Discontinuous Solutions to Nonlinear Evolutionary Partial Differential Equations

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Abstract

We analyze some recent developments in studying discontinuous solutions to nonlinear evolutionary partial differential equations. The central problems include the existence, compactness, and large-time behavior of discontinuous solutions. The nonlinear equations we discuss include nonlinear hyperbolic systems of conservation laws (especially the compressible Euler equations) and the compressible Navier-Stokes equations. Some of recent ideas, approaches, and methods are also discussed.

1. Introduction

We are concerned with global solutions of nonlinear evolutionary partial differential equations:
\[ \partial_t U + A U = 0, \quad U \in \mathbb{R}^n, \]  
(1.1)

with discontinuous initial data:
\[ U|_{t=0} = U_0(x), \]  
(1.2)

and certain appropriate boundary conditions provided the domain under consideration is a subset of \( \mathbb{R}^n \), where \( A \) is a nonlinear differential operator in \( x \in \mathbb{R}^n \) and \( U_0(x) \) is a discontinuous function of large oscillation.

The discontinuous problems arise from many physical situations involving sharp fronts and singularities as observed in nature, such as shock waves and vortices. Mathematically, such problems are generic in the sense that solutions (even starting from smooth initial data) generally develop singularities in a finite time because of strong nonlinearity of the equations without enough dissipation, and hence only discontinuous solutions may exist in the large. One of the main challenges in handling discontinuous solutions is that many powerful analytical techniques, which predominate in the theory of dissipative partial differential equations, may not be
directly applied. Novel mathematical ideas, approaches, and methods need to be developed efficiently to handle discontinuous solutions.

The mathematical study of discontinuous solutions started with the pioneering work of Riemann [58], who first solved a special initial value problem, called the Riemann problem, for the one-dimensional isentropic Euler equations. The Riemann problem plays an essential role in the theory of hyperbolic conservation laws and related numerical analysis and computation of discontinuous solutions (see Glimm [34]; also see [22, 35, 36, 45, 51, 59, 60] and the references cited therein).

Many Chinese mathematicians have made important contributions to the Riemann problem and related problems. See Gu-Li-Yu-Hou [38] and Ding-Chang-Wang-Hsiao-Li [25]. Also see recent contributions in Chang-Hsiao [4], Ding-Liu [27], Li-Zhang-Yang [49], Li-Yu [50], and the references cited therein.

The problems we are interested for discontinuous initial data include:

(1). Existence and solution spaces: For hyperbolic systems of conservation laws with small initial data, the $BV$ space is well-posed for Glimm solutions [34, 54] (also see [2, 3]). Recent examples show that, for large initial data, the total variation of solutions may blow up in a finite time, even for strictly hyperbolic systems [41]. Therefore, one expects at most that the solutions are in $L^\infty$ or weighted $L^p$ in general for large initial data.

(2). Compactness and regularity as $t > 0$: If the initial data are a large oscillatory sequence, then the question is whether the corresponding solution sequence is compact as long as $t > 0$. That is, the solution operator is compact.

(3). Uniqueness and stability: Given a class of solutions containing an entropy solution, we examine whether the solution is unique and stable in this class.

(4). Asymptotic behavior: Given an entropy solution without apriori reference to the method for construction of the solution, the question is how the solution behaves asymptotically as $t \to \infty$.

In general, evolutionary nonlinear partial differential equations can be classified into three types:

(1). Nondissipative systems: The prototype of such systems is nonlinear hyperbolic systems of conservation laws including the compressible Euler equations and Hamilton-Jacobi equations.

(2). Weakly dissipative or nonuniform dissipative systems such as hyperbolic-parabolic systems: The prototype of such systems is the compressible Navier-Stokes equations.

(3). Strongly dissipative systems: Such systems include uniform parabolic equations or systems. In general, solutions of such systems with discontinuous initial data become instantaneously regular as long as $t > 0$.

In this paper, we focus on types (1)-(2), mainly on hyperbolic systems of conservation laws and the compressible Navier-Stokes equations, and discuss some recent developments and advances through these prototypical systems.

2. Hyperbolic Conservation Laws

We are concerned with a hyperbolic system of conservation laws:

$$\partial_t U + \nabla_x \cdot F(U) = 0, \quad U \in \mathbb{R}^m, x \in \mathbb{R}^n,$$

where $F : \mathbb{R}^m \to (\mathbb{R}^m)^n$ is a nonlinear mapping. Since the conservation law is a fundamental law in nature, most of partial differential equations arising from
physical and engineering sciences can be formulated into form (2.1) or its variants. The hyperbolicity of system (2.1) requires that, for all $\eta \in S^{n-1}$, the matrix 

$\left( \sum_{k=1}^{m} \xi_k \nabla F_k (U) \right)_{m \times m}$ 

have $m$ real eigenvalues $\lambda_j (U, \xi), j = 1, 2, \cdots, m$, and be diagonalizable.

