On necessary multiplier conditions for Laguerre expansions II

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Dedicated to Dick Askey and Frank Olver

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Abstract. The necessary multiplier conditions for Laguerre expansions derived in Gasper and Trebels [3] are supplemented and modified. This allows us to place Markett's Cohen type inequality [6] (up to the log–case) in the general framework of necessary conditions.

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1. Introduction. The purpose of this sequel to [3] is to obtain a better insight into the structure of Laguerre multipliers on L^p spaces from the point of view of necessary conditions. We recall that in [3] there occurs the annoying phenomenon that, e.g., the optimal necessary conditions in the case $p = 1$ do not give the "right" unboundedness behavior of the Cesàro means. By slightly modifying these conditions we can not only remedy this defect but can also derive Markett's Cohen type inequality [6] (up to the log–case) as an immediate consequence.
For the convenience of the reader we briefly repeat the notation; we consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f : \| f \|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma \, dx)^{1/p} < \infty\} \ , \quad 1 \le p < \infty,$$

denote the classical Laguerre polynomials by $L_n^\alpha(x),\ \alpha > -1,\ n \in \mathbf{N}_0$ (see Szegö [8, p. 100]), and set

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \quad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Associate to f its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier Laguerre coefficients of f are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x} \, dx \qquad (1)$$

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(if the integrals exist). A sequence $m = \{m_k\}$ is called a (bounded) multiplier on $L^p_{w(\gamma)}$, notation $m \in M^p_{w(\gamma)}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^p_{w(\gamma)}} \leq C \| f \|_{L^p_{w(\gamma)}}$$

for all polynomials f; the smallest constant C for which this holds is called the multiplier norm $\| m \|_{M^p_{\alpha,\gamma}}$. The necessary conditions will be given in certain "smoothness" properties of the multiplier sequence in question. To this end we introduce a fractional difference operator of order δ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the sum converges), the first order difference operator Δ_2 with increment 2 by

$$\Delta_2 m_k = m_k - m_{k+2},$$

and the notation

$$\Delta_2 \Delta^\delta m_k = \Delta^{\delta+1} m_k + \Delta^{\delta+1} m_{k+1}.$$

Generic positive constants that are independent of the functions (and sequences) will be denoted by C. Within the setting of the $L^p_{w(\gamma)}$-spaces our main results now read (with $1/p + 1/q = 1$):

THEOREM 1.1. *Let* α, $a > -1$ *and* $\alpha + a > -1$. *If* $f \in L^p_{w(\gamma)}$, $1 \leq p < 2$, $\gamma > -1$, *then*

$$\left(\sum_{k=0}^{\infty} |(k+1)^{(\gamma+1)/p-1/2} \Delta_2 \Delta^a \hat{f}_\alpha(k)|^q \right)^{1/q} \leq C \| f \|_{L^p_{w(\gamma)}}, \tag{2}$$

provided

$$\frac{\gamma+1}{p} \leq \frac{\alpha+a}{p} + 1 \qquad \text{if } \alpha + a \leq 1/2,$$

$$\frac{\gamma+1}{p} \leq \frac{\alpha+a}{2} + 1 + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right) \quad \text{if } \alpha + a > 1/2.$$

If we note that

$$|m_k| \| L_k^\alpha \|_{L^p_{w(\gamma)}} = \| m_k L_k^\alpha \|_{L^p_{w(\gamma)}} \leq \| m \|_{M^p_{\alpha,\gamma}} \| L_k^\alpha \|_{L^p_{w(\gamma)}}, \quad \gamma > -1,$$

implies $M^p_{w(\gamma)} \subset l^\infty$, we immediately obtain as in [3] (see there the proof of Lemma 2.3)

THEOREM 1.2. *Let* $m = \{m_k\} \in M^p_{w(\gamma)}$, $1 \leq p < 2$, *and let* α, γ, *and* a *be as in Theorem 1.1. Then*

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta_2 \Delta^a m_k|^q \frac{1}{k+1} \right)^{1/q} \leq C \| m \|_{M^p_{\alpha,\gamma}} . \tag{3}$$

An extension of Theorem 1.2 to $2 < p < \infty$ easily follows by duality

$$M^p_{w(\gamma)} = M^q_{w(\alpha q - \gamma q/p)}\ , \quad -1 < \gamma < p(\alpha+1) - 1, \quad 1 < p < \infty.$$

In view of the results in [6], [3] and for an easy comparison we want to emphasize the cases $\gamma = \alpha$ and $\gamma = \alpha p/2$. Therefore, we state

COROLLARY 1.3.

