AN EXTENSION OF GROSS’S LOG-SOBOLEV INEQUALITY
FOR THE LOOP SPACE OF A COMPACT LIE GROUP

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Let $G$ be a compact Lie group, and for $T > 0$, let $L_sG$ be the space of based loops of length $T$

$$L_sG = \left\{ \gamma : [0, T] \rightarrow G \mid \gamma(0) = \gamma(T) = e \right\}.$$ 

We denote the expectation with respect to the Wiener measure on $L_sG$ (also known as the Brownian bridge) by $\langle F \rangle_s$. There is a regular Dirichlet form on $L_sG$, which was constructed using the Malliavin calculus in [1] and [2], and may be represented by the formula

$$\mathcal{E}(F, F) = \langle |d_s F|^2 \rangle_s.$$ 

Consider the function on $L_sG$ given by the Stratonovitch stochastic integral

$$V(\gamma) = T^{-1} \left| \int_0^T \dot{\gamma}(t) \gamma(t)^{-1} \, dt \right|^2 + 1.$$ 

Gross has proved the following theorem for $T = 1$ in [4], and our goal in this paper is to extend his proof to show that the constant $C$ may be chosen independent of $T$ for $T$ sufficiently small.

**Theorem.** There is a constant $C = 1 + O(T)$ such that for all $F \in W^\infty(L_sG)$,

$$\langle F^2 \log |F| \rangle_s \leq C \langle |d_s F|^2 + V|F|^2 \rangle_s + \frac{1}{2} \langle F^2 \rangle_s \log \langle F^2 \rangle_s,$$

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uniformly in $T$ for small $T$.

The above result is to be compared with Gross’s logarithmic Sobolev inequality for a Wiener space $B$, proved in [3], that for $F \in W^\infty(B)$,

$$\langle F^2 \log |F| \rangle \leq \langle |dF|^2 \rangle + \frac{1}{2} \langle F^2 \rangle \log \langle F^2 \rangle.$$

The proof of the above theorem uses this inequality at a key point, just as in Gross’s proof for $T = 1$.

It is tempting to attempt to get rid of the potential $V$ in the inequality above by means of the Hausdorff-Young inequality: for all $x \in \mathbb{R}$ and $y > 0$,

$$xy \leq e^x + y \log y - y.$$ 

This shows that for $\langle |F|^2 \rangle_* = 1$,

$$(1 - C\varepsilon/2)\langle F^2 \log |F| \rangle_* \leq C\langle |d_*F|^2 \rangle_* + C\langle e^{V/\varepsilon} \rangle_* + C(\log \varepsilon - 1).$$

In order to obtain a logarithmic Sobolev inequality from this, we need to be able to take $\varepsilon < 2/C$. In Proposition 2.10, we will see that this value of $\varepsilon$ is marginally too small for $\langle e^{V/\varepsilon} \rangle_*$ to be finite. It is not clear whether some other method will allow the omission of $V$ from the inequality.

In Section 1, we recall some of the results of [2], and prove some abstract lemmas that will be called upon later. In Section 2, we specialize to $L_*G$, where $G$ is a compact Lie group. In Section 3, we discuss the technique of Gross in which he effectively constructs a smooth tubular neighbourhood of $L_*G$ in Wiener space. In Section 4, we give our proof of the main theorem.

Throughout this paper, we will make frequent use of the Hausdorff-Young inequality:

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1. The geometry of Malliavin maps

Let $\mathfrak{g}$ be a real vector space with inner-product, and let $B$ be the classical Wiener space $B = C_*([0, T], \mathfrak{g}) = \{\gamma \in C([0, T], \mathfrak{g}) \mid \gamma(0) = 0\}$. The Wiener measure on $B$ is the unique probability measure such that

$$\int_B \exp \left( i \int_0^T (\alpha(t), \gamma(t)) dt \right) d\mu(\gamma) = e^{-C(\alpha, \alpha)/2} \quad \text{for all } \alpha \in L^\infty([0, T], \mathfrak{g}),$$

where

$$C(\alpha, \alpha) = \int_0^T \int_0^T \min(s, t) \langle \alpha(s), \alpha(t) \rangle ds dt.$$
Let $H \subset B$ be the Hilbert space of finite-energy paths $L^{2,1}_{s}(\mathbb{R}, \mathbb{R})$, with inner product

$$|\gamma|^2 = \int_0^T |\dot{\gamma}(t)|^2 \, dt.$$ 

A **cylinder function** on $B$ is a function of the form

$$F(\gamma(t_1), \ldots, \gamma(t_k)),$$

where $0 < t_1 < \ldots < t_k < T$, and $F \in C^\infty_c(\mathbb{R}^k)$. The space of all cylinder functions, written $C^\infty_c(B)$, is dense in $L^p(B) = L^p(B, d\mu)$ for each $p < \infty$. If $\tau : B \to \mathbb{R}^k$ denotes the map

$$\tau(\gamma) = (\gamma(t_1), \ldots, \gamma(t_k)),$$

then the above cylinder function may be written $\tau^* F$.

We will often make use of the Hilbert tensor product $H_1 \otimes_2 H_2$ of two Hilbert spaces $H_1$ and $H_2$; this is completion of the algebraic tensor product with respect to the quadratic form

$$|v \otimes_2 w|^2 = |v|^2 \cdot |w|^2.$$ 

We also let $\text{HS}(H)$ denote the space of Hilbert-Schmidt operators on $H$.

If $f = \tau^* F$ is a cylinder function, its gradient, denoted by $df$, is the element of $C^\infty_c(B) \otimes H$ defined by applying the map

$$\tau^* : C^\infty_c(\mathbb{R}^k) \otimes \mathbb{R}^k \to C^\infty_c(B) \otimes H$$

to $dF \in C^\infty_c(\mathbb{R}^k) \otimes \mathbb{R}^k$. Forming the closure of this operator, we obtain an unbounded operator from $L^p(B)$ to $L^p(B, H)$, which we will also denote by $d$; its adjoint $d^*$ is then a closed unbounded operator from $L^p(B, H)$ to $L^p(B)$.

The composition of the operators $d$ and $d^*$ acting on the cylinder functions is the Ornstein-Uhlenbeck operator $L = d^* d$, which is essentially self-adjoint with core $C^\infty_c(B)$. If $\mathcal{H}$ is a Hilbert space, the Sobolev space $L^p_s(B, \mathcal{H})$, where $1 < p < \infty$ and $s \in \mathbb{R}$, is the domain of $L^{s/2}$ on $L^p(B, \mathcal{H})$. The space of Malliavin test functions is

$$W^\infty(B, \mathcal{H}) = \bigcap_{p, s < \infty} L^p_s(B, \mathcal{H}).$$

Meyer has proved that $d$ is bounded from $L^p_s(B)$ to $L^p_{s-1}(B, H)$ and that $d^*$ is bounded from $L^p_s(B, H)$ to $L^p_{s-1}(B)$, for all $s \in \mathbb{R}$ and $p < \infty$. It is important to note that $W^\infty$-functions need not be continuous.

We can also define $W^\infty$-maps from a Wiener space $B$ to a compact Riemannian manifold $M$. 

Definition 1.1. A map $\pi : B \rightarrow M$ is in $W^\infty(B, M)$ if it is measurable and if the pull-back map $\pi^* : C^\infty(M) \rightarrow W^\infty(B)$ is bounded.

An equivalent definition can be made by choosing an embedding $\rho : M \rightarrow \mathbb{R}^n$ of $M$ in a Euclidean space $\mathbb{R}^n$: then $\pi \in W^\infty(B, M)$ if and only if $\rho \circ \pi \in W^\infty(B, \mathbb{R}^n)$.

