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Regularity of solutions to regular shock reflection for potential flow

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Abstract. The shock reflection problem is one of the most important problems in mathematical fluid dynamics, since this problem not only arises in many important physical situations but also is fundamental for the mathematical theory of multidimensional conservation laws that is still largely incomplete. However, most of the fundamental issues for shock reflection have not been understood, including the regularity and transition of different patterns of shock reflection configurations. Therefore, it is important to establish the regularity of solutions to shock reflection in order to understand fully the phenomena of shock reflection. On the other hand, for a regular reflection configuration, the potential flow governs the exact behavior of the solution in $C^{1,1}$ across the pseudo-sonic circle even starting from the full Euler flow, that is, both of the nonlinear systems are actually the same in a physically significant region near the pseudo-sonic circle; thus, it becomes essential to understand the optimal regularity of solutions for the potential flow across the pseudo-sonic circle (the transonic boundary from the elliptic to hyperbolic region) and at the point where the pseudo-sonic circle (the degenerate elliptic curve) meets the reflected shock (a free boundary connecting the elliptic to hyperbolic region). In this paper, we study the regularity of solutions to regular shock reflection for potential flow. In particular, we prove that the $C^{1,1}$-regularity is *optimal* for the solution across the pseudo-sonic circle and at the point where the pseudo-sonic circle meets the reflected shock. We also obtain the $C^{2,\alpha}$ regularity of the solution up

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to the pseudo-sonic circle in the pseudo-subsonic region. The problem involves two types of transonic flow: one is a continuous transition through the pseudo-sonic circle from the pseudo-supersonic region to the pseudo-subsonic region; the other a jump transition through the transonic shock as a free boundary from another pseudo-supersonic region to the pseudo-subsonic region. The techniques and ideas developed in this paper will be useful to other regularity problems for nonlinear degenerate equations involving similar difficulties.

1. Introduction

We are concerned with the regularity of global solutions to shock wave reflection by wedges. The shock reflection problem is one of the most important problems in mathematical fluid dynamics, which not only arises in many important physical situations but also is fundamental for the mathematical theory of multidimensional conservation laws that is still largely incomplete; its solutions are building blocks and asymptotic attractors of general solutions to the multidimensional Euler equations for compressible fluids (cf. Mach [30], Courant–Friedrichs [14], von Neumann [36], Glimm–Majda [21], and Morawetz [32]; also see [1,7,20,22,27,33,34]).

In Chen–Feldman [9,10], the first global existence theory of shock reflection configurations for potential flow has been established when the wedge angle θ_w is large, which converge to the unique solution of the normal reflection when θ_w tends to $\pi/2$. However, most of the fundamental issues for shock reflection by wedges have not been understood, including the regularity and transition of different patterns of shock reflection configurations. Therefore, it is important to establish the regularity of solutions to shock reflection in order to understand fully the phenomena of shock reflection, including the case of potential flow which is widely used in aerodynamics (cf. [2,13,21,31,32]). On the other hand, for the regular reflection configuration as in Fig. 1, the potential flow governs the exact behavior of solutions in $C^{1,1}$ across the pseudo-sonic circle P_1P_4 even starting from the full Euler flow, that is, both of the nonlinear systems are actually the same in a physically significant region near the pseudo-sonic circle; thus, it becomes essential to understand the optimal regularity of solutions for the potential flow across the pseudo-sonic circle P_1P_4 and at the point P_1 where the pseudo-sonic circle meets the reflected shock.

In this paper, we develop a mathematical approach in Sects. 2–4 to establish the regularity of solutions to regular shock reflection with the configuration as in Fig. 1 for potential flow. In particular, we prove that the $C^{1,1}$-regularity is *optimal* for the solution across the part $\overline{P_1P_4} \setminus \{P_1\}$ of the pseudo-sonic circle (the degenerate elliptic curve) and at the point P_1 where the pseudo-sonic circle meets the reflected shock (as a free boundary). The problem involves two types of transonic flow: one is a continuous transition through the pseudo-sonic circle P_1P_4 separating the pseudo-

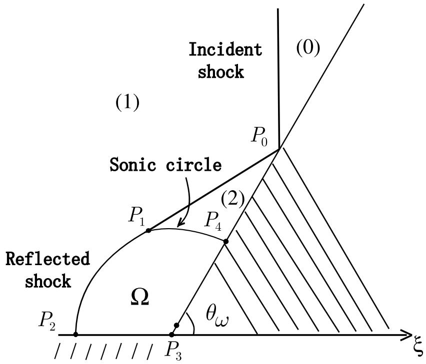

Fig. 1. Regular reflection configuration

supersonic region (2) from the pseudo-subsonic region Ω; the other is a jump transition through the transonic shock as a free boundary separating the pseudo-supersonic region (1) from the pseudo-subsonic region Ω. To achieve the optimal regularity, one of the main difficulties is that P_1P_4 is the transonic boundary separating the elliptic region from the hyperbolic region. Near P_1P_4, the solution is governed by a nonlinear equation, whose main part has the form:

$$(2x - a\psi_x)\psi_{xx} + b\psi_{yy} - \psi_x = 0, \tag{1.1}$$

where $a, b > 0$ are constants. For our solution ψ, (1.1) is elliptic in $\{x > 0\}$; also $\psi > 0$ in $\{x > 0\}$ and $\psi = 0$ on $\{x = 0\}$. We analyze the features of equations modeled by (1.1) and prove the $C^{2,\alpha}$ regularity of solutions of shock reflection problem in the elliptic region up to the part $\overline{P_1P_4} \setminus \{P_1\}$ of the pseudo-sonic circle. As a corollary, we establish that the $C^{1,1}$-regularity is actually optimal across the transonic boundary P_1P_4 from the elliptic to hyperbolic region. Since the reflected shock P_1P_2 is regarded as a free boundary connecting the hyperbolic region (1) with the elliptic region Ω for the nonlinear second-order equation of mixed type, another difficulty for the optimal regularity of the solution is that the point P_1 is exactly the one where the degenerate elliptic curve P_1P_4 meets a transonic free boundary for the nonlinear partial differential equation of second order. As far as we know, this is the first rigorous, optimal regularity result for solutions to a free boundary problem of nonlinear degenerate elliptic equations at the point where an elliptic degenerate curve meets the free boundary. For an asymptotic understanding of the regularity, see Morawetz [32].

We note that the global theory of existence and regularity of regular reflection configurations for the polytropic case $\gamma > 1$, established in [9,10] and Sects. 2–4, applies to the isothermal case $\gamma = 1$ as well. The techniques and ideas developed in this paper will be useful to other regularity problems for nonlinear degenerate equations involving similar difficulties.

The regularity for certain degenerate elliptic and parabolic equations has been studied (cf. [3,15,29,37,38] and the references cited therein). The main feature that distinguishes (1.1) from the equations in Daskalopoulos–Hamilton [15] and Lin–Wang [29] is the crucial role of the nonlinear term $-a\psi_x\psi_{xx}$; the importance of such a nonlinear term in a different context was pointed out and used in von Karman [35]. Indeed, if $a = 0$, then (1.1) becomes a linear equation

$$2x\psi_{xx} + b\psi_{yy} - \psi_x = 0. \tag{1.2}$$

Then $\psi_0(x, y) := cx^{3/2}$ is a solution of (1.2), and ψ_0 with $c > 0$ also satisfies the conditions:

$$\psi > 0 \quad \text{in } \{x > 0\}, \tag{1.3}$$

$$\psi = 0 \quad \text{on } \{x = 0\}, \tag{1.4}$$

$$\partial_y\psi = 0 \quad \text{on } \{y = \pm 1\}. \tag{1.5}$$

Let ψ be a solution of (1.2) in $Q_1 := (0, 1) \times (-1, 1)$ satisfying (1.3)–(1.5). Then the comparison principle implies that $\psi \geq \psi_0$ in Q_1 for sufficiently small $c > 0$. It follows that the solutions of (1.2) satisfying (1.3)–(1.5) are not even $C^{1,1}$ up to $\{x = 0\}$. On the other hand, for the nonlinear equation (1.1) with $a > 0$, the function $\psi(x, y) = \frac{1}{2a}x^2$ is a smooth solution of (1.1) up to $\{x = 0\}$ satisfying (1.3)–(1.5). More general $C^{1,1}$ solutions of (1.1) satisfying (1.3)–(1.5) and the condition

$$-Mx \leq \psi_x \leq \frac{2-\beta}{a}x \quad \text{in } x > 0, \tag{1.6}$$

with $M > 0$ and $\beta \in (0, 1)$, which implies the ellipticity of (1.1), can be constructed by using the methods of [9,10], and their $C^{2,\alpha}$ regularity up to $\{x = 0\}$ follows from this paper. Another feature of the present case is that, for solutions of (1.1) satisfying (1.3), (1.4), and (1.6), we compute explicitly $D^2\psi$ on $\{x = 0\}$ to find that $\psi_{xx}(0, y) = \frac{1}{a}$ and $\psi_{xy}(0, y) = \psi_{yy}(0, y) = 0$ for all such solutions. Thus, all the solutions are separated in $C^{2,\alpha}$ from the solution $\psi \equiv 0$, although it is easy to construct a sequence of solutions of (1.1) satisfying (1.3), (1.4), and (1.6) which converges to $\psi \equiv 0$ in $C^{1,\alpha}$. This shows that the $C^{2,\alpha}$-regularity of (1.1) with conditions (1.3), (1.4), and (1.6) is a truly nonlinear phenomenon of degenerate elliptic equations.

Some efforts have been also made mathematically for the shock reflection problem via simplified models, including the unsteady transonic small-disturbance (UTSD) equation (cf. Keller–Blank [26], Hunter–Keller [25], Hunter [24], Morawetz [32], Gamba–Rosales–Tabak [18], and Canić–Keyfitz–Kim [5]) and the pressure gradient equation or the nonlinear wave system (cf. Zheng [39] and Canić–Keyfitz–Kim [6]). On the other hand, in order to understand the existence and regularity of solutions near the important physical points and for the reflection problem, some asymptotic methods have been also developed (cf. Lighthill [28], Keller–Blank [26], Hunter–Keller [25], Harabetian [23], and Morawetz [32]). Also see Chen [12] for a linear approximation of shock reflection when the wedge angle is close to $\frac{\pi}{2}$ and Serre [33] for an apriori analysis of solutions of shock reflection and related discussions in the context of the Euler equations for isentropic and adiabatic fluids. We further refer the reader to Zheng [39] for regular shock reflection for the pressure gradient equation and to Elling–Liu [16] for the problem of supersonic flow onto a solid wedge, so called the Prandtl problem, for which a related transonic flow problem arises, treated with the aid of the ellipticity principle developed in [17].

We remark that our regularity results for potential flow near the pseudo-sonic circle confirm rigorously the asymptotic scalings used by Hunter–Keller [25], Harabetian [23], and Morawetz [32]. Indeed, the $C^{2,\alpha}$ regularity up to the pseudo-sonic circle away from P_1 and its proof based on the comparison with an ordinary differential equation in the radial direction confirm their asymptotic scaling via the differential equation in that region. Moreover, it deserves to be emphasized that the jump of ψ_{xx} across the pseudo-sonic arc is exactly $\frac{1}{\gamma+1}$, $\gamma \geq 1$, which *is independent of state* (0)*, state* (1)*, and the wedge angle*! The optimal $C^{1,1}$ regularity at P_1 shows that the asymptotic scaling does not work there, i.e., the angular derivatives become large, as stated in [32].

The organization of this paper is the following. In Sect. 2, we describe the shock reflection problem by a wedge and its solution with regular reflection configuration when the wedge angle is suitably large. In Sect. 3, we establish a regularity theory for solutions near the degenerate boundary with Dirichlet data for a class of nonlinear degenerate elliptic equations, in order to study the regularity of solutions to the regular reflection problem. Then we employ the regularity theory developed in Sect. 3 to establish the optimal regularity of solutions for $\gamma > 1$ across the pseudo-sonic circle P_1P_4 and at the point P_1 where the pseudo-sonic circle P_1P_4 meets the reflected shock P_1P_2 in Sect. 4. We also establish the $C^{2,\alpha}$-regularity of solutions in the pseudo-subsonic region up to the pseudo-sonic circle P_1P_4. We further observe that the existence and regularity results for regular reflection configurations for the polytropic case $\gamma > 1$ apply to the isothermal case $\gamma = 1$.

We remark in passing that there may exist a global regular reflection configuration when state (2) is pseudo-subsonic, which is in a very narrow regime (see [14, 36]). In this case, the regularity of the solution behind the reflected shock is direct, and the main difficulty of elliptic degeneracy does not occur. Therefore, in this paper, we focus on the difficult case for the regularity problem when state (2) is pseudo-supersonic, which will be simply called a regular reflection configuration, throughout this paper.

2. Shock reflection problem and regular reflection configurations

In this section, we describe the shock reflection problem by a wedge and its solution with regular reflection configuration when the wedge angle is suitably large.

The Euler equations for potential flow consist of the conservation law of mass and the Bernoulli law for the density ρ and the velocity potential Φ:

$$\partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \nabla_{\mathbf{x}} \Phi) = 0, \tag{2.1}$$

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, \tag{2.2}$$

where B_0 is the Bernoulli constant determined by the incoming flow and/or boundary conditions, and

$$i'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho}$$

with $c(\rho)$ being the sound speed. For polytropic gas, by scaling,

$$p(\rho) = \frac{\rho^\gamma}{\gamma}, \quad c^2(\rho) = \rho^{\gamma-1}, \quad i(\rho) = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}, \quad \gamma > 1. \tag{2.3}$$

2.1. Shock reflection problem. When a plane shock in the $(\mathbf{x}, t)$–coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left-state $(\rho, \nabla_{\mathbf{x}} \Phi) = (\rho_1, u_1, 0)$ and right-state $(\rho_0, 0, 0)$, $u_1 > 0$, $\rho_0 < \rho_1$, hits a symmetric wedge

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_w, \ x_1 > 0\}$$

head on, it experiences a reflection-diffraction process, where $\theta_w \in (0, \frac{\pi}{2})$ is the wedge half-angle. Then the Bernoulli law (2.2) becomes

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = i(\rho_0). \tag{2.4}$$

This reflection problem can be formulated as the following mathematical problem.

