q-Analogues of Some Multivariable Biorthogonal Polynomials

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Abstract

In 1989 M.V. Tratnik found a pair of multivariable biorthogonal polynomials $P_{\mathbf{n}}(\mathbf{x})$ and $\bar{P}_{\mathbf{m}}(\mathbf{x})$, which is not necessarily the complex conjugate of $P_{\mathbf{m}}(\mathbf{x})$, such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w(\mathbf{x}) P_{\mathbf{n}}(\mathbf{x}) \bar{P}_{\mathbf{m}}(\mathbf{x}) \prod_{j=1}^{p} dx_j = \mu_{\mathbf{n},\mathbf{m}} \delta_{N,M},$$

where $\mathbf{x} = (x_1, \ldots, x_p)$, $\mathbf{n} = (n_1, \ldots, n_p)$, $\mathbf{m} = (m_1, \ldots, m_p)$, $N = \sum_{j=1}^{p} n_j$, $M = \sum_{j=1}^{p} m_j$, $\mu_{\mathbf{n},\mathbf{m}}$ is the constant of biorthogonality (which Tratnik did not evaluate),

$$w(\mathbf{x}) = \Gamma(A - iX)\Gamma(B + iX) \left| \frac{\Gamma(c + iX)\Gamma(d + iX)}{\Gamma(2iX)} \right|^2 \prod_{k=1}^{p} \Gamma(a_k + ix_k)\Gamma(b_k - ix_k),$$

$$X = \sum_{k=1}^{p} x_k, \quad A = \sum_{k=1}^{p} a_k, \quad B = \sum_{k=1}^{p} b_k,$$

and the a's, b's, x's, c and d are real. In the q-case we find that the appropriate weight function is a product of a multivariable version of the integrand in the Askey-Roy integral and of the Askey-Wilson weight function in a single variable that depends on $x_1, \ldots, x_p$.

In a related problem we find a discrete 2-variable Racah type biorthogonality:

$$\sum_{x=0}^{N} \sum_{y=0}^{N} w_N(x,y) F_{m,n}(x,y) G_{m',n'}(x,y) = \nu_{m,n} \delta_{m,m'} \delta_{n,n'},$$

where

$$\begin{aligned} w_N(x,y) = {} & \frac{(\alpha q/\gamma\gamma',\ \gamma'/c,\ \alpha c q/\gamma';\ q)_N}{(\alpha q,\ 1/c,\ \alpha c q/\gamma\gamma';\ q)_N} \\ & \times \frac{\left(1 - \frac{\gamma\gamma' q^{2x-N-1}}{\alpha c}\right)(1 - cq^{2y-N}) \left(\frac{\gamma\gamma' q^{-N-1}}{\alpha c}, \gamma; q\right)_x (cq^{-N}, \gamma'; q)_y}{\left(1 - \frac{\gamma\gamma' q^{-N-1}}{\alpha c}\right)(1 - cq^{-N}) \left(q, \frac{\gamma' q^{-N}}{\alpha c}; q\right)_x \left(q, \frac{cq^{1-N}}{\gamma'}; q\right)_y} \\ & \times \frac{(1/c; q)_{x-y} (q^{-N}; q)_{x+y}}{\left(\frac{\gamma\gamma'}{\alpha c}; q\right)_{x-y} \left(\frac{\gamma\gamma' q^{-N}}{\alpha}; q\right)_{x+y}} \alpha^{-x} (\gamma')^{x-y}, \end{aligned}$$

and $F_{m,n}(x,y)$, $G_{m',n'}(x,y)$ are certain bivariate extensions of the q-Racah polynomials.

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1 Introduction

Wilson polynomials [13], defined by

$$(1.1)\qquad P_n(x) = (a+b)_n(a+c)_n(a+d)_n\ {}_4F_3\left[\begin{matrix} -n, n+a+b+c+d-1, a-ix, a+ix \\ a+b, a+c, a+d \end{matrix};1\right]$$

satisfy an orthogonality relation on the real line

$$(1.2)\qquad \int_{-\infty}^{\infty} P_n(x)P_m(x)w(x)dx = h_n\delta_{n,m},$$

where

$$(1.3)\qquad w(x) = \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2$$

is the positive weight function (under the assumption that a, b, c, d are real or occur in complex conjugate pairs), and

$$(1.4)\ h_n = 4\pi n!\,(n+a+b+c+d-1)_n \frac{\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}$$

is the normalization constant. By Whipple's transformation it is easy to see that $P_n(x)$ is symmetric in a, b, c, d, and that

$$\begin{aligned}(1.5)\quad P_n(x) &= (a+b)_n(c-ix)_n(d-ix)_n\ {}_4F_3\left[\begin{matrix} -n, 1-c-d-n, a+ix, b+ix \\ a+b, 1-c-n+ix, 1-d-n+ix \end{matrix};1\right] \\ &= (b+a)_n(c+ix)_n(d+ix)_n\ {}_4F_3\left[\begin{matrix} -n, 1-c-d-n, a-ix, b-ix \\ a+b, 1-c-n-ix, 1-d-n-ix \end{matrix};1\right].\end{aligned}$$

Corresponding to each of these forms M.V. Tratnik [10] introduced a multivariable polynomial:

$$\begin{aligned}(1.6)\qquad P_n(\mathbf{x}) &= (A+c)_N(A+d)_N\prod_{k=1}^{p}(a_k+b_k)_{n_k} \\ &\times\sum_{\mathbf{j}}\frac{(N+A+B+c+d-1)_J(A-iX)_J}{(A+c)_J(A+d)_J}\prod_{k=1}^{p}\frac{(-n_k)_{j_k}(a_k+ix_k)_{j_k}}{(a_k+b_k)_{j_k}j_k!},\end{aligned}$$

$$\begin{aligned}(1.7)\qquad \bar{P}_n(\mathbf{x}) &= (B+c)_N(B+d)_N\prod_{k=1}^{p}(b_k+a_k)_{n_k} \\ &\times\sum_{\mathbf{j}}\frac{(N+A+B+c+d-1)_J(B+iX)_J}{(B+c)_J(B+d)_J}\prod_{k=1}^{p}\frac{(-n_k)_{j_k}(b_k-ix_k)_{j_k}}{(b_k+a_k)_{j_k}j_k!},\end{aligned}$$

$$\begin{aligned}(1.8)\qquad Q_{\mathbf{n}}(\mathbf{x}) &= (c-iX)_N(d-iX)_N\prod_{k=1}^{p}(a_k+b_k)_{n_k} \\ &\times\sum_{\mathbf{j}}\frac{(1-c-d-N)_J(B+iX)_J}{(1-c-N+iX)_J(1-d-N+iX)_J}\prod_{k=1}^{p}\frac{(-n_k)_{j_k}(a_k+ix_k)_{j_k}}{(a_k+b_k)_{j_k}j_k!},\end{aligned}$$

$$\bar{Q}_{\mathbf{n}}(\mathbf{x}) = (c+iX)_N(d+iX)_N \prod_{k=1}^{p}(b_k+a_k)_{n_k} \times \sum_{\mathbf{j}} \frac{(1-c-d-N)_J(A-iX)_J}{(1-c-N-iX)_J(1-d-N-iX)_J} \prod_{k=1}^{p} \frac{(-n_k)_{j_k}(b_k-ix_k)_{j_k}}{(b_k+a_k)_{j_k} j_k!}, \tag{1.9}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, $\mathbf{n} = (n_1, n_2, \ldots, n_p)$, $\mathbf{j} = (j_1, j_2, \ldots, j_p)$, and $X = \sum_{k=1}^{p} x_k$, $N = \sum_{k=1}^{p} n_k$, $M = \sum_{k=1}^{p} m_k$, $A = \sum_{k=1}^{p} a_k$, $B = \sum_{k=1}^{p} b_k$, $J = \sum_{k=1}^{p} j_k$, and the sums in (1.6)–(1.9) are from $j_k = 0$ to n_k, $k = 1, \ldots, p$. Each of the polynomials in (1.6)–(1.9) is of (total) degree $2N$ in the variables $x_1, x_2, \ldots, x_p$. The overbars in (1.7), (1.9), and in (1.21) below are used to denote distinct systems of polynomials and should not be confused with complex conjugation. Tratnik proved that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_{\mathbf{n}}(\mathbf{x})\bar{P}_{\mathbf{m}}(\mathbf{x})w(\mathbf{x}) \prod_{k=1}^{p} dx_k = 0, \quad \text{if } N \neq M, \tag{1.10}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Q_{\mathbf{n}}(\mathbf{x})\bar{Q}_{\mathbf{m}}(\mathbf{x})w(\mathbf{x}) \prod_{k=1}^{p} dx_k = 0, \quad \text{if } N \neq M, \tag{1.11}$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_{\mathbf{n}}(\mathbf{x})Q_{\mathbf{m}}(\mathbf{x})w(\mathbf{x}) \prod_{k=1}^{p} dx_k = 0, \quad \text{if } \mathbf{n} \neq \mathbf{m}, \tag{1.12}$$

and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \bar{P}_{\mathbf{n}}(\mathbf{x})\bar{Q}_{\mathbf{m}}(\mathbf{x})w(\mathbf{x}) \prod_{k=1}^{p} dx_k = 0, \quad \text{if } \mathbf{n} \neq \mathbf{m}, \tag{1.13}$$

where

$$w(\mathbf{x}) = \left|\frac{\Gamma(c+iX)\Gamma(d+iX)}{\Gamma(2iX)}\right|^2 \Gamma(A-iX)\Gamma(B+iX) \prod_{k=1}^{p} \Gamma(a+ix_k)\Gamma(b-ix_k). \tag{1.14}$$

Note that in (1.12) and (1.13) the biorthogonality holds in all of the indices $n_1, n_2, \ldots, n_p$, while in (1.10) and (1.11) the biorthogonality is for polynomials of different degrees ($N \neq M$).