If $\pi \in W^\infty(B, M)$, the composition

$$C^\infty(M) \xrightarrow{\pi^*} W^\infty(B) \xrightarrow{\mu} \mathbb{R}$$

is a positive linear form on $C^\infty(M)$ which equals 1 on the constant function 1, and hence defines a probability measure on $M$. We will write this measure $\pi_* \mu$.

The tangent map

$$d : W^\infty(M, B) \rightarrow W^\infty(B, \text{Hom}(H, \pi^* TM))$$

may be defined by means of an embedding $\rho : M \rightarrow \mathbb{R}^n$, by the formula

$$d(\rho \circ \pi) = d\rho \circ d\pi \in W^\infty(B, \text{Hom}(H, \mathbb{R}^n)).$$

Definition 1.2. Let $\pi : B \rightarrow M$ be in $W^\infty(B, M)$, with differential $\Pi$, and form $\Gamma = \Pi \Pi^* \in W^\infty(B, \text{End}(\pi^* TM))$. Then $\pi$ is a Malliavin map if the determinant $\det(\Gamma)$ satisfies

$$\det(\Gamma)^{-1} \in L^p(B) \quad \text{for } p < \infty.$$

If $\pi$ is a Malliavin map, then the operator $\Gamma^{-1}$ is in $W^\infty(B, \text{End}(\pi^* TM))$, and $\det(\Gamma)^k$ is in $W^\infty(B)$ for all $k \in \mathbb{Z}$. The operator $N = \Pi \Gamma^{-1} \Pi \in W^\infty(B, \text{HS}(H))$ is a projector in $H$ of rank $n$ for a.e. $x \in B$. We may think of $N$ as the projector on the normal bundle to the fibres of the map $\pi$. Thus the projector $P = 1 - N$ orthogonal to $N$ is the projector onto the tangent bundle to the fibres of $\pi$.

We say that an operator $A$ on $W^\infty(B)$ acts along the fibres of $\pi$ if it satisfies the formula

$$A((\pi^* f) F) = (\pi^* f)(AF),$$

for all $f \in C^\infty(M)$ and $F \in W^\infty(B)$. Using the projector $P$, we may construct the exterior differential along the fibres

$$d_* : W^\infty(B) \rightarrow W^\infty(B, H),$$

defined by the formula $d_* F = P(dF)$, and the Ornstein-Uhlenbeck operator along the fibres $L_* : W^\infty(B) \rightarrow W^\infty(B)$, associated to the Dirichlet form $\langle |d_* F|^2 \rangle$, given by the formula $L_* F = d^* P d F$. 
The adjoint of the pull-back $\pi^* : C^\infty(M) \to W^\infty(B)$ is a bounded map $(\pi_*)' : W^\infty(B)' \to C^\infty(M)'$. We define a push-forward map $\pi_* : W^\infty(B) \to C^\infty(M)$ in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
W^\infty(B) & \xrightarrow{\pi^*} & C^\infty(M) \\
F \mapsto F d\mu & & f \mapsto f d(\pi_* \mu) \\
\downarrow & & \downarrow \\
W^\infty(B)' & \xrightarrow{(\pi_*)'} & C^\infty(M)'
\end{array}
$$

The following basic result, due to Malliavin, shows that the map $\pi_* : W^\infty(B) \to C^\infty(M)$ is well-defined.

**Proposition 1.3.** If $\pi$ is a Malliavin map, integration along the fibres of $\pi$ defines a bounded map $\pi_* : W^\infty(B) \to C^\infty(M)$.

Note that $\pi_* \circ \pi^*$ is the identity, and in particular, that $\pi_* 1 = 1$. If $F \in W^\infty(B)$, we define

$$
\langle F \rangle = \int_B F d\mu;
$$

similarly, if $f \in C^\infty(M)$, we define

$$
\langle f \rangle = \int_M f d(\pi_* \mu).
$$

These integrals are related by the formula $\langle F \rangle = \langle \pi_* F \rangle$.

**Definition 1.4.** If $X$ is a vector field on $M$, its horizontal lift $\tilde{X} \in W^\infty(B,H)$ is the unique vector field on $B$ such that

1. if $f \in C^\infty(M)$, $\tilde{X}(\pi^* f) = \pi^* (X(f))$;
2. $\tilde{X}$ is horizontal, that is, $P \tilde{X} = 0$.

It may be checked that the vector field $\Pi^* \Gamma^{-1} \pi^* X$ satisfies the above requirements, so that we obtain an explicit formula for the horizontal lift:

$$
\tilde{X} = \Pi^* \Gamma^{-1} \pi^* X.
$$

We will denote by $\text{div}_{\pi_* \mu} : \Gamma(M,TM) \to C^\infty(M)$ the adjoint of the exterior differential $d : C^\infty(M) \to \Gamma(M,T^*M)$ with respect to the pairings

$$
(f,g) = \int_M fg d(\pi_* \mu) \quad \text{and} \quad (X,\omega) = \int_M \langle X,\omega \rangle d(\pi_* \mu).
$$

In terms of the divergence operator $\text{div}$ defined with respect to the Riemannian volume form $dx$, $\text{div}_{\pi_* \mu}$ is given by the formula

$$
\text{div}_{\pi_* \mu} X = \text{div} X + X \left( \log \left( \frac{d(\pi_* \mu)}{dx} \right) \right).
$$
Definition 1.5. If $X$ is a vector field on $M$, we define the $W^\infty$-function $\alpha(X)$ by the formula

$$\alpha(X) = d^* \tilde{X} - \pi^* (\text{div}_{\pi_* \mu} X).$$

It is easy to see that $\alpha$ satisfies the formula

(1.6) $\alpha(fX) = (\pi^* f) \alpha(X)$ for all $f \in C^\infty(M)$.

The following proposition explains our reason for introducing $\alpha(X)$.

Proposition 1.7. If $X$ is a vector field on $M$ and $F \in W^\infty(B)$, then

$$X(\pi_* F) = \tilde{X}(F) - \alpha(X) F.$$

Proof. If $F \in W^\infty(B)$ and $f \in C^\infty(M)$, then

$$\langle fX(\pi_* F) \rangle = \langle (-X(f) + (\text{div}_{\pi_* \mu} X) f) \pi_* F \rangle = \langle \pi^* (-X(f) + (\text{div}_{\pi_* \mu} X) f) F \rangle.$$

We now use the fact that $\pi^* (X(f)) = \tilde{X}(\pi^* f)$, which gives

$$\langle fX(\pi_* F) \rangle = \langle (-\tilde{X}(\pi^* f) + \pi^* (\text{div}_{\pi_* \mu} X) \pi^* f) F \rangle = \langle \pi^* f (\tilde{X}(F) - d^* \tilde{X} F + \pi^* (\text{div}_{\pi_* \mu} X) F) \rangle,$$

which proves the lemma, since $f$ was arbitrary. \qed

Corollary 1.8. If $F \in W^\infty(B)$ satisfies the formula $\tilde{X}(F) = \frac{1}{2} \alpha(X) F$, then

1. $X \pi_* (|F|^2) = 0$;
2. $X \pi_* (|F|^2 \log |F|^2) = \pi_* (\alpha(X) |F|^2)$.

Proof. To prove (1), we observe that $\tilde{X}(|F|^2) = 2F \tilde{X}(F) = \alpha(X) |F|^2$. Since

$$X(\pi_* |F|^2) = \pi_* (\tilde{X}(|F|^2)) - \pi_* (\alpha(X) |F|^2),$$

(1) follows.

