THE CYCLIC HOMOLOGY OF CROSSED PRODUCT ALGEBRAS, I.

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INTRODUCTION

In their article [9] on cyclic homology, Feigin and Tsygan have given a spectral sequence for the cyclic homology of a crossed product algebra, generalizing Burghelea’s calculation [4] of the cyclic homology of a group algebra. For an analogous spectral sequence for the Hochschild homology of a crossed product algebra, see Brylinski [2], [3].

In this article, we give a new derivation of this spectral sequence, and generalize it to negative and periodic cyclic homology $HC_{-}^\bullet(A)$ and $HP_\bullet(A)$. The method of proof is itself of interest, since it involves a natural generalization of the notion of a cyclic module, in which the condition that the morphism $\tau \in \Lambda(n,n)$ is cyclic of order $n+1$ is relaxed to the condition that it be invertible. We call this category the paracyclic category.

Given a paracyclic module $P$, we can define a chain complex $C(P)$, with differentials $b$ and $B$, which respectively lower and raise degree. The condition that the module $P$ is paracyclic translates to the condition on $C(P)$ that $1 - (bB + Bb)$ is invertible. Our main result is to show that there is an analogue of the Eilenberg-Zilber theorem for bi-paracyclic modules. It is then easy to obtain a new expression for the cyclic homology of a crossed product algebra which leads immediately to the spectral sequence of Feigin and Tsygan.

If $M$ is a module over a commutative ring $k$, we will denote by $M^{(k)}$ the iterated tensor product, defined by $M^{(0)} = k$ and $M^{(k+1)} = M^{(k)} \otimes M$. If $M$ and $N$ are graded modules, we will denote by $M \otimes N$ their graded tensor product.

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1. Paracyclic modules and crossed product algebras

Let $A$ be a unital algebra over a fixed commutative ring $k$. Let $G$ be a (discrete) group which acts on $A$ by automorphisms (which we suppose to fix the identity). Recall the definition of the crossed product algebra $A \rtimes G$: the underlying $k$-module is $A \otimes k[G]$, that is, functions from $G$ to $A$ with finite support, and the product is given on elementary tensor products $a \otimes g$ by the formula

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = (a_1(g_1a_2)) \otimes (g_1g_2).$$

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It is easy to check that this product is associative and unital. In the special case \( k \times G \), we obtain the group ring \( k[G] \).

A cyclic module \( P(n) \) has an underlying simplicial structure, with face morphisms \( d_i : P(n + 1) \to P(n), 0 \leq i \leq n \), and degeneracy morphisms \( s_i : P(n - 1) \to P(n), 0 \leq i \leq n \). In addition, it has morphisms \( t : P(n) \to P(n) \) for each \( n \), such that \( t^{n+1} = 1 \), and \( t \cdot d_i \cdot t^{-1} = d_{i+1}, t \cdot s_i \cdot t^{-1} = s_{i-1} \). We denote the morphism \( d_n : P(n) \to P(n) \) by \( d \), and the morphism \( t \cdot s_0 \cdot t^{-1} : P(n - 1) \to P(n) \) by \( s \). The composition \( d \cdot s \) is equal to \( t \); it follows that together, the morphisms \( d \) and \( s \) generate the action of the cyclic category on \( P \).

Connes defines in [5] a cyclic module \( B^2 \) for any unital algebra \( B \). This cyclic module has as its \( n \)-th space \( B^2(n) \) the \( k \)-module \( B^{(n+1)} \), and the actions of \( d \), \( s \) and \( t = d \cdot s \) are given by the following formulas:

\[
\begin{align*}
    d(a_0, \ldots, a_n) &= (a_n a_0, a_1, \ldots, a_{n-1}), \\
    s(a_0, \ldots, a_n) &= (1, a_0, \ldots, a_n), \\
    t(a_0, \ldots, a_n) &= (a_n, a_0, \ldots, a_{n-1}).
\end{align*}
\]

The goal of this article is to understand the cyclic module \((A \rtimes G)^2\) associated to the crossed product algebra \( A \rtimes G \). This is the cyclic module whose \( n \)-th space \((A \rtimes G)^2(n)\) is \( k[G^{n+1}] \otimes A^{(n+1)} \). Denote the elementary tensor

\[
(a_0 \otimes g_0) \otimes \ldots \otimes (a_n \otimes g_n) \in (A \rtimes G)^2(n)
\]

by \((g_0, \ldots, g_n|_{h_0^{-1} a_0, \ldots, h_n^{-1} a_n})\), where \( h_i = g_i \ldots g_n \). This notation is motivated by the fact that in reordering the tensor product so that all of the factors \( k[G] \) occur to the left, we must pass the group elements \( g_i, \ldots, g_n \) past the algebra element \( a_i \in A \).