Since Whipple's ${}_4F_3$ transformation does not apply for $p \geq 2$ the P's and Q's are no longer equivalent and hence the orthogonality in a single variable becomes biorthogonality in many variables.

We were curious to see what their q-analogues would be. At first sight it might appear that they could be found in a pretty straightforward manner. We were in for a surprise. The first hurdle is an appropriate analogue of the weight function in (1.14). There are many possible candidates but the one that works for a q-analogue of (1.10) is:

$$w^{(p)}(\mathbf{x};q) := \frac{1}{(2\pi)^p} \frac{(e^{2i\Theta}, e^{-2i\Theta};q)_\infty}{(Ae^{-i\Theta}, Be^{i\Theta};q)_\infty h(\cos\Theta;c,d;q)\left(\frac{\beta b_1}{B}e^{i\Theta}, \frac{qB}{\beta b_1}e^{-i\Theta};q\right)_\infty} \times \prod_{k=1}^{p} \frac{(\beta_k e^{i\theta_k}, q\beta_k^{-1}e^{-i\theta_k};q)_\infty}{(a_k e^{i\theta_k}, b_k e^{-i\theta_k};q)_\infty}, \quad p \geq 2, \tag{1.15}$$

where $-\pi \le \theta_k \le \pi$, $\theta_k = x_k \log q$ so that $e^{i\theta_k} = q^{ix_k}$ for $k = 1, \dots, p$, $\Theta = \sum_{j=1}^p \theta_j$, $A = \prod_{j=1}^p a_j$, $B = \prod_{j=1}^p b_j$, $h(\cos\Theta; c, d; q)$ is defined as in [2, (6.1.2)], β is an arbitrary complex parameter such that $\beta \ne q^{\pm n}$ for $n = 0, 1, \dots$, and

$$\beta_{k+1} = \frac{\beta_k}{a_k b_{k+1}}, \quad k = 1, 2, \dots, p-1, \tag{1.16}$$

with $\beta_1 = \beta$. By making repeated use of the Askey-Roy integral [2, (4.11.1)] followed by the use of the Askey-Wilson integral, we shall prove in section 2 that

$$\begin{aligned} W^{(p)}(q) &:= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} w^{(p)}(\mathbf{x}; q) \prod_{k=1}^p d\theta_k \\ &= \frac{2(ABcd; q)_\infty \prod_{k=2}^p (b_k\beta_k, q/b_k\beta_k; q)_\infty}{(q;q)_\infty^p (Ac, Ad, Bc, Bd, cd; q)_\infty \prod_{k=1}^p (a_k b_k; q)_\infty}, \end{aligned} \tag{1.17}$$

which is also valid for $p = 1$. It is understood that the $(p-2)$-fold product in the numerator is taken to be 1 when $p = 1$.

Let

$$\begin{aligned} &A_j = \prod_{k=j}^p a_k,\ B_j = \prod_{k=j}^p b_k,\ J_j = \sum_{k=j}^p j_k,\ K_j = \sum_{r=j}^p k_r, \\ &N_j = \sum_{k=j}^p n_k,\ M_j = \sum_{k=j}^p m_k,\ \Theta_j = \sum_{k=j}^p \theta_k, \end{aligned} \tag{1.18}$$

so that

$$A_1 = A,\ B_1 = B,\ J_1 = J,\ K_1 = K,\ N_1 = N,\ M_1 = M,\ \Theta_1 = \Theta. \tag{1.19}$$

Analogous to Tratnik's polynomials in (1.6) and (1.7) we introduce the functions

$$\begin{aligned} P_{\mathbf{n}}(\mathbf{x}; q) &= (Ac, Ad; q)_N \prod_{k=1}^p (a_k b_k; q)_{n_k} \\ &\times \sum_{\mathbf{j}} \frac{(ABcdq^{N-1}, Ae^{-i\Theta}; q)_J}{(Ac, Ad; q)_J} q^J \prod_{k=1}^p \frac{(q^{-n_k}, a_k e^{i\theta_k}; q)_{j_k}}{(q, a_k b_k; q)_{j_k}} \\ &\times \frac{e^{i(j_1\Theta_2 + \cdots + j_{p-1}\Theta_p)}}{B_2^{j_1} \cdots B_p^{j_{p-1}}} q^{-(N_2 j_1 + N_3 j_2 + \cdots + N_p j_{p-1})}, \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} \bar{P}_{\mathbf{m}}(\mathbf{x}; q) &= (Bc, Bd; q)_M \prod_{k=1}^p (a_k b_k; q)_{m_k} \\ &\times \sum_{\mathbf{k}} \frac{(ABcdq^{M-1}, Be^{i\Theta}; q)_K}{(Bc, Bd; q)_K} q^K \prod_{r=1}^p \frac{(q^{-m_r}, b_r e^{-i\theta_r}; q)_{k_r}}{(q, a_r b_r; q)_{k_r}} \\ &\times \frac{e^{i(k_2(\Theta_2 - \Theta) + \cdots + k_p(\Theta_p - \Theta))}}{a_1^{K_2} a_2^{K_3} \cdots a_{p-1}^{K_p}} q^{-\sum_{r=2}^p k_r (M - M_r)}, \end{aligned} \tag{1.21}$$

Both $P_{\mathbf{n}}(\mathbf{x};q)$ and $\bar{P}_{\mathbf{m}}(\mathbf{x};q)$ are Laurent polynomials in the variables $q^{ix_1},\dots,q^{ix_p}$. Note that if we divide $P_{\mathbf{n}}(\mathbf{x};q)$ by $(1-q)^{3N}$ and replace its parameters $a_1,\dots,a_p,b_1,\dots,b_p,c,d$, respectively, by $q^{a_1},\dots,q^{a_p},q^{b_1},\dots,q^{b_p},q^c,q^d$, and then let $q\to 1$, we obtain $P_{\mathbf{n}}(\mathbf{x})$ as a limit case. Similarly, we see that $\bar{P}_{\mathbf{m}}(\mathbf{x})$ is limit case of $\bar{P}_{\mathbf{m}}(\mathbf{x};q)$. In section 3 we shall do the integration and in section 4 prove the following q-analogue of (1.10):

$$(1.22)\qquad P_{\mathbf{n}}\cdot\bar{P}_{\mathbf{m}}:=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}P_{\mathbf{n}}(\mathbf{x};q)\bar{P}_{\mathbf{m}}(\mathbf{x};q)\,w^{(p)}(\mathbf{x};q)\prod_{k=1}^{p}d\theta_k=0,\quad\text{if } N\neq M,$$

where $w^{(p)}(\mathbf{x};q)$ is given by (1.15), and

$$(1.23)\qquad \begin{aligned}P_{\mathbf{n}}\cdot\bar{P}_{\mathbf{m}} &= L_p\sum_{k_1=0}^{m_1}\cdots\sum_{k_{p-1}=0}^{m_{p-1}}q^{\sum_{j=1}^{p-1}k_j\sum_{r=0}^{j-1}(n_r-m_r)}\\ &\quad\times\frac{\left(ABcdq^{N-1},\frac{ABcdq^{N}}{a_pb_p};q\right)_{k_1+\cdots+k_{p-1}}}{\left(ABcdq^{N+m_p},\frac{ABcdq^{N-n_p}}{a_pb_p};q\right)_{k_1+\cdots+k_{p-1}}}\prod_{r=1}^{p-1}\frac{(q^{-m_r},a_rb_rq^{n_r};q)_{k_r}}{(q,a_rb_r;q)_{k_r}},\end{aligned}$$

when $N=M$, with $n_0=1$ and $m_0=0$, and L_p is as defined in (3.7).