To prove (2), let us calculate $\tilde{X}(|F|^2 \log |F|^2)$:

$$\tilde{X}(|F|^2 \log |F|^2) = 2F \tilde{X}(F) \log |F|^2 + 2F \tilde{X}(F)$$

$$= \alpha(X) |F|^2 \log |F|^2 + \alpha(X) |F|^2.$$

Since $X \pi_* (|F|^2 \log |F|^2) = \pi_* (\tilde{X}(|F|^2 \log |F|^2)) - \pi_* (\alpha(X) |F|^2 \log |F|^2)$, the proof of (2) follows. \qed
2. Based loops in a compact Lie group

In this section, we will discuss a particular case of a Malliavin map, in which $M$ is a compact Lie group $G$, with Lie algebra $\mathfrak{g}$, and the map $\pi$ is the so-called path-ordered exponential. This case is very special, since the Malliavin covariance $\Gamma = \Pi^*\Pi$ is equal to a constant multiple $T \text{id}$ of the identity operator, which makes many of the calculations easier.

To simplify the formulation of the results, we will assume that the group $G$ is a linear group; of course, this is no restriction, since every compact Lie group has a faithful linear representation. We also suppose chosen an invariant Riemannian metric on $G$, which induces an inner product on $\mathfrak{g}$ invariant under the adjoint action $\text{Ad}(g)$ for $g \in G$. Denote the dimension of $G$ by $n$, and the identity of $G$ by $e$. We will always identify a Lie algebra element $X \in \mathfrak{g}$ with the corresponding left-invariant vector field on $G$.

Let $(B, H)$ be the classical Wiener space $(C([0, T], \mathfrak{g}), L^2([0, T], \mathfrak{g}))$. If $x \in H$, we solve the ordinary differential equation for $\gamma(t) : [0, T] \to G$ with initial condition $\gamma(0) = e$,

$$\gamma(t)^{-1}\dot{\gamma}(t) = \dot{x}(t).$$

The solution of this equation is known as the path-ordered exponential, and we will write it as $\gamma[x]$, or simply as $\gamma$ if $x$ is implicit.

The path-ordered exponential identifies $H$ with the space of finite-energy based paths in $L^2([0, T], G)$. Let $\pi$ be the map $x \mapsto \gamma[x](T)$; the fibre of $\pi$ over $e$ can be identified with the space $L^*G$ of finite-energy paths in $G$ which return to the identity at time $T$, that is, the based loop space of $G$.

If $F \in W^\infty(B)$, we will denote by $\langle F \rangle_*$ the integral of $F$ over the fibre $\pi^{-1}(e)$, that is,

$$\langle F \rangle_* = \pi_*(F)(e).$$

When we say that two functions $F_1$ and $F_2$ are equal on $\pi^{-1}(e)$, we mean that $\langle (F_1 - F_2)^2 \rangle_* = 0$.

The map $\gamma(t) : B \to G$ is extended to a family of $W^\infty$-maps from the Wiener space $B$ to $G$, by introducing a mollifier on $B$:

$$x_\varepsilon(t) = \varepsilon^{-1} \int_0^1 \lambda(\varepsilon^{-1}(s-t)) x(s) \, ds,$$

where $\lambda$ is any positive symmetric function in $C^\infty_c(-1, 1)$ such that $\int_{(-1,1)} \lambda \, dt = 1$. The following proposition is a consequence of the theory of Stratanovitch stochastic differential equations.

**Proposition 2.1.**

1. For each $\varepsilon > 0$, the map $\pi_\varepsilon(x) = \pi(x_\varepsilon)$ is a $W^\infty$-map from $B$ to $G$.
2. As $\varepsilon \to 0$, the maps $\pi_\varepsilon$ converge in $W^\infty(B, G)$ to a map $\pi$. 
We now calculate the differential \( d\pi \), and the Malliavin covariance matrix \( \Gamma = (d\pi)(d\pi)^\ast \), of the map \( \pi \) explicitly.

**Proposition 2.2.**

1. \( \Pi = (d\pi)\pi^{-1} \in W^\infty(B, \text{Hom}(H, g)) \) is given by the formula
   \[
   (\Pi(h_t)) = \int_0^T \text{Ad}(\gamma(t))\dot{h}_t \, dt.
   \]

2. The adjoint \( \Pi^*(X) \in W^\infty(B, \text{Hom}(g, H)) \) of \( \Pi \) is given by the formula
   \[
   (\Pi^*X)_t = \int_0^t \text{Ad}(\gamma(s))^{-1}X \, ds.
   \]

3. The Malliavin covariance matrix \( \Gamma = \Pi\Pi^\ast \) equals \( T \) times the identity of \( g \); in particular, the map \( \pi \) satisfies the Malliavin condition, since \( \det(\Gamma) = T^n \) is a constant, and \( N \) is given by the formula \( N = T^{-1}\Pi\Pi^\ast \).

**Proof.** We will calculate \( \Pi_\varepsilon = (d\pi_\varepsilon)(\pi_\varepsilon)^{-1} \), and then take the limit \( \varepsilon \to 0 \). For \( \varepsilon > 0 \), the map \( \pi_\varepsilon \) is smooth, so we can calculate \( \Pi_\varepsilon \) path by path.

By du Hamel’s formula, \( (d\pi_\varepsilon)(\pi_\varepsilon)^{-1} \) equals
\[
(d\pi_\varepsilon)(\pi_\varepsilon)^{-1} = (d\gamma_\varepsilon(T))\gamma_\varepsilon(T)^{-1} = \int_0^T \text{Ad}(\gamma_\varepsilon(t))\dot{h}_\varepsilon(t) \, dt,
\]
from which (1) follows, by sending \( \varepsilon \to 0 \).

Since the metric on \( g \) is invariant, it follows that if \( X \in g \), then
\[
(X, \Pi(h(t))) = \int_0^T \left( X, \text{Ad}(\gamma(t))\dot{h}(t) \right) dt = \int_0^T \left( \text{Ad}(\gamma(t))^{-1}X, \dot{h}(t) \right) dt,
\]
from which we obtain the formula for \( \Pi^*(X) \). It is clear from this that \( \Pi\Pi^\ast = T \). \( \square \)

**Corollary 2.3.** If \( X \in g \), then its horizontal lift \( \tilde{X} \in W^\infty(B, H) \) is given by the formula
\[
\tilde{X}(t) = T^{-1} \int_0^t \text{Ad}(\gamma(s))^{-1}X \, ds,
\]
and \( d^*\tilde{X} \) is given by the formula
\[
d^*\tilde{X} = T^{-1}(d^*\Pi, X),
\]
where \( d^*\Pi \in W^\infty(B, g) \) is the divergence of \( \Pi \).

It follows from Proposition 2.2 that if \( f \in C^\infty(G) \), then \( d(\pi^*f) \) satisfies
\[
|d(\pi^*f)| = T^{1/2}|df|.
\]

The next proposition collects a number of useful formulas.
Proposition 2.5.