a) Let $m \in M^p_{w(\alpha)}$, $1 \le p < 2$, $\alpha > -1$, *and let* $\lambda := (2\alpha+1)(1/p - 1/2)$. *Then, with* $\lambda > 0$,

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{\lambda} \Delta_2 \Delta^{\lambda-1} m_k|^q \frac{1}{k+1} \right)^{1/q} \le C \parallel m \parallel_{M^p_{w(\alpha)}}$$

if $\alpha + \lambda \ge 3/2$; *if* $\alpha + \lambda < 3/2$ *it has additionally to be assumed that* $\lambda \ge 2 - p$. *In the case* $\alpha + \lambda < 3/2$ *and* $\lambda < 2 - p$ *an analogous result holds when the difference operator* $\Delta_2 \Delta^{\lambda-1}$ *is replaced by* Δ_2.

b) Let $m \in M^p_{w(\alpha p/2)}$, $1 \le p < 4/3$, *and* $(\alpha - 1)(1/p - 1/2) \ge -1/2$. *Then*

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)^{1/p-1/2} \Delta_2 m_k|^q \frac{1}{k+1} \right)^{1/q} \le C \parallel m \parallel_{M^p_{w(\alpha p/2)}} .$$

Remarks. 1) For polynomial $f(x) = \sum_{k=0}^n c_k L_k^\alpha(x)$ Theorem 1.1 yields, by taking only the $(k = n)$–term on the left hand side of (2),

$$|c_n|(n+1)^{(\gamma+1)/p-1/2} \le C \parallel f \parallel_{L^p_{w(\gamma)}}, \quad 1 \le p < 2$$

(under the restrictions on γ of Theorem 1.1). In particular, if we choose $\gamma = \alpha$, this comprises formula (1.13) in Markett [6] for his basic case $\beta = \alpha$. For $\gamma = \alpha p/2$, it even extends formula (1.14) in [6] to negative α's as described in Corollary 1.3, b). The case $2 < p < \infty$ can be done by an application of a Nikolskii inequality, see [6].

2) Analogously, Cohen type inequalities follow from Theorem 1.2; in particular, Corollary 1.3 yields

COROLLARY 1.4. *Let* $m = \{m_k\}_{k=0}^n$ *be a finite sequence,* $1 \le p < 2$, *and* $\alpha > -1$.

a) If $m \in M^p_{w(\alpha)}$ *then*

$$(n+1)^{(2\alpha+2)(1/p-1/2)-1/2} |m_n| \le C \parallel m \parallel_{M^p_{w(\alpha)}}, \quad 1 \le p < \frac{4\alpha+4}{2\alpha+3}.$$

b) If $m \in M^p_{w(\alpha p/2)}$ *and* $(\alpha - 1)(1/p - 1/2) \ge -1/2$, *then*

$$(n+1)^{2/p-3/2} |m_n| \le C \parallel m \parallel_{M^p_{w(\alpha p/2)}}, \quad 1 \le p < 4/3.$$

With the exception of the crucial log–case, i.e. $p_0 = (4\alpha+4)/(2\alpha+3)$ or $p_0 = 4/3$, resp., Corollary 1.4 contains Markett's Theorem 1 in [6] and extends it to negative

α's. In particular we obtain for the Cesàro means of order $\delta \geq 0$, represented by its multiplier sequence $m^\delta_{k,n} = A^\delta_{n-k}/A^\delta_n$, the "right" unboundedness behavior (see [4])

$$\| \{m^\delta_{k,n}\} \|_{M^p_{w(\alpha)}} \geq C(n+1)^{(2\alpha+2)(1/p-1/2)-1/2-\delta}, \quad 1 \leq p < \frac{4\alpha+4}{2\alpha+3+2\delta}.$$

3) There arises the question, in how far the type of necessary conditions in [3] are comparable with the present ones. Let $\lambda > 1$. Since $\Delta_2 m_k = \Delta m_k + \Delta m_{k+1}$ we obviously have

$$(4) \qquad \sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta_2 \Delta^{\lambda-1} m_k|^q \frac{1}{k+1} \right)^{1/q}$$

$$\leq C \sup_n \left(\sum_{k=n}^{2n} |(k+1)^{(2\gamma+1)/p-(2\alpha+1)/2} \Delta^{\lambda} m_k|^q \frac{1}{k+1} \right)^{1/q}.$$

In general, a converse cannot hold as can be seen by the following example: choose $\gamma = \alpha,\ \lambda = (2\alpha+1)(1/p-1/2)$ and $m_k = (-1)^k(k+1)^{-\varepsilon},\ 0 < \varepsilon < 1$. Then

$$\sup_n \left(\sum_{k=n}^{2n} |(k+1)\Delta m_k|^q \frac{1}{k+1} \right)^{1/q} = \infty$$

and hence by the embedding properties of the wbv–spaces, see [2], the right hand side of (4) cannot be finite for all $\lambda > 1$. But since $\Delta_2 \Delta^{\lambda-1} m_k = \Delta^{\lambda-1}\Delta_2 m_k \sim (k+1)^{-\varepsilon-\lambda}$, the left hand side of (4) is finite for all $\lambda > 1$.