Problem 1 (Initial-boundary value problem)**.** *Seek a solution of the system of* (2.1) *and* (2.4)*, the initial condition at* $t = 0$*:*

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \text{for } |x_2| > x_1 \tan\theta_w, \ x_1 > 0, \\ (\rho_1, u_1 x_1) & \text{for } x_1 < 0, \end{cases} \tag{2.5}$$

and the slip boundary condition along the wedge boundary ∂W*:*

$$\nabla \Phi \cdot \nu|_{\partial W} = 0, \tag{2.6}$$

where ν *is the exterior unit normal to* ∂W *(see Fig. 2).*

Notice that the initial-boundary value problem (2.1) and (2.4)–(2.6) is invariant under the self-similar scaling:

$$(\mathbf{x}, t) \to (\alpha \mathbf{x}, \alpha t), \quad (\rho, \Phi) \to \left(\rho, \frac{\Phi}{\alpha}\right) \quad \text{for } \alpha \neq 0.$$

Thus, we seek self-similar solutions with the form

$$\rho(\mathbf{x}, t) = \rho(\xi, \eta), \quad \Phi(\mathbf{x}, t) = t\psi(\xi, \eta) \quad \text{for } (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

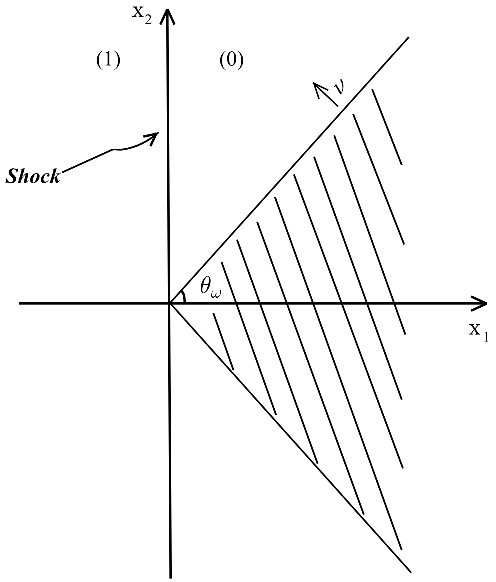

Fig. 2. Initial-boundary value problem

Then the pseudo-potential function $\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)$ is governed by the following potential flow equation of second order:

$$\operatorname{div}(\rho(|D\varphi|^2, \varphi)D\varphi) + 2\rho(|D\varphi|^2, \varphi) = 0 \tag{2.7}$$

with

$$\rho(|D\varphi|^2, \varphi) = \left(\rho_0^{\gamma-1} - (\gamma - 1)\left(\varphi + \frac{1}{2}|D\varphi|^2\right)\right)^{\frac{1}{\gamma-1}}, \tag{2.8}$$

where the divergence div and gradient D are with respect to the self-similar variables (ξ, η). Then we have

$$c^2 = c^2\big(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}\big) = \rho_0^{\gamma-1} - (\gamma - 1)\left(\frac{1}{2}|D\varphi|^2 + \varphi\right). \tag{2.9}$$

Equation (2.7) is a nonlinear equation of mixed elliptic-hyperbolic type. It is elliptic if and only if

$$|D\varphi| < c\big(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}\big), \tag{2.10}$$

which is equivalent to

$$|D\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2}{\gamma + 1}\big(\rho_0^{\gamma-1} - (\gamma - 1)\varphi\big)}. \tag{2.11}$$

Shocks are discontinuities in the pseudo-velocity $D\varphi$. That is, if Ω^+ and $\Omega^- := \Omega\backslash\overline{\Omega^+}$ are two nonempty open subsets of $\Omega \subset \mathbb{R}^2$ and $S := \partial\Omega^+ \cap \Omega$ is a C^1-curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{loc}(\Omega) \cap C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of (2.7) in Ω if and only if φ is in $W^{1,\infty}_{loc}(\Omega)$ and satisfies (2.7) in $\Omega^\pm$ and the Rankine–Hugoniot conditions on S:

$$[\varphi]_S = 0, \tag{2.12}$$

$$[\rho(|D\varphi|^2, \varphi)D\varphi \cdot \nu]_S = 0. \tag{2.13}$$

Fix constants $\rho_1 > \rho_0 > 0$. The plane incident shock solution in the $(\mathbf{x}, t)$-coordinates with states $(\rho, \nabla_{\mathbf{x}}\Psi) = (\rho_0, 0, 0)$ and $(\rho_1, u_1, 0)$ corresponds to a continuous weak solution φ of (2.7) in the self-similar coordinates (ξ, η) with the following form:

$$\varphi_0(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) \quad \text{for } \xi > \xi_0, \tag{2.14}$$

$$\varphi_1(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_1(\xi - \xi_0) \quad \text{for } \xi < \xi_0, \tag{2.15}$$

respectively, where

$$u_1 = (\rho_1 - \rho_0)\sqrt{\frac{2\left(\rho_1^{\gamma-1} - \rho_0^{\gamma-1}\right)}{(\gamma-1)\left(\rho_1^2 - \rho_0^2\right)}} > 0, \tag{2.16}$$

$$\xi_0 = \rho_1\sqrt{\frac{2\left(\rho_1^{\gamma-1} - \rho_0^{\gamma-1}\right)}{(\gamma-1)\left(\rho_1^2 - \rho_0^2\right)}} = \frac{\rho_1 u_1}{\rho_1 - \rho_0} > 0 \tag{2.17}$$

are the velocity of state (1) and the location of the incident shock, uniquely determined by (ρ_0, ρ_1, γ) through (2.13). Then $P_0 = (\xi_0, \xi_0 \tan\theta_w)$ in Fig. 1. Since the problem is symmetric with respect to the axis $\eta = 0$, it suffices to consider the problem in the half-plane $\eta > 0$ outside the half-wedge

$$\Lambda := \{\xi < 0, \eta > 0\} \cup \{\eta > \xi \tan\theta_w,\ \xi > 0\}.$$

Then the initial-boundary value problem (2.1) and (2.4)–(2.6) in the $(\mathbf{x}, t)$-coordinates can be formulated as the following boundary value problem in the self-similar coordinates (ξ, η).

Problem 2 (Boundary value problem (see Fig. 1)). *Seek a solution φ of* (2.7) *in the self-similar domain Λ with the slip boundary condition on $\partial\Lambda$:*

$$D\varphi \cdot \nu|_{\partial\Lambda} = 0 \tag{2.18}$$

and the asymptotic boundary condition at infinity:

$$\varphi \to \bar{\varphi} := \begin{cases} \varphi_0 & \text{for } \xi > \xi_0,\ \eta > \xi\tan\theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0,\ \eta > 0, \end{cases} \quad \text{when } \xi^2 + \eta^2 \to \infty, \tag{2.19}$$

where (2.19) *holds in the sense that* $\lim_{R\to\infty} \|\varphi - \overline{\varphi}\|_{C(\Lambda\backslash B_R(0))} = 0$.

2.2. Existence of regular reflection configurations. Since φ_1 does not satisfy the slip boundary condition (2.18), the solution must differ from φ_1 in $\{\xi < \xi_0\} \cap \Lambda$ and thus a shock diffraction by the wedge occurs. In [9, 10], the existence of global solution φ to Problem 2 has been established when the wedge angle θ_w is large, and the corresponding structure of solution is as follows (see Fig. 1): The vertical line is the incident shock $S = \{\xi = \xi_0\}$ that hits the wedge at the point $P_0 = (\xi_0, \xi_0 \tan\theta_w)$, and state (0) and state (1) ahead of and behind S are given by φ_0 and φ_1 defined in (2.14) and (2.15), respectively. The solutions φ and φ_1 differ within $\{\xi < \xi_0\}$ only in the domain $P_0P_1P_2P_3$ because of shock diffraction by the wedge vertex, where the curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. State (2) behind P_0P_1 is of the form:

$$\varphi_2(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_2(\xi-\xi_0) + (\eta - \xi_0\tan\theta_w)u_2\tan\theta_w, \quad (2.20)$$

which satisfies

$$D\varphi\cdot\nu = 0 \quad \text{on } \partial\Lambda \cap \{\xi > 0\};$$

the constant velocity u_2 and the angle between P_0P_1 and the ξ-axis are determined by $(\theta_w, \rho_0, \rho_1, \gamma)$ from the two algebraic equations expressing (2.12) and (2.13) for φ_1 and φ_2 across P_0P_1. Moreover, the constant density ρ_2 of state (2) satisfies $\rho_2 > \rho_1$, and state (2) is pseudo-supersonic at the point P_0. In addition, $u_2 > 0$ when $\theta_w < \frac{\pi}{2}$. The solution φ is pseudo-subsonic within the pseudo-sonic circle for state (2) with center $(u_2, u_2\tan\theta_w)$ and radius $c_2 = \rho_2^{(\gamma-1)/2} > 0$ (the sonic speed of state (2)), and φ is pseudo-supersonic outside this circle containing the arc P_1P_4 in Fig. 1, so that φ_2 is the unique solution in the domain $P_0P_1P_4$, as argued in [7, 33]. Then φ differs from φ_2 in the domain $\Omega = P_1P_2P_3P_4$, where the equation is elliptic.

Introduce the polar coordinates (r,θ) with respect to the center $(u_2, u_2\tan\theta_w)$ of the pseudo-sonic circle of state (2), that is,

$$\xi - u_2 = r\cos\theta, \qquad \eta - u_2\tan\theta_w = r\sin\theta. \quad (2.21)$$

Then, for $\varepsilon \in (0, c_2)$, we denote by $\Omega_\varepsilon := \Omega \cap \{(r,\theta) : 0 < c_2 - r < \varepsilon\}$ the ε-neighborhood of the pseudo-sonic circle P_1P_4 within Ω. In Ω_ε, we introduce the coordinates:

$$x = c_2 - r, \qquad y = \theta - \theta_w. \quad (2.22)$$

This implies that $\Omega_\varepsilon \subset \{0 < x < \varepsilon,\ y > 0\}$ and $P_1P_4 \subset \{x = 0,\ y > 0\}$. Also we introduce the following notation for various parts of $\partial\Omega$:

$$\begin{aligned}\Gamma_{sonic} &:= \partial\Omega \cap \partial B_{c_2}((u_2, u_2\tan\theta_w)) \equiv P_1P_4;\\ \Gamma_{shock} &:= P_1P_2;\\ \Gamma_{wedge} &:= \partial\Omega \cap \partial\Lambda \equiv P_3P_4.\end{aligned}$$

Then the global theory established in [9, 10] indicates that there exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1)$ such that, when $\theta_w \in [\theta_c, \frac{\pi}{2})$, there exists a global self-similar solution:

$$\Phi(\mathbf{x}, t) = t\varphi\left(\frac{\mathbf{x}}{t}\right) + \frac{|\mathbf{x}|^2}{2t} \quad \text{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t > 0,$$

with

$$\rho(\mathbf{x}, t) = \left(\rho_0^{\gamma-1} - (\gamma - 1)\left(\Phi_t + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\right)\right)^{\frac{1}{\gamma-1}}$$

of Problem 1 (equivalently, Problem 2) for shock reflection by the wedge, which satisfies that, for $(\xi, \eta) = \frac{\mathbf{x}}{t}$,

$$\begin{gathered} \varphi \in C^{0,1}(\Lambda), \\ \varphi \in C^{\infty}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \\ \varphi = \begin{cases} \varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan\theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflection shock } P_0P_1P_2, \\ \varphi_2 & \text{in } P_0P_1P_4. \end{cases} \end{gathered} \tag{2.23}$$

Moreover,

(i) (2.7) is elliptic in Ω;
(ii) $\varphi \geq \varphi_2$ in Ω;
(iii) the reflected shock $P_0P_1P_2$ is C^2 at P_1 and C^∞ except P_1;
(iv) there exists $\varepsilon_0 \in (0, \frac{c_2}{2})$ such that $\varphi \in C^{1,1}(\overline{\Omega_{\varepsilon_0}}) \cap C^2(\overline{\Omega_{\varepsilon_0}} \setminus \overline{\Gamma_{sonic}})$; moreover, in the coordinates (2.22),

$$\|\varphi - \varphi_2\|^{(par)}_{2,0,\Omega_{\varepsilon_0}} := \sum_{0\leq k+l\leq 2}\ \sup_{(x,y)\in\Omega_{\varepsilon_0}} \left(x^{k+\frac{l}{2}-2}\left|\partial_x^k\partial_y^l(\varphi - \varphi_2)(x, y)\right|\right) < \infty; \tag{2.24}$$

(v) there exists $\delta_0 > 0$ so that, in the coordinates (2.22),

$$|\partial_x(\varphi - \varphi_2)(x, y)| \leq \frac{2 - \delta_0}{\gamma + 1}x \quad \text{in } \Omega_{\varepsilon_0}; \tag{2.25}$$

(vi) there exist $\omega > 0$ and a function $y = \hat{f}(x)$ such that, in the coordinates (2.22),

$$\begin{gathered} \Omega_{\varepsilon_0} = \{(x, y) : x \in (0, \varepsilon_0),\ 0 < y < \hat{f}(x)\}, \\ \Gamma_{shock} \cap \partial\Omega_{\varepsilon_0} = \{(x, y) : x \in (0, \varepsilon_0),\ y = \hat{f}(x)\}, \end{gathered} \tag{2.26}$$

and

$$\|\hat{f}\|_{C^{1,1}([0,\varepsilon_0])} < \infty, \qquad \frac{d\hat{f}}{dx} \geq \omega > 0 \quad \text{for } 0 < x < \varepsilon_0. \tag{2.27}$$

The existence of state (2) of form (2.20) with constant velocity $(u_2, u_2 \tan\theta_w)$, $u_2 > 0$, and constant density $\rho_2 > \rho_1$, satisfying (2.12) and (2.13) on P_0P_1, is shown in [10, Sect. 3] for $\theta_w \in [\theta_c, \frac{\pi}{2})$. The existence of a solution φ of Problem 2, satisfying (2.23) and property (iv) follows from [10, Main theorem]. Property (i) follows from Lemma 5.2 and Proposition 7.1 in [10]. Property (ii) follows from Proposition 7.1 and Sect. 9 in [10], which assert that $\varphi - \varphi_2 \in \mathcal{K}$, where the set $\mathcal{K}$ defined by (5.15) in [10]. Property (v) follows from Propositions 8.1 and 8.2 and Sect. 9 in [10]. Property (vi) follows from (5.7) and (5.25)–(5.27) in [10] and the fact that $\varphi - \varphi_2 \in \mathcal{K}$.

These results have been extended in [11] to other wedge-angle cases.

3. Regularity near the degenerate boundary for nonlinear degenerate elliptic equations of second order

In order to study the regularity of solutions to the regular reflection problem, in this section we first study the regularity of solutions near a degenerate boundary for a class of nonlinear degenerate elliptic equations of second order.