1. The gradient $d\Pi \in W^\infty(B, \text{Hom}(H \otimes_2 H, g))$ is given by the formula
   $$d\Pi(a, b) = \int_{0 \leq s \leq t \leq T} [\text{Ad}(\gamma(s))\dot{a}(s), \text{Ad}(\gamma(t))\dot{b}(t)] ds dt.$$  

2. The divergence $d^*\Pi \in W^\infty(B, g)$ is given by the Stratanovitch integral
   $$d^*\Pi = \int_0^T \text{Ad}(\gamma(t))\dot{x}(t) dt.$$  

3. If $a \in H$, then
   $$dd^*\Pi(a) = \int_{0 \leq t \leq T} \text{Ad}(\gamma(t))\dot{a}(t) dt + \int_{0 \leq s \leq t \leq T} [\text{Ad}(\gamma(s))\dot{a}(s), \text{Ad}(\gamma(t))\dot{b}(t)] ds dt.$$  

Proof. The gradient of $\Pi_\varepsilon$ is given by the formula
$$d\Pi_\varepsilon(a, b) = \int_{0 \leq s \leq t \leq T} [\text{Ad}(\gamma_\varepsilon(s))\dot{a}(s), \text{Ad}(\gamma_\varepsilon(t))\dot{b}(t)] ds dt.$$  

The formula for $d\Pi$ follows by taking $\varepsilon \to 0$.  

In a finite-dimensional Wiener space $V$, $d^*\Pi$ would be given by the formula
$$d^*\Pi = -\text{Tr}(d\Pi) + \Pi x,$$  

where $x$ is the identity map from $V$ to itself. In our infinite-dimensional situation, this formula makes sense if we replace $\Pi$ by its approximation $\Pi_\varepsilon$:
$$d^*\Pi_\varepsilon = -\text{Tr}_H(d\Pi_\varepsilon) + \Pi_\varepsilon x.$$  

Using the above formula for $d\Pi_\varepsilon$, it is easy to see that $\text{Tr}_H(d\Pi_\varepsilon)$ vanishes. On the other hand, $\Pi_\varepsilon x$ equals
$$\Pi_\varepsilon x = \int_0^T \text{Ad}(\gamma_\varepsilon(t))\dot{x}(t) dt,$$  

which converges to
$$\int_0^T \text{Ad}(\gamma_\varepsilon(t))\dot{x}(t) dt$$  

as $\varepsilon \to 0$. This proves (2). The proof of (3) is similar to the proof of (1). \qed

Let $y(t)$ be the Stratanovitch stochastic integral
$$y(t) = \int_0^t \text{Ad}(\gamma(s))dx(s) = \lim_{\varepsilon \to 0} \int_0^t \text{Ad}(\gamma(s))\dot{x}_\varepsilon(s) ds,$$  

so that $d^*\Pi = y(T)$. In the rest of this section, we will study the properties of the stochastic process $y(t)$. The following lemma will be basic to this study.
Lemma 2.6. The stochastic process $t \mapsto y(t)$ is a Wiener process; that is, the map from $B$ to itself given by sending $x$ to $y$ is measure-preserving.

Proof. Let us denote the Stratonovitch stochastic differential by $dx(t)$, and the Ito stochastic differential by $\delta x(t)$. The relationship between the two differentials shows that $\gamma^{-t} \delta \gamma(t) = \delta x(t)$. It follows that

$$y(t) = \int_0^t \Ad(\gamma(s)) \delta x(s) + \frac{1}{2} \sum_{ijk} c_{ijk} \int_0^t \Ad(\gamma(t)) X_i d\langle x^j, x^k \rangle$$

and

$$= \int_0^t \Ad(\gamma(s)) \delta x(s),$$

since the quadratic variation $\langle x^j, x^k \rangle$ is symmetric in $j$ and $k$, while the structure coefficients $c_{ijk}$ are antisymmetric. Thus, we see that the quadratic variation process $\langle y, y \rangle$ equals $t$ times the inner product on $g$, and hence that $y(t)$ is a Wiener process on $g$. □

There is a more geometric way to see that $t \mapsto y(t)$ is a Wiener process, for which we will give only the outline. Consider the diagram

$$B \xrightarrow{x \mapsto x} B \xrightarrow{x \mapsto x[\gamma]} P_\gamma G \xrightarrow{\gamma \mapsto \gamma^{-1}} P_\gamma G \xrightarrow{\gamma \mapsto x} B.$$ 

It turns out that the composition of these maps is precisely the map $x \mapsto y[x]$. Since each map is measure-preserving, their composition is, proving that $y$ is a Wiener process.

Corollary 2.7.

1. $\left\langle \exp\left(\frac{\lambda}{2} |y(T)|^2\right) \right\rangle = (1 - T\lambda)^{-n/2}$

2. $\left\langle \exp\left(\frac{\lambda}{2} \int_0^T |y(t)|^2 \, dt\right) \right\rangle = (\cos T\lambda^{1/2})^{-n/2}$

Proof. Since $y(t)$ is a Brownian process, (1) follows from the calculation of the following integral:

$$(2\pi T)^{-n/2} \int_g e^{-|\xi|^2/2T + \lambda |\xi|^2/2} \, d\xi = (1 - T\lambda)^{-n/2}.$$

By the Feynman-Kac formula, the left-hand side of (2) is given by the integral of the heat-kernel

$$\langle \xi | e^{-T(\Delta - \lambda |\xi|^2)/2} | 0 \rangle = (2\pi T)^{-n/2} \left( \frac{T\lambda^{1/2}}{\sin T\lambda^{1/2}} \right)^{n/2} e^{-(\lambda^{1/2} \cot T\lambda^{1/2}) |\xi|^2/2},$$
with respect to $\xi$, which is
\[
\left( \frac{T \lambda^{1/2}}{T \lambda^{1/2} \cot T \lambda^{1/2} \sin T \lambda^{1/2}} \right)^{n/2} = (\cos T \lambda^{1/2})^{-n/2}. \tag*{□}
\]

Let $\{X, Y\}$ denote the Killing form on $g$, given by the formula
\[
\{X, Y\} = -\text{Tr}_g(\text{ad}(X) \text{ad}(Y)),
\]
and let $\|X\|^2 = \{X, X\}$.

**Lemma 2.8.** The differential $dy(T)$ of $y(T)$ satisfies the estimate
\[
|dy(T)|^2 = T + \frac{1}{2} \int_{0 \leq s, t \leq T} \min(s, t) \{\dot{y}(s), \dot{y}(t)\} \, ds \, dt
\leq T + T|y(T)|^2 + \int_0^T \|y(t)\|^2 \, dt.
\]

**Proof.** The proof makes use of the same mollification method as in the proof of Proposition 2.5; hence, we will tacitly suppose that $x(t)$ is smooth.

The formula for $|dy(T)|^2$ easily follows from the formula for $dy(T)$ in Proposition 2.5 (3). Pretending that $x(t)$ is smooth, we integrate twice by parts:
\[
\int_{0 \leq s, t \leq T} \min(s, t) \{\dot{y}(s), \dot{y}(t)\} \, ds \, dt
= T\|y(T)\|^2 - 2 \int_0^T \{y(t), y(T)\} \, dt + \int_0^T \|y(t)\|^2 \, dt
\leq 2T|y(T)|^2 + 2 \int_{0 \leq t \leq T} \|y(t)\|^2 \, ds. \tag*{□}
\]

Ito’s formula shows that the measure $\pi_* \mu$ is determined by the formula
\[
\frac{d(\pi_* \mu)}{dg} = k(T, g),
\]
where $k(T, g) = \langle g | e^{-T\Delta} | e \rangle$ is the heat-kernel for the invariant Laplacian $\Delta$ on $G$.