We have the following formulas for \( d \), \( s \) and \( t = d \cdot s \) acting on the cyclic \( k \)-module \((A \rtimes G)^2\):

\[
\begin{align*}
    d(g_0, \ldots, g_n|a_0, \ldots, a_n) &= (g_n g_0, g_1, \ldots, g_{n-1}|_{g_{n-1} (g_{n-1}^{-1} a_n) a_0}, g_n a_1, \ldots, g_n a_{n-1}), \\
    s(g_0, \ldots, g_n|a_0, \ldots, a_n) &= (1, g_0, \ldots, g_n|_{1, a_0}, \ldots, a_n), \\
    t(g_0, \ldots, g_n|a_0, \ldots, a_n) &= (g_n, g_0, \ldots, g_{n-1}|_{g_n g_{n-1}^{-1} a_n, g_n a_0, \ldots, a_{n-1}}),
\end{align*}
\]

where \( g = g_0 \ldots g_n \).

Inspired by the above formulas, we would like to define a bi-cyclic module \( A_{2}G \) whose diagonal is the above cyclic module. Denote the two sets of generators by \((\bar{d}, \bar{s}, \bar{t})\) and \((d, s, t)\) respectively; the barred and unbarred maps commute with each other. We define \( A_{2}G(p, q) \) to be \( k[G^{p+1}] \otimes A^{(q+1)} \), spanned by elementary tensor products which we denote \((g_0, \ldots, g_p|a_0, \ldots, a_q)\). The two sets of generators for the bi-cyclic structure should be defined in such a way as to factor each of the generators of the cyclic structure on \( C(A \rtimes G) \) into two pieces, the barred ones acting within \( k[G^{p+1}] \) and the unbarred ones within
$\mathbf{k}[G^{q+1}]$. The natural formulas for the action of $(d, s, t)$ and $(d, s, t)$ are as follows:

\[
\bar{d}(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_p, g_0, g_1, \ldots, g_{p-1}|g_p a_0, \ldots, g_p a_q),
\]

\[
\bar{s}(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_0, \ldots, g_p|1, a_0, \ldots, a_q),
\]

\[
\bar{t}(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_p, g_0, \ldots, g_{p-1}|g_p a_0, \ldots, g_p a_q),
\]

\[
d(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_0, \ldots, g_p|g^{-1} a_q a_0, a_1, \ldots, a_{q-1}),
\]

\[
s(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_0, \ldots, g_p|1, a_0, \ldots, a_q),
\]

\[
t(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_0, \ldots, g_p|g^{-1} a_q a_0, a_1, \ldots, a_{q-1}),
\]

where $g = g_0 \ldots g_p$. However, it is easy to see that this does not define a bi-cyclic structure: on $A_G^1$, the operators $\bar{d}^{q+1}$ and $t^{q+1}$ are not equal to the identity, although $\bar{d}^{q+1} = t^{-q-1}$. Let $T = \bar{d}^{q+1} = t^{-q-1}$: it is given by the formula

\[
T(g_0, \ldots, g_p|a_0, \ldots, a_q) = (g_0, \ldots, g_p|g a_0, \ldots, g a_q).
\]

In order to understand the structure of $A_G^1$, we use a category related to the cyclic category $\Lambda$ of Connes. This category $\Lambda_{\infty}$, which we call the paracyclic category has the same set of objects as the simplicial category $\Delta$, namely the natural numbers $n$. Recall that the morphisms $\Delta(n, m)$ from $n$ to $m$ are the monotonically increasing maps from the set $\{0, \ldots, n\}$ to the set $\{0, \ldots, m\}$. Similarly, the morphisms $\Lambda_{\infty}(m, n)$ from $m$ to $n$ in the paracyclic category $\Lambda_{\infty}$ are monotonically increasing maps $f$ from $\mathbb{Z}$ to itself such that

\[
f(i + k(m + 1)) = f(i) + k(n + 1)
\]

for all $k \in \mathbb{Z}$. We identify $\Delta$ with the subcategory of $\Lambda_{\infty}$ such that $f \in \Lambda_{\infty}(m, n)$ lies in $\Delta$ if and only if $f$ maps $\{0, \ldots, m\} \subset \mathbb{Z}$ into $\{0, \ldots, n\}$. The paracyclic category has been studied by Fiedorowicz and Loday [8] and Nistor [15]. Dwyer and Kan [7] study the duplicial category, similar to the paracyclic category, except that $\mathbb{Z}$ is replaced by $\mathbb{N}$. The cyclic category $\Lambda$ of Connes [5] is the quotient of $\Lambda_{\infty}$ by the relation $T = 1$, while the categories $\Lambda_{\infty}$ of Feigin-Tsygan and Bökstedt-Hsiang-Madsen [1] are the quotient of $\Lambda_{\infty}$ by the relation $T^r = 1$.