2.1. Entropy Solutions. Let $(\eta, q) : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^n$ be an entropy-entropy flux pair (an entropy pair, for short), determined by the linear hyperbolic systems:

$$\nabla q_k (U) = \nabla \eta (U) \nabla F_k (U), \quad k = 1, 2, \cdots, n.$$ 

An entropy pair is called convex if $\nabla^2 \eta (U) \geq 0$. If the inequality is strict, then $(\eta, q)$ is called a strictly convex entropy pair. Friedrichs-Lax [32] observed that most of the conservative systems that result from continuum mechanics are endowed with a globally defined, strictly convex entropy. An $L^\infty$ function $U(x, t)$ is called an entropy solution if, for any convex entropy pair $(\eta, q)$,

$$(2.3) \quad \int_0^\infty \int_{\mathbb{R}^n} \left( \nabla \eta (U) \partial_t \varphi + q(U) \cdot \nabla_x \varphi \right) dx dt + \int_{\mathbb{R}^n} \eta (U_0 (x)) \varphi (x, 0) dx \geq 0,$$ 

for any nonnegative test function $\varphi (x, t) \in C_0^\infty (\mathbb{R}^{n+1}_+)$.

2.2. Difficulties. Two of the main difficulties to solve (2.1) are singularity of the solutions and resonance of the systems.

Singularity: Since $F(U)$ is a nonlinear mapping, solutions of the Cauchy problem (even starting from smooth initial data) generally develop singularities in a finite time, and then the solutions become discontinuous functions. This feature reflects the physical phenomenon of breaking of waves and development of shock waves. For this reason, attention must focus on solutions in the space of discontinuous functions satisfying (2.3).

Resonance: A system is called nonstrictly hyperbolic or resonant if there exist some $\xi_0 \in S^{n-1}$ and $U_0 \in \mathbb{R}^m$ such that $\lambda_i (U_0, \xi_0) = \lambda_j (U_0, \xi_0)$ for some $i \neq j$. Such a degeneracy is quite generic. For example, when $n = 3, m = 2 \pmod{4}$, Lax [44] showed that any system must be nonstrictly hyperbolic. The same situation occurs for $m = \pm 2, \pm 3, \pm 4 (\pmod{8})$ (see [33]).

The problems we consider include the existence, compactness, and large-time behavior of entropy solutions. The compactness and large-time behavior are more challenging, especially for nonstrictly hyperbolic systems, since the solutions are discontinuous and all the information about the solutions is the entropy inequality (2.3), only in the sense of distributions.

2.3. Decay via Compactness and Entropy. It is well known that, for the linear case with a nonzero propagation speed and periodic initial data, the solution is periodic in time $t$. However, this is not true for the genuinely nonlinear case. The observation is that the genuine nonlinearity of equations forces the nonlinear waves of each characteristic family to interact vigorously and to cancel each other. The theory of Glimm-Lax [35], for $2 \times 2$ strictly hyperbolic and genuinely nonlinear systems, has indicated that the resulting mutual cancellation of interacting shock and rarefaction waves of the same family induces the decay of periodic Glimm solutions, provided the initial oscillation is small. Recently, Dafermos [24] showed that any periodic solution with small locally bounded variation and certain regularity for the $2 \times 2$ systems decays asymptotically. A further problem is whether the decay phenomenon holds for more general cases: (a) any $L^\infty$ large
periodic solutions without restrictions of either small oscillation or local bounded variation, and (b) more general nonlinear hyperbolic systems, especially nonstrictly hyperbolic systems and multidimensional scalar equations. The counterexample of Greenberg-Rasce [37] indicates that the asymptotic behavior is very sensitive with respect to the smoothness of the flux functions.

2.3.1. Framework. One of our main observations in Chen-Frid [9] is that the compactness of $L^\infty$ solution operator in $L^1_{loc}$, coupling with the weak convergence of periodic initial data to the mean, yields the decay of the $L^\infty$ periodic solution $U(x, t)$ in $L^1$, provided that (2.1) is endowed with a strictly convex entropy. This is achieved via the compactness of the self-similar scaling sequence $U^T(x, t) = U(Tx, Tt)$.