The asymptotic expansion for the heat kernel shows that $k(T, g)$ may be written for small $T$ as
\[
k(T, g) = (4\pi T)^{-\dim(G)/2} e^{-\delta(g)^2/4T} \left( \sum_{i < N} T^i a_i(g) + r_N(T, g) \right),
\]
where
\[
\delta(g) = \text{Tr}_g(\text{ad}(X) \text{ad}(Y)) = -\text{Tr}_g(\text{ad}(X) \text{ad}(Y)).
\]
where $\delta(g)$ is the Riemannian distance between $g$ and the identity, $a_i \in C^\infty(G)$, and $r_N \in C^\infty((0,\varepsilon] \times G)$ satisfies the estimates

$$|\partial_T^k \partial_g^\alpha r_N(T, g)| \leq C(k, \alpha) T^{N-k-|\alpha|}$$

for $N \geq 2k - |\alpha|$. It follows that for small $T$,

$$C_1 T^{-n/2} e^{-\delta(g)^2/4T} \leq k(T, g) \leq C_2 T^{-n/2} e^{-\delta(g)^2/4T},$$

(2.9)

We close this section with an estimate which differs from Corollary 2.7 in that it estimates an integral over one fibre of $\pi$, and not over all of $B$.

**Proposition 2.10.**

$$\langle \exp(\frac{\lambda}{2} |y(T)|^2) \rangle_* = \frac{\text{vol}(G/T)}{k(T, e)} \left( \frac{2\pi}{\lambda} \right)^{n/2} \int_t k(T, e^X) e^{(T-\lambda^{-1})|X|^2/2} \det_g (1 + \text{ad}(X)) dX$$

**Proof.** We will use the formula

$$\langle \exp(\frac{\lambda}{2} |y(T)|^2) \rangle_* = \left( \frac{2\pi}{\lambda} \right)^{n/2} \int_g \langle \exp(X, y(T)) \rangle_* e^{-|X|^2/2\lambda} dX$$

This may be rewritten as an integral over the Cartan subalgebra $t$ by the change of variables formula

$$\int_g f(X) dX = \int_{G/T} \left( \int_t f(\text{Ad}(g)X) \det_g (1 + \text{ad}(X)) dX \right) dg,$$

where

$$1 \leq \det_g (1 + \text{ad}(X)) \leq O(|X|^{\dim(g/t)}).$$

Using the fact that $\langle \exp(X, y(T)) \rangle_*$ is invariant under conjugation $X \mapsto \text{Ad}(g)X$, we see that

$$\langle \exp(\frac{\lambda}{2} |y(T)|^2) \rangle_* = \text{vol}(G/T) \left( \frac{2\pi}{\lambda} \right)^{n/2} \int_t \langle \exp(X, y(T)) \rangle_* e^{-|X|^2/2\lambda} \det_g (1 + \text{ad}(X)) dX.$$
We now apply the result of Lemma 2.6. By the Ito formula, we see that the Ito stochastic differential
\[ \delta \left\{ f(\gamma(t))e^{(X,y(t))} \right\} = (df(\gamma(t)) + X, \delta x(t))e^{(X,y(t))} 
+ \left( -\frac{1}{2} \Delta f(\gamma(t)) + X(f)(\gamma(t)) + \frac{1}{2} |X|^2 \right)e^{(X,y(t))} \]
From this, it follows that \( \langle e^{(X,y(T))} \rangle_\ast \) is given by the ratio of heat kernels
\[ \frac{\langle e| \exp T\left( -\frac{1}{2} \Delta + X + \frac{1}{2} |X|^2 \right) |e \rangle}{\langle e| \exp T\left( -\frac{1}{2} \Delta \right) |e \rangle} = \frac{e^{T|X|^2/2k(T,e^TX)}}{k(T,e)}, \]
since the vector field \( X \) commutes with the Laplacian \( \Delta \). \( \square \)
Note that it is an easy consequence of this proposition that
\[ \langle \exp \left( \frac{\lambda}{2} |y(T)|^2 \right) \rangle_\ast < \infty \]
if and only if \( \lambda < T^{-1} \).

3. THE TUBULAR NEIGHBOURHOOD OF A FIBRE

In this section, we will explain Gross's idea of constructing a tubular neighbourhood in \( B \) of the fibre \( \pi^{-1}(e) \) of the map \( \pi \) above the identity element of \( G \). Introduce the family of balls
\[ B_r = \{ \exp(Y) \mid |Y| < T^{1/2}r \} \subset G, \]
where \( r \) and \( T \) are small. On such a ball, we will use radial coordinates; thus, we will write \( h(x) \) instead of \( h(\exp x) \) when \( h \) is a function on \( B_r \).
Let \( R \) be a smooth vector field on \( G \) which on \( B_r \) equals the radial vector field of \( g \). We introduce the vector field \( R \) because on \( B_r \), its integral curves are the one-parameter semigroups of \( G \).
Define a map \( \varphi : g \times B \rightarrow B \) by
\[ \varphi(X,x) = x + T^{-1} \int_0^t \text{Ad}(\gamma(s))^{-1}X \, ds. \]
Since \( \gamma[\varphi(X,x)] = \exp(tX/T)\gamma(t) \), we see that
\[ \pi(\varphi(X,x)) = \pi(x) \exp(X), \]
and hence that the map \( \varphi \) defines a tubular neighbourhood of the fibre \( \pi^{-1}(e) \).

Given an element \( x \in B \) such that \( \pi(x) = \exp(X) \in B_{2r} \), we may form the path in \( B \)
\[
\sigma \in [0, 1] \mapsto x_\sigma = \varphi((\sigma - 1)X, x),
\]
which covers the path \( \exp(\sigma X) \) in \( G \); in particular, \( x_0 \) lies in the fibre \( \pi^{-1}(e) \). It is easy to check that \( x_\sigma \) is the integral curve for the vector field \( \tilde{R} \).

If \( F \in W^\infty(B) \), define the function \( \tilde{F} \) to take the value
\[
(3.1) \quad \tilde{F}(x) = \pi^* \psi F(x_0) \exp \left( \frac{1}{2} \int_0^1 \alpha(R)(x_\sigma) \, d\sigma \right)
\]
at the path \( x \), where \( \psi \in C^\infty_c(B_{2r}) \) is a smooth cut-off function which equals 1 on the ball \( B_r \). It is clear that \( \tilde{F} \) and \( F \) are equal on \( \pi^{-1}(e) \), and that \( \tilde{F} \) satisfies the differential equation
\[
(3.2) \quad \tilde{R}(\tilde{F}) = \frac{\alpha(R)}{2} F
\]
on \( \pi^{-1}(B_r) \). In the remainder of this section, we will prove the following result, which expresses the fact that the tubular neighbourhood constructed above has a certain amount of regularity.

**Theorem 3.3.** The function \( \tilde{F} \) defined above lies in \( W^\infty(B) \).

The first step in the proof that \( \tilde{F} \in W^\infty(B) \) is the special case where \( F = 1 \). If \( G_\sigma \) is a family of measurable functions on \( B \), then by Leibniz’s rule,
\[
\exp \left( \int_0^1 G_\sigma \, d\sigma \right) \in W^\infty(B)
\]
if \( \exp(G_\sigma) \) is uniformly in \( L^p(B) \), and \( G_\sigma \) is uniformly in \( W^\infty(B) \), for all \( \sigma \in [0, 1] \).
In our case, \( G_\sigma = \frac{1}{2} \alpha(R)(x_\sigma) \). Thus, it suffices to prove the following lemma.