The category $\Lambda_{\infty}$ is generated by morphisms $\partial : n \rightarrow n + 1$,

\[
\partial(k) = k + 1 \quad \text{if} \quad 0 \leq k \leq n,
\]

and $\sigma : n \rightarrow n - 1$,

\[
\sigma(k) = k \quad \text{if} \quad 0 \leq k \leq n.
\]

The map $\partial$ is the face map $\partial_n$ in the simplicial category $\Delta$, while $\sigma$ does not lie in $\Delta$. Denote $\sigma \partial$ by $\tau$; it corresponds to the map

\[
\tau(k) = k + 1 \quad \text{for all} \quad k \in \mathbb{Z}.
\]

The face and degeneracy maps of the simplicial category $\Delta$ embedded in $\Lambda_{\infty}$ are given by the formulas

\[
\partial_i = \tau^{-i-1} \cdot \partial \cdot \tau^i : n - 1 \rightarrow n, \quad 0 \leq i \leq n,
\]

\[
\sigma_i = \tau^i \cdot \sigma \cdot \tau^{-i-1} : n + 1 \rightarrow n, \quad 0 \leq i \leq n.
\]
Since \( \sigma = \tau^{-1} \cdot \sigma_0 \cdot \tau \), we may think of \( \sigma \) as an extra degeneracy \( \sigma_{-1} \).

Each object \( n \) in the category \( \Lambda_\infty \) has an automorphism \( T = \tau^{n+1} \), and it is easily seen that if \( f \) is any morphism in \( \Lambda_\infty \), then \( T \circ f = f \circ T \). This shows that \( T \) induces an invertible automorphism of the category \( \Lambda_\infty \).

A paracyclic \( k \)-module \( P \) is a contravariant functor from \( \Lambda_\infty \) to the category of \( k \)-modules. In particular, a paracyclic module may be considered as a simplicial module, by the inclusion \( \Delta \subset \Lambda_\infty \). Denote the actions of \( \partial_i \), \( \sigma_i \) and \( \tau \) on a paracyclic module \( P \) by \( d_i \), \( s_i \) and \( t \), and of \( \partial_i \) and \( \sigma_i \) by \( d_i \) and \( s_i \). The category of paracyclic modules has an automorphism \( T \), induced by the automorphism \( T \) of the category \( \Lambda_\infty \).

We now see that \( A_\mathbb{Z}G \) is a bi-paracyclic module which satisfies the extra relation \( T = T^{-1} \); this implies that the diagonal paracyclic module of \( A_\mathbb{Z}G \) is cyclic. We call a bi-paracyclic module satisfying this extra relation a cylindrical module. The cylindrical category \( \Sigma \) is the quotient of \( \Lambda_\infty \times \Lambda_\infty \) by the relation \( T = T^{-1} \); a cylindrical module is a contravariant functor from \( \Sigma \) to the category of modules.

Later, we will be interested in the paracyclic module \( A_\mathbb{Z}G(0, n) \) which forms the bottom row of the bi-paracyclic module \( A_\mathbb{Z}G \). This paracyclic module, which we denote \( A^G_n \), is given explicitly by \( n \mapsto \mathbb{Z}[G] \otimes A^{(n+1)} \). The group \( G \) acts on \( A^G_n \) by the formula

\[
h \cdot (g|a_0, \ldots, a_n) = (hgh^{-1}|ha_0, \ldots, ha_n).
\]

Let \( \Lambda^n_\infty \) be the paracyclic set \( m \mapsto \Lambda_\infty(m, n) \), and let \( |\Lambda^n_\infty| \) be the geometric realization of the simplicial set underlying \( \Lambda^n_\infty \). In the following proposition, we parametrize the \( n \)-simplex \( \Delta^n \) by

\[
\Delta^n = \{0 \leq t_1 \leq \cdots \leq t_n \leq 1\}.
\]

(This result is similar to Proposition 2.7 of Dwyer-Hopkins-Kan [6] and Theorem 3.4 of Jones [12], and may be proved in the same way.)

**Proposition 1.1.** The geometric realization \( |\Lambda^n_\infty| \) is homeomorphic to \( \mathbb{R} \times \Delta^n \), with non-degenerate \( n+1 \)-simplices given, for \( 0 \leq j \leq n \) and \( k \in \mathbb{Z} \), by

\[
S^n_k = \{(t|t_1, \ldots, t_n) \in \mathbb{R} \times \Delta^n \mid t_j \leq t + k \leq t_{j+1}\}.
\]

Each simplex \( S^n_k \) is identified with \( \Delta^{n+1} \) by the map

\[
(t_1, \ldots, t_{n+1}) \mapsto (t_j + k|t_{j+1}, \ldots, t_{n+1}, t_1, \ldots, t_{j-1})
\]

The maps \( \partial : |\Lambda^n_\infty| \to |\Lambda^{n+1}_\infty| \), \( \sigma : |\Lambda^n_\infty| \to |\Lambda^{n-1}_\infty| \), \( \tau : |\Lambda^n_\infty| \to |\Lambda^n_\infty| \) and \( T : |\Lambda^n_\infty| \to |\Lambda^n_\infty| \) are given by the formulas

\[
\partial(t|t_1, \ldots, t_n) = (t|t_1, \ldots, t_n),
\]

\[
\sigma(t|t_1, \ldots, t_n) = (t + t_1|t_2 - t_1, \ldots, t_n - t_1, t_n),
\]

\[
\tau(t|t_1, \ldots, t_n) = (t + t_1|t_2 - t_1, \ldots, t_n - t_1),
\]

\[
T(t|t_1, \ldots, t_n) = (t + 1|t_1, \ldots, t_n).
\]
2. Parachain complexes

The following definition is inspired by the definition of a duchain complex due to Dwyer and Kan [7].