Theorem 1.1 will be proved in Section 2 by interpolating between (L^1, l^∞)– and (L^2, l^2)–estimates. The $a \neq 0$ case is an easy consequence of the case $a = 0$ when one uses the basic formula (see formula (3) in [3] and Remark 3 preceding Section 3 there)

$$(5) \quad \Delta^a R^\alpha_k(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+a+1)} x^a R^{\alpha+a}_k(x), \quad x > 0,\ a > -1 - \min\{\alpha, \alpha/2 - 1/4\},$$

where in the case $a > -(2\alpha+1)/4$ the series for the fractional difference converges absolutely. In Section 3, a necessary (L^1, l^1)–estimate is derived and it is compared with a corresponding sufficient (l^1, L^1)–estimate.

2. Proof of Theorem 1.1. Let us first handle the (L^2, l^2)–estimate. Since

$$\Delta_2 \Delta^a \hat{f}_\alpha(k) = \Delta^{1+a} \hat{f}_\alpha(k) + \Delta^{1+a} \hat{f}_\alpha(k+1)$$

it follows from the Parseval formula preceding Corollary 2.5 in [3] that

$$(6) \quad \left(\sum_{k=0}^{\infty} |\sqrt{A^{\alpha+1+a}_k} \Delta_2 \Delta^a \hat{f}_\alpha(k)|^2 \right)^{1/2} \leq C \left(\int_0^\infty |f(t) e^{-t/2} t^{(\alpha+1+a)/2}|^2 \, dt \right)^{1/2}.$$

Concerning the (L^1, l^∞)–estimate we first restrict ourselves to the case $a = 0$. Define $\mu \in \mathbf{R}$ by

$$2\left(\frac{1}{p} - \frac{1}{2}\right)\mu = \frac{\gamma}{p} - \frac{\alpha+1}{2};$$

with the notation $\mathcal{L}_k^\alpha(t) = (A_k^\alpha/\Gamma(\alpha+1))^{1/2} R_k^\alpha(t) e^{-t/2} t^{\alpha/2}$ it follows that

$$|\Delta_2 \hat{f}_\alpha(k)| = C|\int_0^\infty f(t)\{\mathcal{L}_k^\alpha(t)/\sqrt{A_k^\alpha} - \mathcal{L}_{k+2}^\alpha(t)/\sqrt{A_{k+2}^\alpha}\} e^{-t/2} t^{\alpha/2}\, dt|$$

$$\leq C(k+1)^{-1-\alpha/2} \int_0^\infty |f(t)||t^{-\mu-1/2} \mathcal{L}_k^\alpha(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt$$

$$+C(k+1)^{-\alpha/2} \int_0^\infty |f(t)||t^{-\mu-1/2}\{\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)\}| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt = I + II.$$

We distinguish the two cases $\alpha \leq 1/2$ and $\alpha > 1/2$:

First consider the case $\alpha \leq 1/2$. By the asymptotic estimates for $\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)$ in Askey and Wainger [1, p. 699], see formula (2.12) in [6], it follows for $\gamma \leq \alpha + p - 1$ that

$$\| t^{-\mu-1/2}\{\mathcal{L}_k^\alpha(t) - \mathcal{L}_{k+2}^\alpha(t)\} \|_\infty \leq C(k+1)^{-1-\mu}$$

so that

$$II \leq C(k+1)^{-1-\mu-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad \gamma \leq \alpha + p - 1. \tag{7}$$

By Lemma 1, 4th case, in [5]

$$\| t^{-\mu-1/2} \mathcal{L}_k^\alpha(t) \|_\infty \leq C(k+1)^{-\mu-5/6}$$

so that trivially

$$I \leq C(k+1)^{-1-\mu-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad \frac{\gamma+1}{p} \leq \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3}.$$