We adopt the following definitions for ellipticity and uniform ellipticity: Let $\Omega \subset \mathbb{R}^2$ be open, $u \in C^2(\Omega)$, and

$$\mathcal{L}u = \sum_{i,j=1}^{2} A_{ij}(x, Du)D^2_{ij}u + B(x, Du), \tag{3.1}$$

where $A_{ij}(x, p)$ and $B(x, p)$ are continuous on $\overline{\Omega} \times \mathbb{R}^2$ and $D^2_{ij} = \partial^2_{x_ix_j}$. The operator $\mathcal{L}$ is elliptic with respect to u in Ω if the coefficient matrix

$$A(x, Du(x)) := [A_{ij}(x, Du(x))]$$

is positive for every $x \in \Omega$. Furthermore, $\mathcal{L}$ is uniformly elliptic with respect to u in Ω if

$$\lambda I \le A(x, Du(x)) \le \lambda^{-1} I \quad \text{for every } x \in \Omega,$$

where $\lambda > 0$ is a constant, and I is the 2×2 identity matrix.

The following standard comparison principle for the operator $\mathcal{L}$ follows from [19, Theorem 10.1].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that the operator $\mathcal{L}$ is elliptic in Ω with respect to either u or v. Let $\mathcal{L}u \le \mathcal{L}v$ in Ω and $u \ge v$ on $\partial\Omega$. Then $u \ge v$ in Ω.*

3.1. Nonlinear degenerate elliptic equations and regularity theorem. We now study the regularity of positive solutions near the degenerate boundary with Dirichlet data for the class of nonlinear degenerate elliptic equations of the form:

$$\begin{aligned}\mathcal{L}_1\psi := (2x - a\psi_x + O_1)\psi_{xx} + O_2\psi_{xy} + (b + O_3)\psi_{yy} \\ - (1 + O_4)\psi_x + O_5\psi_y = 0 \quad \text{in } Q^+_{r,R},\end{aligned} \tag{3.2}$$

$$\psi > 0 \quad \text{in } Q^+_{r,R}, \tag{3.3}$$

$$\psi = 0 \quad \text{on } \partial Q^+_{r,R} \cap \{x = 0\}, \tag{3.4}$$

where $a, b > 0$ are constants and, for $r, R > 0$,

$$Q^+_{r,R} := \{(x, y) : x \in (0, r),\ |y| < R\} \subset \mathbb{R}^2, \tag{3.5}$$

and the terms $O_i(x, y), i = 1, \dots, 5$, are continuously differentiable and

$$\frac{|O_1(x, y)|}{x^2},\ \frac{|O_k(x, y)|}{x} \le N \quad \text{for } k = 2, \dots, 5, \tag{3.6}$$

$$\frac{|DO_1(x, y)|}{x},\ |DO_k(x, y)| \le N \quad \text{for } k = 2, \dots, 5, \tag{3.7}$$

in $\{x > 0\}$ for some constant N.

Conditions (3.6) and (3.7) imply that the terms O_i, $i = 1, \dots, 5$, are "small"; the precise meaning of which can be seen in Sect. 4 for the shock reflection problem below (also see the estimates in [10]). Thus, the main terms of (3.2) form the following equation:

$$(2x - a\psi_x)\psi_{xx} + b\psi_{yy} - \psi_x = 0 \quad \text{in } Q^+_{r,R}. \tag{3.8}$$

Equation (3.8) is elliptic with respect to ψ in $\{x > 0\}$ if $\psi_x < \frac{2x}{a}$. In this paper, we consider the solutions that satisfy

$$-Mx \le \psi_x \le \frac{2-\beta}{a}x \quad \text{in } Q^+_{r,R} \tag{3.9}$$

for some constants $M \ge 0$ and $\beta \in (0, 1)$. Then (3.8) is uniformly elliptic in every subdomain $\{x > \delta\}$ with $\delta > 0$. The same is true for (3.2) in $Q^+_{r,R}$ if r is sufficiently small.

Remark 3.1. If $\hat{r}$ is sufficiently small, depending only on a, b, and N, then (3.6), (3.7), and (3.9) imply that (3.2) is uniformly elliptic with respect to ψ in $Q^+_{\hat{r},R} \cap \{x > \delta\}$ for any $\delta \in (0, \frac{\hat{r}}{2})$. We will always assume such a choice of $\hat{r}$ hereafter.

Let $\psi \in C^2(Q^+_{\hat{r},R})$ be a solution of (3.2) satisfying (3.9). Remark 3.1 implies that the interior regularity

$$\psi \in C^{2,\alpha}\big(Q^+_{\hat{r},R}\big) \quad \text{for all } \alpha \in (0, 1) \tag{3.10}$$

follows first from the linear elliptic theory in two-dimensions (cf. [19, Chap. 12]) to conclude the solution in $C^{1,\alpha}$ which leads that the coefficient becomes C^α and then from the Schauder theory to get the $C^{2,\alpha}$ estimate (cf. [19, Chap. 6]), where we have used the fact $O_i \in C^1(\{x > 0\})$.

Therefore, we focus on the regularity of ψ near the boundary $\{x=0\}\cap\partial Q^+_{\hat{r},R}$ where the ellipticity of (3.2) degenerates.

Theorem 3.1 (Regularity theorem)**.** *Let $a, b, M, N, R > 0$ and $\beta \in (0, \frac{1}{4})$ be constants. Let $\psi \in C(\overline{Q^+_{\hat{r},R}}) \cap C^2(Q^+_{\hat{r},R})$ satisfy* (3.3)*,* (3.4)*,* (3.9)*, and* (3.2) *in $Q^+_{\hat{r},R}$ with $O_i = O_i(x, y)$ satisfying $O_i \in C^1(\overline{Q^+_{\hat{r},R}})$,* (3.6)*, and* (3.7)*. Then*

$$\psi \in C^{2,\alpha}\big(\overline{Q^+_{\hat{r}/2,R/2}}\big) \quad \textit{for any } \alpha \in (0,1),$$

with

$$\psi_{xx}(0,y) = \frac{1}{a}, \quad \psi_{xy}(0,y) = \psi_{yy}(0,y) = 0 \quad \textit{for all } |y| < \frac{R}{2}.$$

To prove Theorem 3.1, it suffices to show that, for any given $\alpha \in (0, 1)$,

$$\psi \in C^{2,\alpha}\big(\overline{Q^+_{r,R/2}}\big) \quad \text{for some } r \in (0, \hat{r}/2), \tag{3.11}$$

since ψ belongs to $C^{2,\alpha}(\overline{Q^+_{\hat{r}/2,R/2} \cap \{x > r/2\}})$ by (3.10).

Note that, by (3.3), (3.4), and (3.9), it follows that

$$0 < \psi(x,y) \le \frac{2-\beta}{2a}x^2 \quad \text{for all } (x,y) \in Q^+_{\hat{r},R}. \tag{3.12}$$

The essential part of the proof of Theorem 3.1 is to show that, if a solution ψ satisfies (3.12), then, for any given $\alpha \in (0, 1)$, there exists $r \in (0, \hat{r}/2]$ such that

$$\left|\psi(x,y) - \frac{1}{2a}x^2\right| \le Cx^{2+\alpha} \quad \text{for all } (x,y) \in Q^+_{r,7R/8}. \tag{3.13}$$

Notice that, although $\psi^{(0)} \equiv 0$ is a solution of (3.2), it satisfies neither (3.13) nor the conclusion $\psi^{(0)}_{xx}(0, y) = \frac{1}{a}$ of Theorem 3.1. Thus it is necessary to improve first the lower bound of ψ in (3.12) to separate our solution from the trivial solution $\psi^{(0)} \equiv 0$.

3.2. Quadratic lower bound of ψ.

By Remark 3.1, (3.2) is uniformly elliptic with respect to ψ inside $Q^+_{\hat{r},R}$. Thus, our idea is to construct a positive subsolution of (3.2), which provides our desired lower bound of ψ.

Proposition 3.1. *Let ψ satisfy the assumptions in Theorem 3.1. Then there exist $r \in (0, \hat{r}/2]$ and $\mu > 0$, depending only on $a, b, N, R, \hat{r}, \beta$, and $\inf_{Q^+_{\hat{r},R}\cap\{x>\hat{r}/2\}} \psi$, such that*

$$\psi(x,y) \ge \mu x^2 \quad \textit{on } Q^+_{r,15R/16}.$$

Proof. In this proof, all the constants below depend only on the data, i.e., a, b, M, N, R, $\hat{r}$, β, and $\inf_{Q^+_{\hat{r},R}\cap\{x>\hat{r}/2\}}\psi$, unless otherwise is stated.

Fix y_0 with $|y_0| \le \frac{15R}{16}$. We now prove that

$$\psi(x, y_0) \ge \mu x^2 \quad \text{for } x \in (0, r). \tag{3.14}$$

We first note that, without loss of generality, we may assume that $R = 2$ and $y_0 = 0$. Otherwise, we set $\tilde{\psi}(x, y) := \psi(x, y_0 + \frac{R}{32}y)$ for all $(x, y) \in Q^+_{\hat{r},2}$. Then $\tilde{\psi} \in C(\overline{Q^+_{\hat{r},2}}) \cap C^2(Q^+_{\hat{r},2})$ satisfies (3.2) with (3.6) and conditions (3.3), (3.4), and (3.9) in $Q^+_{\hat{r},2}$, with some modified constants a, b, N, β and functions O_i, depending only on the corresponding quantities in the original equation and on R. Moreover, $\inf_{Q^+_{\hat{r},2}\cap\{x>\hat{r}/2\}}\tilde{\psi} = \inf_{Q^+_{\hat{r},R}\cap\{x>\hat{r}/2\}}\psi$. Then (3.14) for ψ follows from (3.14) for $\tilde{\psi}$ with $y_0 = 0$ and $R = 2$. Thus we will keep the original notation with $y_0 = 0$ and $R = 2$. Then it suffices to prove

$$\psi(x, 0) \ge \mu x^2 \quad \text{for } x \in (0, r). \tag{3.15}$$

By Remark 3.1 and the Harnack inequality, we conclude that, for any $r \in (0, \hat{r}/2)$, there exists $\sigma = \sigma(r) > 0$ depending only on r and the data a, b, N, R, $\hat{r}$, β, and $\inf_{Q^+_{\hat{r},R}\cap\{x>\hat{r}/2\}}\psi$, such that

$$\psi \ge \sigma \quad \text{on } Q^+_{\hat{r},3/2} \cap \{x > r\}. \tag{3.16}$$

Let $r \in (0, \hat{r}/2)$, $k > 0$, and

$$0 < \mu \le \frac{\sigma(r)}{r^2} \tag{3.17}$$

to be chosen. Set

$$w(x, y) := \mu x^2(1 - y^2) - kxy^2. \tag{3.18}$$

Then, using (3.16) and (3.17), we obtain that, for all $x \in (0, r)$ and $|y| < 1$,

$$\begin{cases} w(0, y) = 0 \le \psi(0, y), \\ w(r, y) \le \mu r^2 \le \psi(r, y), \\ w(x, \pm 1) = -kx \le 0 \le \psi(x, \pm 1). \end{cases}$$

Therefore, we have

$$w \le \psi \quad \text{on } \partial Q^+_{r,1}. \tag{3.19}$$

Next, we show that w is a strict subsolution $\mathcal{L}_1 w > 0$ in $Q^+_{r,1}$, if the parameters are chosen appropriately. In order to estimate $\mathcal{L}_1 w$, we denote

$$A_0 := \frac{k}{\mu} \tag{3.20}$$

and notice that

$$\begin{aligned} w_{yy} &= -2x(\mu x + k) = -2x(\mu x + k)((1 - y^2) + y^2) \\ &= -2\mu x(1 - y^2)(x + A_0) - 2ky^2 x\left(\frac{x}{A_0} + 1\right). \end{aligned}$$

Then, by a direct calculation and simplification, we obtain

$$\mathcal{L}_1 w = 2\mu x(1 - y^2)I_1 + ky^2 I_2, \tag{3.21}$$

where

$$\begin{aligned} I_1 &= 1 - 2\mu a(1 - y^2) - O_4 + \frac{O_1}{x} - ((b + O_3) + yO_5)(x + A_0) \\ &\quad - \frac{y(2x + A_0)}{x} O_2, \\ I_2 &= (1 + O_4) + 2\mu a(1 - y^2) - 2(b + O_3)x\left(\frac{x}{A_0} + 1\right) \\ &\quad - 2yx\left(\frac{x}{A_0} + 1\right)O_5 - 2y\left(\frac{2x}{A_0} + 1\right)O_2. \end{aligned}$$

Now we choose r and μ so that $\mathcal{L}_1 w \geq 0$ holds. Clearly, $\mathcal{L}_1 w \geq 0$ if $I_1, I_2 \geq 0$. By (3.6), we find that, in $Q^+_{r,1}$,

$$\begin{aligned} I_1 &\geq 1 - 2\mu a - C_0 r - (b + N + C_0 r)A_0, \\ I_2 &\geq 1 - C_0 r - \frac{r}{A_0} C_0 r. \end{aligned} \tag{3.22}$$

Choose r_0 to satisfy the smallness assumptions stated above and

$$0 < r_0 \leq \min\left\{\frac{1}{4C_0}, \frac{b + N}{C_0}, \frac{1}{8\sqrt{C_0(b + N)}}, \frac{\hat{r}}{2}\right\}, \tag{3.23}$$

where C_0 is the constant in (3.22). For such a fixed r_0, we choose μ_0 to satisfy (3.17) and

$$\mu_0 \leq \frac{1}{8a}, \tag{3.24}$$

and A_0 to satisfy

$$4C_0 r_0^2 < A_0 < \frac{1}{8(b + N)}, \tag{3.25}$$

where we have used (3.23) to see that $4C_0 r_0^2 < \frac{1}{8(b+N)}$ in (3.25). Then k is defined from (3.20). From (3.22)–(3.25),

$$I_1, I_2 > 0,$$

which implies that

$$\mathcal{L}_1 w > 0 \quad \text{in } \mathcal{Q}^+_{r,1} \tag{3.26}$$

whenever $r \in (0, r_0]$ and $\mu \in (0, \mu_0]$.

By (3.19), (3.26), Remark 3.1, and Lemma 3.1, we have

$$\psi(x, y) \geq w(x, y) = \mu x^2(1 - y^2) - kxy^2 \quad \text{in } \mathcal{Q}^+_{r,1}.$$

In particular,

$$\psi(x, 0) \geq \mu x^2 \quad \text{for } x \in [0, r]. \tag{3.27}$$

This implies (3.15), thus (3.14). The proof is completed. □

With Proposition 3.1, we now make the $C^{2,\alpha}$ estimate of ψ.