**Definition 2.1.** A **parachain complex** is a graded $k$-module $(V_i)_{i \in \mathbb{N}}$ with two operators $b : V_i \to V_{i-1}$ and $B : V_i \to V_{i+1}$, such that

1. $b^2 = B^2 = 0$, and
2. the operator $T = 1 - (bB + Bb)$ is invertible.

It may be easily checked that $T$ commutes with both $b$ and $B$. When $T$ is the identity, the two differentials $b$ and $B$ commute; such a parachain complex is called a **mixed complex**.

In the definition of a duchain complex, there is no condition on $T$: parachain complexes bear the same relationship to paracyclic modules that duchain complexes bear to duplicial modules.

If $V_\bullet$ is a graded vector space, let $V_\bullet[u]$ be the graded vector space of formal power series in a variable $u$ of degree $-2$ with coefficients in $V_\bullet$. If $V_\bullet$ is a mixed complex, one considers the associated complex $V_\bullet[u]$ with differential $b + uB$; this motivates considering the operator $b + uB$ on $V_\bullet[u]$ even when $V_\bullet$ is only a paracyclic module.

**Definition 2.2.** A morphism between parachain complexes $V_\bullet$ and $\tilde{V}_\bullet$ is a map from $V_\bullet[u]$ to $\tilde{V}_\bullet[u]$ homogeneous of degree 0,

$$f = \sum_{k=0}^{\infty} u^k f_k,$$

such that $(\tilde{b} + u\tilde{B}) \cdot f = f \cdot (b + uB)$.

Without introducing the operator $b + uB$, a morphism $f : V_\bullet \to \tilde{V}_\bullet$ may be defined as a sequence of maps $f_k : V_i \to \tilde{V}_{i+2k}, k \geq 0$, such that

$$\tilde{b} \cdot f_k + \tilde{B} \cdot f_{k-1} = f_k \cdot b + f_{k-1} \cdot B.$$

The composition of two parachain complex maps is a parachain complex map, and a map of parachain complexes $f$ satisfies $\tilde{T} \cdot f_i = f_i \cdot T$. Thus, the operator $T$ defines an action of $\mathbb{Z}$ on the category of parachain complexes.

There is a functor $C$ from paracyclic modules to parachain complexes, with underlying graded module $C_n(P) = P(n)$ and operators $b = \sum_{i=0}^{n}(-1)^i d_i$ and $B = (1 - (-1)^{n+1}) sN$; here $N$ is the norm operator $N = \sum_{i=0}^{n}(-1)^i t^i$.

The proof of the following theorem is close to the discussion of Section 1 of [14].

**Theorem 2.3.** The functor $P \mapsto C(P)$ is a $\mathbb{Z}$-equivariant functor from the category of paracyclic $k$-modules to the category of parachain complexes over $k$, that is,

1. $b^2 = B^2 = 0$,
2. $bB + Bb = 1 - T$, and
3. it intertwines the natural transformations $T$ of these two categories.
Proof. The proof that \( b^2 = 0 \) is the same as usual, since it only depends on the underlying simplicial module structure on \( P \).

The operator \( B^2 : C_n(P) \to C_{n+2}(P) \) is given by

\[
B^2 = (1 - (-1)^{n+2}t)sN(1 - (-1)^{n+1}t)sN
= (1 - (-1)^{n+2}t)s(1 - T)sN
= (1 - (-1)^{n+2}t)ssN(1 - T)
= (s_n - (-1)^{n+2}s_{n+1})sN(1 - T),
\]

which shows that \( B^2 \) is zero in the associated chain complex. Here, we have used the formulas

\[
(1 - (-1)^n t)N = 1 - T, \quad ss = s_n s, \quad ts = T - (-1)^n t - n - 1.
\]

To calculate \( bB + Bb \), we introduce the operator \( b' = \sum_{i=1}^n (-1)^i d_i \) on \( C_n(P) \).

Lemma 2.4. If \( P \) is a paracyclic module, then on \( C(P)_n \) we have the formulas

1. \( b(1 - (-1)^n t) = (1 - (-1)^{n-1} t)b' \),
2. \( Nb = b'N \), and
3. \( sb' + b's = 1 \).

Proof. The first formula is proved in the same way as in [14]. The operators \( b \) and \( b' \) on \( C_n(P) \) are given by

\[
b = \sum_{i=0}^n (-1)^i t^i dt^{-i-1}, \quad b' = \sum_{i=0}^{n-1} (-1)^i t^i dt^{-i-1}.
\]

Thus,

\[
b(1 - (-1)^n t) = \sum_{i=0}^n (-1)^{n-i} t^i d(t^{-i-1} - (-1)^{n-t^{-i}})
= -(-1)^n d + (1 - (-1)^{n-1} t) \sum_{i=0}^{n-1} (-1)^i t^i dt^{-i-1} + (-1)^n t^n dt^{-n-1}.
\]

However, \( t^n dt^{-n-1} = d \), and the formula follows.

