# Northwestern Math/Physics Seminar: Introduction to de Rham Cohomology

Justin D. Thomas

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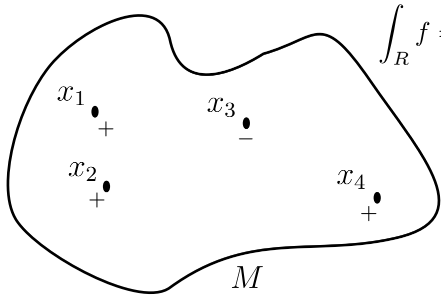
## 1 De Rham Cohomology

Motivation: Differential forms are things you integrate.

To integrate, need 2 things: A space, or region, to integrate over, and a form to integrate. The dimension of the space must match the rank of the form. Let's pretend we are working in some ambient manifold $M$. If you like, $M = \mathbb{R}^3$.

### 1.1 dim = 0 case

What is our region? A 0-manifold. What is a 0-manifold? It is a bunch of points. Here I will add the condition, which we will need for all dimensions, that the region is compact and oriented. What does this mean for a 0-manifold? We have a finite number of points labeled with either a + or a −. See figure 1.1 below.



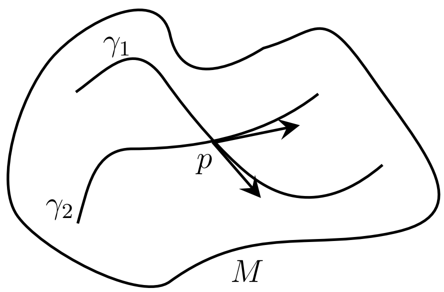

$$\int_R f = f(x_1) + f(x_2) - f(x_3) + f(x_4)$$

$$R = \{x_1, x_2, x_3, x_4\}$$

Figure 1: Integrating a 0-form over a 0-manifold.

### 1.2 dim = 1 case

Now our region is a compact oriented 1-manifold. More precisely, it is a parametrized curve in the space. We can throw in "parametrized" as one of our conditions on the regions we integrate over. A 1-form is something we want to integrate over any given curve. So it must depend on the curve itself and not just the points on the curve. Suppose $\gamma_1$ and $\gamma_2$ are two such parametrized curves $[0,1] \to M$ which intersect at some point $p$. In figure 1.2 we see that the tangent vectors are distinct at $p$. Thus, for a 1-form to distinguish between these two curves at the point $p$, it must depend on tangent vectors. Indeed it is enough to have a map $\omega\colon T_pM \to \mathbb{R}$ for every $p \in M$. These maps must vary smoothly over $M$. Together they define a 1-form $\omega$. In this way, we can define $\int_\gamma \omega$ as $\int_0^1 \omega_{\gamma(t)}(\gamma'(t))\,dt$.




$$\int_{\gamma_1} \omega = \int_0^1 \omega_{\gamma_1(t)}(\gamma_1'(t))\,dt$$

$$\gamma_1\colon [0,1] \to M$$

Figure 2: Integrating a 1-form over a parametrized curve.

## 1.3 Relating 0-forms and 1-forms

We take a moment to examine how 0-forms give us 1-forms and how the fundamental theorem of calculus relates integrating certain 1-forms to integrating 0-forms.

### 1.3.1 From 0-forms to 1-forms

We already know several examples of 1-forms. Recall that if $f$ is any function $f\colon M \to \mathbb{R}$ (always read *smooth function* when you see just function) and if $v \in T_pM$ is a vector at $p \in M$, then $vf \in \mathbb{R}$ is the directional derivative of $f$ in the direction $v$. Define the 1-form $df$ by $(df)_p(v) = vf$.

This is easier to see by specializing to $\mathbb{R}^3$. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$. Let $v = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ be a vector field. Here I am already expanding what I meant by $v$ above. There it was a vector at some fixed $p \in M$. Now we have just chosen a vector for every $p \in M = \mathbb{R}^3$. Recall that when we have a coordinate system, it gives us a basis for vectors. In the standard $(x, y, z)$ coordinates of $\mathbb{R}^3$ we have $\partial/\partial x = \mathbf{i}$, $\partial/\partial y = \mathbf{j}$, and $\partial/\partial z = \mathbf{k}$. So, if we compute $df(v) = vf$ we now get a function, rather than just a number since $p$ varies over all of $\mathbb{R}^3$. We see that

$$vf = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}.$$

On the other hand, note that $x$ is the function $\mathbb{R}^3 \to \mathbb{R}$, sending $(x, y, z) \to x$. So $dx(\partial/\partial x) = 1$ while $dx(\partial/\partial y) = 0$ and $dx(\partial/\partial z) = 0$. Similar statements hold for $dy$ and $dz$. What this means is that $\{dx, dy, dz\}$ is the dual basis to $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$. The latter gives a basis for each $T_p\mathbb{R}^3$ and the former gives a basis for each $T_p^*\mathbb{R}^3$, the dual of $T_p\mathbb{R}^3$. This means we can write $df$ as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Which is, of course, what you should expect from calculus.