**Theorem 2.1.** [9]. Suppose that system (2.1) is endowed with a strictly convex entropy. Assume that $U(x, t) \in L^\infty(\mathbb{R}_+^{n+1})$ is a periodic solution of (2.1) and (1.2) with period $P \subset \mathbb{R}^n$, and its scaling sequence $\{U^T\}$ is compact in $L^1_{loc}(\mathbb{R}_+^{n+1})$. Then $U(x, t)$ asymptotically decays to $\bar{U} \equiv \frac{1}{|P|} \int_P U_0(x) dx$ in $L^1$:

$$\text{ess lim}_{t \to -\infty} \int_P |U(x, t) - \bar{U}| dx = 0. \quad (2.4)$$

In Theorem 2.1, we assume that $\{U^T\}$ is compact in $L^1_{loc}(\mathbb{R}_+^{n+1})$, which is a corollary of the compactness of solution operator. Such a compactness can be achieved by the method of compensated compactness, the averaging method, and other analytical techniques. For example, we have

**Theorem 2.2.** [9, 12]. Consider system (2.1) with a strictly convex entropy pair $(\eta_*, q_*)$. Assume that the uniformly bounded sequence $U^T(x, t) \in L^\infty(\mathbb{R}_+^{n+1})$ satisfies (2.3) for any convex entropy pair $(\eta, q) \in \Lambda$, where $\Lambda$ is a linear space of entropy pairs of (2.1) including $(\eta_*, q_*)$. Then the measure sequence

$$\partial_t \eta(U^T) + \nabla_x \cdot q(U^T) \quad \text{is compact in } H^{-1}_{\text{loc}}(\mathbb{R}_+^{n+1}), \quad (2.5)$$

for any entropy pair $(\eta, q) \in \Lambda \cap C^2(\mathbb{R}^m; \mathbb{R}_+^{n+1})$.

Therefore, the method of compensated compactness (e.g. [61, 57, 7]) is one of the efficient methods to achieve the compactness with the aid of Theorem 2.2.

2.3.2. Multidimensional Scalar Conservation Laws. For multidimensional scalar conservation laws with $C^2$ flux function $f : \mathbb{R} \to \mathbb{R}^n$:

$$\partial_t u + \nabla_x \cdot f(u) = 0, \quad (2.6)$$

and initial data $u_0(x) \in L^\infty(\mathbb{R}^n)$ with period $P$, the existence of global entropy solutions of the Cauchy problem is due to Kruzkov [43]. Combining Theorem 2.1 with a theorem by Lions-Péthame-Tadmor [46], we conclude

**Theorem 2.3.** [9, 12]. Assume that the flux function $f(u)$ is $C^2$ satisfies

$$\text{meas} \left\{ u \in \mathbb{R} \mid \tau + f'(u) \cdot \vec{k} = 0 \right\} = 0, \quad \text{for any } (\tau, \vec{k}) \in S^n. \quad (2.7)$$

Let $u(x, t) \in L^\infty(\mathbb{R}_+^{n+1})$ be an entropy solution with periodic initial data $u_0(x)$. Then $u(x, t)$ asymptotically decays to $\bar{u} = \frac{1}{|P|} \int_P u_0(x) dx$ in $L^1$.

2.3.3. The Isentropic Euler Equations. Consider the isentropic Euler equations for compressible fluids:

$$\partial_t \rho + \partial_x m = 0, \quad \partial_t m + \partial_x (m^2 / \rho + p(\rho)) = 0, \quad (2.8)$$
where \( \rho, m, \) and \( p \) are the density, the momentum, and the pressure, respectively. In the non-vacuum state \( (\rho > 0) \), \( v = m/\rho \) is the velocity, \( p = p(\rho) \in C^1(0, \infty) \) is a given function of \( \rho \) depending on compressible fluids under consideration. Strict hyperbolicity and genuine nonlinearity away from the vacuum require that
\[
(2.9) \quad p'(\rho) > 0, \quad pp''(\rho) + 2p'\rho > 0, \quad \rho > 0.
\]
Near the vacuum, \( p(\rho) \) is allowed very singular: The principal singular part of \( p(\rho) \) coincide with the \( \gamma \)-law pressure for some \( \gamma > 1 \), but additional singularities are allowed (as real gases). That is, for sufficiently small \( \rho \ll 1 \),
\[
(2.10) \quad p(\rho) = \kappa \rho^\gamma (1 + P(\rho)), \quad |P^{(k)}(\rho)| \leq C \rho^{1-k}, \quad 0 \leq k \leq 4.
\]
Consider the Cauchy problem for (2.8) with initial data
\[
(2.11) \quad (\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \quad 0 \leq \rho_0(x) \leq C_0, \quad |m_0(x)/\rho_0(x)| \leq C_0 < \infty.
\]
The main difficulty of this system is that strict hyperbolicity fails, and the flux function is only Lipschitz continuous at the vacuum state \( \rho = 0 \), which occurs in fluid mechanics.