Discrete multivariable extensions of the Racah polynomials were considered in Tratnik [12] as well as in van Diejen and Stokman [1] and in Gustafson [5]. For other related works see, for instance, [4, 6, 9, 11]. We have found q-extensions of Tratnik's systems of multivariable Racah and Wilson polynomials, complete with their orthogonality relations, see this Proceedings [3] for our multivariable extension of the Askey-Wilson polynomials. However, there seems to be at least one more extension that, to our knowledge, has not yet been investigated. The seed of this extension lies in Rosengren's [8] multivariable extension of the q-Hahn polynomials as well as in Rahman's [7] 2-variable discrete biorthogonal system. In sections 5 and 6 we shall prove the following 2-variable extension of the q-Racah polynomial orthogonality [2, (7.2.18)]:

$$(1.24)\qquad \sum_{x=0}^{N}\sum_{y=0}^{N}w_N(x,y)F_{m,n}(x,y)G_{m',n'}(x,y)=\nu_{m,n}\delta_{m,m'}\delta_{n,n'},$$

where $0\le m,n,m',n'\le N$,

$$(1.25)\qquad \begin{aligned}&F_{m,n}(x,y)\\ &=\frac{\left(\frac{\alpha q^{N+1-x-y}}{\gamma\gamma'};q\right)_{m+n}(q^{x-y}/c;q)_n(\alpha cq^{1+y-x}/\gamma\gamma';q)_m}{(q^{-N};q)_{m+n}(\alpha cq/\gamma\gamma';q)_n(1/c;q)_m}c^{n-m}q^{mx+ny}\\ &\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m},\gamma q^x,\gamma\gamma'q^{x-N-1}/\alpha c;q)_i(q^{-n},\gamma'q^y,cq^{y-N};q)_j(\gamma\gamma'q^{-M-n}/\alpha;q)_{i+j}}{(q,\gamma,\gamma\gamma'q^{x-y-m}/\alpha c;q)_i(q,\gamma',cq^{1+y-x-n};q)_j(\gamma\gamma'q^{x+y-N-m-n}/\alpha;q)_{i+j}}q^{i+j},\end{aligned}$$

$$(1.26)\qquad \begin{aligned}&G_{m,n}(x,y)\\ &=\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(q^{-m},q^{-x},\gamma\gamma'q^{x-N-1}/\alpha c;q)_i(q^{-n},q^{-y},cq^{y-N};q)_j(\alpha q^{m+n};q)_{i+j}}{(q,\gamma,\gamma'q^n/c;q)_i(q,\gamma',\alpha cq^{m+1}/\gamma';q)_j(q^{-N};q)_{i+j}}q^{i+j},\end{aligned}$$

and the weight function is

$$(1.27)\quad w_N(x,y) = \frac{(\alpha q/\gamma\gamma', \gamma'/c, \alpha cq/\gamma'; q)_N}{(\alpha q, 1/c, \alpha cq/\gamma\gamma'; q)_N} \times \frac{(1-\gamma\gamma' q^{2x-N-1}/\alpha c)(1-cq^{2y-N})(\gamma\gamma' q^{-N-1}/\alpha c, \gamma; q)_x (cq^{-N}, \gamma'; q)_y}{(1-\gamma\gamma' q^{-N-1}/\alpha c)(1-cq^{-N})(q, \gamma' q^{-N}/\alpha c; q)_x (q, cq^{1-N}/\gamma'; q)_y} \times \frac{(1/c;q)_{x-y}(q^{-N};q)_{x+y}}{(\gamma\gamma'/\alpha c; q)_{x-y}(\gamma\gamma' q^{-N}/\alpha; q)_{x+y}} \alpha^{-x}(\gamma')^{x-y}.$$

The normalization constant in (1.24) is given by

$$(1.28)\qquad \nu_{m,n} = \frac{1-\alpha}{1-\alpha q^{2m+2n}} \frac{(q, \alpha cq/\gamma'; q)_m (q, \gamma'/c; q)_n (\alpha q/\gamma\gamma', \alpha q^{N+1}; q)_{m+n}}{(\gamma, 1/c; q)_m (\gamma', \alpha cq/\gamma\gamma'; q)_n (\alpha, q^{-N}; q)_{m+n}} c^{n-m} q^{mn}.$$

Notice that both $F_{m,n}(x,y)$ and $G_{m,n}(x,y)$ are Laurent polynomials in the variables q^x and q^y, and $G_{m,n}(x,y)$ is a polynomial of (total) degree $n+m$ in the variables $q^{-x} + \gamma\gamma' q^{x-N-1}/\alpha c$ and $q^{-y} + cq^{y-N}$.

We wish to make the observation that the summation in (1.24) is over the square of length N, although the vanishing of the weight function above the main diagonal, because of the factor $(q^{-N};q)_{x+y}$ in the numerator, makes it effectively over the triangle $0 \le x+y \le N$. A very innocuous observation but it will help simplify the calculations somewhat as we shall see in section 6.

It seems reasonable to expect that there is a multivariable extension of (1.24), but we were unable to find it, mainly because an extension of the q-shifted factorials of the type $(a;q)_{x-y}$ doesn't appear too obvious to us.

2 Calculation of $W^{(p)}(q)$

The key to the proof of (1.17) is to observe that by periodicity we can change $\theta_1, \theta_2, \ldots, \theta_p$ to, say, $\Theta, \theta_2, \ldots, \theta_p$ (so that $\theta_1 = \Theta - \Theta_2$), with the limits of integration unchanged. So the total weight transforms to

$$(2.1)\qquad W^{(p)}(q) = \frac{1}{(2\pi)^{p-1}} \int_{-\pi}^{\pi} \frac{(e^{2i\Theta}, e^{-2i\Theta}; q)_\infty d\Theta}{(Ae^{-i\Theta}, Be^{i\Theta}; q)_\infty\, h(\cos\Theta; c, d; q) (\frac{\beta}{B_2} e^{i\Theta}, \frac{qB_2}{\beta} e^{-i\Theta}; q)_\infty} \times \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} I_2(\theta_3, \ldots, \theta_p) \prod_{k=3}^{p} \frac{(\beta_k e^{i\theta_k}, qe^{-i\theta_k}/\beta_k; q)_\infty}{(a_k e^{i\theta_k}, b_k e^{-i\theta_k}; q)_\infty} d\theta_3 \cdots d\theta_p,$$

where

$$(2.2)\qquad I_2(\theta_3, \ldots, \theta_p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\beta_2 e^{i\theta_2}, qe^{i(\Theta_3-\Theta)+i\theta_2}/\beta, qe^{-i\theta_2}/\beta_2, \beta e^{i(\Theta-\Theta_3)-i\theta_2}; q)_\infty}{(a_2 e^{i\theta_2}, b_1 e^{i(\Theta_3-\Theta)+i\theta_2}, b_2 e^{-i\theta_2}, a_1 e^{i(\Theta-\Theta_3)-i\theta_2}; q)_\infty} d\theta_2.$$

However, this integral matches exactly with the Askey-Roy integral [2, (4.11.1)], provided we assume that $\max(|a_1|, |b_1|, |a_2|, |b_2|) < 1$ (with, of course, $|q| < 1$). By [2, (4.11.1)], it then follows that

$$(2.3)\qquad I_2(\theta_3, \ldots, \theta_p) = \frac{(b_2\beta_2, q/b_2\beta_2, a_1a_2b_1b_2, a_1\beta_2 e^{i(\Theta-\Theta_3)}, qe^{i(\Theta_3-\Theta)}/a_1\beta_2; q)_\infty}{(q, a_1b_1, a_2b_2, a_1a_2 e^{i(\Theta-\Theta_3)}, b_1b_2 e^{i(\Theta_3-\Theta)}; q)_\infty}.$$

Substitution of (2.3) into (2.1) makes it clear that the integration over θ_4 presents exactly the same situation, and so does the remaining integrations up to and including θ_p. Finally, one is left with an Askey-Wilson integral over Θ:

$$\begin{aligned}(2.4)\qquad W^{(p)}(q) &= \frac{(AB;q)_\infty \prod_{k=2}^p (b_k\beta_k, q/b_k\beta_k;q)_\infty}{(q;q)_\infty^{p-1}\prod_{k=1}^p(a_kb_k;q)_\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(e^{2i\Theta},e^{-2i\Theta};q)_\infty}{h(\cos\Theta;\,A,B,c,d;q)}\,d\Theta \\ &= \frac{2(ABcd;q)_\infty\prod_{k=2}^p(b_k\beta_k,q/b_k\,\beta_k;q)_\infty}{(q;q)_\infty^p(Ac,Ad,Bc,Bd,cd;q)_\infty\prod_{k=1}^p(a_kb_k;q)_\infty},\end{aligned}$$

by [2,(6.1.1)], which completes the proof of (1.17).