**Lemma 3.4.** Let \( \psi \in C^\infty_c(B_{2r}) \) be such that \( |\psi| \leq 1 \).

1. The functional
\[
\pi^* \psi \alpha(R)(x_\sigma)
\]
is in \( W^\infty(B) \), uniformly in \( \sigma \in [0, 1] \).

2. The functional
\[
\pi^* \psi \exp(\alpha(R)(x_\sigma))
\]
is in \( L^p(B) \) for all \( p < \infty \), uniformly in \( \sigma \in [0, 1] \).
Proof. If $X \in \mathfrak{g}$, then

$$
\alpha(X) = d^* \tilde{X} + \pi^*(X(\log k(T)))
$$

and hence

$$
\alpha(R)[x_\sigma] = T^{-1}(y_\sigma(T), \pi^*_\sigma R) + \pi^*_\sigma(R(\log k(T))),
$$

where $y_\sigma = y[x_\sigma]$ and $\pi_\sigma(x) = \pi(x_\sigma)$. On $\pi^{-1}(B_{2r})$ the maps $\pi_\sigma$ are uniformly $W^\infty$, showing that $\pi^* \psi \pi_\sigma^*(R(\log k(T)))$ is uniformly in $W^\infty(B)$. To prove (1), we must prove that $y_\sigma(T)$ is uniformly in $W^\infty(B)$.

By the same argument as was used to prove Lemma 2.6 (1), we see that $y_\sigma$ is given by the Ito integral

$$
y_\sigma(t) = \int_0^t \text{Ad}((\sigma - 1)sX/T) \text{Ad}(\gamma(s)) \delta x_s + \frac{t(\sigma - 1)}{T} X.
$$

It follows from Theorem 2.19 of Kusuoka and Stroock [5] that $y_\sigma(T)$ is in $W^\infty(B)$, since this theorem shows that stochastic differential equations with smooth data have $W^\infty$ solutions.

Let us now prove (2). If $X \in \mathfrak{g}$, then on inverse image by $\pi$ of the ball $B_{2r}$,

$$
|\alpha(X)| \leq |d^* \tilde{X}| + \sup_{g \in B_{2r}} |X(\log k(T))|
$$

$$
\leq \frac{|X|}{T} |y(T)| + \frac{Cr}{T^{1/2}}
$$

It follows by (1.6) that on $\pi^{-1}(B_{2r})$,

$$
|\alpha(R)[x_\sigma]| \leq \frac{Cr}{T^{1/2}} |y_\sigma(T)| + Cr^2.
$$

By (*), we see that $y_\sigma(t) - t(\sigma - 1)X/T$ is a Wiener process, and hence that

$$
\langle e^p|y_\sigma(T)| \rangle \leq e^{p(1-\sigma)}|X| \langle e^p|x(T)| \rangle < \infty,
$$

proving (2). □

It remains to be proved that $\pi^* \psi F(x_0)$ lies in $W^\infty(B)$ for any $\psi \in C_0^\infty(B_{2r})$. Observe that

$$
\pi_* \left( \left| F(x_0) \exp \left( \frac{1}{p} \int_0^1 \alpha(R)(x_\sigma) \, d\sigma \right) \right|^p \right)
$$
is constant on the ball $B_{2r}$, and is equal to its value at the identity, namely $\langle |F|^p \rangle_*$. This follows by the same method as was used to prove Corollary 1.8. This shows that the function
\[ \pi^* \psi F(x_0) \exp \left( \frac{1}{p} \int_0^1 \alpha(R)(x_\sigma) \, d\sigma \right) \]
is in $L^p(B)$. It follows from Lemma 3.4 (2) that $\pi^* \psi(x) F(x_0) \in L^p(B)$ for all $p < \infty$. A similar argument shows that $\pi^* \psi |d^k F|^2(x_0) \in L^p(B)$ for all $p < \infty$, where $k \in \mathbb{N}$ and $d^k F \in W^\infty(B, H \otimes^k)$ is the tensor of $k$-th derivatives of $F$.

Denote the map $x \mapsto x_0$ by $H$; restricted to $\pi^{-1}(B_{2r})$, it is a Wiener map, that is, it lies in $I^+W^\infty(\pi^{-1}(B_{2r}), H)$. The chain rule now shows that $\pi^* \psi |d^k F|^2(x_0) \in L^p(B)$ for all $p < \infty$, and hence that $F \in W^\infty(B)$. To give an example, the second derivatives of $F(x_\sigma)$ are given by the formula
\[ d^2 F(x_0) = H^* (d^2 F) \circ (dH \otimes dH) + H^* (dF) \circ d^2 H. \]
This completes the proof of Theorem 3.3.

4. The rough logarithmic Sobolev inequalities

Our goal in this section is to prove the following logarithmic Sobolev inequality.

**Theorem 4.1.** There is a constant $C$ such that for $F \in W^\infty(B)$, uniformly for small $T$,
\[ \langle F^2 \log F \rangle_* \leq C \langle |d_* F|^2 + (T^{-1} |y(T)|^2 + 1) F^2 \rangle_* + \frac{1}{2} \langle F^2 \rangle_* \log \langle F^2 \rangle_* . \]

The idea of the proof is as follows. If $F$ is in $W^\infty(B)$, we use Theorem 3.3 to replace it by another $W^\infty$ function $\tilde{F}$ equal to $F$ on $\pi^{-1}(e)$ but which satisfies the ordinary differential equation
\[ \dot{\tilde{R}}(F) = \frac{\alpha(R)}{2} \tilde{F}. \]
It follows that $d_* F = d_* \tilde{F}$ on $\pi^{-1}(e)$, so that we may replace $F$ by $\tilde{F}$ in proving the theorem.

Along the fibre $\pi^{-1}(e)$, the horizontal part $Nd\tilde{F}$ of the differential $d\tilde{F}$ may be identified by (4.2):
\[ \dot{X}\tilde{F}|_{\pi^{-1}(e)} = \frac{1}{2} \alpha(X) \tilde{F}|_{\pi^{-1}(e)} \]
\[ = \frac{1}{2T} (y(T), X) F|_{\pi^{-1}(e)}. \]
From this, we see that
\[ \langle |d\tilde{F}|^2 \rangle_* = \langle |d_* F|^2 \rangle_* + \frac{1}{4T} \langle |y(T)|^2 F^2 \rangle_* . \]
Thus, the proof of Theorem 4.1 is reduced to that of the following result.
Theorem 4.3. There is a constant $C$ such that for positive $F \in W^\infty(B)$ satisfying (4.2), uniformly for small $T$,

$$\langle F^2 \log F \rangle_s \leq C(|dF|^2 + F^2)_s + \frac{1}{2} \langle F^2 \rangle_s \log \langle F^2 \rangle_s.$$ 

If $u$ is a smooth positive function on the unit ball $\{x \in g \mid |x| < 1\}$ such that

$$\int_G u^2 \, dx = (4\pi)^{n/2},$$

we define $u_r$ to be the rescaled function $u_r(\exp x) = r^{-n/2} u(x/T^{1/2} r)$ on $B_r$. We show that the logarithmic Sobolev inequality for $(\pi^* u_r) F$ on $B$ implies the logarithmic Sobolev inequality for $F$ on the fibre $\pi^{-1}(e)$, once $r$ is chosen sufficiently small. This is done by using Gronwall’s inequality applied to the ordinary differential equation (4.2) to relate the integrals over the fibre $\pi^{-1}(x)$, for $x \in B_r$,

$$\pi_s(F^2 \log F)(x) \text{ and } \pi_s(|dF|^2)(x),$$

to the analogous integrals over the fibre $\pi^{-1}(e)$,

$$\langle F^2 \log F \rangle_s \text{ and } \langle |dF|^2 \rangle_s.$$ 

Lemma 4.4. Let $F \in W^\infty(B)$ be a positive function satisfying (4.2) and such that $\langle F^2 \rangle = 1$. Then there is a constant $C$ such that the following inequality holds uniformly for small $T$ and $r$:

$$\langle F^2 \log F \rangle_s \leq (1 + O(T + r)) \int_G u_r^2 \pi_s(F^2 \log F) \, d(\pi_s \mu) + O(1).$$

Proof. Denote by $\varphi(x)$ the function $x^2 \log x + 1$; we introduce the function $\varphi$ because it is positive on the positive real interval.