**Theorem 2.4.** [18]. Consider the isentropic Euler equations (2.8)-(2.10). Then
(i). There exists a global solution \( (\rho(x,t), m(x,t)) \) of (2.8)-(2.11), satisfying
\[
0 \leq \rho(x,t) \leq C, \quad |m(x,t)/\rho(x,t)| \leq C, \quad \text{for some } C \text{ depending only on } C_0 \text{ and } \gamma, \text{ and }
\]
\[
\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leq 0 \text{ in the sense of distributions for any convex weak entropy pairs } (\eta, q).
\]
If \( (\rho_0(x), m_0(x)) \) is periodic with period \( P \), then \( (\rho(x,t), m(x,t)) \) is also periodic in \( x \) with same period \( P \).
(ii). The solution operator \( (\rho, m) (\cdot, t) = S_t(\rho_0, m_0)(\cdot) \), determined by (a), is compact in \( L^1_{loc}(\mathbb{R}^2_+ \) for \( t > 0 \).
(iii). Let \( (\rho(x,t), m(x,t)), 0 \leq \rho(x,t) \leq C, \quad |m(x,t)/\rho(x,t)| \leq C \), be a periodic entropy solution of (2.8)-(2.11) with period \( P \). Then \( (\rho(x,t), m(x,t)) \) asymptotically decays to \( \frac{1}{|P|} \int_P (\rho_0(x), m_0(x))dx \) in \( L^1 \).

For polytropic gases, the existence result was proved by DiPerna [29] for the case \( \gamma = 1 + 1/N, N \geq 5 \) odd, for \( L^2 \cap L^\infty(\mathbb{R}) \) initial data, by Ding-Chen-Luo [26] and Chen [6] for \( 1 \leq \gamma \leq 5/3 \) for usual gases with general \( L^\infty \) initial data. The existence result was also obtained by Lions-Perthame-Tadmor [47] for \( \gamma \geq 3 \) and by Lions-Perthame-Souganidis [48] for \( 5/3 < \gamma < 3 \). More general pressure law will be handled in [19].

**Remark 2.1.** The same decay results hold for periodic entropy solutions in \( L^\infty \) with large oscillation for some important systems (see [9]). The approach described here can be easily generalized to the decay problem for \( L^p \) entropy solutions and to the asymptotic problems for entropy solutions of hyperbolic conservation laws with relaxation (see [9, 20]).

### 2.4. Asymptotic Stability via Compactness and Entropy

The ideas and observations in §2.3 can be refined into new approaches for solving the asymptotic stability of entropy solutions \( U(x,t) \in L^\infty(\mathbb{R}^{n+1}_+ \) or \( L^\infty \cap BV_{loc}(\mathbb{R}^{n+1}_+) \) of (2.1) and (1.2) with
\[
(2.12) \quad U_0(x) = R_0(x/|x|) + P_0(x), \quad P_0(x) \in L^1(\mathbb{R}^n).
\]
Let \( R(x,t) \) be the classical Riemann solution of (2.1) with Riemann data
\[
(2.13) \quad R|_{t=0} = R_0(x/|x|).
\]
2.4.1. Framework. The first approach is to invoke the compactness of the scaling sequences of the perturbation solutions and the uniqueness of the Riemann solution in a class of solutions which includes all possible limits of the scaling sequences.

**Theorem 2.5.** [10, 12]. Let $S(\mathbb{R}^{n+1}_+)$ denote a class of functions in $L^\infty$ defined on $\mathbb{R}^{n+1}_+$. Assume that the Cauchy problems (2.1) and (2.12)-(2.13) satisfy the following.

(i). The Riemann solution is unique in the class $S(\mathbb{R}^{n+1}_+)$;

(ii). Given an entropy solution of (2.1) and (2.12), $U \in S(\mathbb{R}^{n+1}_+)$, the self-similar scaling sequence $U^T(x,t)$ is compact in $L_{loc}^1(\mathbb{R}^{n+1}_+)$, and any limit function of its subsequences is still in $L_{loc}^1 \cap S(\mathbb{R}^{n+1}_+)$. 