We leave the proof of the second formula to the reader. To prove the third formula, we use the fact that on \( n, \)

\[
\tau^{-i-1} \partial \tau^i \sigma = \sigma \tau^{-i-2} \partial \tau^{i+1}
\]

for \( 0 \leq i \leq n - 1 \). Thus, it follows that

\[
sb' = \sum_{i=0}^{n-1} (-1)^i st^i dt^{-i-1} = \sum_{i=0}^{n-1} (-1)^i t^{i+1} dt^{-i-2} s = 1 - b's. \quad \square
\]
As a corollary of this lemma, we see that on $C_n(P)$,

$$bB = (1 - (-1)^n t)b'sN,$$
and

$$Bb = (1 - (-1)^n t)sb'N,$$

and hence that

$$Bb + bB = (1 - (-1)^n t)(sb' + b's)N = (1 - (-1)^n t)N
\quad = 1 - t^{n+1} = 1 - T.$$

This completes the proof of the theorem. □

A multi-parachain complex is a $\mathbb{N}^k$-graded module $V_{n_1...n_k}$ with operators

$$b_i : V_{n_1...n_i...n_k} \rightarrow V_{n_1...n_i-1...n_k},$$

$$B_i : V_{n_1...n_i...n_k} \rightarrow V_{n_1...n_i+1...n_k}.$$

The operators $\{b_i, B_i\}$ and $\{b_j, B_j\}$ are required to (graded) commute if $i$ and $j$ are not equal, while $T_i = 1 - (b_iB_i + B_ib_i)$ is required to be invertible.

There is a functor $V \mapsto \text{Tot}(V)$ from multiparachain complexes to parachain complexes, which we will call the **total parachain complex**. It is formed by setting

$$\text{Tot}_n(V) = \sum_{n_1 + \cdots + n_k = n} V_{n_1...n_k},$$

with operators

$$\text{Tot}(b) = \sum_{i=1}^k b_i,$$
$$\text{Tot}(B) = \sum_{i=1}^n T_{i+1} \cdots T_k B_i.$$

The definition of $\text{Tot}(B)$ on $\text{Tot}(V)$ may seem a little strange, but is justified by the following lemma, which shows that $\text{Tot}(V)$ is a parachain complex.

**Lemma 2.5.** The total $T$-operator $\text{Tot}(T) = T_1 \cdots T_k$.

**Proof.** The proof uses the fact that $\{b_i, B_i\}$ and $\{b_j, B_j\}$ commute for $i \neq j$. Thus, we see that

$$1 - (\text{Tot}(b) \text{Tot}(B) + \text{Tot}(B) \text{Tot}(b)) = 1 - \sum_{i=1}^k [b_i, B_i]T_{i+1} \cdots T_k$$
$$= 1 - \sum_{i=1}^k (1 - T_i)T_{i+1} \cdots T_k,$$
from which the lemma follows. □

We are most interested in the special case of biparachain complexes. We will denote $b_1$ and $b_2$ by $\bar{b}$ and $b$, and $B_1$ and $B_2$ by $\bar{B}$ and $B$. When $T = T^{-1}$, we call a bi-parachain complex $V$ a **cylindrical complex**; in this case, the above lemma shows that $\text{Tot}(T) = 1$, that is, $\text{Tot}(V)$ is a mixed complex.

Finally, we have the normalized chain functor $N$ from paracyclic modules to parachain complexes, with underlying graded module

$$N_n(P) = P(n)/\sum_{i=0}^{n} \text{im}(s_i),$$

and operators $b, B$ induced by those on $C(P)$. It is a standard result that the quotient map $(C(P), b) \rightarrow (N(P), b)$ is a quasi-isomorphism of complexes. More generally, if $P$ is a multi-paracyclic module, we denote by $N(P)$ the multi-paracyclic complex obtained by normalizing successively in all directions.

### 3. The Eilenberg-Zilber Theorem for Paracyclic Modules

Let $P(p, q)$ be a bi-paracyclic module, and let $C(P)$ be the biparachain complex obtained by forming the chain complex successively in both directions. Let $\text{Tot}(C(P))$ be the total parachain complex of $C(P)$: by the above results, if $P$ is a cylindrical module, $\text{Tot}(C(P))$ is a mixed complex. Using the diagonal embedding of $\Lambda_\infty$ into $\Lambda_\infty \times \Lambda_\infty$, we see that the diagonal $n \mapsto P(n, n)$ is a paracyclic object, which we will denote by $\Delta P(n)$. The action of $\Lambda_\infty$ on $\Delta P(n)$ is generated by the maps $\bar{dd}, \bar{ss}$ and $\bar{tt}$.

The shuffle product is a natural map from the total complex $(\text{Tot}(C(P)), \text{Tot}(b))$ to the chain complex of the diagonal $(C(\Delta P), b)$, which is an equivalence of complexes; this is proved using the method of acyclic models. This product was extended to a map of mixed complexes by Hood and Jones [11] when $P$ is bi-cyclic; see also our paper [10], where we give explicit formulas for this map. We will give explicit formulas on normalized chains; to extend these results to the unnormalized chains, we may apply the results of Kassel [13], who shows how to construct a homotopy inverse to the normalization map.