3 Computation of the integral in (1.22)

We shall carry out the integrations in (1.22) in much the same way as we did in the previous section. We transform the integration variables $\theta_1,\dots,\theta_p$ to $\theta_2,\dots,\theta_p$ and Θ as before; then we isolate the θ_2-integral by observing that the factors $(a_1e^{i(\Theta-\Theta_3)-i\theta_2};q)_{j_1}\,(a_2e^{i\theta_2};q)_{j_2}\,(b_1e^{i(\Theta_3-\Theta)+i\theta_2};q)_{k_1}$ $\times(b_2e^{-i\theta_2};q)_{k_2}\,e^{i\theta_2(j_1+k_2)+ik_2(\Theta_3-\Theta)+ij_1\Theta_3}$ can be glued on to the integrand of $W^{(p)}(q)$, to get

$$(-\beta)^{j_1+k_2}q^{\binom{j_1+k_2}{2}}e^{ij_1\Theta}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\beta_2e^{i\theta_2},\,q^{1-j_1-k_2}\,e^{i(\Theta_3-\Theta)+i\theta_2}/\beta,\beta q^{j_1+k_2}e^{i(\Theta-\Theta_3)-i\theta_2},qe^{-i\theta_2}/\beta_2;q)_\infty}{(a_2q^{j_2}e^{i\theta_2},b_1q^{k_1}e^{i(\Theta_3-\Theta)+i\theta_2},b_2q^{k_2}e^{-i\theta_2},a_1q^{j_1}e^{i(\Theta-\Theta_3)-i\theta_2};q)_\infty}d\theta_2$$

which via [2, (4.11.1)] equals, on a bit of simplification,

$$\begin{aligned}(3.1)\qquad & a_1^{k_2}b_2^{j_1}q^{j_1k_2}e^{ij_1\Theta_3}\frac{(b_2\beta_2,q/b_2\beta_2,a_1a_2b_1b_2q^{j_1+j_2+k_1+k_2};q)_\infty}{(q,a_1b_1q^{j_1+k_1},a_2b_2q^{j_2+k_2};q)_\infty} \\ &\times\frac{(a_1\beta_2e^{i(\Theta-\Theta_3)},qe^{i(\Theta_3-\Theta)}/a_1\beta_2;q)_\infty}{(a_1a_2q^{j_1+j_2}e^{i(\Theta-\Theta_3)},b_1b_2q^{k_1+k_2}e^{i(\Theta_3-\Theta)};q)_\infty}.\end{aligned}$$

Since $\Theta_3=\theta_3+\Theta_4$, we may now isolate the θ_3–integral in exactly the same way, carry out a similar integration, simplify, and obtain

$$\begin{aligned}(3.2)\qquad & a_1^{k_2}(a_1a_2)^{k_3}(b_2b_3)^{j_1}b_3^{j_2}e^{i(j_1+j_2)\Theta_4}q^{j_1k_2+(j_1+j_2)k_3} \\ &\times\frac{\left(b_2\beta_2,q/b_2\beta_2,b_3\beta_3,q/b_3\beta_3,a_1a_2a_3b_1b_2b_3q^{j_1+j_2+j_3+k_1+k_2+k_3};q\right)_\infty}{\left(q,q,a_1b_1q^{j_1+k_1},a_2b_2q^{j_2+k_2},a_3b_3q^{j_3+k_3};q\right)_\infty} \\ &\times\frac{\left(a_1a_2\beta_3e^{i(\Theta-\Theta_4)},qe^{i(\Theta_4-\Theta)}/a_1a_2\beta_3;q\right)_\infty}{\left(a_1a_2a_3q^{j_1+j_2+j_3}\,e^{i(\Theta-\Theta_4)},b_1b_2b_3q^{k_1+k_2+k_3}\,e^{i(\Theta_4-\Theta)};q\right)_\infty}.\end{aligned}$$

A clear pattern is now emerging. The θ_p integral is

$$\begin{aligned}(3.3)\qquad & q^{j_1k_2+(j_1+j_2)k_3+\cdots+(j_1+\cdots+j_{p-2})k_{p-1}}(a_1^{k_2+\cdots+k_{p-1}}a_2^{k_3+\cdots+k_{p-1}}\cdots a_{p-2}^{k_{p-1}})(b_2^{j_1}\,b_3^{j_1+j_2}\cdots b_{p-1}^{j_1+\cdots+j_{p-2}}) \\ &\times\frac{(\frac{AB}{a_pb_p}q^{J+K-j_p-k_p};q)_\infty\prod_{r=2}^{p-1}(b_r\beta_r,q/b_r\beta_r;q)_\infty}{(q;q)_\infty^{p-2}\prod_{r=1}^{p-1}(a_rb_rq^{j_r+k_r};q)_\infty} \\ &\times\left[\frac{e^{-i\Theta k_p}}{2\pi}\int_{-\pi}^{\pi}\frac{\left(\beta_pe^{i\theta_p},\frac{qe^{i(\theta_p-\Theta)}}{(a_1\cdots a_{p-2})\beta_{p-1}},\,qe^{-i\theta_p}/\beta_p,\,(a_1\cdots a_{p-2})\beta_{p-1}e^{i(\Theta-\theta_p)};q\right)_\infty}{\left(a_pq^{j_p}e^{i\theta_p},\frac{B}{b_p}q^{K-k_p}e^{i(\theta_p-\Theta)},b_pq^{k_p}e^{-i\theta_p},\frac{A}{a_p}q^{J-j_p}e^{i(\Theta-\theta_p)};q\right)_\infty}e^{i\theta_p(J-j_p+k_p)}d\theta_p\right].\end{aligned}$$

The expression in [ ] above can, once again, be computed by use of [2, (4.11.1)], and simplified to

$$\left(\frac{Aq^{J-j_p}}{a_p}\right)^{k_p} \frac{b_p^{J-j_p}\left(b_p\beta_p, q/b_p\beta_p, \frac{A\beta_p e^{i\Theta}}{a_p}, \frac{qa_p}{A\beta_p}e^{-i\Theta}, ABq^{J+K}; q\right)_\infty}{\left(q, a_pb_pq^{j_p+k_p}, Aq^Je^{i\Theta}, Bq^Ke^{-i\Theta}, \frac{AB}{a_pb_p}q^{J+K-j_p-k_p}; q\right)_\infty}. \tag{3.4}$$

Since, by repeated application of (1.16) we get $A\beta_p/a_p = \beta b_1/B$, the Θ-integral simply becomes the Askey-Wilson integral

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(e^{2i\Theta}, e^{-2i\Theta}; q\right)_\infty}{h(\cos\Theta; Aq^J, Bq^K, c, d; q)}\, d\Theta \\ &\quad = \frac{2(ABcd\, q^{J+K}; q)_\infty}{(q, cd, ABq^{J+K}, Acq^J, Adq^J, Bcq^K, Bdq^K; q)_\infty}.\end{aligned} \tag{3.5}$$

Collecting these results and substituting into the integral in (1.22), we find that

$$\begin{aligned} P_{\mathbf{n}}\cdot \bar{P}_{\mathbf{m}} &= L_p \sum_{\mathbf{j}}\sum_{\mathbf{k}} \frac{(ABcdq^{N-1};q)_J(ABcdq^{M-1};q)_K}{(ABcd;q)_{J+K}} q^{J+K} \\ &\quad\times \prod_{r=1}^{p} \frac{(q^{-n_r};q)_{j_r}(q^{-m_r};q)_{k_r}(a_rb_r;q)_{j_r+k_r}}{(q, a_rb_r;q)_{j_r}(q, a_rb_r;q)_{k_r}} \\ &\quad\times\ q^{\sum_{s=2}^{p}[j_{s-1}(K_s-N_s)+k_s(M_s-M)]}, \end{aligned} \tag{3.6}$$

where

$$L_p = (Ac, Ad; q)_N (Bc, Bd; q)_M W^{(p)}(q) \prod_{r=1}^{p} (a_rb_r;q)_{m_r}(a_rb_r;q)_{n_r}. \tag{3.7}$$

4 Biorthogonality

The sum over j_1 and k_1 in (3.6) gives

$$\begin{aligned}&\frac{(ABcdq^{N-1};q)_{J_2}(ABcdq^{M-1};q)_{K_2}}{(ABcd;q)_{J_2+K_2}} q^{J_2+K_2} \\ &\times \sum_{k_1=0}^{m_1} \frac{(q^{-m_1}, ABcdq^{M+K_2-1};q)_{k_1}}{(q, ABcdq^{J_2+K_2};q)_{k_1}} q^{k_1}\, {}_3\phi_2\left[\begin{matrix} q^{-n_1}, & ABcdq^{N+J_2-1}, & a_1b_1q^{k_1} \\ & ABcdq^{J_2+K_2+k_1}, & a_1b_1\end{matrix}; q, q^{1+K_2-N_2}\right].\end{aligned} \tag{4.1}$$

Since, by [2, (3.2.7)], the above ${}_3\phi_2$ equals

$$\frac{(ABcd;q)_{J_2+K_2+k_1}(q^{1+K_2-N};q)_{n_1}}{(ABcd;q)_{J_2+K_2+n_1}(q^{1+K_2-N})_{k_1}}\, {}_3\phi_2\left[\begin{matrix} q^{-k_1}, & ABcdq^{N+J_2-1}, & a_1b_1q^{n_1} \\ & ABcdq^{n_1+J_2+K_2}, & a_1b_1\end{matrix}; q, q^{1+K-N}\right],$$

we can now do the summation over k_1 via [2, (1.5.3)] to obtain that the expression in (4.1) reduces to

$$\begin{aligned}&\frac{(ABcdq^{N-1};q)_{J_2}(ABcdq^{M-1};q)_{K_2}(ABcdq^{N+M_2-1};q)_{m_1}(q^{1+K_2-N};q)_{n_1}}{(ABcd;q)_{n_1+J_2+K_2}(q^{1+K_2-N};q)_{m_1}} \\ &\times\ (-1)^{m_1}q^{\binom{m_1}{2}+(1+K_2-N)m_1+J_2+K_2} \\ &\times\ {}_4\phi_3\left[\begin{matrix} q^{-m_1}, & a_1b_1q^{n_1}, & ABcdq^{N+J_2-1}, & ABcdq^{M+K_2-1} \\ & a_1b_1, & ABcdq^{N+M_2-1}, & ABcdq^{n_1+J_2+K_2}\end{matrix}; q, q\right].\end{aligned} \tag{4.2}$$