3.3. $C^{2,\alpha}$ estimate of ψ. If ψ satisfies (3.2)–(3.4) and (3.9), it is expected that ψ is "very close" to $\frac{x^2}{2a}$, which is a solution to (3.8). More precisely, we now prove (3.13). To achieve this, we study the function

$$W(x, y) := \frac{x^2}{2a} - \psi(x, y). \tag{3.28}$$

By (3.2), W satisfies

$$\begin{aligned} \mathcal{L}_2 W &:= (x + aW_x + O_1)W_{xx} + O_2 W_{xy} \\ &\quad + (b + O_3)W_{yy} - (2 + O_4)W_x + O_5 W_y \\ &= \frac{O_1 - xO_4}{a} \quad \text{in } \mathcal{Q}^+_{\hat{r},R}, \end{aligned} \tag{3.29}$$

$$W(0, y) = 0 \quad \text{on } \partial\mathcal{Q}^+_{\hat{r},R} \cap \{x = 0\}, \tag{3.30}$$

$$-\frac{1-\beta}{a}x \leq W_x(x, y) \leq \left(M + \frac{1}{a}\right)x \quad \text{in } \mathcal{Q}^+_{\hat{r},R}. \tag{3.31}$$

Lemma 3.2. *Let $a, b, N, R, \hat{r}, \beta$, and O_i be as in Theorem 3.1. Let μ be the constant determined in Proposition 3.1. Then there exist $\alpha_1 \in (0, 1)$ and $r_1 > 0$ such that, if $W \in C(\overline{\mathcal{Q}^+_{\hat{r},R}}) \cap C^2(\mathcal{Q}^+_{\hat{r},R})$ satisfies (3.29)–(3.31), then*

$$W(x, y) \leq \frac{1 - \mu_1}{2ar^\alpha} x^{2+\alpha} \quad \textit{in } \mathcal{Q}^+_{r,7R/8}, \tag{3.32}$$

whenever $\alpha \in (0, \alpha_1]$ and $r \in (0, r_1]$ with $\mu_1 := \min(2a\mu, 1/2)$.

Proof. In the proof below, all the constants depend only on the data, i.e., $a, b, N, \beta, R, \hat{r}, \inf_{\mathcal{Q}^+_{\hat{r},R} \cap \{x > \hat{r}/2\}} \psi$, unless otherwise is stated.

By Proposition 3.1,

$$W(x,y) \le \frac{1-\mu_1}{2a}x^2 \quad \text{in } Q^+_{r_0,15R/16}, \tag{3.33}$$

where r_0 depends only on a, b, N, R, $\hat{r}$, and β.

Fix y_0 with $|y_0| \le \frac{7R}{8}$. We now prove that

$$W(x,y_0) \le \frac{1-\mu_1}{2ar^\alpha}x^{2+\alpha} \quad \text{for } x \in (0,r).$$

By a scaling argument similar to the one in the beginning of proof of Lemma 3.1, i.e., considering the function $\tilde{\psi}(x,y) = \psi(x, y_0 + \frac{R}{32}y)$ in $Q^+_{\hat{r},2}$, we conclude that, without loss of generality, we can assume that $y_0 = 0$ and $R = 2$. That is, it suffices to prove that

$$W(x,0) \le \frac{1-\mu_1}{2ar^\alpha}x^{2+\alpha} \quad \text{for } x \in (0,r) \tag{3.34}$$

for some $r \in (0,r_0)$ and $\alpha \in (0,\alpha_1)$, under the assumptions that (3.29)–(3.31) hold in $Q^+_{\hat{r},2}$ and (3.33) holds in $Q^+_{r_0,2}$.

For any given $r \in (0,r_0)$, let

$$A_1 r^\alpha = \frac{1-\mu_1}{2a}, \qquad B_1 = \frac{1-\mu_1}{2a}, \tag{3.35}$$

$$v(x,y) = A_1 x^{2+\alpha}(1-y^2) + B_1 x^2 y^2. \tag{3.36}$$

Since (3.30) holds on $\partial Q^+_{\hat{r},2} \cap \{x=0\}$ and (3.33) holds in $Q^+_{r_0,2}$, then, for all $x \in (0,r)$ and $|y| \le 1$, we obtain

$$\begin{cases} v(0,y) = 0 = W(0,y), \\ v(r,y) = \big(A_1 r^\alpha (1-y^2) + B_1 y^2\big) r^2 = \frac{1-\mu_1}{2a} r^2 \ge W(r,y), \\ v(x,\pm 1) = B_1 x^2 = \frac{1-\mu_1}{2a}x^2 \ge W(x,\pm 1). \end{cases}$$

Thus,

$$W \le v \quad \text{on } \partial Q^+_{r,1}. \tag{3.37}$$

We now show that $\mathcal{L}_2 v < \mathcal{L}_2 W$ in $Q^+_{r,1}$. From (3.29),

$$\mathcal{L}_2 v - \mathcal{L}_2 W = \mathcal{L}_2 v - \frac{O_1 - xO_4}{a}.$$

In order to rewrite the right-hand side in a convenient form, we write the term v_{yy} in the expression of $\mathcal{L}_2 v$ as $(1-y^2)v_{yy} + y^2 v_{yy}$ and use similar expressions for the terms v_{xy} and v_y. Then a direct calculation yields

$$\mathcal{L}_2 v - \frac{O_1 - xO_4}{a} = (2+\alpha)A_1 x^{1+\alpha}(1-y^2)J_1 + 2B_1 x y^2 J_2,$$

where

$$\begin{aligned}
J_1 &= (1+\alpha)\left(1 + a\big((2+\alpha)A_1x^\alpha(1-y^2) + 2B_1y^2\big) + \frac{O_1}{x}\right)\\
&\quad - (2+O_4) + T_1,\\
J_2 &= 1 + a\big((2+\alpha)A_1x^\alpha(1-y^2) + 2B_1y^2\big) + \frac{O_1}{x} - (2+O_4) + T_2,\\
T_1 &= \frac{1}{(2+\alpha)A_1x^{1+\alpha}}\Big(2O_2xy\big(2B_1 - (2+\alpha)A_1x^\alpha\big)\\
&\qquad\qquad + 2x^2\big(B_1 - A_1x^\alpha\big)((b+O_3) + O_5y) - \frac{O_1 - xO_4}{a}\Big),\\
T_2 &= \frac{(2+\alpha)A_1x^{1+\alpha}}{2B_1x}T_1.
\end{aligned}$$

Thus, in $Q^+_{r,1}$,

$$\mathcal{L}_2 v - \mathcal{L}_2 W < 0 \quad \text{if } J_1, J_2 < 0. \tag{3.38}$$

By (3.6) and (3.35), we obtain

$$|T_1|, |T_2| \le Cr^{1-\alpha} \quad \text{in } Q^+_{r,1},$$

so that, in $Q^+_{r,1}$,

$$J_1 \le (1+\alpha)\left(1 + \frac{2+\alpha}{2}(1-\mu_1)\right) - 2 + Cr^{1-\alpha}, \tag{3.39}$$

$$J_2 \le 1 + \frac{2+\alpha}{2}(1-\mu_1) - 2 + Cr^{1-\alpha}. \tag{3.40}$$

Choose $\alpha_1 > 0$, depending only on μ_1, so that, if $0 < \alpha \le \alpha_1$,

$$(1+\alpha)\left(1 + \frac{2+\alpha}{2}(1-\mu_1)\right) - 2 \le -\frac{\mu_1}{4}. \tag{3.41}$$

Such a choice of $\alpha_1 > 0$ is possible because we have the strict inequality in (3.41) when $\alpha = 0$, and the left-hand side is an increasing function of $\alpha > 0$ (where we have used $0 < \mu_1 \le 1/2$ by reducing μ if necessary). Now, choosing $r_1 > 0$ so that

$$r_1 < \min\left\{\left(\frac{\mu_1}{4C}\right)^{\frac{1}{1-\alpha}}, r_0\right\} \tag{3.42}$$

is satisfied, we use (3.39)–(3.41) to obtain

$$J_1, J_2 < 0 \quad \text{in } Q^+_{r,1}.$$

Then, by (3.38), we obtain

$$\mathcal{L}_2 v < \mathcal{L}_2 W \quad \text{in } \ Q^+_{r,1} \tag{3.43}$$

whenever $r \in (0,\ r_1]$ and $\alpha \in (0,\ \alpha_1]$. By (3.37), (3.43), Remark 3.1, and the standard comparison principle (Lemma 3.1), we obtain

$$W \le v \quad \text{in } \ Q^+_{r,1}. \tag{3.44}$$

In particular, using (3.35) and (3.36) with $y = 0$, we arrive at (3.34). □

Using Lemma 3.2, we now generalize the result (3.32) for any $\alpha \in (0, 1)$.

Proposition 3.2. *Let $a, b, N, R, \hat{r}, \beta$, and O_i be as in Theorem 3.1. Then, for any $\alpha \in (0, 1)$, there exist positive constants r and A which depend only on $a, b, N, R, \hat{r}, \beta$, and α so that, if $W \in C(\overline{Q^+_{\hat{r},R}}) \cap C^2(Q^+_{\hat{r},R})$ satisfies* (3.29)–(3.31)*, then*

$$W(x, y) \le A\, x^{2+\alpha} \quad \textit{in } \ Q^+_{r,3R/4}. \tag{3.45}$$

Proof. As argued before, without loss of generality, we may assume that $R = 2$ and it suffices to show that

$$W(x, 0) \le A x^{2+\alpha} \quad \text{for } \ x \in [0, r]. \tag{3.46}$$

By Lemma 3.2, it suffices to prove (3.46) for the case $\alpha > \alpha_1$. Fix any $\alpha \in (\alpha_1, 1)$ and set the following comparison function:

$$u(x, y) = \frac{1-\mu_1}{2ar_1^{\alpha_1} r^{\alpha-\alpha_1}} x^{2+\alpha}(1 - y^2) + \frac{1-\mu_1}{2ar_1^{\alpha_1}} x^{2+\alpha_1} y^2. \tag{3.47}$$

By Lemma 3.2,

$$W \le u \quad \text{on } \ \partial Q^+_{r,1} \quad \text{for } \ r \in (0, r_1]. \tag{3.48}$$

As in the proof of Lemma 3.2, we write

$$\begin{aligned}\mathcal{L}_2 u - \frac{O_1 - xO_4}{a} &= (2+\alpha)\frac{(1-\mu_1)x^{1+\alpha}}{2ar_1^{\alpha_1} r^{\alpha-\alpha_1}}(1 - y^2)\hat{J}_1 \\ &\quad + (2+\alpha_1)\frac{(1-\mu_1)x^{1+\alpha_1}}{2ar_1^{\alpha_1}} y^2 \hat{J}_2,\end{aligned}$$

where

$$\begin{aligned} D_0 &= \frac{1-\mu_1}{2}\left((1-y^2)(2+\alpha)\left(\frac{x}{r}\right)^{\alpha} + y^2(2+\alpha_1)\left(\frac{x}{r}\right)^{\alpha_1}\right), \\ \hat{J}_1 &= (1+\alpha)\left(1 + \left(\frac{r}{r_1}\right)^{\alpha_1} D_0\right) - 2 + \hat{T}_1, \end{aligned}$$

$$\begin{aligned}
\hat{J}_2 &= (1+\alpha_1)\left(1+\left(\frac{r}{r_1}\right)^{\alpha_1} D_0\right) - 2 + \hat{T}_2,\\
\hat{T}_1 &= \frac{2ar_1^{\alpha_1} r^{\alpha-\alpha_1}}{(2+\alpha(1-\mu_1))x^{1+\alpha}}\\
&\qquad\qquad \times \left(\mathcal{L}_2 u - ((x+au_x)u_{xx} - 2u_x) - \frac{O_1 - xO_4}{a}\right),\\
\hat{T}_2 &= \frac{2ar_1^{\alpha_1}}{(a+\alpha_1)(1-\mu_1)x^{1+\alpha_1}}\\
&\qquad\qquad \times \left(\mathcal{L}_2 u - ((x+au_x)u_{xx} - 2u_x) - \frac{O_1 - xO_4}{a}\right).
\end{aligned}$$

By (3.6), we have

$$|\hat{T}_1|, |\hat{T}_2| \le Cr^{1-\alpha_1}$$

for some positive constant C depending only on a, b, N, β, r_1, and α_1. Thus, we have

$$\max\{\hat{J}_1, \hat{J}_2\} \le (1+\alpha)\left(1+\left(\frac{r}{r_1}\right)^{\alpha_1}(2+\alpha)\frac{1-\mu_1}{2}\right) - 2 + Cr^{1-\alpha_1} \quad \text{for } (x,y) \in Q^+_{r,1}. \tag{3.49}$$

Choosing $r > 0$ sufficiently small, depending only on r_1, α, and C, we obtain

$$\mathcal{L}_2 u - \mathcal{L}_2 W = \mathcal{L}_2 u - \frac{O_1 - xO_4}{a} < 0 \quad \text{in } Q^+_{r,1}.$$

Then Lemma 3.1 implies that

$$W \le u \quad \text{in } Q^+_{r,1}.$$

Thus, (3.46) holds with

$$A = \frac{1-\mu_1}{2ar_1^{\alpha_1} r^{\alpha-\alpha_1}}.$$

□

Lemma 3.3. *Let $a, b, N, R, \hat{r}, \beta$, and O_i be as in Theorem 3.1. Then there exist $r_2 > 0$ and $\alpha_2 \in (0,1)$ such that, if $W \in C(\overline{Q^+_{\hat{r},R}}) \cap C^2(Q^+_{\hat{r},R})$ satisfies* (3.29)–(3.31)*, we have*

$$W(x,y) \ge -\frac{1-\beta}{2ar^{\alpha}} x^{2+\alpha} \quad \text{in } Q^+_{r,7R/8} \tag{3.50}$$

whenever $\alpha \in (0, \alpha_2]$ and $r \in (0, r_2]$.

Proof. By (3.30) and (3.31), it can be easily verified that $W(x,y) \ge -\frac{1-\beta}{2a}x^2$ in $Q^+_{\hat{r},R}$. Now, similar to the proof of Lemma 3.2, it suffices to prove that, with the assumption $R = 2$,

$$W(x, 0) \geq -\frac{1-\beta}{2ar^{\alpha}} x^{2+\alpha} \quad \text{for } x \in (0, r)$$

for some $r > 0$ and $\alpha \in (0, \alpha_2)$.