**Theorem 3.1.** Let $P$ be a bi-paracyclic module. There is a natural quasi-isomorphism $f_0 + uf_1 : \text{Tot}(C(P)) \rightarrow C(\Delta P)$ of parachain complexes such that $f_0 : \text{Tot}(C(P)) \rightarrow C(\Delta P)$ is the shuffle map.

**Proof.** We must construct a map

$$f_1 : \text{Tot}(C(P)) \rightarrow C_{*+2}(\Delta P)$$

to satisfy the following two formulas:

$$b \cdot f_1 = f_1 \cdot (b + \bar{b}) - B \cdot f_0 + f_0 \cdot (B + \bar{B}),$$

$$B \cdot f_1 = f_1 \cdot (B + \bar{B}).$$

The fact that $f = f_0 + uf_1$ is a quasi-isomorphism then follows by a standard argument from the fact that it is true for the shuffle product $f_0$. 

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Let \( \iota_n \) be the non-degenerate \( n \)-simplex in \( \Lambda^\infty_n \), corresponding to the identity map on the object \( n \in \Lambda^\infty \). This simplex corresponds to the geometric simplex

\[
\{(0|t_1, \ldots, t_n) \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\} \subset |\Lambda^\infty_n|,
\]

in the geometric realization of \( \Lambda^\infty_n \), as described in Proposition 1.1. By definition, the non-degenerate simplices of \( \Lambda^\infty_n \) are in one-to-one correspondence with the morphisms of \( \Lambda^\infty \) with range \( n \), and these simplices are obtained by applying the corresponding morphism of the opposite category \( \Lambda^\infty_{op} \) to \( \iota_n \).

If \( X \) is a paracyclic set, the chains on \( X \) with values in \( k \), written \( k[X] \), form a paracyclic module in an evident way. Similarly, the module of chains on the bi-paracyclic set \( \Lambda^\infty_p \times \Lambda^\infty_q \) is a bi-paracyclic module, which we denote by \( k[\Lambda^\infty_p \times \Lambda^\infty_q] \). The following result is the analogue of Lemma 2.1 of Hood and Jones [11].

**Lemma 3.3.** If \( P \) is a bi-paracyclic module and \( x \in P(p,q) \), there is a unique map of bi-paracyclic modules \( i_x : k[\Lambda^\infty_p \times \Lambda^\infty_q] \to V \) such that \( i_x(\iota_p \times \iota_q) = x \).

From this lemma and the fact that \( f_1 \) is to be natural, we see that it suffices to define \( f_1 \) on the elements \( \iota_p \times \iota_q \in k[\Lambda^\infty_p \times \Lambda^\infty_q] \). The following argument may be better understood by reference to Figure 3.1.

The image of \( \iota_p \times \iota_q \) under the map \( B \) is the chain

\[
\{0\} \times [0,1] \times \Delta^p \times \Delta^q \subset \mathbb{R}^2 \times \Delta^p \times \Delta^q.
\]
Similarly, its image under the map $T\bar{B}$ is the chain

$$[0, 1] \times \{1\} \times \Delta_p \times \Delta_q.$$ 

Finally, $B \cdot f_0(\iota_p \times \iota_q)$ is the chain

$$\{(t, t) \mid t \in [0, 1]\} \times \Delta_p \times \Delta_q.$$ 

From this, we see that $f_0 \cdot (T\bar{B} + B) - B \cdot f_0$ applied to $\iota_p \times \iota_q$ is the chain

$$\partial K \times \Delta_p \times \Delta_q,$$

where $K$ is the triangle $\{(s, t) \mid 0 \leq t \leq s \leq 1\} \subset \mathbb{R}^2$. It is now obvious that in order for the formula

$$b \cdot f_1 - f_1 \cdot (b + \bar{b}) = f_0 \cdot (T\bar{B} + B) - B \cdot f_0$$

to hold when applied to $\iota_p \times \iota_q$, we must choose $f_1(\iota_p \times \iota_q)$ to equal the simplicial chain corresponding to the geometric chain

$$K \times \Delta_p \times \Delta_q.$$ 

This may be done uniquely, because we work in the normalized chain complex. An explicit formula for this chain may be given in terms of the cyclic shuffles introduced in [10]; we see from these formulas or by a geometric argument that $f_1 \cdot B = f_1 \cdot \bar{B} = B \cdot f_1 = 0$ modulo degenerate chains. 

