g \in L^\infty_{w(\beta)}$ with $\hat{g}_\beta(k) = \hat{f}_\alpha(k)$ and $\|g\|_{L^\infty_{w(\beta)}} \le \liminf_{k\to\infty}\|g_{r_k}\|_{L^\infty_{w(\beta)}}$ for a suitable sequence $r_k \to 1^-$; hence also the assertion in the case $p = \infty$.

Theorem 2.2 *For $\alpha \ge 0$ there holds*
i) $\quad M^{1,p}_{\alpha;\alpha} = M^{p',\infty}_{\alpha;\alpha} = (L^p_{w(\alpha)})^\frown, \quad 1 < p \le \infty,$
ii) $\quad M^{1,1}_{\alpha;\alpha} = M^{\infty,\infty}_{\alpha;\alpha} = \{m = \{m_k\}_{k\in\mathbf{N}_0} : \|P_r(m)\|_{L^1_{w(\alpha)}} = O(1),\ r \to 1^-\},$
where $P_r(m)(x) = (\Gamma(\alpha+1))^{-1}\sum_k r^k m_k L_k^\alpha(x)$.

Proof

The first equalities in *i)* and *ii)* are the standard duality statements. Let us briefly indicate the second equalities (which are also more or less standard).
If $m = \{m_k\}_{k\in\mathbf{N}_0}$ are the Fourier Laguerre coefficients of an $L^p_{w(\alpha)}$–function, $1 < p \le \infty$, or in the case $p = 1$ of a bounded measure with respect to the weight $e^{-x/2}x^\alpha$, then

Young's inequality in Görlich and Markett [7] (or a slight extension of it to measures in the case $p=1$) shows that $m \in M^{p',\infty}_{\alpha;\alpha}$.

Conversely, associate formally to a sequence $m=\{m_k\}$ an operator T_m by

$$T_m f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha(x). \tag{7}$$

Then, in essentially the notation of Görlich and Markett [7],

$$T_m(P_r f)(x) = P_r(m) * f(x) = \int_0^\infty T_x^\alpha(P_r(m)(y)) f(y) e^{-y} y^\alpha dy,$$

where T_x^α is the Laguerre translation operator. If $\|f\|_{L^{p'}_{w(\alpha)}} = 1$ then

$$\|T_m(P_r f)\|_{L^\infty_{w(\alpha)}} \le \|m\|_{M^{p',\infty}_{\alpha;\,\alpha}} \|P_r f\|_{L^{p'}_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}} ,$$

and hence, by the converse of Hölder's inequality,

$$\sup_{\|f\|_{L^{p'}_{w(\alpha)}}=1} \left| \int_0^\infty T_x^\alpha(P_r(m)(y)) e^{-y/2} y^{\alpha/p} f(y) e^{-y/2} y^{\alpha/p'} dy \right|$$

$$= \|T_x^\alpha(P_r(m))\|_{L^p_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}}$$

for $x \ge 0,\ 0 \le r < 1$. In particular, for $x=0$ we obtain

$$\|P_r(m)\|_{L^p_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\,\alpha}} , \quad 0 \le r < 1.$$

Now weak* compactness gives the desired converse embedding.

3 Proof of Theorems 1.4 and 1.5

The proof relies heavily on the Parseval formula

$$\frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} A_k^\alpha |\hat{f}_\alpha(k)|^2 = \int_0^\infty |f(x)e^{-x/2}|^2 x^\alpha dx \tag{8}$$

and its extension

$$\sum_{k=0}^{\infty} A_k^{\alpha+\lambda} |\Delta^\lambda \hat{f}_\alpha(k)|^2 \approx \int_0^\infty |f(x)e^{-x/2}|^2 x^{\alpha+\lambda} dx, \quad \lambda \ge 0, \tag{9}$$

which is a consequence of the formula

$$\Delta^\lambda \hat{f}_\alpha(k) = C_{\alpha,\lambda} \hat{f}_{\alpha+\lambda}(k) \tag{10}$$

(see e.g. the proof of Lemma 2.1 in [6]). For the proof of Theorem 1.4 we further need the following discrete analog of the $p = 2$ case of a weighted Hardy inequality in Muckenhoupt [11] whose proof can at once be read off from [11] by replacing the integrals there by sums and using the fact that

$$a \le 2(a+b)^{1/2}[(a+b)^{1/2} - b^{1/2}]$$

when a, $b \ge 0$; also see the extensions in [1, Sec. 4].