For this, we use the comparison function:

$$v(x, y) := -Lx^{2+\alpha}(1 - y^2) - Kx^2y^2, \quad \text{with } Lr^{\alpha} = K = \frac{1-\beta}{2a}.$$

Then we follow the same procedure as the proof of Lemma 3.2, except that $\mathcal{L}_2 v > \mathcal{L}_2 W$, to find that the conditions for the choice of $\alpha, r > 0$ are inequalities (3.41) and (3.42) with μ_1, r_1 replaced by β, r_2 and with an appropriate constant C. □

Using Lemma 3.3, we now generalize the result (3.50) for any $\alpha \in (0, 1)$.

Proposition 3.3. *Let $a, b, N, R, \hat{r}, \beta$, and O_i be as in Theorem 3.1. Then, for any $\alpha \in (0, 1)$, there exist positive constants r and B depending on $a, b, N, R, \hat{r}, \beta$, and α so that, if $W \in C(\overline{Q^+_{\hat{r},R}}) \cap C^2(Q^+_{\hat{r},R})$ satisfies* (3.29)–(3.31)*, then*

$$W(x, y) \geq -Bx^{2+\alpha} \quad \text{in } Q^+_{r,3R/4}. \tag{3.51}$$

Proof. For fixed $\alpha \in (\alpha_2, 1)$, we set the comparison function:

$$u_-(x, y) = -\frac{1-\beta}{2ar_2^{\alpha_2} r^{\alpha-\alpha_2}} x^{2+\alpha}(1 - y^2) - \frac{1-\beta}{2ar_2^{\alpha_2}} x^{2+\alpha_2} y^2.$$

Then, using the argument as in the proof of Proposition 3.2, we can choose $r > 0$ appropriately small so that

$$\mathcal{L}_2 u_- - \mathcal{L}_2 W = \mathcal{L}_2 u_- - \frac{O_1 - xO_4}{a} > 0$$

holds for all $(x, y) \in Q^+_{r,1}$. □

With Propositions 3.2 and 3.3, we now prove Theorem 3.1.

3.4. Proof of Theorem 3.1.

We divide the proof into four steps.

Step 1. Let ψ be a solution of (3.2) in $Q^+_{\hat{r},R}$ for $\hat{r}$ as in Remark 3.1, and let the assumptions of Theorem 3.1 hold. Then ψ satisfies (3.10). Thus it suffices to show that, for any given $\alpha \in (0, 1)$, there exists $r > 0$ so that $\psi \in C^{2,\alpha}(\overline{Q^+_{r,R/2}})$ and $\psi_{xx}(0, y) = \frac{1}{a}$, $\psi_{xy}(0, y) = \psi_{yy}(0, y) = 0$ for all $|y| < \frac{R}{2}$.

Let $W(x, y)$ be defined by (3.28). Then, in order to prove Theorem 3.1, it suffices to show that, for any given $\alpha \in (0, 1)$, there exists $r > 0$ so that

(i) $W \in C^{2,\alpha}(\overline{Q^+_{r,R/2}})$;
(ii) $D^2W(0, y) = 0$ for all $|y| < \frac{R}{2}$.

Step 2. By definition, W satisfies (3.29)–(3.31). For any given $\alpha \in (0,1)$, there exists $r > 0$ so that both (3.45) and (3.51) hold in $Q^+_{r,3R/4}$ by Propositions 3.2 and 3.3. Fix such $r > 0$.

Furthermore, since W satisfies estimate (3.31), we can introduce a cutoff function into the nonlinear term of (3.29), i.e., modify the nonlinear term away from the values determined by (3.31) to make the term bounded in W_x/x. Namely, fix $\zeta \in C^\infty(\mathbb{R})$ satisfying

$$\begin{aligned} -\frac{1-\beta/2}{a} \le \zeta \le M + \frac{2}{a} \quad &\text{on } \mathbb{R}; \\ \zeta(s) = s \quad \text{on } &\left(-\frac{1-\beta}{a}, M + \frac{1}{a}\right); \\ \zeta \equiv 0 \quad \text{on } \mathbb{R} \setminus &\left(-\frac{2-\beta}{a}, M + \frac{4}{a}\right). \end{aligned} \tag{3.52}$$

Then, from (3.29) and (3.31), it follows that W satisfies

$$\begin{aligned} x\left(1 + a\zeta\left(\frac{W_x}{x}\right) + \frac{O_1}{x}\right) W_{xx} + O_2 W_{xy} + (b + O_3) W_{yy} \\ -(2 + O_4) W_x + O_5 W_y = \frac{O_1 - xO_4}{a} \quad \text{in } Q^+_{\hat{r},R}. \end{aligned} \tag{3.53}$$

Step 3. For $z := (x, y) \in Q^+_{r/2,R/2}$, define

$$R_z := \left\{(s,t) : |s - x| < \frac{x}{8},\ |t - y| < \frac{\sqrt{x}}{8}\right\}. \tag{3.54}$$

Then

$$R_z \subset Q^+_{r,3R/4} \quad \text{for any } z = (x,y) \in Q^+_{r/2,R/2}. \tag{3.55}$$

Fix $z_0 = (x_0, y_0) \in Q^+_{r/2,R/2}$. Rescale W in R_{z_0} by defining

$$W^{(z_0)}(S,T) = \frac{1}{x_0^{2+\alpha}} W\left(x_0 + \frac{x_0}{8}S, y_0 + \frac{\sqrt{x_0}}{8}T\right) \quad \text{for } (S,T) \in Q_1, \tag{3.56}$$

where $Q_h = (-h,h)^2$ for $h > 0$. Then, by (3.45), (3.51), and (3.55), we have

$$\|W^{(z_0)}\|_{C^0(\overline{Q_1})} \le \frac{1}{ar^\alpha}. \tag{3.57}$$

Moreover, since W satisfies (3.29), $W^{(z_0)}$ satisfies the following equation for $(S,T)\in Q_1$:

$$\begin{aligned}&\left(1+\frac{S}{8}\right)\left(1+a\zeta\left(\frac{8x_0^\alpha W_S^{(z_0)}}{1+\frac{S}{8}}\right)+\tilde{O}_1^{(z_0)}\right)W_{SS}^{(z_0)}+\tilde{O}_2^{(z_0)}W_{ST}^{(z_0)}\\&\qquad+\big(b+\tilde{O}_3^{(z_0)}\big)W_{TT}^{(z_0)}-\frac{1}{8}\big(2+\tilde{O}_4^{(z_0)}\big)W_S^{(z_0)}+\frac{1}{8}\tilde{O}_5^{(z_0)}W_T^{(z_0)}\\&\quad=\frac{\big(1+\frac{S}{8}\big)}{64ax_0^\alpha}\big(\tilde{O}_1^{(z_0)}-\tilde{O}_4^{(z_0)}\big),\end{aligned}\tag{3.58}$$

where

$$\begin{gathered}\tilde{O}_1^{(z_0)}(S,T)=\frac{1}{x_0\left(1+\frac{S}{8}\right)}O_1(x,y),\qquad \tilde{O}_2^{(z_0)}(S,T)=\frac{1}{\sqrt{x_0}}O_2(x,y),\\ \tilde{O}_3^{(z_0)}(S,T)=O_3(x,y),\qquad \tilde{O}_4^{(z_0)}(S,T)=O_4(x,y),\\ \tilde{O}_5^{(z_0)}(S,T)=\sqrt{x_0}\,O_5(x,y),\end{gathered}$$

with $x=x_0(1+\frac{S}{8})$ and $y=y_0+\frac{\sqrt{x_0}}{8}T$. Then, from (3.6) and (3.7), we find that, for all $(S,T)\in\overline{Q_1}$ and $z_0\in Q^+_{r/2,R/2}$ with $r\le 1$,

$$\begin{aligned}\big|\tilde{O}_k^{(z_0)}(S,T)\big|&\le 2N\sqrt{r}\quad\text{for }\ k=1,\dots,5,\\ \big|D\tilde{O}_k^{(z_0)}(S,T)\big|&\le 2N\sqrt{r}\quad\text{for }\ k\ne 2,\\ \big|D\tilde{O}_2^{(z_0)}(S,T)\big|&\le 2N.\end{aligned}\tag{3.59}$$

Also, denoting the right-hand side of (3.58) by $F^{(z_0)}(S,T)$, we obtain from (3.59) that, for all $(S,T)\in\overline{Q_1}$ and $z_0\in Q^+_{r/2,R/2}$,

$$|F^{(z_0)}(S,T)|\le Cr^{1-\alpha},\qquad |DF^{(z_0)}(S,T)|\le Cr^{\frac{1}{2}-\alpha},\tag{3.60}$$

where C depends only on N and a.

Now, writing (3.58) as

$$\sum_{i,j=1}^{2}A_{ij}(DW^{(z_0)},S,T)D_{ij}^2W^{(z_0)}+\sum_{i=1}^{2}B_i(S,T)D_iW^{(z_0)}=F^{(z_0)}\quad\text{in }\ Q_1,\tag{3.61}$$

we get from (3.52) and (3.58)–(3.60) that, if $r>0$ is sufficiently small, depending only on the data, then (3.61) is uniformly elliptic with elliptic constants depending only on b but independent of z_0, and that the coefficients $A_{ij}(p,S,T)$, $B_i(S,T)$, and $F^{(z_0)}(S,T)$, for $p\in\mathbb{R}^2$, $(S,T)\in Q_1$, satisfy

$$\|A_{ij}\|_{C^1(\mathbb{R}^2\times\overline{Q_1})}\le C,\qquad \left\|\left(B_i,\frac{F^{(z_0)}}{r^{1/2-\alpha}}\right)\right\|_{C^1(\overline{Q_1})}\le C,$$

where C depends only on the data and is independent of z_0. Then, by [10, Theorem A1] and (3.57),

$$\begin{aligned} \|W^{(z_0)}\|_{C^{2,\alpha}(\overline{Q_{1/2}})} &\le C\big(\|W^{(z_0)}\|_{C^0(\overline{Q_1})} + \|F^{(z_0)}\|_{C^\alpha(\overline{Q_1})}\big) \\ &\le C\left(\frac{1}{r^\alpha a} + r^{1/2-\alpha}\right) =: \hat{C}, \end{aligned} \tag{3.62}$$

where C depends only on the data and α in this case. From (3.62),

$$\begin{aligned} &\big|D_x^i D_y^j W(x_0, y_0)\big| \le C x_0^{2+\alpha-i-j/2} \\ &\qquad\qquad \text{for all } \ (x_0, y_0) \in Q^+_{r/2,R/2}, 0 \le i+j \le 2. \end{aligned} \tag{3.63}$$

Step 4. It remains to prove the C^α-continuity of D^2W in $\overline{Q^+_{r/2,R/2}}$.

For two distinct points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in Q^+_{r/2,R/2}$, consider

$$A := \frac{|W_{xx}(z_1) - W_{xx}(z_2)|}{|z_1 - z_2|^\alpha}.$$

Without loss of generality, assume that $x_1 \le x_2$. There are two cases:

Case 1. $z_1 \in R_{z_2}$. Then

$$x_1 = x_2 + \frac{x_2}{8}S, \quad y_1 = y_2 + \frac{\sqrt{x_2}}{8}T \quad \text{for some } \ (S, T) \in Q_1.$$

By (3.62),

$$\frac{\big|W_{SS}^{(z_2)}(S,T) - W_{SS}^{(z_2)}(0,0)\big|}{(S^2+T^2)^{\alpha/2}} \le \hat{C},$$

which is

$$\frac{|W_{xx}(x_1, y_1) - W_{xx}(x_2, y_2)|}{\big((x_1-x_2)^2 + x_2(y_1-y_2)^2\big)^{\alpha/2}} \le \hat{C}.$$

Since $x_2 \in (0, r)$ and $r \le 1$, the last estimate implies

$$\frac{|W_{xx}(x_1, y_1) - W_{xx}(x_2, y_2)|}{\big((x_1-x_2)^2 + (y_1-y_2)^2\big)^{\alpha/2}} \le \hat{C}.$$

Case 2. $z_1 \notin R_{z_2}$. Then, either $|x_1 - x_2| > \frac{x_2}{8}$ or $|y_1 - y_2| > \frac{\sqrt{x_2}}{8}$. Since $0 \le x_2 \le r \le 1$, we find

$$|z_1 - z_2|^\alpha \ge \left(\frac{x_2}{8}\right)^\alpha.$$

Thus, using (3.63) and $x_1 \le x_2$, we obtain

$$\frac{|W_{xx}(z_1) - W_{xx}(z_2)|}{|z_1 - z_2|^\alpha} \le \frac{|W_{xx}(z_1)| + |W_{xx}(z_2)|}{|z_1 - z_2|^\alpha} \le \hat{C}\frac{x_1^\alpha + x_2^\alpha}{x_2^\alpha} \le 2\hat{C}.$$

Therefore, $A \le 2\hat{C}$ in both cases, where $\hat{C}$ depends on α, r, and the data. Since $z_1 \neq z_2$ are arbitrary points of $Q^+_{r/2,R/2}$, we obtain

$$[W_{xx}]_{C^\alpha(\overline{Q^+_{r/2,R/2}})} \le 2\hat{C}. \tag{3.64}$$

The estimates for W_{xy} and W_{yy} can be obtained similarly. In fact, for these derivatives, we obtain the stronger estimates: For any $\delta \in (0, r/2]$,

$$[W_{xy}]_{C^\alpha(\overline{Q^+_{\delta,R/2}})} \le \hat{C}\sqrt{\delta}, \quad [W_{yy}]_{C^\alpha(\overline{Q^+_{\delta,R/2}})} \le \hat{C}\delta,$$

where $\hat{C}$ depends on α, r, and the data, but is independent of $\delta > 0$ and z_0.

Thus, $W \in C^{2,\alpha}(\overline{Q^+_{r,R/2}})$ with $\|W\|_{C^{2,\alpha}(\overline{Q^+_{r,R/2}})}$ depending only on the data because $r > 0$ depends on the data. Moreover, (3.63) implies $D^2W(0, y) = 0$ for any $|y| \le R/2$. This concludes the proof of Theorem 3.1. □

4. Optimal regularity of solutions to regular shock reflection across the sonic circle

As we indicated in Sect. 2, the global solution φ constructed in [9–11] is at least $C^{1,1}$ near the pseudo-sonic circle P_1P_4. On the other hand, the behavior of solutions to regular shock reflection has not been understood completely; so it is essential to understand first the regularity of regular reflection solutions. In this section, we prove that $C^{1,1}$ is in fact the optimal regularity of any solution φ across the pseudo-sonic circle P_1P_4 in the class of standard regular reflection solutions. Our main results include the following three ingredients:

(i) There is no regular reflection solution that is C^2 across the pseudo-sonic circle;
(ii) For the solutions constructed in [9–11] or, more generally, for any regular reflection solution satisfying properties (ii) and (iv)–(vi) at the end of Sect. 2, φ is $C^{2,\alpha}$ in the pseudo-subsonic region Ω up to the pseudo-sonic circle P_1P_4, excluding the endpoint P_1, but $D^2\varphi$ has a jump across P_1P_4;
(iii) In addition, $D^2\varphi$ does not have a limit at P_1 from Ω.