(iii). The Riemann solution $R(x/t) = R(\xi)$ is piecewise Lipschitz in $\xi \in \mathbb{R}^n$. Then the Riemann solution $R(x/t)$ is asymptotically stable in $S(\mathbb{R}^{n+1}_+)$ with respect to the corresponding initial perturbation $P_0(x) \in L^1(\mathbb{R}^n)$:

$$\text{ess lim}_{t \to \infty} \int_{|\xi| \leq K} |U(\xi,t) - R(\xi)|d\xi = 0, \quad \text{for any} \quad K \in (0, \infty).$$

For $BV$ solutions, the compactness of the scaling sequence is obtained through the following observation.

**Theorem 2.6.** [10]. Assume that $U(x,t) \in BV_{loc}(\mathbb{R}^{n+1}_+)$ satisfies

$$TV_{|x| \leq cT_0} \times (0,T_0)(U) \leq CT_0^\alpha, \quad \text{for} \ c > 0, T_0 > 0, \text{ and } C > 0 \text{ indept. of } T_0.$$

Then $U^T(x,t)$ also satisfies (2.15) with the same constant $C$.

This condition is satisfied by the entropy solutions possessing total variation in $x$ uniformly bounded for all $t > 0$, which is the case for the solutions constructed by Glimm’s method (see [34, 35]). Hence, the compactness follows from the uniform boundedness of the sequence in $BV$. For $L^\infty$ solutions, the method of compensated compactness and the Young measures can be applied to yield the compactness of uniformly bounded sequences of entropy solutions (see [7, 8, 28, 31, 59]).

The uniqueness of Riemann solutions in the class of $BV$ solutions for $2 \times 2$ systems is due to DiPerna [30] and for the $3 \times 3$ system of compressible Euler equations with a general pressure law to Chen-Frid [11] (also see [10]).

The second approach is to estimate the difference between the perturbation solution $U$ and the Riemann solution $R$ in $L^p$: Assume that $U(x,t), R(x/t) \in L^\infty(\mathbb{R}^{n+1}_+)$ satisfy that there exist $C > 0$, independent of $t$, and $\gamma \in [0,1)$ such that

$$\|U - R\|_{L^p(\mathbb{R}^n)}(t) \leq Ct^{n-1+\gamma}, \quad \text{for some } 1 \leq p < \infty.$$ 

Then the Riemann solution $R(x/t)$ is asymptotically stable in $L^1$ with respect to the corresponding initial perturbation $P_0(x)$ in the sense of (2.14).

Another approach is to understand directly the asymptotic behavior of the solution along the rays $\xi = x/t, \xi \in \mathbb{R}$, with the aid of the theory of divergence-measure fields (see [10, 13]).

2.4.2. Multidimensional Scalar Conservation Laws. A direct application is the asymptotic stability of Riemann solutions $R(x/t)$ of the multidimensional scalar conservation law (2.6) under initial perturbation (2.12), with the aid of Kruzkov’s Theorem [43].
Theorem 2.7. Any Riemann solution \( R(x/t) \) of (2.6) with data \( R_0(x/|x|) \) is asymptotically stable with respect to large \( L^\infty \cap L^1 \) perturbation \( P_0(x) \) in the sense of (2.14).

2.4.3. The \( p \)-System. Consider the \( p \)-system

\[
\begin{align*}
\partial_t v - \partial_x u &= 0, \\
\partial_t u + \partial_x p &= 0,
\end{align*}
\]

where \( p \in C^2(a,b) \) satisfies \( p'(v) < 0 \) and \( p''(v) > 0 \). Combining Theorems 2.5-2.6 with the uniqueness of Riemann solutions by DiPerna [30], we have

Theorem 2.8. [10]. Any Riemann solution \( R(x/t) \) of (2.17) with Riemann data \( R_0(x/|x|) \) is asymptotically stable with respect to large \( L^1 \) perturbation \( P_0(x) \in L^\infty \cap BV_{loc}(\mathbb{R}) \) in the sense of (2.14). Any Riemann solution consisting of only rarefaction waves is asymptotically stable even with respect to large \( L^\infty \cap L^1 \) perturbation \( P_0(x) \) in the sense of (2.14).