### 1.3.2 The Fundamental Theorem of Calculus

grad$f$ Let $\gamma\colon [0, 1] \to M$ be a parametrized curve, and let $f$ be a 0-form on $M$. Let $\partial\gamma$ be the boundary of $\gamma$, namely $\{\gamma(0), \gamma(1)\}$. Orient $\partial\gamma$ via the orientation of $\gamma$: say $\gamma$ is oriented from $\gamma(0)$ to $\gamma(1)$, then $\gamma(0)$ is labeled $-$ and $\gamma(1)$ is labeled $+$. The fundamental theorem of calculus computes

$$\begin{aligned}\int_\gamma df &= \int_0^1 (df)_{\gamma(t)}(\gamma'(t))\,dt = \int_0^1 \frac{d}{dt} f \circ \gamma(t)\,dt \\ &= f(\gamma(1)) - f(\gamma(0)) = \int_{\partial\gamma} f\end{aligned}$$

So we can trade the $d$ on $f$ for a boundary $\partial$ on $\gamma$. We shall see that this is the simplest version of Stoke's theorem.

## 1.4 dim=2 case

Let $M = \mathbb{R}^2$ viewed as a manifold with the following two coordinate charts. First, we have global coordinates $(x, y)$. Second, we have polar coordinates $(r, \theta)$ for $-\pi < \theta < \pi$ and $r > 0$. There are no such restrictions on $x$ and $y$. All we need now to describe this manifold is the transition function between some open subsets $U, V$ of the two charts. We use for $U$ the entire $(r, \theta)$ chart and for $V$ we delete the negative $x$-axis from the $(x, y)$ chart ($(0, 0)$ also deleted). The transition function from the $U$ to $V$ chart is the usual change of coordinates $x = r\cos\theta$, $y = r\sin\theta$. Let $R \subset M$ be a parametrized surface which lies in $U$. Let $R_{r,\theta}$ be the region in the $(r, \theta)$ coordinate system and let $R_{x,y}$ be the region in the $(x, y)$ coordinate system. Let $f$ be a 0-form on $M$. We know from calculus that

$$\int_{R_{x,y}} f(x, y)\,dxdy = \int_{R_{r,\theta}} f(r\cos\theta, r\sin\theta)\,rdrd\theta.$$

However, in calculus, $dx$, $dy$, etc. were nothing more than symbols. Now let's interpret them as 1-forms. We have

$$dx = \cos\theta dr - r\sin\theta d\theta \qquad dy = \sin\theta dr + r\cos\theta d\theta.$$

Now, what could $dxdy$ possibly mean? Well we know what the answer should be: $rdrd\theta$. What if we just multiply $dx$ and $dy$? Let's allow the functions $\cos\theta, r$, etc. to commute with everything, but let's be careful about $dr$ and $d\theta$ since multiplying these things is new to us. We compute

$$dxdy = \cos\theta\sin\theta\, drdr + r\cos^2\theta\, drd\theta - r\sin^2\theta\, d\theta dr - r^2\sin\theta\cos\theta\, d\theta d\theta.$$

Now note that if we require 1-forms to *anticommute*, that is $\omega\eta = -\eta\omega$ for 1-forms $\eta$ and $\omega$, then the above expression simplifies to $r\, drd\theta$! This is exactly what it should be. This is called the wedge product. To remember the anti-commutativity, one often writes $dx \wedge dy$. We treat this so-called *exterior product* just as we did above. To be concrete, the rules in addition to $dx \wedge dy = -dy \wedge dx$ are as follows. If $f$ is a 0-form, then

$$f(dx \wedge dy) = (fdx) \wedge dy = dx \wedge fdy. \tag{1}$$

Also,

$$(dx + dy) \wedge dz = dx \wedge dz + dy \wedge dz. \tag{2}$$

Of course, these rules apply to all 1-forms, not just those that look like $dx$ where $x$ is a 0-form. In fact, you can think of $fdx$ as the wedge product of a 0-form and a 1-form. You may object that these commute, at least that's what we assumed in the above calculation. However we can just impose a graded commutativity: $\omega \wedge \eta = (-1)^{mn}\eta \wedge \omega$ where $\omega$ is an $m$-form and $\eta$ is an $n$-form. Up to now we have only defined this concept for $m, n \in \{0, 1\}$, but we will see that it is true in more generality. The result is an $(m+n)$-form.