4. APPLICATION TO THE CYCLIC HOMOLOGY OF CROSSED PRODUCT ALGEBRAS

Recall the definition of the cyclic homology of a mixed complex $(V, b, B)$. Let $W$ be a graded module over the polynomial ring $k[u]$, where $\deg(u) = -2$; we will always assume that $W$ has finite homological dimension. If $C_\bullet$ is a mixed complex, we denote $C_\bullet[u] \otimes_k W$ by $C_\bullet \boxdot W$. We define the cyclic homology of the mixed complex $C_\bullet$ with coefficients in $W$ to be

$$\text{HC}(C_\bullet; W) = H_\bullet(C_\bullet \boxdot W, b + uB).$$

In the particular case where $V = C(A^2)$, we write

$$\text{HC}_\bullet(A; W) = \text{HC}_\bullet(C(A^2); W).$$

If $f : C_\bullet \to \tilde{C}_\bullet$ is a map of mixed complexes, it induces a map of cyclic homology

$$f : \text{HC}(C_\bullet; W) \to \text{HC}(\tilde{C}_\bullet; W).$$

We say that $f$ is a quasi-isomorphism (and write $f : C_\bullet \simeq \tilde{C}_\bullet$) if $f$ induces an isomorphism of homology

$$f : H_\bullet(C_\bullet, b) \cong H_\bullet(\tilde{C}_\bullet, \bar{b}).$$
If \( f : C_\bullet \simeq \tilde{C}_\bullet \) is a quasi-isomorphism of mixed complexes, and \( W \) is a graded \( k[u] \)-module of finite homological dimension, we obtain isomorphisms of cyclic homology

\[
 f : \text{HC}(C_\bullet; W) \cong \text{HC}(\tilde{C}_\bullet; W).
\]

Let us list some examples of cyclic homology with different coefficients \( W \):

1. \( W = k[u] \) gives negative cyclic homology \( \text{HC}_-^\bullet (A) \);
2. \( W = k[u, u^{-1}] \) gives periodic cyclic homology \( \text{HP}_\bullet (A) \);
3. \( W = k[u, u^{-1}]/uk[u] \) gives cyclic homology \( \text{HC}_\bullet (A) \);
4. \( W = k[u]/uk[u] \) gives the Hochschild homology \( \text{HH}_\bullet (A) \).

Using the Eilenberg-Zilber Theorem for parachain complexes (Theorem 3.1), we obtain the following theorem.

**Theorem 4.1.** Let \( A \) be a unital algebra over the commutative ring \( k \), and let \( G \) be a discrete group which acts on \( A \). There is a quasi-isomorphism of mixed complexes

\[
 f_0 + uf_1 : \text{Tot}(\mathcal{N}(A\sharp G)) \simeq \mathcal{N}((A \rtimes G)^\natural).
\]

Thus, we obtain isomorphism of cyclic homology groups

\[
 \text{HC}_\bullet (A \rtimes G; W) = \text{HC}_\bullet (\text{Tot}(\mathcal{N}(A\sharp G)); W).
\]

It is also possible to take the unnormalized chain complex \( \mathcal{C}(A\sharp G) \) in this theorem, since this is quasi-isomorphic to the normalized chain complex. This allows us to restate our result in the following more explicit form.

**Corollary 4.2.** There are operators \( b, \bar{b}, B \) and \( \bar{B} \) on the complex

\[
 \text{Tot}_n(\mathcal{C}(A\sharp G)) = \sum_{p+q=n} k[G^{p+1}] \otimes A^{(q+1)}
\]

such that the homology of the complex

\[
 (\text{Tot}(\mathcal{C}(A\sharp G)) \boxtimes W, b + \bar{b} + u(B + T\bar{B}))
\]

is the cyclic homology \( \text{HC}_\bullet (A \rtimes G; W) \).

The above theorem leads to the spectral sequence of Feigin and Tsygan, converging to \( \text{HC}_\bullet (A \rtimes G; W) \) (see Appendix 6 of [9]). We filter the complex \( \mathcal{C}(A\sharp G) \) by subspaces

\[
 F_{pq}^i \text{Tot}(\mathcal{C}(A\sharp G)) \boxtimes W = \sum_{q \leq i} k[G^{p+1}] \otimes A^{(q+1)} \boxtimes W.
\]

Recall the paracyclic module \( A_\sharp^\bullet (\mathfrak{n}) = A\sharp G(\mathfrak{n}, \mathfrak{n}) \cong k[G] \otimes A^{(n+1)} \) of Section 1. If \( M \) is a \( G \)-module, let \( C_p(G, M) = k[G^p] \otimes M \) be the space of \( p \)-chains on \( G \) with values in \( M \), with boundary \( \delta : C_p(G, M) \to C_{p-1}(G, M) \).
Lemma 4.3. The $E^0$-term of the spectral sequence is isomorphic to the complex

$$E^0_{pq} = C_p(G, C_q(A^{\natural}_G) \boxtimes W).$$

**Proof.** Consider the map $\beta$ from $C_{pq}(A\natural G)$ to $C_p(G, C_q(A^{\natural}_G))$ given by the formula

$$(g_0, \ldots, g_p|a_0, \ldots, a_q) \mapsto (g_1, \ldots, g_p|g|a_0, \ldots, a_q),$$

where $g = g_0 \ldots g_p$. It is easily seen that

$$(\bar{b}\beta^{-1})(g_1, \ldots, g_p|g|a_0, \ldots, a_q) = (g_2, \ldots, g_p|g|a_0, \ldots, a_q)$$

$$+ \sum_{i=1}^{p-1} (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_p|g|a_0, \ldots, a_q)$$

$$+ (-1)^p (g_1, \ldots, g_{p-1}|g_p g_p^{-1}|g_p a_0, \ldots, g_p a_q),$$

which is just the boundary for group homology with coefficients in $C_q(A^{\natural}_G)$. □