Lemma 3.1 *Let $\{u_k\}_{k\in\mathbf{N}_0}$, $\{v_k\}_{k\in\mathbf{N}_0}$ be non-negative sequences (if $v_k = 0$ we set $v_k^{-1} = 0$). Then*

a) $$\sum_{k=0}^{\infty}\Big|\sum_{j=0}^{k} a_j\Big|^2 u_k \le C\sup_N\Big(\sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1}\Big)\sum_{j=0}^{\infty}|a_j|^2 v_j.$$

b) $$\sum_{k=0}^{\infty}\Big|\sum_{j=k}^{\infty} a_j\Big|^2 u_k \le C\sup_N\Big(\sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1}\Big)\sum_{j=0}^{\infty}|a_j|^2 v_j.$$

Proof of Theorem 1.4. Using (9) and the operator T_m defined in (7), we obtain

$$\int_0^\infty |T_m f(x)e^{-x/2}|^2 x^{\alpha+1}dx \approx \sum_{k=0}^{\infty} A_k^{\alpha+1}|\Delta(m_k\hat{f}_\alpha(k))|^2.$$

Since

$$\Delta(m_k\hat{f}_\alpha(k)) = m_k\Delta\hat{f}_\alpha(k) + \hat{f}_\alpha(k+1)\Delta m_k \tag{11}$$

we first observe that

$$\sum_{k=0}^{\infty} A_k^{\alpha+1}|m_k|^2|\Delta\hat{f}_\alpha(k)|^2 \le \|m\|_\infty^2 \sum_{k=0}^{\infty} A_k^{\alpha+1}|\Delta\hat{f}_\alpha(k)|^2 \le C\|m\|_\infty^2\|f\|^2_{L^2_{w(\alpha+1)}}.$$

To dominate the term containing Δm_k we deduce from (8) that for $\alpha \ge 0$ the Fourier Laguerre coefficients tend to zero as $k \to \infty$. Hence

$$\sum_{k=0}^{\infty} A_k^{\alpha+1}|\hat{f}_\alpha(k+1)\Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1}|\Delta m_k|^2\Big|\sum_{j=k+1}^{\infty}\Delta\hat{f}_\alpha(j)\Big|^2 =: I.$$

In order to apply Lemma 3.1 b), we choose $u_k = A_k^{\alpha+1}|\Delta m_k|^2$ and $v_k = A_k^{\alpha+1}$, and observe that when $M \in \mathbf{N}$, $2^{M-1} \le N < 2^M$, we have that

$$\begin{aligned}\Big(\sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1}\Big) &\le C(N+1)^{-\alpha}\sum_{j=0}^{M}\sum_{k=2^j-1}^{2^{j+1}-2}(k+1)|\Delta m_k|^2\frac{A_k^{\alpha+1}}{k+1}\\ &\le C(N+1)^{-\alpha}\sum_{j=0}^{M}(2^j)^\alpha\|m\|_{2,1}^2 \le C\|m\|_{2,1}^2\end{aligned}$$

uniformly in N if $\alpha > 0$. Then Lemma 3.1 b) gives

$$I \le C\|m\|_{2,1}^2 \sum_{j=0}^{\infty} A_j^{\alpha+1} |\Delta \hat{f}_\alpha(j)|^2 \le C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2$$

by (9). Thus there remains to consider the case $-1 < \alpha < 0$. For the same choice of u_k and v_k one easily obtains

$$\Big(\sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1} \Big) \le C\|m\|_{2,1}^2 \,.$$

Now assume that $\hat{f}(0) = 0$. Then we have

$$\sum_{k=0}^{\infty} A_k^{\alpha+1} |\hat{f}_\alpha(k+1)\Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta m_k|^2 \Big| \sum_{j=0}^{k} \Delta \hat{f}_\alpha(j) \Big|^2 \le C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2 \,,$$

where the last estimate follows by Lemma 3.1 a); thus Theorem 1.4 is established.

The **proof of Theorem 1.5** is essentially contained in [6]. As in [6], consider a monotone decreasing C^∞-function $\phi(x)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x \ge 4 \end{cases} \,, \quad \phi_i(x) = \phi(x/2^i).$$

Then the $\phi_i(k)$ are the Fourier Laguerre coefficients of an $L^2_{w(\alpha+1)}$-function $\Phi^{(i)}$ with norm $\|\Phi^{(i)}\|_{L^2_{w(\alpha+1)}} \le C\,(2^i)^{\alpha/2}$ and

$$\sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta m_k|^2 = \sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta(m_k \phi_i(k))|^2 \le \sum_{k=0}^{2^{i+2}} A_k^{\alpha+1} |\Delta(m_k \phi_i(k))|^2$$

$$\le C\|T_m \Phi^{(i)}\|_{L^2_{w(\alpha+1)}}^2 \le C\|m\|_{M^2_{\alpha;\,\alpha+1}}^2 \|\Phi^{(i)}\|_{L^2_{w(\alpha+1)}}^2 \le C 2^{i\alpha} \|m\|_{M^2_{\alpha;\,\alpha+1}}^2 .$$

This immediately leads to

$$\|m\|_\infty + \left(\sum_{2^i}^{2^{i+1}} |(k+1)\Delta m_k|^2 \frac{1}{k+1} \right)^{1/2} \le C\|m\|_{M^2_{\alpha;\,\alpha+1}},$$

uniformly in i, since by [6, (10)] there holds $\|m\|_\infty \le C\|m\|_{M^2_{\alpha;\,\alpha+1}}$; thus Theorem 1.5 is established.