## 1.5 General $n$-forms and the Wedge product

We can extract from the discussion what the definition of a 2-form should be, after all $f(x,y)dx \wedge dy$ should be a canonical example of a 2-form as it is an integrand when integrating over the 2-dimensional region $R_{x,y}$.

Before we make the definition, let's introduce some notation. Let $\mathfrak{X}(M)$ be the ($\mathbb{R}$-linear) space of vector fields on $M$. Let $\Omega^n(M)$ be the n-forms on $M$ (also $\mathbb{R}$-linear); so far I have only defined $n$-forms for $n = 0$ and $n = 1$. At $n = 0$ they are functions on $M$ and at $n = 1$ they are linear maps $\mathfrak{X}(M) \to \Omega^0(M)$.

Creating 2-forms is simply a matter of wedging 1-forms together. Define $\Omega^2(M) = \Omega^1(M) \wedge \Omega^1(M)$. This means nothing more than what we encountered above. In general, if $w_1, \dots, w_n$ form a basis of some vector space $W$, then $\{w_i \wedge w_j \mid i < j\}$ forms a basis for $W \wedge W$. We only need to remember that $w_i \wedge w_j = -w_j \wedge w_i$ and $(aw_i + bw_j) \wedge w_k = aw_i \wedge w_k + bw_j \wedge w_k$. In particular, 2-forms on $\mathbb{R}^2$ can all be written as $fdx \wedge dy$ for some 0-form $f$. This easily extends to define $\Omega^n(M)$ to be $\bigwedge^n \Omega^1(M)$. Here we have expressions of the form $w_{i_1} \wedge \cdots \wedge w_{i_n}$, subject to similar rules.

A little linear algebra tells us that the rule we can interpret the 2-form $dx \wedge dy$ as a linear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \Omega^0(M)$ given by

$$(dx \wedge dy)(X, Y) = \frac{1}{2}(dx(X)dy(Y) - dx(Y)dy(X)) \quad \text{where} \quad X, Y \in \mathfrak{X}(M).$$

The formula above is used because it is antisymmetric in $X$ and $Y$. In general, $\Omega^n(M)$ consists of maps $\omega\colon \mathfrak{X}(M)^{\times n} \to \omega^0(M)$ satisfying

$$\omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = (-1)^\sigma \omega(X_1, \dots, X_n) \quad \sigma \in S_n$$

This is the generalized antisymmetry condition. The symbol $(-1)^\sigma$ is notation for $\mathsf{sign}(\sigma)$. From this point of view, if $\omega$ is a $p$-form and $\eta$ is a $q$-form then $\omega \wedge \eta$ is a $(p+q)$-form. As a map $\mathfrak{X}(M)^{\times(p+q)} \to \Omega^0(M)$ we have

$$(\omega \wedge \eta)(X_1, \dots, X_{p+q}) = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})$$

As a final note on the wedge product, also called the exterior product of forms, suppose that we have an $n \times n$ matrix $A = (a^i_j)$. Let $\{w^1, \dots, w^n\}$ be a basis of some $n$-dimensional vector space $W$. Put $v^i = a^i_j w^j$ (using summation convention). Then one can show that

$$v^1 \wedge \cdots \wedge v^n = \det(A) w^1 \wedge w^n.$$

This is why we could we found $dx \wedge dy = r\,dr \wedge d\theta$ in the last section simply by using the anti-commutativity rule. The factor $r$ is the Jacobian, the determinant of the matrix expressing $dx, dy$ in terms of $dr, d\theta$.