By Lemma 1, 5th case, in [5]

$$\| t^{-\mu-1/2} \mathcal{L}_k^\alpha(t) \|_\infty \leq C(k+1)^{\mu+1/2}$$

so that

$$I \leq C(k+1)^{\mu-1/2-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt,$$

$$\leq C(k+1)^{-1-\mu-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad \frac{\gamma+1}{p} > \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3},$$

provided that $\mu - (\alpha+1)/2 \leq -1 - \mu - \alpha/2$ which is equivalent to $\mu \leq -1/4$ or $\gamma \leq 3p/4 - 1/2 + \alpha p/2$. But this is no further restriction since for $\alpha \leq 1/2$ there holds

$\alpha + p - 1 \le 3p/4 - 1/2 + \alpha p/2$. Summarizing, for $-1 < \alpha \le 1/2$, $\gamma \le \alpha + p - 1$ and $\mu = (\gamma/p - (\alpha+1)/2)/2(1/p - 1/2)$ we have that

$$\sup_k |(k+1)^{1+\mu+\alpha/2} \Delta_2 \hat{f}_\alpha(k)| \le C \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt. \tag{8}$$

Now consider the case $\alpha > 1/2$. Then, by formula (2.12) in [6], (7) is obviously true when $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$. Again, the application of Lemma 1 in [5] requires $\gamma \le \alpha + p - 1$, which for $\alpha > 1/2$ is less restrictive than $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$. Its 4th case now leads to

$$I \le C(k+1)^{-11/6-\mu-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \qquad \frac{\gamma+1}{p} \le \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3},$$

and its 5th case to

$$I \le C(k+1)^{\mu-1/2-\alpha/2} \int_0^\infty |f(t)| e^{-t/2} t^{(\alpha+1)/2+\mu}\, dt, \quad (\gamma+1)/p > \frac{\alpha+1}{2} - \frac{1}{3p} + \frac{2}{3}.$$

But $\mu - 1/2 - \alpha/2 \le -\mu - 1 - \alpha/2$ if $(\gamma+1)/p \le \alpha/2 + 1 + (1/p - 1/2)/2$; so that, summarizing, (8) also holds under this restriction for $\alpha > 1/2$.

Now an application of the Stein and Weiss interpolation theorem (see [7]) with $Tf = \{Tf(k)\}$and $Tf(k) = \sqrt{A_k^{\alpha+1}} \Delta_2 \hat{f}_\alpha(k)$ gives the assertion of Theorem 1.1 in the case $a = 0$.

If $a \ne 0$ then by (1), the definition of $\Delta_2 \Delta^a$, and by (5)

$$\Delta_2 \Delta^a \hat{f}_\alpha(k) = C\{\Delta \hat{f}_{\alpha+a}(k) + \Delta \hat{f}_{\alpha+a}(k+1)\} = C \Delta_2 \hat{f}_{\alpha+a}(k),$$

since already the condition $\gamma < \alpha + a + 1$ (which implies no new restriction) gives absolute convergence of the infinite sum and integral involved (see the formula following (9) in [3]) and Fubini's Theorem can be applied. Hence all the previous estimates remain valid when α is replaced by $\alpha + a$.

3. A variant for integrable functions. Theorem 1.1 gives a necessary condition for a sequence $\{f_k\}$ to generate with respect to L_k^α an $L^1_{w(\gamma)}$–function. But this condition is hardly comparable with the following sufficient one which is a slight modification of Lemma 2.2 in [3].

THEOREM 3.1. *Let $\alpha > -1$ and $\delta > 2\gamma - \alpha + 1/2 \ge 0$. If $\{f_k\}$ is a bounded sequence with $\lim_{k\to\infty} f_k = 0$ and*

$$\sum_{k=0}^\infty (k+1)^{\delta+\alpha-\gamma} |\Delta^{\delta+1} f_k| \le K_{\{f_k\}},$$

then there exists a function $f \in L^1_{w(\gamma)}$ with $\hat{f}_\alpha(k) = f_k$ for all $k \in \mathbf{N}_0$ and

$$\| f \|_{L^1_{w(\gamma)}} \le C\, K_{\{f_k\}}$$

for some constant C independent of the sequence $\{f_k\}$.