Note that the ${}_4\phi_3$ series is balanced. Now, the sum over j_2 and k_2 gives

$$
\begin{aligned}
(4.3)\qquad & \frac{(ABcdq^{N-1};q)_{J_3}(ABcdq^{M-1};q)_{K_3}(ABcdq^{N+M_2-1};q)_{m_1}(q^{1+K_3-N};q)_{n_1}}{(ABcd;q)_{n_1+J_3+K_3}(q^{1+K_3-N};q)_{m_1}} \\
& \times (-1)^{m_1}q^{\binom{m_1}{2}+(1+K_3-N)m_1+J_3+K_3} \\
& \times \sum_{k_1=0}^{m_1} \frac{(q^{-m_1},a_1b_1q^{n_1},ABcdq^{N+J_3-1},ABcdq^{M+K_3-1};q)_{k_1}}{(q,a_1b_1,ABcdq^{n_1+J_3+K_3},ABcdq^{M_2+N-1};q)_{k_1}}q^{k_1} \\
& \times \sum_{k_2=0}^{m_2} \frac{(q^{-m_2},ABcdq^{M+K_3+k_1-1},q^{1+K_3-N_2};q)_{k_2}}{(q,ABcdq^{n_1+J_3+K_3+k_1},q^{1+K_3-N+m_1};q)_{k_2}}q^{k_2} \\
& \times {}_3\phi_2\left[\begin{matrix} q^{-n_2}, & ABcdq^{N+J_3+k_1-1}, & a_2b_2q^{k_2} \\ & ABcdq^{n_1+J_3+K}, & a_2b_2 \end{matrix};q,q^{1+K_3-N_3}\right].
\end{aligned}
$$

As in the previous step we apply [2, (3.2.7)] to the ${}_3\phi_2$ series above, use [2, (1.5.3)] to do the k_2 sum and simplify the coefficients to reduce (4.3) to the following expression

$$
\begin{aligned}
(4.4)\qquad & \frac{(ABcdq^{N-1};q)_{J_3}(ABcdq^{M-1};q)_{K_3}(ABcdq^{N+M_3-1};q)_{m_1+m_2}(q^{1+K_3-N};q)_{n_1+n_2}}{(ABcd;q)_{n_1+n_2+J_3+K_3}(q^{1+K_3-N};q)_{m_1+m_2}} \\
& \times (-1)^{m_1+m_2}q^{\binom{m_1+m_2}{2}+(1+K_3-N)(m_1+m_2)+J_3+K_3} \\
& \times \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2} \frac{(q^{-m_1},a_1b_1q^{n_1};q)_{k_1}(q^{-m_2},a_2b_2q^{n_2};q)_{k_2}}{(q,a_1b_1;q)_{k_1}(q,a_2b_2;q)_{k_2}}q^{(n_1-m_1)k_2} \\
& \times \frac{(ABcdq^{N+J_3-1},ABcdq^{M+K_3-1};q)_{k_1+k_2}}{(ABcdq^{n_1+n_2+J_3+K_3},ABcdq^{M_3+N-1};q)_{k_1+k_2}}q^{k_1+k_2}.
\end{aligned}
$$

A clear pattern of terms is now emerging, and by induction we find that at the $(p-1)$–th step the sum over $j_1,k_1,\dots,j_{p-1},k_{p-1}$ in (3.6) equals

$$
\begin{aligned}
(4.5)\qquad & \frac{(ABcdq^{N-1};q)_{J_p}(ABcdq^{M-1};q)_{K_p}(ABcdq^{N+M_p-1};q)_{M-m_p}(q^{1+K_p-N};q)_{N-n_p}}{(ABcd;q)_{N-N_p+J_p+K_p}(q^{1+K_p-N};q)_{M-m_p}} \\
& \times (-1)^{M-m_p}q^{\binom{M-m_p}{2}+(1+K_p-N)(M-m_p)+J_p+K_p} \\
& \times \sum_{k_1,\dots,k_{p-1}} \left[\prod_{r=1}^{p-1}\frac{(q^{-m_r},a_rb_rq^{n_r};q)_{k_r}}{(q,a_rb_r;q)_{k_r}}\right] \frac{(ABcdq^{N+J_p-1},ABcdq^{M+K_p-1};q)_{K-k_p}}{(ABcdq^{N-n_p+J_p+K_p},ABcdq^{M_p+N-1};q)_{K-k_p}} \\
& \times q^{k_1+k_2(1+n_1-m_1)+\cdots+k_{p-1}(1+n_1+\cdots+n_{p-2}-m_1-\cdots-m_{p-2})}.
\end{aligned}
$$

Using (4.5) we obtain that the sum over $\mathbf{j}$ and $\mathbf{k}$ in (3.6) equals

$$
\begin{aligned}
(4.6)\qquad & \frac{(ABcdq^{N+m_p-1};q)_{M-m_p}}{(ABcd;q)_{N-n_p}}(-1)^{M-m_p}q^{\binom{M-m_p}{2}+(1-N)(M-m_p)} \\
& \times \sum_{k_1=0}^{m_1}\cdots\sum_{k_{p-1}=0}^{m_{p-1}} q^{k_1+k_2(1+n_1-m_1)+\cdots+k_{p-1}(1+n_1+\cdots+n_{p-2}-m_1-\cdots-m_{p-2})} \\
& \times \left[\prod_{r=1}^{p-1}\frac{(q^{-m_r},a_rb_rq^{n_r};q)_{k_r}}{(q,a_rb_r;q)_{k_r}}\right] \frac{(ABcdq^{N-1},ABcdq^{M-1};q)_{k_1+\cdots+k_{p-1}}}{(ABcdq^{N-n_p},ABcdq^{N+m_p-1};q)_{k_1+\cdots+k_{p-1}}}S_p,
\end{aligned}
$$

where

$$S_p = \sum_{k_p=0}^{m_p} \frac{(q^{-m_p}, ABcdq^{M+k_1+\cdots+k_{p-1}-1}; q)_{k_p} (q^{1+M-N-m_p+k_p}; q)_\infty}{(q, ABcdq^{N-n_p+k_1+\cdots+k_{p-1}}; q)_{k_p} (q^{1+k_p-n_p}; q)_\infty} q^{k_p} \times {}_3\phi_2 \left[\begin{matrix} q^{-n_p}, & ABcdq^{N+k_1+\cdots+k_{p-1}-1}, & a_p b_p q^{k_p} \\ & ABcdq^{N-n_p+k_1+\cdots+k_{p-1}+k_p}, & a_p b_p \end{matrix} ; q, q \right]. \tag{4.7}$$

Note that the ${}_3\phi_2$ series is balanced, so by [2, (II.12)] it has the sum

$$\frac{(q^{1+k_p-n_p}, \frac{ABcd}{a_p b_p} q^{N-n_p+k_1+\cdots+k_{p-1}}; q)_{n_p}}{(\frac{q^{1-n_p}}{a_p b_p}, ABcdq^{N-n_p+K}; q)_{n_p}}.$$

Hence,

$$S_p = \frac{\left(\frac{ABcd}{a_p b_p} q^{N-n_p+k_1+\cdots+k_{p-1}}; q\right)_{n_p}}{\left(\frac{q^{1-n_p}}{a_p b_p}, ABcdq^{N+k_1+\cdots+k_{p-1}-n_p}; q\right)_{n_p}} \times \sum_{k_p=0}^{m_p} \frac{\left(q^{-m_p}, ABcdq^{M+k_1+\cdots+k_{p-1}-1}; q\right)_{k_p} \left(q^{1+M-N-m_p+k_p}; q\right)_\infty}{\left(ABcdq^{N+k_1+\cdots+k_{p-1}}; q\right)_{k_p} (q; q)_\infty} q^{k_p}. \tag{4.8}$$

First, let us suppose that $N \geq M \geq 0$. Then it is clear from the right side of (4.8) that S_p is zero unless $k_p \geq N - M + m_p$, as well as $m_p \geq k_p$. So, we must have

$$m_p + (N - M) \leq k_p \leq m_p. \tag{4.9}$$

This is a contradiction unless $N = M$, and then $k_p = m_p$. In that case

$$S_p = q^{m_p} \frac{\left(\frac{ABcd}{a_p b_p} q^{N-n_p+k_1+\cdots+k_{p-1}}; q\right)_{n_p} \left(q^{-m_p}, ABcdq^{N+k_1+\cdots+k_{p-1}-1}; q\right)_{m_p}}{\left(\frac{q^{1-n_p}}{a_p b_p}; q\right)_{n_p} \left(ABcdq^{N+k_1+\cdots+k_{p-1}-n_p}; q\right)_{m_p+n_p}}. \tag{4.10}$$