## 1.6 Extending $d$ to All Forms

We have a map $d\colon \Omega^0(M) \to \Omega^1(M)$. We will extend this to have maps $d\colon \Omega^n(M) \to \Omega^{n+1}(M)$. Recall the definition of $d$. Let $X \in \mathfrak{X}(M)$, and let $f \in \Omega^0(M)$. Then $(df)(X) = X(f)$, by definition. Now recall the Liebniz rule for differentiating products, $X(fg) = gX(f) + fX(g)$. In terms of $d$, this means $(d(fg))(X) = g\,df(X) + fdg(X)$. Taking away the test vector $X$, we see the Liebniz rule also holds for $d$: $d(fg) = g\,df + fdg$.

The wedge product of 0-forms is just multiplication of functions. We could write the Liebniz rule again,

$$d(f \wedge g) = (df) \wedge g + f \wedge (dg).$$

To extend $d$ to higher forms, we ask that this rule be true when $f$ and $g$ are not just 0-forms. To do this in a well-defined manner we put

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^\omega \omega \wedge d\eta,$$

where $(-1)^\omega = (-1)^n$, $\omega \in \Omega^n(M)$. You can check that this computes the same result for $(-1)^{\omega\eta} \eta \wedge \omega$. More concretely, if $\omega = fdx + gdy$ is a 1-form on on $\mathbb{R}^2$, then $d\omega$ is given by

$$df \wedge dx + dg \wedge dy = (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy = (g_x - f_y) dx \wedge dy$$

One can check that this formula is invariant under change of coordinates. Thus this is a well defined notion of differentiating forms on a manifold. In the next section we will completely study the case of $M = \mathbb{R}^3$.

## 1.7 $M = \mathbb{R}^3$: The grad, curl, div story

We shall examine the de Rham complex of $\mathbb{R}^3$. This is the top row of the following diagram. The point of this section is that this complex should already be familiar to the reader. Actually, the bottom row is familiar, but equivalent to the top row.

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\| & & \downarrow \cdot & & \downarrow * & & \downarrow * \\
\| & & \downarrow & & \Omega^1(\mathbb{R}^3) & & \Omega^0(\mathbb{R}^3) \\
\| & & \downarrow & & \downarrow \cdot & & \| \\
\mathsf{f^ns} & \xrightarrow{\nabla} & \mathsf{VecFlds} & \xrightarrow{\nabla\times} & \mathsf{VecFlds} & \xrightarrow{\nabla\cdot} & \mathsf{f^ns}
\end{array}$$

Recall that $\mathsf{curl}(\mathsf{grad} f) = 0$ for any function $f$ on $\mathbb{R}^3$. Also $\mathsf{div}(\mathsf{curl}\, v) = 0$ for any vector field $v$. That is, the two compositions of maps on the bottom row are zero. On the top row, this is summarized as $d^2 = 0$. You can check that $d^2 = 0$ when $M$ is any manifold just by using the definition of $d$ in the previous section. We shall come back to this fundamental identity later.

It is important to understand why the above diagram commutes. Consider the leftmost rectangle. Zero-forms and functions (denoted $\mathsf{f^ns}$) are the same thing, thus the vertical equality sign. Let $f$ be a 0-form on $\mathbb{R}^3$. On the one hand, we already saw that

$$df = f_x dx + f_y dy + f_z dz,$$

On the other hand, we know that the gradient of $f$ is given by

$$\mathsf{grad} f = \nabla f = f_x \frac{\partial}{\partial x} + f_y \frac{\partial}{\partial y} + f_z \frac{\partial}{\partial z}.$$

These formulas are the same if we identify $dx$ with $\partial/\partial x$, etc. This is what is meant by the label $\cdot$ on the vertical arrow: the standard dot product in $\mathbb{R}^3$. The identification is made by the inner product. Indeed, if $v = f\partial/\partial x + g\partial/\partial y + h\partial/\partial z$ is a vector field, then $v \cdot \partial/\partial x = f$. But also $dx(v) = f$, so the map $v \mapsto v \cdot \partial/\partial x$ is just the map $v \mapsto dx(v)$. Vector fields (denoted above as $\mathsf{VecFlds}$, or earlier as $\mathfrak{X}(\mathbb{R}^3)$) are dual to 1-forms, and there are many isomorphisms $\Omega^0 \to \mathfrak{X}$. The point of the dot product is that it singles out one of these isomorphisms as special.