In order to state these results, we first define the class of regular reflection solutions. As proved in [9,10], when the wedge angle θ_w is large, such a regular reflection configuration exists; and in [11], we extend this to other wedge angles for which state (2) of form (2.20) exists.

Now we define the class of regular reflection solutions.

Definition 4.1. *Let* $\gamma > 1$, $\rho_1 > \rho_0 > 0$ *and* $\theta_w \in (0, \frac{\pi}{2})$ *be constants, let* u_1 *and* ξ_0 *be defined by* (2.16) *and* (2.17). *Let the incident shock* $S = \{\xi = \xi_0\}$ *hit the wedge at the point* $P_0 = (\xi_0, \xi_0 \tan\theta_w)$, *and let state* (0) *and state* (1) *ahead of and behind* S *be given by* (2.14) *and* (2.15), *respectively. The function* $\varphi \in C^{0,1}(\Lambda)$ *is a regular reflection solution if* φ *is a solution to Problem* 2 *(see Fig. 1) such that*

(a) *there exists state* (2) *of form* (2.20) *with* $u_2 > 0$, *satisfying the entropy condition* $\rho_2 > \rho_1$ *and the Rankine–Hugoniot condition* $(\rho_1 D\varphi_1 - \rho_2 D\varphi_2) \cdot \nu = 0$ *along the line* $S_1 := \{\varphi_1 = \varphi_2\}$ *which contains the points* P_0 *and* P_1, *such that* $P_1 \in \Lambda$ *is on the pseudo-sonic circle of state* (2), *and state* (2) *is pseudo-supersonic along* P_0P_1*;*

(b) *there exists an open, connected domain* $\Omega := P_1P_2P_3P_4 \subset \Lambda$ *such that* (2.23) *holds and* (2.7) *is elliptic in* Ω*;*

(c) $\varphi \geq \varphi_2$ *on the part* $P_1P_2 = \Gamma_{shock}$ *of the reflected shock.*

Remark 4.1. The global solution constructed in [9–11] is a regular reflection solution, which is a part of the assertions at the end of Sect. 2.

Remark 4.2. We note that, in the case $\theta_w = \frac{\pi}{2}$, the regular reflection becomes the normal reflection, in which $u_2 = 0$ and the solution is smooth across the sonic line of state (2), see [10, Sect. 3.1]. Condition $\theta_w \in (0, \frac{\pi}{2})$ in Definition 4.1 rules out this case. Moreover, for $\theta_w \in (0, \frac{\pi}{2})$, the property $u_2 > 0$ in part (a) of Definition 4.1 is always true for state (2) of form (2.20), satisfying the entropy condition $\rho_2 > \rho_1$ and the Rankine–Hugoniot condition along the line $S_1 := \{\varphi_1 = \varphi_2\}$ which contains the point P_0. These are readily derived from the calculations in [10, Sect. 3.2].

Remark 4.3. If state (2) exists and is pseudo-supersonic at P_0, then the line $S_1 = \{\varphi_1 = \varphi_2\}$ necessarily intersects the pseudo-sonic circle of state (2); see the argument in [10] starting from (3.5) there. Thus, the only assumption regarding the point P_1 is that S_1 intersects the pseudo-sonic circle within Λ.

Remark 4.4. There may exist a global regular reflection configuration when state (2) is pseudo-subsonic at P_0 which is a very narrow regime [14,36]. Such a case does not involve the difficulty of elliptic degeneracy which we are facing in our case.

Remark 4.5. Since $\varphi = \varphi_1$ on Γ_{shock} by (2.23), Condition (c) in Definition 4.1 is equivalent to

$$\Gamma_{shock} \subset \{\varphi_2 \leq \varphi_1\},$$

that is, Γ_{shock} is below S_1.

Furthermore, we have

Lemma 4.1. *For any regular reflection solution* φ *in the sense of Definition 4.1,*

$$\varphi > \varphi_2 \quad \text{in } \Omega. \tag{4.1}$$

Proof. By (2.7), (2.8), and (2.20), $\psi := \varphi - \varphi_2$ satisfies

$$\begin{aligned}&\big(\tilde{c}^2 - (\psi_\xi - \xi + u_2)^2\big)\psi_{\xi\xi} + \big(\tilde{c}^2 - (\psi_\eta - \eta + u_2\tan\theta_w)^2\big)\psi_{\eta\eta}\\ &\qquad -2(\psi_\xi - \xi + u_2)(\psi_\eta - \eta + u_2\tan\theta_w)\psi_{\xi\eta} = 0 \quad \text{in } \Omega,\end{aligned}\tag{4.2}$$

where $\tilde{c}^2(D\psi, \psi, \xi, \eta) = c_2^2 + (\gamma - 1)((\xi - u_2)\psi_\xi + (\eta - u_2\tan\theta_w)\psi_\eta - \frac{1}{2}|D\psi|^2 - \psi)$. We regard that the coefficients of (4.2) are computed on ψ as fixed, and hence we can consider (4.2) as a linear equation with respect to the second derivative of ψ.

Since (4.2) is elliptic inside Ω and φ is smooth inside Ω, it follows that (4.2) is uniformly elliptic in any compact subset of Ω. Furthermore, we have

$$\begin{aligned}\psi &= 0 && \text{on } \Gamma_{sonic},\\ D\psi\cdot(-\sin\theta_w, \cos\theta_w) &= 0 && \text{on } \Gamma_{wedge},\\ \psi_\eta = -u_2\tan\theta_w &< 0 && \text{on } \partial\Omega \cap \{\eta = 0\},\\ \psi &\geq 0 && \text{on } \Gamma_{shock} \quad \text{(by Definition 4.1 (c))}.\end{aligned}$$

Then the strong maximum principle implies

$$\psi > 0 \quad \text{in } \Omega,$$

which is (4.1). This completes the proof. □

Now we first show that any regular reflection solution in our case cannot be C^2 across the pseudo-sonic circle $\Gamma_{sonic} := P_1P_4$.

Theorem 4.1. *Let φ be a regular reflection solution in the sense of Definition 4.1. Then φ cannot be C^2 across the pseudo-sonic circle Γ_{sonic}.*

Proof. On the contrary, assume that φ is C^2 across Γ_{sonic}. Then $\psi = \varphi - \varphi_2$ is also C^2 across Γ_{sonic}, where φ_2 is given by (2.20). Moreover, since $\psi \equiv 0$ in $P_0P_1P_4$ by (2.23), we have $D^2\psi(\xi, \eta) = 0$ at all $(\xi, \eta) \in \Gamma_{sonic}$.

Now substituting $\varphi = \psi + \varphi_2$ into (2.7) and writing the resulting equation in the (x, y)-coordinates (2.22) in the domain $\Omega_{\varepsilon_0} := \Omega \cap \{0 < x < \varepsilon\}$ as defined in Sect. 2, we find by an explicit calculation that $\psi(x, y)$ satisfies (3.2) in Ω_{ε_0} with $a = \gamma + 1$ and $b = \frac{1}{c_2}$ and with $O_j = O_j(x, y, \psi, \psi_x, \psi_y)$, $j = 1, \ldots, 5$, given by

$$\begin{aligned}O_1(D\psi, \psi, x) &= -\frac{x^2}{c_2} + \frac{\gamma + 1}{2c_2}(2x - \psi_x)\psi_x\\ &\qquad - \frac{\gamma - 1}{c_2}\left(\psi + \frac{1}{2(c_2 - x)^2}\psi_y^2\right),\\ O_2(D\psi, \psi, x) &= -\frac{2}{c_2(c_2 - x)^2}(\psi_x + c_2 - x)\psi_y,\end{aligned}$$

$$
\begin{aligned}
O_3(D\psi,\psi,x) &= \frac{1}{c_2(c_2-x)^2}\bigg(x(2c_2-x)-\frac{\gamma+1}{2(c_2-x)^2}\psi_y^2\\
&\qquad\qquad -(\gamma-1)\Big(\psi+(c_2-x)\psi_x+\frac{1}{2}\psi_x^2\Big)\bigg),\\
O_4(D\psi,\psi,x) &= \frac{1}{c_2-x}\bigg(x-\frac{\gamma-1}{c_2}\Big(\psi+(c_2-x)\psi_x\\
&\qquad\qquad +\frac{1}{2}\psi_x^2+\frac{(\gamma+1)\psi_y^2}{2(\gamma-1)(c_2-x)^2}\Big)\bigg),\\
O_5(D\psi,\psi,x) &= -\frac{2}{c_2(c_2-x)^3}\big(\psi_x+c_2-x\big)\psi_y.
\end{aligned}
\tag{4.3}
$$

Let $(0, y_0)$ be a point in the relative interior of Γ_{sonic}. Then $(0, y_0) + Q^+_{r,R} \subset \Omega_{\varepsilon_0}$ if $r, R > 0$ are sufficiently small. By shifting the coordinates $(x, y) \to (x, y - y_0)$, we can assume $(0, y_0) = (0, 0)$ and $Q^+_{r,R} \subset \Omega_{\varepsilon_0}$. Note that the shifting coordinates in the y-direction does not change the expressions in (4.3).

Since $\psi \in C^2(\Omega_{\varepsilon_0} \cup \Gamma_{sonic})$ with $D^2\psi \equiv 0$ on Γ_{sonic}, reducing r if necessary, we get $|D\psi| \le \delta x$ in $Q^+_{r,R}$, where $\delta > 0$ is so small that (3.9) holds in $Q^+_{r,R}$, with $\beta = M = 1$, and that the terms O_i defined by (4.3) satisfy (3.6) and (3.7) with $M = 1$. Also, from Definition 4.1, we obtain that $\psi = \varphi - \varphi_2 > 0$ in $Q^+_{r,R}$. Now we can apply Proposition 3.1 to conclude

$$\psi(x, y) \ge \mu x^2 \quad \text{on } Q^+_{r,15R/16}$$

for some $\mu, r > 0$. This contradicts the fact that $D^2\psi(0, y) = 0$ for all $y \in (-R, R)$, that is, $D^2\psi(\xi, \eta) = 0$ at any $(\xi, \eta) \in \Gamma_{sonic}$. □

In the following theorem, we study more detailed regularity of ψ near the pseudo-sonic circle in the case of $C^{1,1}$ regular reflection solutions. Note that this class of solutions especially includes the solutions constructed in [9–11].

From now on, we use a localized version of Ω_ε: For given neighborhood $\mathcal{N}(\Gamma_{sonic})$ of Γ_{sonic} and $\varepsilon > 0$, define

$$\Omega_\varepsilon := \Omega \cap \mathcal{N}(\Gamma_{sonic}) \cap \{x < \varepsilon\}.$$

Since $\mathcal{N}(\Gamma_{sonic})$ will be fixed in the following theorem, we do not specify the dependence of Ω_ε on $\mathcal{N}(\Gamma_{sonic})$.

Theorem 4.2. *Let φ be a regular reflection solution in the sense of Definition 4.1 and satisfy the properties: There exists a neighborhood $\mathcal{N}(\Gamma_{sonic})$ of Γ_{sonic} such that*

(a) *φ is $C^{1,1}$ across the part Γ_{sonic} of the pseudo-sonic circle, i.e., $\varphi \in C^{1,1}(\overline{P_0P_1P_2P_3} \cap \mathcal{N}(\Gamma_{sonic}))$;*

(b) *there exists* $\delta_0 > 0$ *so that, in the coordinates* (2.22),

$$|\partial_x(\varphi - \varphi_2)(x, y)| \le \frac{2-\delta_0}{\gamma+1} x \quad in\ \ \Omega \cap \mathcal{N}(\Gamma_{sonic}); \tag{4.4}$$

(c) *there exist* $\varepsilon_0 > 0$*,* $\omega > 0$ *and a function* $y = \hat{f}(x)$ *such that, in the coordinates* (2.22),

$$\begin{aligned}\Omega_{\varepsilon_0} &= \{(x, y) \,:\, x \in (0, \varepsilon_0),\ 0 < y < \hat{f}(x)\},\\ \Gamma_{shock} \cap \partial\Omega_{\varepsilon_0} &= \{(x, y) \,:\, x \in (0, \varepsilon_0),\ y = \hat{f}(x)\},\end{aligned} \tag{4.5}$$

and

$$\|\hat{f}\|_{C^{1,1}([0,\varepsilon_0])} < \infty, \qquad \frac{d\hat{f}}{dx} \ge \omega > 0 \quad for\ \ 0 < x < \varepsilon_0. \tag{4.6}$$

Then we have

(i) φ *is* $C^{2,\alpha}$ *in* Ω *up to* Γ_{sonic} *away from the point* P_1 *for any* $\alpha \in (0, 1)$*. That is, for any* $\alpha \in (0, 1)$ *and any given* $(\xi_0, \eta_0) \in \overline{\Gamma_{sonic}} \setminus \{P_1\}$*, there exists* $K < \infty$ *depending only on* ρ_0*,* ρ_1*,* γ*,* ε_0*,* α*,* $\|\varphi\|_{C^{1,1}(\Omega_{\varepsilon_0})}$*, and* $d = dist((\xi_0, \eta_0), \Gamma_{shock})$ *so that*

$$\|\varphi\|_{2,\alpha;\overline{B_{d/2}(\xi_0,\eta_0)\cap\Omega_{\varepsilon_0/2}}} \le K;$$

(ii) *for any* $(\xi_0, \eta_0) \in \Gamma_{sonic} \setminus \{P_1\}$*,*

$$\lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\ (\xi,\eta)\in\Omega}} (D_{rr}\varphi - D_{rr}\varphi_2) = \frac{1}{\gamma+1};$$

(iii) $D^2\varphi$ *has a jump across* Γ_{sonic}*: For any* $(\xi_0, \eta_0) \in \Gamma_{sonic} \setminus \{P_1\}$*,*

$$\lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\ (\xi,\eta)\in\Omega}} D_{rr}\varphi - \lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\ (\xi,\eta)\in\Lambda\setminus\Omega}} D_{rr}\varphi = \frac{1}{\gamma+1};$$

(iv) *the limit* $\lim_{\substack{(\xi,\eta)\to P_1\\ (\xi,\eta)\in\Omega}} D^2\varphi$ *does not exist.*

Proof. The proof consists of seven steps.