2.4.4. The Compressible Euler Equations. The Euler system for compressible fluids in Lagrangian coordinates reads

\[
\begin{align*}
\partial_t v - \partial_x u &= 0, \\
\partial_t u + \partial_x p &= 0, \\
\partial_t (e + \frac{u^2}{2}) + \partial_x (pu) &= 0,
\end{align*}
\]

where \( u, p, v, \) and \( e \) represent the velocity, the pressure, the specific volume \( (v = 1/\rho, \rho \) the density), and the internal energy of the fluids, respectively. Other important physical variables are the temperature \( \theta \) and the entropy \( S \). To close system (2.18), one needs the basic law of thermodynamics which translates into the differential equation: \( \frac{dc}{dS} = \theta \frac{dS}{dv} - p \frac{dv}{dS} \). We choose \( (v, S) \) as the independent variables to get the constitutive equations of state \( p = p(v, S), \theta = \theta(v, S) \), and \( e = e(v, S) \) satisfying

\[
\begin{align*}
e_v(v, S) &= -p(v, S), \\
e_S(v, S) &= \theta(v, S), \\
\theta_v(v, S) &= -p_S(v, S).
\end{align*}
\]

As usual, we assume that the function \( p = p(v, S) \) satisfies

\[
(2.20) \quad p_v(v, S) < 0, \quad p_{vv}(v, S) > 0, \quad \nabla_{v, S}^2 p(v, S) \geq 0, \quad \text{in the domain } v > 0.
\]

For ideal polytropic gases, \( pv = R_0 \theta, e = c_v \theta, p(v, S) = \kappa e^{S/c_v} v^{-\gamma}, \gamma = 1+R/c_v > 1 \), where \( R \) and \( c_v \) are positive constants.

System (2.18) can be written in the conservation form (2.1) with \( U = (v, u, E) \) and \( F(U) = (-u, p, pu) \), where \( E = \frac{4}{3}u^2 + e, \) and \( p \) is a function of \( U \). We are interested in the large-time behavior of solutions in \( L^\infty \cap BV_{loc} \) of the Cauchy problem for (2.18) with initial data:

\[
(2.21) \quad U|_{t=0} = (v_0, u_0, E_0) \equiv R_0(x) + P_0(x), \quad \text{with } P_0(x) \in L^1(\mathbb{R}).
\]

The entropy-entropy flux pair \( \eta_* = \frac{\kappa^2}{2} - \int_{v_0}^v p(\sigma, S) d\sigma + K(S), \eta_* = up(v, S) \) is a strictly convex pair in \( (v, u, S) \) in the bounded domain \( V \subset \{v > 0\} \) where \( (v, u, S)(x, t) \) in the solutions assume their values, provided that

\[
(2.22) \quad K''(S) > \int_{v_0}^v p_{SS}(\sigma, S) dv - \frac{p^2_u(v, S)}{p_v(v, S)} , \quad (v, u, S) \in V.
\]

Observe that any entropy solution \( U(x, t) \in L^\infty(\mathbb{R}^2_+), v(x, t) > 0, \) satisfies the entropy inequality

\[
(2.23) \quad \int_0^\infty \int_{-\infty}^\infty \{\eta_*(U)\partial_t \varphi + q_*(U)\partial_x \varphi\} dx \, dt + \int_{-\infty}^\infty \eta_*(U_0(x)) \varphi(x, 0) \, dx \geq 0,
\]
for any nonnegative \( \varphi \in C^1_0(\mathbb{R}^2) \), provided \( K(S), v_0 > 0 \) are chosen such that
\[
K'(S) \leq \theta(v_0, S).
\]

We choose \( K(S), v_0 > 0 \) such that, besides (2.22), both (2.24) and
\[
K'(S) \geq \theta(v_0, S) - \theta(v, S)
\]
hold in \( \mathcal{V} \subset \{ (v, u, S) \mid v > 0 \} \subset \mathbb{R}^3 \). It is easy to check for polytropic gases that there exist \( K(S), v_0 > 0 \), depending on \( \mathcal{V} \), such that (2.22) and (2.24)-(2.25) hold.

By the approach developed in Chen-Frid \([10]\), we have

**Theorem 2.9.** \([11]\). Let \( U(x, t) = (v(x, t), u(x, t), E(x, t)) \in BV(\mathbb{R}^2_+; \mathcal{V}) \) be an entropy solution of (2.18) and (2.21) in \( \mathbb{R}^2_+ \) satisfying \( TV(-\varepsilon \mathcal{T} \times [0, T])U \leq CT \), for some \( c, C > 0 \) and any \( T > 0 \). Assume (2.20), (2.22), and (2.24)-(2.25) are satisfied. Then \( U(x, t) \) asymptotically tends to the corresponding Riemann solution \( R(x/t) \) in the sense of (2.14), provided that \( \mathcal{V} \) is small. Moreover, if the Riemann solution \( R(x/t) \) consists of at most rarefaction waves and contact discontinuity, then any entropy solution \( U(x, t) \in L^\infty(\mathbb{R}^2_+; \mathcal{V}) \) asymptotically tends to the corresponding Riemann solution \( R(x/t) \) in the sense of (2.14), for arbitrarily large \( \mathcal{V} \subset \{ v > 0 \} \).