Although we do not use it, let us state the formula for $\beta\bar{B}\beta^{-1}$:

$$(\beta\bar{B}\beta^{-1})(g_1, \ldots, g_p|g|a_0, \ldots, a_q) = (1, g_1, \ldots, g_p|g|a_0, \ldots, a_q)$$

$$+ \sum_{i=1}^{p} (-1)^p i (g_i \ldots g_p) \cdot (1, g_1, \ldots, g_p, (g_1 \ldots g_p)^{-1}, g_1, \ldots, g_{i-1}|g|a_0, \ldots, a_q).$$

It follows from Lemma 4.3 that the $E^1$-term of the spectral sequence is

$$E^1_{pq} = H_p(G, C_q(A^{\natural}_G) \boxtimes W).$$

The following lemma enables us to give the differential $d^1$ a natural interpretation.

**Lemma 4.4.** The homology spaces $H_p(G, A^{\natural}_G)$ are cyclic modules, with respect to the cyclic structure induced by the maps

$$d(g|a_0, \ldots, a_q) = (g|g^{-1}a_q)a_0, a_1, \ldots, a_{q-1}),$$

$$s(g|a_0, \ldots, a_q) = (g|1, a_0, \ldots, a_q),$$

acting on $A^{\natural}_G$.

**Proof.** If we apply the chain functor $C$ along the $G$-axis of the bi-paracyclic module $A\natural G$, we obtain the paracyclic parachain chain complex $(C_p(G, A^{\natural}_G), \bar{b}, \bar{B})$, where $\bar{b}$ is the homology boundary. The operator $T\bar{B}$ gives a chain homotopy of $T$ to the identity, since

$$\bar{b}\bar{B} + \bar{B}\bar{b} = 1 - T = 1 - T^{-1},$$

showing that $H_p(G, A^{\natural}_G)$ is a cyclic module for each $p$. □

We see that the differential $d^1$ is just the differential $b + uB$ associated to the cyclic module $H_\bullet(G, A^{\natural}_G)$. This completes the proof of the following theorem.
Theorem 4.5. By means of the isomorphism

\[ H_p(G, C_q(A_G^\mathbb{b}) \boxtimes W) \cong C_q(H_p(G, A_G^\mathbb{b})) \boxtimes W, \]

the \(E^2\)-term of the spectral sequence may be identified with the cyclic homology

\[ HC_q(H_p(G, A_G^\mathbb{b}); W) \]

of the cyclic module \(H_\bullet(G, A_G^\mathbb{b})\).

To see the relationship between this spectral sequence and that of Feigin and Tsygan, we observe that the \(G\)-module \(A_G^\mathbb{b}\) decomposes into a direct sum over the conjugacy classes \([g] = \{gh^{-1} \mid h \in G\}\) of \(G\):

\[ A_G^\mathbb{b} = \sum_{[g]} A_{[g]}^\mathbb{b}, \]

where \(A_{[g]}^\mathbb{b}\) is the paracyclic \(G\)-module such that \(A_{[g]}^\mathbb{b}(n)\) consists of all functions from the conjugacy class \([g]\) to \(A^{(n+1)}\). Choose an arbitrary element \(g \in [g]\), and let \(A_g^\mathbb{b}\) be the stalk of \(A_G^\mathbb{b}\) over \(g\). This paracyclic module is acted on by the centralizer \(G^g\) of \(g\), and it is easily seen that

\[ A_{[g]}^\mathbb{b} \cong \text{Ind}_{G^g}^{G} A_g^\mathbb{b} \]

is an induced module. Shapiro’s Lemma now shows that

\[ H_p(G, A_G^\mathbb{b}) \cong \sum_{[g]} H_p(G^g, A_g^\mathbb{b}), \]

from which Feigin and Tsygan’s form of the spectral sequence follows easily.

Now suppose the order \(|G|\) of the group \(G\) is finite and invertible in \(k\). It follows that \(E_{pq}^2 = 0\) if \(p > 0\), and our spectral sequence collapses. The only remaining contribution to \(E^2\) comes from the cyclic module \(H_0(G, A_G^\mathbb{b})\) of coinvariants in \(A_G^\mathbb{b}\), introduced by Brylinski [2].

Proposition 4.6. If \(G\) is finite and \(|G|\) is invertible in \(k\), then there is a natural isomorphism of cyclic homology and

\[ HC_\bullet(A \rtimes G; W) = HC_\bullet(H_0(G, A_G^\mathbb{b}); W), \]

where \(H_0(G, A_G^\mathbb{b})\) is the cyclic module

\[ H_0(G, A_G^\mathbb{b})(n) = H_0(G, k[G] \otimes A^{(n+1)}). \]
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