On the other hand, if $M \geq N \geq 0$ then

$$m_p - (M - N) \leq k_p \leq m_p. \tag{4.11}$$

So we get

$$S_p = q^{m_p+N-M} \frac{\left(\frac{ABcd}{a_p b_p} q^{N-n_p+k_1+\cdots+k_{p-1}}; q\right)_{n_p}}{\left(\frac{q^{1-n_p}}{a_p b_p}, ABcdq^{N-n_p+k_1+\cdots+k_{p-1}}; q\right)_{n_p}} \times \frac{\left(q^{-m_p}, ABcdq^{M+k_1+\cdots+k_{p-1}-1}; q\right)_{m_p+N-M}}{\left(ABcdq^{N+k_1+\cdots+k_{p-1}}; q\right)_{m_p+N-M}} \times {}_2\phi_1 \left[\begin{matrix} q^{N-M}, & ABcdq^{N+m_p+k_1+\cdots k_{p-1}-1} \\ & ABcdq^{2N-M+m_p+k_1+\cdots+k_{p-1}} \end{matrix} ; q, q \right]. \tag{4.12}$$

However, the above ${}_2\phi_1$ equals

$$(4.13)\qquad \frac{\left(q^{1+N-M};q\right)_{M-N}}{(ABcdq^{2N-M+m_p+k_1+\cdots+k_{p-1}};q)_{M-N}}(ABcdq^{N+m_p+k_1+\cdots+k_{p-1}-1})^{M-N},$$

which vanishes unless $N=M$. This completes the proof of (1.22).

Also, with $N=M$, (3.6), (4.6) and (4.10) give

$$\begin{aligned}(4.14)\quad P_{\mathbf{n}}\cdot\bar{P}_{\mathbf{m}} &= L_p\frac{(ABcdq^{N-1};q)_N\left(\frac{a_pb_pq^{1-N}}{ABcd};q\right)_{n_p}}{(ABcd;q)_{N+m_p}(a_pb_p;q)_{n_p}}(-1)^N q^{-\binom{N}{2}-m_p-n_p}(ABcdq^N)^{n_p}\\ &\times\sum_{k_1=0}^{m_1}\cdots\sum_{k_{p-1}=0}^{m_{p-1}} q^{k_1+k_2(1+n_1-m_1)+\cdots+k_{p-1}(1+n_1+\cdots+n_{p-2}-m_1-m_2-\cdots-m_{p-2})}\\ &\times\frac{\left(ABcdq^{N-1},\frac{ABcdq^N}{a_pb_p};q\right)_{k_1+\cdots+k_{p-1}}}{\left(ABcdq^{N+m_p},\frac{ABcdq^{N-n_p}}{a_pb_p};q\right)_{k_1+\cdots+k_{p-1}}}\prod_{r=1}^{p-1}\frac{(q^{-m_r},a_rb_rq^{n_r};q)_{k_r}}{(q,a_rb_r;q)_{k_r}},\end{aligned}$$

which is, of course, the same as (1.23). By taking p=2, e.g., in which case the series on the right hand side of (4.14) becomes a terminating balanced ${}_4\phi_3$ series, it is easily seen that in general the above inner product does not vanish when $N=M$ and $\mathbf{n}\neq\mathbf{m}$.

In closing this section we would like to point out that unlike the $q\to 1$ case that corresponds to the Tratnik biorthogonalities, the q-analogues of $P_{\mathbf{n}}\cdot Q_{\mathbf{m}}$, $P_{\mathbf{n}}\cdot\bar{Q}_{\mathbf{m}}$ or $Q_{\mathbf{n}}\cdot\bar{Q}_{\mathbf{m}}$ do not seem to work out the same way as $P_{\mathbf{n}}\cdot\bar{P}_{\mathbf{m}}$.

5 Transformations of $F_{m,n}(x,y)$ and $G_{m,n}(x,y)$

We shall now address the problem of proving the biorthogonality relation (1.24). First of all, it is very simple to use [2, (II.20)] to prove that

$$(5.1)\qquad \sum_{x=0}^{N}\sum_{y=0}^{N}W_N(x,y)=1.$$

The forms of $F_{m,n}(x,y)$ and $G_{m,n}(x,y)$ that turn out to be most convenient for the summations in (1.24) are as follows:

$$\begin{aligned}(5.2)\qquad F_{m,n}(x,y) &= \frac{\left(\frac{\gamma\gamma' q^{-N}}{\alpha};q\right)_{x+y}\left(\frac{\gamma\gamma'}{\alpha c};q\right)_{x-y}}{(q^{-N};q)_{x+y}(c^{-1};q)_{x-y}}\left(\frac{\alpha}{\gamma\gamma'}\right)^x\left(\frac{\alpha q^{N+n+1}}{\gamma\gamma'}\right)^m q^{Nn}\\ &\times\sum_{j=0}^{x}\sum_{k=0}^{y}\frac{\left(\frac{\gamma\gamma'}{\alpha}q^{-m-n};q\right)_{j+k}\left(q^{-x},\frac{\gamma\gamma'}{\alpha c}q^{x-N-1},\gamma q^m;q\right)_j}{\left(\frac{\gamma\gamma'}{\alpha}q^{-N};q\right)_{j+k}\left(q,\gamma,\frac{\gamma\gamma' q^{-n}}{\alpha c};q\right)_j}\\ &\times\frac{\left(q^{-y},cq^{y-N},\gamma' q^n;q\right)_k}{(q,cq^{1-m},\gamma';q)_k}q^{j+k},\end{aligned}$$

and

$$G_{m,n}(x,y) = \frac{(\alpha q^{N+1};q)_{m+n}\left(\frac{\alpha cq}{\gamma'};q\right)_m\left(\frac{\gamma'}{c};q\right)_n}{(q^{-N};q)_{m+n}\left(\frac{\gamma'}{c};q\right)_m\left(\frac{\alpha cq}{\gamma'};q\right)_m}\left(\frac{\gamma' q^{-N-1}}{\alpha c}\right)^m\left(\frac{cq^{-N}}{\gamma'}\right)^n$$
$$\times\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(\alpha q^{m+n};q)_{j+k}(q^{-m},\gamma q^x,\frac{\alpha c}{\gamma'}q^{N-x+1};q)_j}{(\alpha q^{N+1};q)_{j+k}(q,\gamma,\frac{\alpha c}{\gamma'}q^{n+1};q)_j}$$
$$\times\frac{(q^{-n},\gamma' q^y,\frac{\gamma' q^{N-y}}{c};q)_k}{(q,\gamma',\frac{\gamma' q^m}{c};q)_k}q^{j+k},\quad \text{assuming } 0\le m+n\le N. \tag{5.3}$$

Since

$$ {}_4\phi_3\left[\begin{matrix} q^{-m}, & \alpha q^{j+m+n}, & q^{-x}, & \frac{\gamma\gamma'}{\alpha c}q^{x-N-1} \\ & \gamma, & \frac{\gamma' q^n}{c}, & q^{j-N}\end{matrix};q,q\right]$$
$$=\frac{\left(\frac{\alpha cq^{j+1}}{\gamma'},\alpha q^{N+n+1};q\right)_m}{\left(\frac{\gamma' q^n}{c},q^{j-N};q\right)_m}\left(\frac{\gamma' q^{-N-1}}{\alpha c}\right)^m {}_4\phi_3\left[\begin{matrix} q^{-m}, & \alpha q^{j+m+n}, & \gamma q^x, & \frac{\alpha c}{\gamma'}q^{N-x+1} \\ & \gamma, & \frac{\alpha cq^{j+1}}{\gamma'}, & \alpha q^{N+n+1}\end{matrix};q,q\right]$$

and

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, & \alpha q^{i+m+n}, & q^{-y}, & cq^{y-N} \\ & \gamma', & \frac{\alpha cq^{i+1}}{\gamma'}, & q^{m-N}\end{matrix};q,q\right]$$
$$=\frac{\left(\frac{\gamma' q^m}{c},\alpha q^{N+1+i};q\right)_n}{\left(\frac{\alpha cq^{i+1}}{\gamma'},q^{m-N};q\right)_n}\left(\frac{cq^{-N}}{\gamma'}\right)^n {}_4\phi_3\left[\begin{matrix} q^{-n}, & \alpha q^{m+n+i}, & \gamma' q^y, & \frac{\gamma' q^{N-y}}{c} \\ & \gamma', & \frac{\gamma' q^m}{c}, & \alpha q^{N+1+i}\end{matrix};q,q\right]$$

by [2, (III.15)], (5.3) follows from (1.26) with a bit of simplification.