The proof follows along the lines of Lemma 2.2 in [3] since the norm of the Cesàro kernel

$$\chi_n^{\alpha,\delta}(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} \sum_{k=0}^{n} A_{n-k}^\delta L_k^\alpha(x) = (A_n^\delta \Gamma(\alpha+1))^{-1} L_n^{\alpha+\delta+1}(x)$$

can be estimated with the aid of Lemma 1 in [5] by

$$\| \chi_k^{\alpha,\delta} \|_{L^1_{w(\gamma)}} \leq C(k+1)^{\alpha-\gamma}, \quad \delta > 2\gamma - \alpha + 1/2.$$

The variant of Theorem 1.1 in the case $p = 1$ is

THEOREM 3.2. *If* $\alpha > -1$ *and* $\gamma > \max\{-1/3, \alpha/2 - 1/6\}$*, then*

$$\sum_{k=0}^{\infty} (k+1)^{\gamma-2/3} |\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k)| \leq C \| f \|_{L^1_{w(\gamma)}} .$$

A comparison of the sufficient condition and the necessary one nicely shows where the $L^1_{w(\gamma)}$–functions live; in particular we see that the "smoothness" gap (the difference of the orders of the difference operators) is just greater than 7/6. It is clear that Theorem 3.2 can be modified by using the Δ_2–operator. Theorem 3.2 does not follow from the $p = 1$ case of Lemma 2.1 in [3] since that estimate would lead to the divergent sum $\sum_{k=0}^{\infty}(k+1)^{-1} \| f \|_{L^1_{w(\gamma)}}$.

Proof

By formula (5) we have

$$\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k) = C \int_0^\infty f(t) R_k^{2\gamma+1/3}(t) t^{2\gamma+1/3} e^{-t}\, dt$$

$$= C(k+1)^{-\gamma-1/6} \int_0^\infty f(t) \mathcal{L}_k^{2\gamma+1/3}(t) t^{\gamma+1/6} e^{-t/2}\, dt$$

and hence

$$\sum_{k=0}^{\infty} (k+1)^{\gamma-2/3} |\Delta^{2\gamma-\alpha+1/3} \hat{f}_\alpha(k)|$$

$$\leq C \int_0^\infty |f(t)| \sum_{k=0}^{\infty} (k+1)^{-5/6} |t^{1/6} \mathcal{L}_k^{2\gamma+1/3}(t)| t^\gamma e^{-t/2}\, dt$$

if the right hand side converges. To show this we discuss for $j \in \mathbf{Z}$

$$\sup_{2^j \leq t \leq 2^{j+1}} \sum_{k=0}^{\infty} (k+1)^{-5/6} |t^{1/6} \mathcal{L}_k^{2\gamma+1/3}(t)|$$

and prove that this quantity is uniformly bounded in j, whence the assertion.

First consider those $j \geq 0$ for which there exists a nonnegative integer n such that $0 \leq k \leq 2^n$ implies $3\nu/2 := 3(2k + 2\gamma + 4/3) \leq 2^j$ but such that this inequality fails to hold for $k \geq 2^{n+1}$; the latter assumption in particular implies that essentially $\nu/2 \geq 2^{j+1}$ for $k \geq 2^{n+4}$. Since $\| t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t) \|_\infty \leq C(k+1)^{-1/6}$ by Lemma 1 in [5], we obviously have

$$(9) \qquad \sum_{k=0}^{\infty}(k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq \left(\sum_{k=0}^{2^n} + \sum_{k=2^{n+4}}^{\infty}\right)\ldots + O(1).$$

For $k = 0, \ldots, 2^n$ we can now apply the fourth case of formula (2.5) in [5] to obtain $|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ce^{-\mu 2^j}$ for some positive constant μ and the first sum on the right hand side of (9) is bounded uniformly in j. In consequence of the choice of n the second case of formula (2.5) in [5] can be used for $k \geq 2^{n+4}$, giving

$$\sum_{k=2^{n+4}}^{\infty} (k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ct^{-1/12} \sum_{k=2^{n+4}}^{\infty} (k+1)^{-13/12} = O(1)$$

since $2^j \leq t \leq 2^{j+1}$ and j and n are comparable.
Now consider the remaining j's: We have to split up the sum $\sum_{k=0}^{\infty}\ldots$ into two parts, one where k is such that $2^j\nu \geq 1$ (this contribution has just been seen to be uniformly bounded in j), the other where k is such that $2^j\nu \leq 1$. To deal with the last case choose again n to be the greatest integer such that $2^{n+2} + 4\gamma + 8/3 \leq 2^{-j}$; this time, n and $-j$ are comparable and we obtain by the first case of (2.5) in [5]

$$\sum_{k=0}^{2^n}(k+1)^{-5/6}|t^{1/6}\mathcal{L}_k^{2\gamma+1/3}(t)| \leq Ct^{\gamma+1/3}\sum_{k=0}^{2^n}(k+1)^{\gamma-2/3} = O(1)$$

if $2^j \leq t \leq 2^{j+1}$, $\gamma > -1/3$, which completes the proof.

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