Step 1. Let

$$\psi := \varphi - \varphi_2.$$

By (2.23) and (4.4), we have

$$\psi(0, y) = \psi_x(0, y) = \psi_y(0, y) = 0 \quad \text{for all}\ \ (0, y) \in \Gamma_{sonic}, \tag{4.7}$$

and thus, using also (4.5) and (4.6), we find that

$$|\psi(x, y)| \le Cx^2, \quad |D_{x,y}\psi(x, y)| \le Cx \quad \text{for all}\ \ (x, y) \in \Omega_{\varepsilon_0}, \tag{4.8}$$

where C depends only on $\|\psi\|_{C^{1,1}(\overline{\Omega_{\varepsilon_0}})}$ and $\|\hat{f}\|_{C^1([0,\varepsilon_0])}$.

Recall that, in the (x, y)-coordinates (2.22) in the domain Ω_{ε_0} defined in (c) satisfying (4.5), $\psi(x, y)$ satisfies (3.2) with $O_i = O_i(x, y, \psi, \psi_x, \psi_y)$ given by (4.3). Then it follows from (4.3) and (4.8) that (3.6) and (3.7) hold with N depending only on ε_0, $\|\psi\|_{C^{1,1}(\overline{\Omega_{\varepsilon_0}})}$, and $\|\hat{f}\|_{C^1([0,\varepsilon_0])}$.

Step 2. Now, using (4.4) and reducing ε_0 if necessary, we conclude that (3.2) is uniformly elliptic on $\Omega_{\varepsilon_0} \cap \{x > \delta\}$ for any $\delta \in (0, \varepsilon_0)$. Moreover, by (c), (3.2) with (4.3), considered as a linear elliptic equation, has C^1 coefficients. Furthermore, since the boundary conditions (2.18) hold for φ and φ_2, especially on $\Gamma_{wedge} = \{y = 0\}$, it follows that, in the (x, y)-coordinates, we have

$$\psi_y(x, 0) = 0 \quad \text{for all } x \in (0, \varepsilon_0). \tag{4.9}$$

Then, by the standard regularity theory for the oblique derivative problem for linear, uniformly elliptic equations, ψ is C^2 in Ω_{ε_0} up to $\partial\Omega_{\varepsilon_0} \cap \{0 < x < \varepsilon_0,\ y = 0\}$. From this and (c), we have

$$\psi \in C^{1,1}(\overline{\Omega_{\varepsilon_0}}) \cap C^2\big(\Omega_{\varepsilon_0} \cup \Gamma^{(\varepsilon_0)}_{wedge}\big), \tag{4.10}$$

where $\Gamma^{(\varepsilon_0)}_{wedge} := \Gamma_{wedge} \cap \partial\Omega_{\varepsilon_0} \equiv \{(x, 0) \,:\, 0 < x < \varepsilon_0\}$.

Reflect Ω_{ε_0} with respect to the y-axis, i.e., using (4.5), define

$$\hat{\Omega}_{\varepsilon_0} := \{(x, y) \,:\, x \in (0, \varepsilon_0),\ -\hat{f}(x) < y < \hat{f}(x)\}. \tag{4.11}$$

Extend $\psi(x, y)$ from Ω_{ε_0} to $\hat{\Omega}_{\varepsilon_0}$ by the even reflection, i.e., defining $\psi(x, -y) = \psi(x, y)$ for $(x, y) \in \Omega_{\varepsilon_0}$. Using (4.9) and (4.10), we conclude that the extended function $\psi(x, y)$ satisfies

$$\psi \in C^{1,1}(\overline{\hat{\Omega}_{\varepsilon_0}}) \cap C^2(\hat{\Omega}_{\varepsilon_0}). \tag{4.12}$$

Now we use the explicit expressions (3.2) and (4.3) to find that, if $\psi(x, y)$ satisfies (3.2) with (4.3) in Ω_{ε_0}, then the function $\tilde{\psi}(x, y) := \psi(x, -y)$ also satisfies (3.2) with $O_k(D\tilde{\psi}, \tilde{\psi}, x)$ defined by (4.3) in Ω_{ε_0}. Thus, in the extended domain $\hat{\Omega}_{\varepsilon_0}$, the extended $\psi(x, y)$ satisfies (3.2) with $O_1, \ldots, O_5$ defined by the expressions (4.3) in $\hat{\Omega}_{\varepsilon_0}$.

Moreover, by (2.23), it follows that $\psi = 0$ on Γ_{sonic}. Thus, in the (x, y)-coordinates, for the extended ψ, we obtain

$$\psi(0, y) = 0 \quad \text{for all } y \in (-\hat{f}(0), \hat{f}(0)). \tag{4.13}$$

Also, using $\varphi \geq \varphi_2$ in Ω,

$$\psi(0, y) \geq 0 \quad \text{in } \hat{\Omega}_{\varepsilon_0}. \tag{4.14}$$

Step 3. Let $P = (\xi_*, \eta_*) \in \Gamma_{sonic} \setminus \{P_1\}$. Then, in the (x, y)-coordinates, $P = (0, y_*)$ with $y_* \in [0, \hat{f}(0))$. Then, by (4.6) and (4.11), there exist $r, R > 0$, depending only on ε_0, $c_2 = c_2(\rho_0, \rho_1, u_1, \theta_w)$, and $d = dist((\xi_*, \eta_*), \Gamma_{shock})$, such that

$$(0, y_*) + Q^+_{r,R} \subset \hat{\Omega}_{\varepsilon_0}.$$

Then, in $Q^+_{r,R}$, the function $\hat{\psi}(x, y) := \psi(x, y - y_*)$ satisfies all the conditions of Theorem 3.1. Thus, applying Theorem 3.1 and expressing the results in terms of ψ, we obtain that, for all $y_* \in [0, \hat{f}(0))$,

$$\begin{gathered} \lim_{\substack{(x,y)\to(0,y_*)\\(x,y)\in\Omega}} \psi_{xx}(x, y) = \frac{1}{\gamma+1}, \\ \lim_{\substack{(x,y)\to(0,y_*)\\(x,y)\in\Omega}} \psi_{xy}(x, y) = \lim_{\substack{(x,y)\to(0,y_*)\\(x,y)\in\Omega}} \psi_{yy}(x, y) = 0. \end{gathered} \tag{4.15}$$

Since $\psi_{rr} = \psi_{xx}$ by (2.22), this implies assertions (i) and (ii) of Theorem 4.2.

Now assertion (iii) of Theorem 4.2 follows from (ii) since, by (2.23), $\varphi = \varphi_2$ in $B_\varepsilon(\xi_*, \eta_*) \setminus \Omega$ for small $\varepsilon > 0$ and φ_2 is a C^∞-smooth function in $\mathbb{R}^2$.

Step 4. It remains to show assertion (iv) of Theorem 4.2. We prove this by contradiction. Assume that assertion (iv) is false, i.e., there exists a limit of $D^2\psi$ at P_1 from Ω. Then our strategy is to choose two different sequences of points converging to P_1 and show that the limits of ψ_{xx} along the two sequences are different, which reaches to a contradiction. We note that, in the (x, y)-coordinates, the point $P_1 = (0, \hat{f}(0))$.

Step 5 **(A sequence close to Γ_{sonic})**. Let $\{y^{(1)}_m\}_{m=1}^\infty$ be a sequence such that $y^{(1)}_m \in (0, \hat{f}(0))$ and $\lim_{m\to\infty} y^{(1)}_m = \hat{f}(0)$. By (4.15), there exists $x^{(1)}_m \in (0, \frac{1}{m})$ such that

$$\left|\psi_{xx}\big(x^{(1)}_m, y^{(1)}_m\big) - \frac{1}{\gamma+1}\right| + \left|\psi_{xy}\big(x^{(1)}_m, y^{(1)}_m\big)\right| + \left|\psi_{yy}\big(x^{(1)}_m, y^{(1)}_m\big)\right| < \frac{1}{m}$$

for each $m = 1, 2, 3, \ldots$ Moreover, using (4.6), we have

$$y^{(1)}_m < \hat{f}(0) \le \hat{f}\big(x^{(1)}_m\big).$$

Thus, using (4.5), we have

$$\begin{gathered} \big(x^{(1)}_m, y^{(1)}_m\big) \in \Omega, \quad \lim_{m\to\infty} \big(x^{(1)}_m, y^{(1)}_m\big) = (0, \hat{f}(0)), \\ \lim_{m\to\infty} \psi_{xx}\big(x^{(1)}_m, y^{(1)}_m\big) = \frac{1}{\gamma+1}, \\ \lim_{m\to\infty} \psi_{xy}\big(x^{(1)}_m, y^{(1)}_m\big) = \lim_{m\to\infty} \psi_{yy}\big(x^{(1)}_m, y^{(1)}_m\big) = 0. \end{gathered} \tag{4.16}$$

Step 6 (**The Rankine–Hugoniot conditions on** Γ_{shock}). In order to construct another sequence, we first combine the Rankine–Hugoniot conditions on Γ_{shock} into a condition of the following form:

Lemma 4.2. *There exists* $\varepsilon \in (0, \varepsilon_0)$ *such that* ψ *satisfies*

$$\hat{b}_1(x, y)\psi_x + \hat{b}_2(x, y)\psi_y + \hat{b}_3(x, y)\psi = 0 \quad on\ \Gamma_{shock} \cap \partial\Omega_\varepsilon, \tag{4.17}$$

where $\hat{b}_k \in C(\overline{\Gamma_{shock} \cap \partial\Omega_\varepsilon})$ *and further satisfies*

$$\hat{b}_1(x, y) \geq \lambda, \quad |\hat{b}_2(x, y)| \leq \frac{1}{\lambda}, \quad |\hat{b}_3(x, y)| \leq \frac{1}{\lambda} \quad on\ \Gamma_{shock} \cap \partial\Omega_\varepsilon \tag{4.18}$$

for some constant $\lambda > 0$.

Proof. To prove this, we first work in the (ξ, η)-coordinates. Since

$$\varphi = \varphi_1, \quad \rho D\varphi \cdot \nu = \rho_1 D\varphi_1 \cdot \nu \quad \text{on } \Gamma_{shock},$$

then ν is parallel to $D\varphi_1 - D\varphi$ so that

$$(\rho_1 D\varphi_1 - \rho D\varphi) \cdot (D\varphi_1 - D\varphi) = 0 \quad \text{on } \Gamma_{shock}. \tag{4.19}$$

Since both φ and φ_2 satisfy (2.8) and (2.9) and $\psi := \varphi - \varphi_2$, we have

$$\begin{aligned} \rho &= \rho(D\psi, \psi, \xi, \eta) \\ &= \left(\rho_2^{\gamma-1} + (\gamma - 1)\left((\xi - u_2)\psi_\xi + (\eta - v_2)\psi_\eta - \frac{1}{2}|D\psi|^2 - \psi\right)\right)^{\frac{1}{\gamma-1}}, \\ c^2 &= c^2(D\psi, \psi, \xi, \eta) \\ &= c_2^2 + (\gamma - 1)\left((\xi - u_2)\psi_\xi + (\eta - v_2)\psi_\eta - \frac{1}{2}|D\psi|^2 - \psi\right). \end{aligned} \tag{4.20}$$

Then, writing $\varphi = \varphi_2 + \psi$ and using (2.14) and (2.15), we rewrite (4.19) as

$$E(\psi_\xi, \psi_\eta, \psi, \xi, \eta) = 0 \quad \text{on } \Gamma_{shock}, \tag{4.21}$$

where, for $(p_1, p_2, p_3, \xi, \eta) \in \mathbb{R}^5$,

$$\begin{aligned} &E(p_1, p_2, p_3, \xi, \eta) \\ &\quad = \rho_1((u_1 - \xi)(u_1 - u_2 - p_1) + \eta(v_2 + p_2)) - \rho(p_1, p_2, p_3, \xi, \eta) \\ &\qquad \times ((u_2 - \xi + p_1)(u_1 - u_2 - p_1) - (v_2 - \eta + p_2)(v_2 + p_2)), \end{aligned} \tag{4.22}$$

$$\begin{aligned} &\rho(p_1, p_2, p_3, \xi, \eta) \\ &= \left(\rho_2^{\gamma-1} + (\gamma - 1)\left((\xi - u_2)p_1 + (\eta - v_2)p_2 - \frac{1}{2}\left(p_1^2 + p_2^2\right) - p_3\right)\right)^{\frac{1}{\gamma-1}}, \end{aligned} \tag{4.23}$$

with $v_2 := u_2 \tan\theta_w$.

Since the points P_0 and P_1 both lie on $S_1 = \{\varphi_1 = \varphi_2\}$, we have

$$(u_1 - u_2)(\xi_1 - \xi_0) - v_2(\eta_1 - \eta_0) = 0,$$

where (ξ_1, η_1) are the coordinates of P_1. Now, using the condition $\varphi = \varphi_1$ on Γ_{shock}, i.e., $\psi + \varphi_2 = \varphi_1$ on Γ_{shock}, we have

$$\eta = \frac{(u_1 - u_2)(\xi - \xi_1) - \psi(\xi, \eta)}{v_2} + \eta_1 \quad \text{on } \Gamma_{shock}. \tag{4.24}$$

From (4.21) and (4.24), we conclude

$$F(\psi_\xi, \psi_\eta, \psi, \xi) = 0 \quad \text{on } \Gamma_{shock}, \tag{4.25}$$

where

$$F(p_1, p_2, p_3, \xi) = E\left(p_1, p_2, p_3, \xi, \frac{(u_1 - u_2)(\xi - \xi_1) - p_3}{v_2} + \eta_1\right). \tag{4.26}$$

Now, from (4.22) and (4.23), we obtain that, for any $\xi \in \mathbb{R}$,

$$\begin{aligned} F(0,0,0,\xi) &= E\left(0,0,0,\xi, \frac{(u_1 - u_2)(\xi - \xi_1)}{v_2} + \eta_1\right) \\ &= \rho_1((u_1 - \xi_1)(u_1 - u_2) + v_2\eta_1) \\ &\quad - \rho_2((u_2 - \xi_1)(u_1 - u_2) - v_2(v_2 - \eta_1)) \\ &= (\rho_1 D\varphi_1(\xi_1, \eta_1) - \rho_2 D\varphi_2(\xi_1, \eta_1)) \cdot (u_1 - u_2, -v_2) \\ &= 0, \end{aligned} \tag{4.27}$$

where the last expression is zero since it represents the right-hand side of the Rankine–Hugoniot condition (2.13) at the point P_1 of the shock $S_1 = \{\varphi_1 = \varphi_2\}$ separating state (2) from state (1).