The uniqueness and asymptotic stability of Riemann solutions in the class of entropy solutions for the Cauchy problem (2.18) and (2.21) can be extended to the Euler equations in Eulerian coordinates (see \([11]\)).

We recall recent fundamental results for \( m \times m \) systems in Bressan-Crasta-Piccoli \([2]\), Bressan-Liu-Yang \([3]\), Liu-Yang \([54]\), and the references cited therein on the \( L^1 \)-stability of entropy solutions in \( L^\infty \cap BV \) obtained by the Glimm scheme and the wave front-tracking method, or satisfying an additional regularity, with small total variation. We remark that the uniqueness results described here cannot be directly obtained from these stability results, since the results presented here neither impose smallness restrictions on the total variation and additional regularity of the solutions, nor need specific reference to any particular method for constructing the entropy solutions.

Another important example is Hamilton-Jacobi equations, which can be transformed into a system of conservation laws. A general framework to establish global \( L^\infty \) solutions of Hamilton-Jacobi equations has been established in Chen-Su \([21]\).

### 3. The Compressible Navier-Stokes Equations

In this section we consider the global existence and large-time behavior of discontinuous solutions of the compressible Navier-Stokes equations in Lagrangian coordinates:

\[
\partial_t v - \partial_x u = 0, \quad \partial_t u + \partial_x p = \partial_x \left( \frac{\epsilon \partial_x u}{v} \right), \quad \partial_t (\epsilon + \frac{u^2}{2}) + \partial_x (u p) = \partial_x \left( \frac{\epsilon u \partial_x u + \lambda \partial_x \epsilon}{v} \right),
\]

with large discontinuous initial data, where \( p = p(v, \epsilon) \) is the pressure as in \( \S 2.4.4 \), \( \epsilon \) and \( \lambda \) are fixed positive viscosity parameters, and \( x \) is the Lagrangian coordinate so that \( x = \text{constant} \) corresponds to a particle path. For polytropic gases, the pressure \( p \) is given via the equation of state: \( p = \frac{(\gamma - 1) \epsilon}{v} = \frac{(\gamma - 1) c_v \theta}{v} \), where \( \theta \) is the temperature, \( \epsilon \) is the internal energy, and \( \gamma > 1, c_v > 0 \) are constants. For
concreteness, we focus on the following initial-boundary value problem for (3.1):

\begin{align}
(3.2) & \quad (v, u, e)|_{t=0} = (v_0(x), u_0(x), e_0(x)), \quad 0 \leq x \leq 1, \\
(3.3) & \quad u(i, t) = \partial_x e(i, t) = 0, \quad i = 0, 1, \quad t \geq 0,
\end{align}

where the initial data satisfy that, for some constant \( C_0 > 0 \),

\begin{align}
(3.4) & \quad v_0(x), e_0(x) \geq C_0^{-1}, \quad \text{TV}(v_0, u_0, e_0) \leq C_0.
\end{align}

Such a problem is motivated by the fact that large discontinuous solutions are fundamental in both the physical theory of nonequilibrium thermodynamics and the mathematical theory for compressible fluid flow (also see [39]). These considerations require to seek a rigorous theory to accommodate the large discontinuities. It is worthwhile, at this point, to give a heuristic argument of the convection of discontinuities in solutions of (3.1). If \( v_0 \) is piecewise smooth, having isolated jumps at points \( y_1 < \cdots < y_N \), then the discontinuities in \( v, p, \partial_x u, \) and \( \partial_x e \) convect only along the corresponding particle paths \( x = y_d \) and satisfy the jump conditions:

\begin{align}
(3.5) & \quad \left[ p(v, e) - \frac{\epsilon \partial_x u}{v} \right] = 0, \quad \left[ \frac{\partial_x e}{v} \right] = 0,
\end{align}

where \([w]\) denotes a jump of function \( w \) across \( x = y_d : [w(y_d, t)] = w(y_d + 0, t) - w(y_d - 0, t) \). We remark that the convective contact discontinuities here should not be confused with convective contact discontinuities present in solutions of the Euler equation (2.18) (i.e. (3.1) with \( \epsilon = \lambda = 0 \)). A solution of (2.18) obtained as a limit \( (v, p, \partial_x u, \partial_x e, v) \) can be considered as a solution of (3.1) (as \( \epsilon, \lambda \to 0 \) of solutions of (3.1) is not necessarily continuous, which has often unrelated contact discontinuities of magnitude \( |v_r - v_l| \) at \( x = 0 \).