Corollary 1.8 combined with (4.2) shows that the radial derivative of $\pi_s(\varphi(F))$ in the direction $x \in g$ equals

$$\frac{d}{dt} \pi_s(\varphi(F))(tx) = \frac{|x|}{2} \pi_s(\alpha(\hat{x}) F^2)(tx),$$

where $\hat{x} = |x|^{-1} x$. Since $T^{-1/2} |x| \leq r$ on $B_r$, the Hausdorff-Young inequality shows that

$$\frac{d}{dt} \pi_s(\varphi(F))(tx) \geq -r \pi_s(\varphi(F))(tx) - \frac{r}{2} \pi_s(e^{T^{1/2}|\alpha(\hat{x})|})(tx)$$
on the set $B_r$. By Gronwall’s inequality,

\[(\ast) \quad \pi_*(\varphi(F))(x) \geq e^{-r}\langle \varphi(F) \rangle_* - \frac{r}{2} \int_0^1 \pi_* H(tx) \, dt,\]

where $H = \sup_{x \in S^{n-1}} (e^{T^{1/2} |\alpha(\hat{x})|})$.

By the asymptotic expansion for the heat-kernel $k(T, g)$ on $G$ for small $T$,

\[
\int_G u_r^2 \, d(\pi_* \mu) = 1 + O(T + r^2).
\]

Multiplying $(\ast)$ by $u_r^2$ and integrating over $G$ with respect to the measure $\pi_* \mu$, we see that

\[
\int_G u_r^2 \pi_*(\varphi(F)) \, d(\pi_* \mu) \geq (1 + O(T + r)) \langle \varphi(F) \rangle_* - O(r^{1-n}) \int_{B_r} \left( \int_0^1 \pi_* H(tx) \, dt \right) \, d(\pi_* \mu).
\]

The second term on the right-hand side is estimated by replacing the measure $d(\pi_* \mu)$ by the equivalent measure $T^{-n/2} \, dx$ (see (2.8)), and then changing variables from $x$ to $y = tx$:

\[
\int_{B_r} \left( \int_0^1 (\pi_* H)(tx) \, dt \right) \, d(\pi_* \mu) \leq C_2 T^{-n/2} \int_{B_r} \left( \int_0^1 (\pi_* H)(tx) \, dt \right) \, dx \\
\leq C_2 T^{-n/2} \int_{B_r} \left( \int_{|y|/T^{1/2}} t^{-n} \, dt \right) \, d(\pi_* H)(y) dy \\
\leq C_2 T^{-1/2} r^{n-1} \int_{B_r} |y|^{1-n} (\pi_* H)(y) \, dy.
\]

Hölder’s inequality with respect to the measure $dy$ on $B_r$ now shows that if $s > n$,

\[
\int_{B_r} |y|^{1-n} \pi_* H \, dy \leq C(n, s) (T^{1/2} r)^{1-n/s} \left( \int_{B_r} (\pi_* H)^s \, dy \right)^{1/s}.
\]

Applying Hölder’s inequality along the fibres of $\pi$ shows that

\[
\int_{B_r} (\pi_* H)^s \, dy \leq C_1^{-1} T^{n/2} \int_{B_r} (\pi_* H)^s \, d(\pi_* \mu) \\
\leq C_1^{-1} T^{n/2} \int_{B_r} \pi_*(H^s) \, d(\pi_* \mu) = C_1^{-1} T^{n/2} \|H\|_s^s.
\]
Combining all of this, we see that
\[
\int_{B_r} \left( \int_0^1 \pi_* H(tx) \, dt \right) \, d(\pi_* \mu) \leq Cr^{1-n/s} \|H\|_s,
\]
where \(C\) is a constant depending on \(C_1, C_2, s\) and \(C(n, s)\), but not on \(T\). We may as well choose \(s = 2n\), but any real number greater than \(n\) will do equally well.

It remains to prove that \(\|H\|_s < \infty\). First of all, note that \(H\) may be bounded using an orthonormal basis \(x_i\) of \(g\), as follows:
\[
H = \sup_{\hat{x} \in S^{n-1}} \left( e^{T^{1/2} |\alpha(\hat{x})|} \right) \leq e^{T^{1/2}(|\alpha(x_1)| + \cdots + |\alpha(x_n)|)}.
\]
It follows that
\[
\|H\|_s^s = \int_B \sup_{\hat{x} \in S^{n-1}} e^{sT^{1/2} |\alpha(\hat{x})|} \, d\mu \leq \int_B e^{sT^{1/2}(|\alpha(x_1)| + \cdots + |\alpha(x_n)|)} \, d\mu
\]
\[
\leq \left( \prod_{i=1}^n \int_B e^{nsT^{1/2} |\alpha(x_i)|} \, d\mu \right)^{1/n}.
\]
If \(X \in g\), then
\[
\alpha(X) = d^* \tilde{X} - \pi^*(\text{div}_{\pi_* \mu} X)
\]
\[
= T^{-1}(y(T), X) - \pi^*(X(\log k(T))).
\]
We see by (2.8) that for small \(T > 0\),
\[
T^{1/2} |\alpha(X)| \leq \frac{C \|X\|_0}{T^{1/2}} (|y(T)| + \delta(\gamma(T)))
\]
\[
\leq \frac{\epsilon}{4T} (|y(T)|^2 + \delta(\gamma(T))^2) + \epsilon^{-1} C^2 \|X\|^2,
\]
for some constant \(C\) depending only on \(G\); here, \(\epsilon\) is an arbitrary positive constant. It now follows by the estimates of Corollary 2.7 that the integral \(\langle e^{T^{1/2} |\alpha(X)|} \rangle\) is uniformly bounded for small \(T\), proving that \(\|H\|_s < \infty\).

In this way, we have proved that
\[
\int_G u^2 \pi_* (F^2 \log F) \, d(\pi_* \mu) = \int_G u^2 \pi_* (\varphi(F)) \, d(\pi_* \mu) - (1 + O(T + r^2))
\]
\[
\geq (1 + O(T + r)) \langle F^2 \log F \rangle_* - O(1),
\]
which after a little rearrangement gives the lemma. □

If we apply the logarithmic Sobolev inequality for the Wiener space $B$ to the function $(\pi^* u_r)F \in W^\infty(B)$, which satisfies $\langle |(\pi^* u_r)F|^2 \rangle = 1 + O(T + r^2)$, we obtain the inequality

$$\langle |(\pi^* u_r)F|^2 \log((\pi^* u_r)F) \rangle \leq \langle |d((\pi^* u_r)F)|^2 \rangle + O(T + r^2).$$

Since

$$|(\pi^* u_r)F|^2 \log((\pi^* u_r)F) = (\pi^* u_r)^2 F^2 \log F + \pi^* (u_r^2 \log u_r) F^2$$

and $\pi^*(F^2) = 1$, we see that

$$\int_{B_r} u_r^2 \pi^*(F^2 \log F) d(\pi^* \mu) \leq \int_{B_r} u_r^2 \pi^*(|dF|^2) d(\pi^* \mu)$$

$$+ \int_{B_r} \pi^*(|d(\pi^* u_r)|^2) d(\pi^* \mu) - \int_{B_r} u_r^2 \log u_r d(\pi^* \mu) + O(T + r^2).$$