Remark. 3) (Added on Aug. 10, 1994) The characterization (6) can easily be extended to

$$M^2_{\alpha,\alpha+l} = wbv_{2,l}, \quad \alpha > -1, \quad \alpha \ne 0, \ldots, l-1,\ l \in \mathbf{N}. \tag{12}$$

In the case $\alpha < l-1$ the multiplier operator is defined only on the subspace $\{f \in L^2_{w(\alpha+l)} : \hat{f}_\alpha(k) = 0,\ 0 \le k < (l-1-\alpha)/2\}$.

The necessity part carries over immediately (see also [6]). The sufficiency part will be proved by induction. Thus suppose that (12) is true for $l = 1, \ldots, n$ and α's as indicated. Then, as in the case $n = 1$, by (9)

$$\int_0^\infty |T_m f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx \approx \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n \Delta(m_k \hat{f}_\alpha(k))|^2$$

$$\le C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n (m_k \Delta \hat{f}_\alpha(k))|^2 + C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n (\hat{f}_\alpha(k+1) \Delta m_k)|^2 =: I + II$$

By the assumption and (10)

$$I \le C \|m\|^2_{wbv_{2,n}} \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n \hat{f}_{\alpha+1}(k))|^2 \le C \|m\|^2_{wbv_{2,n+1}} \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx$$

on account of the embedding properties of the *wbv*–spaces [5]. Analogously II can be estimated by

$$II \le C \|\{(k+1)\Delta m_k\}\|^2_{wbv_{2,n}} \sum_{k=0}^\infty A_k^{\alpha+n+1} \Big| \Delta^n \Big(\frac{\hat{f}_\alpha(k+1)}{k+1} \Big) \Big|^2.$$

By the Leibniz formula for differences there holds

$$\begin{aligned} \Delta^n \Big(\frac{\hat{f}_\alpha(k+1)}{k+1} \Big) &\le C \sum_{j=0}^n |\Delta^j \hat{f}_\alpha(k+1)|\, |\Delta^{n-j} \frac{1}{j+k+1}| \\ &\le C \sum_{j=0}^n (j+k+1)^{j-n-1} |\Delta^j \hat{f}_\alpha(k+1)|. \end{aligned}$$

Hence we have to dominate for $j = 0, \ldots, n$

$$II_j := \sum_{k=0}^\infty A_k^{\alpha-n-1+2j} |\Delta^j \hat{f}_\alpha(k+1)|^2.$$

If $\alpha > n$ then $c_j := -\alpha - 2j + n + 1 < 1$ for all $j = 0, \ldots, n$, $\Delta^j \hat{f}_\alpha(k+1) = \sum_{i=k+1}^\infty \Delta^{j+1} \hat{f}_\alpha(i)$, and we can apply [8, Theorem 346] repeatedly to obtain

$$II_j \le C \sum_{k=0}^\infty A_k^{\alpha-n-1+2j} |(k+1) \Delta^{j+1} \hat{f}_\alpha(k+1)|^2 \approx \sum_{k=0}^\infty A_k^{\alpha-n+2j+1} |\Delta^{j+1} \hat{f}_\alpha(k+1)|^2$$

$$\le \ldots \le C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^{n+1} \hat{f}_\alpha(k+1)|^2 \le C \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx\,.$$

Since $\|\{(k+1)\Delta m_k\}\|_{wbv_{2,n}} \leq C\|m\|_{wbv_{2,n+1}}$, this gives the assertion for the weight x^{n+1} in the case $\alpha > n$.

If $\alpha < n,\ \alpha \neq 0, \dots, n$, then some $c_j > 1$. For the application of [8, Theorem 346] one needs $c_j \neq 1$; this is guaranteed by the hypothesis $\alpha \neq 0, \dots, n$ (in the case of an additional weight x^{n+1}). For the j for which $c_j > 1$ we have to use the representation

$$\Delta^j \hat{f}_\alpha(k+1) = -\sum_{i=0}^{k} \Delta^{j+1} \hat{f}_\alpha(i), \quad \text{if } \Delta^j \hat{f}_\alpha(0) = 0,$$

i.e., the first $(j+1)$ Fourier-Laguerre coefficients have to vanish to ensure this representation. But $0 \leq j \leq j_0$, where j_0 is choosen in such a way that $c_{j_0} > 1$ and $c_{j_0+1} < 1$, hence $j_0 = [(n-\alpha)/2]$ (with respect to the additional weight x^{n+1}); here we used the standard notation for $[a],\ a \in \mathbf{R}$, to be the greatest integer $\leq a$. Hence the condition that the first $[(n-\alpha)/2]+1$ Fourier-Laguerre coefficients have to vanish is needed if the additional weight is x^{n+1}. A repeated application of [8, Theorem 346] with appropriate $c > 1$ or $c < 1$ now gives the assertion.

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