**Theorem 3.1 (Well-Posedness and Regularity).** Given initial data \((v_0, u_0, e_0(x)) \) satisfying (3.4), there exists a unique weak solution \((v, u, e)(x, t)\) of (3.1)-(3.3):

\begin{align}
(v, u, e)(x, t) & \in C([0, \infty); L^2(0, 1)), \quad (u, e)(x, t) \in C((0, \infty); H^2(0, 1)),
\end{align}

satisfying that there exists \( M > 0 \), independent of \( t \), such that

\begin{align}
(3.6) & \quad M^{-1} \leq v(x, t) \leq M, \quad e(x, t) \geq M^{-1}, \\
(3.7) & \quad \text{TV}_{[0,1]} (v(\cdot, t)) \leq M, \quad \|v(\cdot, t') - v(\cdot, t)\|_{L^1} \leq M|t' - t|^{1/2}, \\
(3.8) & \quad \|\partial_x u(\cdot, t)\|_{L^2} \leq M\sigma^{-1/4}(t), \quad \|\partial_x e(\cdot, t)\|_{L^2} \leq M\sigma^{-1/4}(t)
\end{align}

hold for all \( t, t' > 0 \), where \( \sigma(s) = \min(s, 1) \). Furthermore, the solution is locally stable with large discontinuous initial data.

The existence results in Theorem 3.1 immediately extend to the Cauchy problem with the aid of estimates (3.6)-(3.8) and the arguments as in [5, 42]; and the BV assumption (3.4) on the initial data can be relaxed (see [15, 16]). We emphasize that the bounds of estimates (3.6)-(3.8) are independent of time, even with large discontinuous initial data, which are crucial to allow for the determination of the large-time behavior of the discontinuous solutions. The time-independent estimate (3.6) for the specific volume indicates that vacuum and concentration do never occur for all time, even starting from large discontinuous initial data. This phenomenon shows the essential difference between the Navier-Stokes equations and the Euler equations for which vacuum may occur for the solutions of large oscillations, even for the Riemann solutions with large Riemann initial data (cf. [4, 14, 26]). With the aid of estimates (3.6)-(3.8), we have
Theorem 3.2 (Discontinuities and Large-Time Behavior). Let the initial data $(v_0, u_0, e_0)(x)$ satisfy (3.4). Then the discontinuous solution $(v, u, e)(x, t)$ has the following behavior: The functions $v(\cdot, t), p(\cdot, t), \partial_x u(\cdot, t)$, and $\partial_x e(\cdot, t)$ have one-sided limits at each discontinuous point $y_d$ for $t > 0$, and the jump conditions (3.5) hold pointwise. Furthermore, when $t \to \infty$, 

\begin{align}
\|(v, p, \partial_x u, \partial_x e)(y_d, t)\| &\leq M e^{-M^{-1}t} \to 0, \tag{3.9} \\
\|(v - \int_0^1 v_0(x)dx, u, e - \int_0^1 (e_0(x) + \frac{u_0^2(x)}{2})dx)(\cdot, t)\|_{L^\infty(0,1)} &\to 0. \tag{3.10}
\end{align}

Theorem 3.2 indicates that the large jumps of initial discontinuities decay exponentially, although they persist all the time. Moreover, all physical variables decay asymptotically in $L^\infty$.

Regarding early work on (3.1) with large discontinuous initial data, we refer to Matsumura-Yanagi [55] for the isothermal case and to Amosov-Zlotnick [1] for the existence of weak solutions and some of their time-dependent estimates for certain initial and boundary data. We also refer to Hoff [39]-[40] for the global well-posedness and large-time behavior for small discontinuous initial data.

The results we obtain for (3.1)-(3.4) can be converted to the ones for the corresponding initial-boundary value problem of the compressible Navier-Stokes equations in Euler coordinates (cf. [5]). The approaches developed here have been applied to establishing the global well-posedness and large-time behavior of discontinuous solutions to the Navier-Stokes equations for a compressible reactive flow, which describe dynamic combustion (see [16]).

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