To derive (5.2) from (1.25) we need two applications of [2, (III.15)] on each of the two ${}_4\phi_3$ series involved in (1.25). First

$$ {}_4\phi_3\left[\begin{matrix} q^{-m}, & \frac{\gamma\gamma'}{\alpha c}q^{x-N-1}, & \gamma q^x, & \frac{\gamma\gamma' q^{j-m-n}}{\alpha} \\ & \gamma, & \frac{\gamma\gamma' q^{x-y-m}}{\alpha c}, & \frac{\gamma\gamma'}{\alpha}q^{x+y-N-m-n+j}\end{matrix};q,q\right] \tag{5.4}$$
$$=\frac{(q^{y-N},c^{-1}q^{n-y-j};q)_m}{\left(\frac{\alpha c}{\gamma\gamma'}q^{1+y-x},\frac{\alpha q^{N-x-y+n+1-j}}{\gamma\gamma'};q\right)_m}\left(\frac{\alpha cq^{N-x+1}}{\gamma\gamma'}\right)^m$$
$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-x}, & \frac{\gamma\gamma' q^{x-N-1}}{\alpha c}, & q^{-m}, & \frac{\alpha}{\gamma'}q^{m+n-j} \\ & \gamma, & q^{y-N}, & c^{-1}q^{n-y-j}\end{matrix};q,q\right]$$
$$=\frac{\left(\frac{\alpha c}{\gamma\gamma'}q^{1+y-x+m},\frac{\alpha q^{N-x-y+1+m+n}}{\gamma\gamma'};q\right)_{x-m}}{(q^{y+m-N},c^{-1}q^{m+n-y};q)_{x-m}}\left(\frac{\gamma\gamma'}{\alpha c}q^{x-N-1}\right)^{x-m}$$
$$\times\frac{\left(\frac{\gamma\gamma'}{\alpha}q^{x+y-N-m-n},cq^{1+y-x-n};q\right)_j}{\left(\frac{\gamma\gamma'}{\alpha}q^{y-N-n},cq^{1+y-m-n};q\right)_j}$$
$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-x}, & \frac{\gamma\gamma'}{\alpha c}q^{x-N-1}, & \gamma q^m, & \frac{\gamma\gamma' q^{j-m-n}}{\alpha} \\ & \gamma, & \frac{\gamma\gamma' q^{-y}}{\alpha c}, & \frac{\gamma\gamma'}{\alpha}q^{j+y-N-n}\end{matrix};q,q\right].$$

Substituted into (1.25) this leads to another balanced ${}_4\phi_3$ series:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, & cq^{y-N}, & \gamma' q^y, & \frac{\gamma\gamma'}{\alpha}q^{i-m-n} \\ & \gamma', & cq^{y-m-n+1}, & \frac{\gamma\gamma'}{\alpha}q^{i+y-N-n} \end{matrix}; q, q\right] $$

which, when transformed twice in the same manner as in (5.4), leads to

$$ (5.5) \qquad \frac{\left(c^{-1}q^{m+n-y}, \frac{\alpha q^{N+1-y+n}}{\gamma\gamma'}; q\right)_{y-n}}{\left(q^{m+n-N}, \frac{\alpha c q^{n+1}}{\gamma\gamma'}; q\right)_{y-n}} \left(cq^{y-N}\right)^{y-n} \frac{\left(\frac{\gamma\gamma'}{\alpha}q^{y-N-n}, \frac{\gamma\gamma'}{\alpha c}q^{-y}; q\right)_i}{\left(\frac{\gamma\gamma'}{\alpha}q^{-N}, \frac{\gamma\gamma'}{\alpha c}q^{-n}; q\right)_i} $$
$$ \times\ {}_4\phi_3\left[\begin{matrix} q^{-y}, & cq^{y-N}, & \gamma' q^n, & \frac{\gamma\gamma'}{\alpha}q^{i-m-n} \\ & \gamma',\ cq^{1-m}, & \frac{\gamma\gamma'}{\alpha}q^{i-N} & \end{matrix}; q, q\right]. $$

After some simplifications (5.4) and (5.5) give (5.2). Denoting the left hand side of (1.24) by $F_{m,n}\cdot G_{m',n'}$, it follows that

$$ (5.6) \quad F_{m,n}\cdot G_{m',n'} = A_{m,n,m',n'} \sum_{x=0}^{N}\sum_{y=0}^{N} \frac{\left(1-\frac{\gamma\gamma'}{\alpha c}q^{2x-N-1}\right)\left(1-cq^{2y-N}\right)\left(\frac{\gamma\gamma' q^{-N-1}}{\alpha c}, \gamma; q\right)_x}{\left(1-\frac{\gamma\gamma'}{\alpha c}q^{-N-1}\right)\left(1-cq^{-N}\right)\left(q, \frac{\gamma' q^{-N}}{\alpha c}; q\right)_x} $$
$$ \times \frac{(cq^{-N}, \gamma'; q)_y}{\left(q, \frac{cq^{1-N}}{\gamma'}; q\right)_y}\gamma^{-x}(\gamma')^{-y} $$
$$ \times \sum_j\sum_k \frac{\left(\frac{\gamma\gamma' q^{-m-n}}{\alpha}; q\right)_{j+k}\left(q^{-x}, \frac{\gamma\gamma'}{\alpha c}q^{x-N-1}, \gamma q^m; q\right)_j}{\left(\frac{\gamma\gamma' q^{-N}}{\alpha}; q\right)_{j+k}\left(q, \gamma, \frac{\gamma\gamma' q^{-n}}{\alpha c}; q\right)_j} $$
$$ \times \frac{(q^{-y}, cq^{y-N}, \gamma' q^n; q)_k}{(q, cq^{1-m}, \gamma'; q)_k} q^{j+k} $$
$$ \times \sum_r\sum_s \frac{(\alpha q^{m'+n'}; q)_{r+s}\left(q^{-m'}, \gamma q^x, \frac{\alpha c q^{N-x+1}}{\gamma'}; q\right)_r}{\left(\alpha q^{N+1}; q\right)_{r+s}(q, \gamma, \frac{\alpha c}{\gamma'}q^{n'+1}; q\Big)_r} $$
$$ \times \frac{\left(q^{-n'}, \gamma' q^y, \frac{\gamma'}{c}q^{N-y}; q\right)_s}{\left(q, \gamma', \frac{\gamma' q^{m'}}{c}; q\right)_s} q^{r+s}, $$

where

$$ (5.7) \qquad A_{m,n,m',n'} = \frac{(\alpha q/\gamma\gamma', \gamma'/c, \alpha c q/\gamma'; q)_N (\alpha q^{N+1}; q)_{m'+n'} \left(\frac{\gamma'}{c}; q\right)_{n'} \left(\frac{\alpha c q}{\gamma'}; q\right)_{m'}}{(\alpha q, 1/c, \alpha c q/\gamma\gamma'; q)_N (q^{-N}; q)_{m'+n'} \left(\frac{\alpha c q}{\gamma'}; q\right)_{n'} \left(\frac{\gamma'}{c}; q\right)_{m'}} $$
$$ \times \left(\frac{\gamma' q^{-N-1}}{\alpha c}\right)^{m'} \left(cq^{-N}/\gamma'\right)^{n'} \left(\frac{\alpha q^{N+n+1}}{\gamma\gamma'}\right)^m q^{Nn}. $$

6 Proof of (1.24)

Since each term in the weight function can be glued on nicely with the x and y dependent terms of the two double series in (5.6), the x, y-sum can be isolated as

$$
\begin{aligned}
& {}_6W_5\left(\frac{\gamma\gamma'}{\alpha c}q^{2j-N-1};\gamma q^{r+j},\frac{\gamma\gamma'}{\alpha c}q^{j},q^{j-N};q,\gamma^{-1}q^{-j-r}\right)\\
&\quad\times\,{}_6W_5\left(cq^{2k-N};\gamma' q^{s+k},cq^{k+1},q^{k-N};q,(\gamma' q^{k+s})^{-1}\right)\\
&=\frac{\left(\frac{\gamma\gamma'}{\alpha c}q^{2j-N},\frac{q^{-N-r}}{\gamma};q\right)_{N-j}\left(cq^{2k-N+1},\frac{q^{-N-s}}{\gamma'};q\right)_{N-k}}{\left(\frac{\gamma' q^{j-r-N}}{\alpha c},q^{j-N};q\right)_{N-j}\left(\frac{cq^{1-N+k-s}}{\gamma'},q^{k-N};q\right)_{N-k}},
\end{aligned}
$$

by [2, (II.21)]. The sum over j, k, r, s in (5.6) now reduces to

$$
\begin{aligned}
(6.1)\qquad F_{m,n}\cdot G_{m',n'} &= A_{m,n,m',n'}\frac{(\gamma q,\gamma' q,\alpha cq/\gamma\gamma',1/c;q)_N}{(q,q,\alpha cq/\gamma',\gamma'/c;q)_N}\\
&\quad\times\sum_j\sum_k\sum_r\sum_s\frac{\left(\frac{\gamma\gamma' q^{-m-n}}{\alpha};q\right)_{j+k}(\alpha q^{m'+n'};q)_{r+s}}{\left(\frac{\gamma\gamma' q^{-N}}{\alpha};q\right)_{j+k}(\alpha q^{N+1};q)_{r+s}}\\
&\quad\times\frac{\left(q^{-N},\frac{\gamma\gamma'}{\alpha c},\gamma q^{m};q\right)_j(q^{-N},\gamma' q^{n},cq;q)_k\left(q^{-m'},\frac{\alpha cq}{\gamma'},\gamma q^{N+1};q\right)_r}{\left(q,\gamma,\frac{\gamma\gamma' q^{-n}}{\alpha c};q\right)_j(q,\gamma',cq^{1-m};q)_k\left(q,\gamma,\frac{\alpha cq^{n'+1}}{\gamma'};q\right)_r}\\
&\quad\times\frac{(q^{-n'},\gamma'/c,\gamma' q^{N+1};q)_s}{\left(q,\gamma',\frac{\gamma' q^{m'}}{c};q\right)_s}\frac{(\gamma;q)_{r+j}(\gamma';q)_{s+k}}{(\gamma q;q)_{r+j}(\gamma' q;q)_{s+k}}q^{j+k+r+s}.
\end{aligned}
$$