Now we write condition (4.25) in the (x, y)-coordinates on $\Gamma_{shock} \cap \partial\Omega_\varepsilon$. By (2.21), (2.22), and (4.25), we have

$$\Psi(\psi_x, \psi_y, \psi, x, y) = 0 \quad \text{on } \Gamma_{shock} \cap \partial\Omega_\varepsilon, \tag{4.28}$$

where

$$\begin{aligned} \Psi(p_1, p_2, p_3, x, y) = F\Big(&-p_1 \cos(y + \theta_w) - \frac{p_2}{c_2 - x}\sin(y + \theta_w), \\ &- p_1 \sin(y + \theta_w) + \frac{p_2}{c_2 - x}\cos(y + \theta_w), \\ &p_3, u_2 + (c_2 - x)\cos(y + \theta_w)\Big). \end{aligned} \tag{4.29}$$

From (4.27) and (4.29), we find

$$\begin{aligned} \Psi(0,0,0,x,y) = F(0,0,0,u_2 + (c_2 - x)\cos(y + \theta_w)) = 0 \\ \text{on } \Gamma_{shock} \cap \partial\Omega_{\varepsilon_0}. \end{aligned} \tag{4.30}$$

By its explicit definition (4.22), (4.23), (4.25), and (4.29), the function $\Psi(p_1, p_2, p_3, x, y)$ is C^∞ on the set $\{|(p_1, p_2, p_3, x)| < \delta\}$, where $\delta > 0$ depends only on $u_2, v_2, \rho_2, \xi_0, \eta_0$, i.e., on the data. Using (4.8) and choosing $\varepsilon > 0$ small, we obtain

$$|x| + |\psi(x, y)| + |D\psi(x, y)| \le \delta \quad \text{for all } (x, y) \in \overline{\Omega_\varepsilon}.$$

Thus, from (4.28)–(4.30), it follows that ψ satisfies (4.17) on $\Gamma_{shock} \cap \partial\Omega_\varepsilon$, where

$$\hat{b}_k(x, y) = \int_0^1 \Psi_{p_k}(t\psi_x(x, y), t\psi_y(x, y), t\psi(x, y), x, y)dt \quad \text{for } k = 1, 2, 3. \tag{4.31}$$

Thus, we have

$$\hat{b}_k \in C(\overline{\Gamma_{shock} \cap \partial\Omega_\varepsilon}), \quad |\hat{b}_k| \le \frac{1}{\lambda} \quad \text{on } \Gamma_{shock} \cap \partial\Omega_\varepsilon, \quad \text{for } k = 1, 2, 3,$$

for some $\lambda > 0$.

It remains to show that $\hat{b}_1 \ge \lambda$ for some $\lambda > 0$. For that, since $\hat{b}_1$ is defined by (4.31), we first show that $\Psi_{p_1}(0, 0, 0, 0, y_1) > 0$, where $(x_1, y_1) = (0, \hat{f}(0))$ are the coordinates of P_1.

In the calculation, we will use that, since $(0, y_1)$ are the (x, y)-coordinates of $P_1 = (\xi_1, \eta_1)$, then, by (2.21) and (2.22),

$$\xi_1 = u_2 + c_2\cos(y_1 + \theta_w), \quad \eta_1 = v_2 + c_2\sin(y_1 + \theta_w),$$

which implies

$$(\xi_1 - u_2)^2 + (\eta_1 - v_2)^2 = c_2^2.$$

Also, $c_2^2 = \rho_2^{\gamma-1}$. Then, by explicit calculation, we obtain

$$\begin{aligned}\Psi_{p_1}(0, 0, 0, 0, y_1) = &\frac{\rho_1}{c_2}((u_1 - \xi_1)(\xi_1 - u_2) - \eta_1(\eta_1 - v_2))\\ &- \frac{\rho_2}{c_2}((u_2 - \xi_1)(\xi_1 - u_2) + (v_2 - \eta_1)(\eta_1 - v_2)).\end{aligned} \tag{4.32}$$

Now, working in the (ξ, η)-coordinates on the right-hand side and noting that $D\varphi_1(\xi_1, \eta_1) = (u_1 - \xi_1, -\eta_1)$ and $D\varphi_2(\xi_1, \eta_1) = (u_2 - \xi_1, v_2 - \eta_1)$, we rewrite (4.32) as

$$\Psi_{p_1}(0, 0, 0, 0, y_1) = -\frac{1}{c_2}(\rho_1 D\varphi_1(\xi_1, \eta_1) - \rho_2 D\varphi_2(\xi_1, \eta_1)) \cdot D\varphi_2(\xi_1, \eta_1),$$

where $D = (\partial_\xi, \partial_\eta)$. Since the point P_1 lies on the shock $S_1 = \{\varphi_1 = \varphi_2\}$ separating state (2) from state (1), then, denoting by τ_0 the unit vector along the line S_1, we have

$$D\varphi_1(\xi_1, \eta_1) \cdot \tau_0 = D\varphi_2(\xi_1, \eta_1) \cdot \tau_0.$$

Now, using the Rankine–Hugoniot condition (2.13) at the point P_1 for φ_1 and φ_2, we obtain

$$\rho_1 D\varphi_1(\xi_1,\eta_1) - \rho_2 D\varphi_2(\xi_1,\eta_1) = (\rho_1-\rho_2)(D\varphi_2(\xi_1,\eta_1)\cdot\tau_0)\tau_0,$$

and thus

$$\Psi_{p_1}(0,0,0,0,y_1) = \frac{1}{c_2}(\rho_2-\rho_1)(D\varphi_2(\xi_1,\eta_1)\cdot\tau_0)^2,$$

where $\rho_2 > \rho_1$ by the assumption of our theorem.

Thus, it remains to prove that $D\varphi_2(\xi_1,\eta_1)\cdot\tau_0 \neq 0$. Note that

$$|D\varphi_2(\xi_1,\eta_1)| = c_2 = \rho_2^{(\gamma-1)/2},$$

since (ξ_1,η_1) is on the pseudo-sonic circle. Thus, on the contrary, if $D\varphi_2(\xi_1,\eta_1)\cdot\tau_0 = 0$, then, using also $D\varphi_1(\xi_1,\eta_1)\cdot\tau_0 = D\varphi_2(\xi_1,\eta_1)\cdot\tau_0$, we can write the Rankine–Hugoniot condition (2.13) at (ξ_1,η_1) in the form:

$$\rho_1|D\varphi_1(\xi_1,\eta_1)| = \rho_2\,\rho_2^{(\gamma-1)/2} = \rho_2^{(\gamma+1)/2}. \tag{4.33}$$

Since both φ_1 and φ_2 satisfy (2.7) and since $\varphi_1(\xi_1,\eta_1) = \varphi_2(\xi_1,\eta_1)$ and $|D\varphi_2(\xi_1,\eta_1)| = c_2$, we have

$$\rho_1^{\gamma-1} + \frac{\gamma-1}{2}|D\varphi_1(\xi_1,\eta_1)|^2 = \rho_2^{\gamma-1} + \frac{\gamma-1}{2}\rho_2^{\gamma-1}.$$

Combining this with (4.33), we obtain

$$\frac{2}{\gamma+1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma-1} + \frac{\gamma-1}{\gamma+1}\left(\frac{\rho_2}{\rho_1}\right)^2 = 1. \tag{4.34}$$

Consider the function

$$g(s) = \frac{2}{\gamma+1}s^{\gamma-1} + \frac{\gamma-1}{\gamma+1}s^{-2} \quad \text{on } (0,\infty).$$

Since $\gamma > 1$, we have

$$g'(s) < 0 \quad \text{on } (0,1); \qquad g'(s) > 0 \quad \text{on } (1,\infty); \qquad g(1) = 1.$$

Thus, $g(s) = 1$ only for $s = 1$. Therefore, (4.34) implies $\rho_1 = \rho_2$, which contradicts the assumption $\rho_1 < \rho_2$ of our theorem. This implies that $D\varphi_2(\xi_1,\eta_1)\cdot\tau_0 \neq 0$, thus $\Psi_{p_1}(0,0,0,0,y_1) > 0$.

Choose $\lambda := \frac{1}{2}\Psi_{p_1}(0,0,0,0,y_1)$. Then $\lambda > 0$. Since the function $\Psi(p_1,p_2,p_3,x,y)$ is C^∞ on the set $\{|(p_1,p_2,p_3,x)| < \delta\}$ and since $\psi \in C^{1,1}(\overline{\Omega_{\varepsilon_0}})$ with $\psi(0,y) = \psi_x(0,y) = \psi_y(0,y) = 0$ by (4.7), we find that, for small $\varepsilon > 0$,

$$\Psi_{p_1}(t\psi_x(x,y), t\psi_y(x,y), t\psi(x,y), x, y) \geq \lambda$$
$$\text{for all } (x,y) \in \Gamma_{shock}\cap\partial\Omega_\varepsilon,\ t\in[0,1].$$

Thus, from (4.31), we find $\hat{b}_1 \geq \lambda$. Lemma 4.2 is proved. □

Step 7 **(A sequence close to Γ_{shock})**. Now we construct the sequence close to Γ_{shock}. Recall that we have assumed that assertion (iv) is false, i.e., $D^2\psi$ has a limit at P_1 from Ω. Then (4.16) implies

$$\lim_{\substack{(x,y)\to(0,\hat{f}(0))\\(x,y)\in\Omega}} \psi_{xy}(x,y) = \lim_{\substack{(x,y)\to(0,\hat{f}(0))\\(x,y)\in\Omega}} \psi_{yy}(x,y) = 0, \tag{4.35}$$

where $(0, \hat{f}(0))$ are the coordinates of P_1 in the (x, y)-plane. Note that, from (4.7),

$$\psi_y(x, \hat{f}(x)) = \int_0^x \psi_{xy}(s, \hat{f}(0))ds + \int_{\hat{f}(0)}^{\hat{f}(x)} \psi_{yy}(x, t)dt,$$

and, from (2.26), all points in the paths of integration are within Ω. Furthermore, by (2.27), $0 < \hat{f}(x) - \hat{f}(0) < Cx$ with C independent of $x \in (0, \varepsilon_0)$. Now, (4.35) implies

$$\lim_{x\to 0+} \frac{\psi_y(x, \hat{f}(x))}{x} = 0. \tag{4.36}$$

Also, by Lemma 4.2,

$$|\psi_x(x, \hat{f}(x))| = \left| \frac{\hat{b}_2\psi_y + \hat{b}_3\psi}{\hat{b}_1} \right| \le C(|\psi_y| + |\psi|) \quad \text{on } (0, \varepsilon),$$

where $\varepsilon > 0$ is from Lemma 4.2. Then, using (4.36) and $|\psi(x, y)| \le Cx^2$ by (4.8), we have

$$\lim_{x\to 0+} \frac{\psi_x(x, \hat{f}(x))}{x} = 0. \tag{4.37}$$

Let

$$\mathcal{F}(x) := \psi_x\left(x, \hat{f}(x) - \frac{\omega}{10}x\right)$$

for some constant $\omega > 0$. Then $\mathcal{F}(x)$ is well-defined and differentiable for $0 < x < \varepsilon_0$ so that

$$\begin{aligned} \mathcal{F}(x) &= \psi_x\left(x, \hat{f}(x) - \frac{\omega}{10}x\right) \\ &= \psi_x(x, \hat{f}(x)) + \int_0^1 \frac{d}{dt}\psi_x\left(x, \hat{f}(x) - \frac{t\omega}{10}x\right)dt \\ &= \psi_x(x, \hat{f}(x)) - \frac{\omega}{10}x\int_0^1 \psi_{xy}\left(x, \hat{f}(x) - \frac{t\omega}{10}x\right)dt. \end{aligned} \tag{4.38}$$

Now (4.35) and (4.37) imply

$$\lim_{x\to 0+} \frac{\mathcal{F}(x)}{x} = 0. \tag{4.39}$$

By (4.10) and since $\hat{f} \in C^{1,1}([0, \varepsilon_0])$, we have

$$\mathcal{F} \in C([0, \varepsilon]) \cap C^1((0, \varepsilon)). \tag{4.40}$$

Then (4.39) and the mean-value theorem imply that there exists a sequence $\{x_k^{(2)}\}$ with $x_k^{(2)} \in (0, \varepsilon)$ and

$$\lim_{k\to\infty} x_k^{(2)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \mathcal{F}'\big(x_k^{(2)}\big) = 0. \tag{4.41}$$

By definition of $\mathcal{F}(x)$,

$$\psi_{xx}(x, g(x)) = \mathcal{F}'(x) - g'(x)\psi_{xy}(x, g(x)) \tag{4.42}$$

where $g(x) := \hat{f}(x) - \frac{\omega}{10}x$.

On the other hand, $|\hat{f}'(x)|$ is bounded. Then, using (4.35), (4.41), and (4.42) yields

$$\lim_{k\to\infty} \psi_{xx}\big(x_k^{(2)}, g\big(x_k^{(2)}\big)\big) = \lim_{k\to\infty} \mathcal{F}'\big(x_k^{(2)}\big) = 0.$$

Note that $\lim_{x\to 0+} g(x) = \hat{f}(0)$. Thus, denoting $y_k^{(2)} = g(x_k^{(2)})$, we conclude

$$\big(x_k^{(2)}, y_k^{(2)}\big) \in \Omega, \quad \lim_{k\to\infty} \big(x_k^{(2)}, y_k^{(2)}\big) = (0, \hat{f}(0)), \quad \lim_{k\to\infty} \psi_{xx}\big(x_k^{(2)}, y_k^{(2)}\big) = 0.$$

Combining this with (4.16), we conclude that ψ_{xx} does not have a limit at P_1 from Ω, which implies assertion (iv). This completes the proof of Theorem 4.2. □

Remark 4.6. For the isothermal case, $\gamma = 1$, there exists a global regular reflection solution in the sense of Definition 4.1 when $\theta_w \in (0, \frac{\pi}{2})$ is close to $\frac{\pi}{2}$. Moreover, the solution has the same properties stated in Theorem 4.2 with $\gamma = 1$. This can be verified by the limiting properties of the solutions for the isentropic case when $\gamma \to 1+$. This is because, when $\gamma \to 1+$,

$$i(\rho) \to \ln\rho, \quad p(\rho) \to \rho, \quad c^2(\rho) \to 1 \quad \text{in (2.3)},$$
$$\rho(|D\varphi|^2, \varphi) \to \rho_0 e^{-(\varphi + \frac{1}{2}|D\varphi|^2)} \quad \text{in (2.8)},$$

and

$$c_*(\varphi, \rho_0, \gamma) \to 1 \quad \text{in (2.11)},$$

in which case the arguments for establishing Theorem 4.2 are even simpler.

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