Again let $v = f\partial/\partial x + g\partial/\partial y + h\partial/\partial z$ be a vector field on $M$. We know that the $\mathsf{curl}$ of $v$ is given by

$$\nabla \times v = (h_y - g_z)\frac{\partial}{\partial x} + (f_z - h_x)\frac{\partial}{\partial y} + (g_x - f_y)\frac{\partial}{\partial z}.$$

On the other hand, $v$ is identified with the 1-form $\nu = f dx + g dy + h dz$. We find that

$$d\nu = (-f_y + g_x) dx \wedge dy + (-f_z + h_x) dx \wedge dz + (-g_z + h_y) dy \wedge dz.$$

This is close to looking like $\nabla \times v$ but it needs some work. First, we introduce the isomorphism $*\colon \Omega^k \to \Omega^{n-k}$, where $n$ is the dimension of the manifold. For a form $\omega \in \Omega^k(\mathbb{R}^3)$, define $*\omega$ as the unique $(3-k)$-form such that $\omega \wedge *\omega = dx \wedge dy \wedge dz = d\mathsf{Vol}$. In particular, to compute $*d\nu$ we need

$$(dx \wedge dy) \wedge dz = d\mathsf{Vol} \quad (dx \wedge dz) \wedge (-dy) = d\mathsf{Vol} \quad (dy \wedge dz) \wedge dx = d\mathsf{Vol}.$$

We extend $*$ to act on $d\nu$ using the above calculations, we get

$$*d\eta = (h_y - g_z)dx + (f_z - h_x)dy + (g_x - f_y)dz$$

which the dot product $\cdot$ identifies with $\nabla \times v$.

You should check for yourself that the rightmost rectangle above commutes. Thus the whole diagram commutes. Since all the vertical arrows are isomorphisms, this says that the top row is *isomorphic to* the bottom row.

Note that the equation $d^2 = 0$ shows that for every form $\nu$ which can be written as $d\omega$ for another form $\omega$ we have $d\nu = 0$. We say that forms which are in the image of $d$ are *exact*. That is, forms that are given by $d\omega$ for some $\omega$. We say that forms which are in the kernel of $d$ are *closed*. That is, forms $\eta$ satisfying $d\eta = 0$. We have shown that every exact form is closed. The point of de Rham cohomology is the converse question: if a form is closed, is it exact? You may remember that if $v$ is a vector field on $\mathbb{R}^3$ such that $\nabla \times v = 0$ then $v = \nabla f$ for some function $f$. Also, if $\nabla \cdot v = 0$ then $v = \nabla \times w$ for some vector field $w$. Thus the answer to our question is *yes* for forms on $\mathbb{R}^3$. However, it does not hold for an arbitrary manifold $M$.

Consider $M = S^1$. Cover $S^1$ with the following two coordinate patches $\alpha\colon (-\pi, \pi) \to S^1$ and $\beta\colon (0, 2\pi) \to S^1$. Both $\alpha$ and $\beta$ are defined as $x \mapsto e^{ix}$. The transition map from $\alpha$ coordinates to $\beta$ coordinates is defined on $(-\pi, 0) \to (\pi, 2\pi)$ as $\beta(\alpha) = \alpha + 2\pi$ and is defined on $(0, \pi) \to (0, \pi)$ as $\beta(\alpha) = \alpha$. There is a 1-form $\omega$ on $S^1$ which is $d\alpha$ in the $\alpha$ coordinate system and $d\beta$ in the $\beta$ coordinate system. This exists because it transforms correctly under the transition map. Now if $\omega = df$ for some function $f$ on $S^1$, then let $\gamma(x) = e^{ix}$ for $x \in [0, 2\pi]$. We would have

$$\int_\gamma \omega = \int \gamma df = f(\gamma(2\pi)) - f(\gamma(0)) = 0.$$

On the other hand, $\omega(\gamma'(t)) = 1$ for all $t$, so the integral must be $2\pi$. We conclude that $\omega$ is not exact.

It should be noted that computations are not normally made in this way. There are powerful tools which allow you to break up a space into pieces on which you know the answer to the question of whether closed forms are exact. You can then assemble this information together to tell you the answer on the whole space.

Formally we define the $n^{th}$ *homology* group $H^n(M)$ to be $Z_n/B_n$, where $Z_n$ consists of the closed $n$-forms and $B_n$ is the exact $n$-forms. These are also called *cycles* and *boundaries* respectively. When $n = 0$, there are no boundaries. The space of cycles has dimension equal to the number of connected components of $M$. When $n > \dim M$, there are no $n$-forms, so $H^n(M) = 0$. In practice, you would use other methods to compute the homology of a space, and this would answer the question about closed and exact forms.