The sum over j is a multiple of

$$
\begin{aligned}
(6.2)\qquad & {}_5\phi_4\left[\begin{matrix}q^{-N}, & \gamma q^{r}, & \frac{\gamma\gamma'}{\alpha c}, & \gamma q^{m}, & \frac{\gamma\gamma'}{\alpha}q^{k-m-n}\\ & \gamma q^{r+1}, & \frac{\gamma\gamma' q^{-n}}{\alpha c}, & \gamma, & \frac{\gamma\gamma'}{\alpha}q^{k-N}\end{matrix};q,q\right]\\
&=\frac{(q;q)_N\left(\frac{\gamma' q^{-n-r}}{\alpha c};q\right)_n(q^{-r};q)_m\left(\frac{\gamma' q^{k-N-r}}{\alpha};q\right)_{N-m-n}}{(\gamma q^{r+1};q)_N\left(\frac{\gamma\gamma' q^{-n}}{\alpha c};q\right)_n(\gamma;q)_m\left(\frac{\gamma\gamma'}{\alpha}q^{k-N};q\right)_{N-m-n}}(\gamma q^{r})^N,
\end{aligned}
$$

by [2, (1.9.10)]. Together with a similar expression for the sum over k we now have

(6.3) $F_{m,n}\cdot G_{m',n'}$

$$
\begin{aligned}
&= A_{m,n,m',n'}\frac{\left(\frac{\alpha cq}{\gamma\gamma'},1/c;q\right)_N}{(\alpha cq/\gamma',\gamma'/c;q)_N}\sum_r\sum_s\frac{(\alpha q^{m'+n'};q)_{r+s}}{(\alpha q^{N+1};q)_{r+s}}\\
&\quad\times\frac{\left(q^{-m'},\frac{\alpha cq}{\gamma'},\gamma q^{N+1};q\right)_r\left(q^{-n'},\frac{\gamma'}{c},\gamma' q^{N+1};q\right)_s}{\left(q,\frac{\alpha cq^{n'+1}}{\gamma'};q\right)_r\left(q,\frac{\gamma' q^{m'}}{c};q\right)_s}q^{r+s}\\
&\quad\times\frac{\left(\frac{\gamma' q^{-n-r}}{\alpha c};q\right)_n\left(\frac{\gamma' q^{-N-r}}{\alpha};q\right)_{N-m-n}(q^{-r};q)_m}{(\gamma q^{N+1};q)_r\left(\frac{\gamma\gamma' q^{-n}}{\alpha c};q\right)_n(\gamma;q)_m\left(\frac{\gamma\gamma'}{\alpha}q^{-N};q\right)_{N-m-n}}(\gamma q^{r})^N
\end{aligned}
$$

$$\times \frac{\left(\frac{cq^{1-m-s}}{\gamma'};q\right)_m \left(\frac{q^{-N-r-s}}{\alpha};q\right)_{N-m-n}(q^{-s};q)_n}{(\gamma' q^{N+1};q)_s(cq^{1-m};q)_m(\gamma';q)_n\left(\frac{\gamma'}{\alpha}q^{-N-r};q\right)_{N-m-n}}(\gamma' q^s)^N$$

$$= A_{m,n,m',n'}\frac{\left(\frac{\alpha cq}{\gamma\gamma'},1/c;q\right)_N}{\left(\frac{\alpha cq}{\gamma'},\frac{\gamma'}{c};q\right)_N}(\gamma\gamma')^N\frac{\left(\frac{\gamma' q^{-n}}{\alpha c};q\right)_n\left(\frac{cq^{1-m}}{\gamma'};q\right)_m(q^{-N}/\alpha;q)_{N-m-n}}{\left(\gamma',\frac{\gamma\gamma' q^{-n}}{\alpha c};q\right)_n(\gamma,cq^{1-m};q)_m\left(\frac{\gamma\gamma'}{\alpha}q^{-N};q\right)_{N-m-n}}$$

$$\times \sum_r\sum_s \frac{(\alpha q^{m'+n'};q)_{r+s}}{(\alpha q^{m+n+1};q)_{r+s}}\frac{\left(q^{-m'},\frac{\alpha cq^{n+1}}{\gamma};q\right)_r\left(q^{-n'},\frac{\gamma' q^m}{c};q\right)_s(q^{-r};q)_m}{\left(q,\frac{\alpha cq^{n'+1}}{\gamma'};q\right)_r\left(q,\frac{\gamma'}{c}q^{m'};q\right)_s}$$

$$\times (q^{-s};q)_n q^{(m+1)r+(n+1)s}.$$

The r,s sum is

$$(6.4)\qquad (-1)^{m+n}q^{\binom{m+1}{2}+\binom{n+1}{2}}\frac{(\alpha q^{m'+n'};q)_{m+n}\left(q^{-m'},\frac{\alpha cq^{n+1}}{\gamma'};q\right)_m\left(q^{-n'},\frac{\gamma' q^m}{c};q\right)_n}{(\alpha q^{m+n+1};q)_{m+n}\left(\frac{\alpha cq^{n'+1}}{\gamma'};q\right)_m\left(\frac{\gamma' q^{m'}}{c};q\right)_n}$$

$$\times \sum_{r=0}^{m'-m}\sum_{s=0}^{n'-n}\frac{(\alpha q^{m+n+m'+n'};q)_{r+s}\left(q^{m-m'},\frac{\alpha cq^{n+m+1}}{\gamma'};q\right)_r\left(q^{n-n'},\frac{\gamma' q^{n+m}}{c};q\right)_s}{(\alpha q^{2m+2n+1};q)_{r+s}\left(q,\frac{\alpha cq^{m+n'+1}}{\gamma'};q\right)_r\left(q,\frac{\gamma' q^{n+m'}}{c};q\right)_s}q^{r+s}$$

which vanishes unless $m'\geq m$ and $n'\geq n$.

The sum in (6.4), via [2, (II.12) and (II.6)], equals

$$\frac{\left(q^{1+m-m'+n-n'},\frac{\alpha c}{\gamma'}q^{m+n+1};q\right)_{n'-n}(q^{1+m-m'};q)_{m'-m}}{\left(\alpha q^{2m+2n+1},\frac{cq^{1-m'-n'}}{\gamma'};q\right)_{n'-n}(\alpha q^{2m+n+n'+1};q)_{m'-m}}\left(\alpha q^{m+n+m'+n'}\right)^{m'-m}$$

which vanishes unless $m'\leq m$ and $n'\leq n$. Thus we must have

$$(6.5)\qquad F_{m,n}\cdot G_{m',n'}=0\quad \text{unless}\quad (m,n)=(m',n'),\quad \text{and then}$$

$$(6.6)\qquad F_{m,n}\cdot G_{m,n}=\frac{1-\alpha}{1-\alpha q^{2m+2n}}\frac{\left(q,\frac{\alpha cq}{\gamma'};q\right)_m\left(q,\frac{\gamma'}{c};q\right)_n\left(\frac{\alpha q}{\gamma\gamma'},\alpha q^{N+1};q\right)_{m+n}}{(\gamma,1/c;q)_m\left(\gamma',\frac{\alpha cq}{\gamma\gamma'};q\right)_n(\alpha,q^{-N};q)_{m+n}}c^{n-m}q^{mn},$$

which completes the proof of (1.24) and (1.28).

It may be mentioned that there are other double series representations for $F_{m,n}(x,y)$ that one could use instead of (5.2) in the derivation of the biorthogonality relation (1.24) which do not contain the factor $1/(q^{-N};q)_{x+y}$ that cancels out the $(q^{-N};q)_{x+y}$ factor in the weight function, but the subsequent computations turn out to be quite tedious, while the final result is, of course, the same.

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