To handle the second term on the right-hand side, we use (2.5), which shows that

$$\int_{B_r} \pi^*(|d(\pi^* u_r)|^2) d(\pi^* \mu) = O(r^{-2}),$$

while to bound the third term, we use the fact that $x^2 \log x \geq -(2e)^{-1}$. Thus, we see that

$$(4.5) \quad \int_{B_r} u_r^2 \pi^*(F^2 \log F) d(\pi^* \mu) \leq \int_{B_r} u_r^2 \pi^*(|dF|^2) d(\pi^* \mu) + O(r^{-2}).$$

To complete the proof of Theorem 4.3, we will imitate the proof of Lemma 4.4 to obtain an upper bound for

$$\int_{B_r} u_r^2 \pi^*(|dF|^2 + \varepsilon F^2 \log F) d(\pi^* \mu)$$

in terms of $\langle |dF|^2 + \varepsilon F^2 \log F \rangle$, where $\varepsilon$ is a small positive constant.

**Lemma 4.6.** Let $F \in W^\infty(B)$ be a positive function satisfying (4.2) and such that $\pi^*(F^2) = 1$ on the ball $B_r$. Then there is a constant $C$ such that the following inequality holds uniformly for small $T$ and $r$:

$$\int_G u_r^2 \pi^*(|dF|^2 + \varepsilon F^2 \log F) d(\pi^* \mu) \leq (1 + O(T + r)) \langle |dF|^2 + \varepsilon F^2 \log F \rangle + O(1).$$
Proof. If $F \in W^\infty(B)$ satisfies the ordinary differential equation (4.2) along the one-parameter semigroup $\exp(tX) \subset G$, it follows that

$$X \pi_* (|dF|^2) = \pi_* (\dot{X}|dF|^2) - \pi_* (\alpha(X)|dF|^2) = 2\pi_* (dF, [\dot{X}, dF]) - \pi_* (F dF, d\alpha(X)).$$

The first term of the right-hand side is bounded by means of the formula

$$[d, \dot{X}] = T^{-1}(\Pi^* \rho(\nabla X)\Pi + X \cdot d\Pi, d),$$

where $\rho(\nabla X)$ is the section of the bundle $\text{End}(TM)$ over $M$ corresponding to $\nabla X \in \Gamma(M, TM \otimes TM)$. It is clear that the Hilbert-Schmidt norm of $T^{-1} \Pi^* \rho(\nabla X)\Pi$ is uniformly bounded for small $T$, and the same is true for $T^{-1} X \cdot d\Pi$ by Proposition 2.6. In this way, we obtain the inequality

$$X \pi_* (|dF|^2) \leq \pi_* (C|dF|^2 + |F| |dF| |d\alpha(X)|).$$

Applying the Cauchy inequality

$$|F| |dF| |d\alpha(X)| \leq T^{-1/2}|dF|^2 + T^{1/2} F^2 |d\alpha(X)|^2,$$

we see that

$$X \pi_* (|dF|^2) \leq \pi_* ((T^{-1/2} + C)|dF|^2 + T^{1/2} F^2 |d\alpha(X)|^2).$$

We can now bound the radial derivative of $\pi_* (|dF|^2 + r\varphi(F))$, in the direction $x \in \mathfrak{g}$, where $\varphi(x) = x^2 \log x + 1$. On the set $B_r$, it satisfies the bound, uniform in $T$ for $T$ small,

$$\frac{d}{dt} \pi_* (|dF|^2 + \varepsilon \varphi(F))(tx) \leq O(r) \pi_* (|dF|^2 + \varepsilon \varphi(F))(tx) + O(r) \pi_* (F^2 (T|d\alpha(\hat{x})|^2 + rT^{1/2}|\alpha(\hat{x})|))(tx) \leq O(r) \pi_* (|dF|^2 + \varepsilon \varphi(F))(tx) + O(\varepsilon r) \pi_* (e^{\varepsilon T} |d\alpha(\hat{x})|^2 + e^{T^{1/2}|\alpha(\hat{x})|})(tx),$$

where we have applied the Hausdorff-Young inequalities

$$F^2 (T|d\alpha(\hat{x})|^2) \leq \frac{\varepsilon}{2} \varphi(F) + \varepsilon e^{-T}|d\alpha(\hat{x})|^2, \quad \text{and}$$

$$F^2 (T^{1/2}|\alpha(\hat{x})|) \leq \frac{1}{2} \varphi(F) + e^{T^{1/2}|\alpha(\hat{x})|).$$
By Gronwall’s inequality,

\[ \pi_s(|dF|^2 + \varepsilon \varphi(F))(x) \geq (1 + O(r)) \langle |dF|^2 + \varepsilon \varphi(F) \rangle_s + O(\varepsilon r) \int_0^1 \pi_s J(tx) \, dt, \]

where

\[ J = \sup_{\hat{x} \in S^{n-1}} (e^{T^{-1} |d\alpha(\hat{x})|^2} + e^{T^{1/2} |d\alpha(\hat{x})|})(tx). \]

The rest of the proof is the same as that of Lemma 4.4, except that we must bound \( \|J\|_s \) instead of \( \|H\|_s \). Let \( X \in g \). Since \( d(\pi^* f) = T^{1/2} \pi^* (df) \), we see that

\[ d\alpha(X) = T^{-1} (d\gamma(T), \pi^* X) - d\pi^* (X (\log k(T))) = T^{-1} (d\gamma(T), X) - T^{1/2} \pi^* (d(X (\log k(T)))) \]

It follows that

\[ T |d\alpha(X)|^2 \leq \frac{2 \|X\|^2}{T} |d\gamma(T)|^2 + 2T^2 |d(X (\log k(T)))|^2 \leq \frac{2 \|X\|^2}{T} |d\gamma(T)|^2 + C_0(G) \|X\|^2_0 + C_1(G) T \|X\|^2_1, \]

where the constants \( C_i(G) \) depend only on the group \( G \). The uniform bound on \( \langle e^{T^{-1} |d\alpha(X)|^2} \rangle \) for \( T \) small enough follows from Corollary 2.7, and we see that \( \|J\|_s < \infty \) uniformly. \( \Box \)

Let us assemble the results obtained so far in this section. Under the conditions on the function \( F \) of Lemma 4.4, we see by combining Lemma 4.4 and (4.5) that

\[ \langle F^2 \log F \rangle_s \leq (1 + O(T + r)) \int_{B_r} u_x^2 \pi_s (|dF|^2) \, d(\pi_s \mu) + O(r^{-2}). \]

Combining this with Lemma 4.6, we see that

\[ \langle F^2 \log F \rangle_s \leq (1 + O(T + r)) \langle |dF|^2 + \varepsilon F^2 \log F \rangle_s + O(r^{-2}). \]

If we choose \( \varepsilon \) sufficiently small (so that \( (1 + O(T + r)) \varepsilon \leq \frac{1}{2} \)), we obtain the logarithmic Sobolev inequality

\[ \langle F^2 \log F \rangle_s \leq (1 + O(T + r)) \langle |dF|^2 \rangle_s + O(r^{-2}) \langle F^2 \rangle_s + \frac{1}{2} \langle F^2 \rangle_s \log \langle F^2 \rangle_s, \]

where we have now removed the condition that \( \langle F^2 \rangle_s = 1 \). This immediately leads to Theorem 4.3.
References