In this lecture we will prove Thom’s Transversality Theorem and apply it to complete the proof of the equivalence $\Omega_n \cong \pi_n MO$ begun in the previous lecture.

1. Transversality

An idea of “general position” seems to have existed very early in topology. This was made precise in the notion of a transverse intersection, which possibly originates in Thom’s thesis in the early 1950s.

**Definition 1.1.** Let $f : P \to E$ be a smooth map of manifolds and $i : M \to E$ a smooth submanifold of $E$. $f$ is transverse to $i$ at a point $x$ of $M$ if, for any $p \in f^{-1}\{x\}$, the induced map

$$df + di : T_pP \oplus T_xM \to T_xE$$

is surjective. If $f$ is transverse to $i$ at every point $x \in M$, then $f$ is transverse to $i$, notated $f \triangleleft i$.

**Remark 1.2.** If $f^{-1}\{x\}$ is the empty set, then $f$ is automatically transverse to $i$ at $x$.

The notion of transversality generalizes that of a regular value of a map $f : P \to E$. That is, we have the following:

**Example 1.3.** Let the submanifold $M$ consist of a single point $M = \ast \to E$. In this case, $f$ is transverse to $x$ if and only $x$ is a regular value of $f$.

As observed by Pontryagin in the 30s, the inverse image of a regular value always has the structure of a smooth manifold; this feature is part of what gives the notion of a regular value its importance. This generalizes.

**Proposition 1.4.** If $f \triangleleft i$, then $f^{-1}M \to P$ is a smooth submanifold.

**Proof.** Apply the inverse function theorem. $\square$

Regular values occur in abundance, as follows from the well-known theorem of Brown, Sard and Morse.

**Theorem 1.5** (Brown–Sard–Morse). For any smooth map of manifolds $f : P \to E$, the regular values of $f$ form a dense subspace of $E$.

Thom’s Transversality Theorem, the key geometric input making the work of [5] go, is a generalization of this result.

**Theorem 1.6** (Thom Transversality). Let $P$ be a smooth manifold, and let $i : M \hookrightarrow E$ be a smooth submanifold. The subspace $\text{Map}_{\text{sm}}^m(P,E) \subset \text{Map}^m(P,E)$, consisting of those maps $f : P \to E$ for which $f$ is transverse $i$, is dense.

**Remark 1.7.** Additionally, every map $P \to E$ can be approximated within arbitrarily small $\varepsilon$ by a smooth; i.e., $\text{Map}^m(P,E)$ is a dense subspace of $\text{Map}(P,E)$. Thus, by composing, we obtain that transverse to $M$ maps, $\text{Map}_{\text{sm}}^m(P,E)$, form a dense subspace of all maps, $\text{Map}(P,E)$.

*Date: Lecture April 5, 2010. Last edited on April 12, 2010.*
We will in fact prove a modification of this theorem, namely, the following statement: For any smooth map \( f : P \to E \) there exists a smooth embedding \( s : M \to E \) arbitrarily close to \( i \), and for which \( f \) is transverse to \( s \). This is easily seen to be equivalent.

**Proof.** First, we will demonstrate that the transversality theorem for a general manifold \( E \) is a consequence of the particular case in which \( E \) has the structure of a vector bundle over \( M \). A tubular neighborhood \( N_i \) of the embedding \( i \) is an open submanifold of \( E \), so the inverse image \( f^{-1}N_i \) therefore defines a smooth open submanifold of \( P \):

\[
P \xrightarrow{f} E \\
\downarrow \hspace{1cm} \downarrow i \\
f^{-1}N_i \xrightarrow{f} N_i \xleftarrow{i} M
\]

Now, let us assume the the transversality theorem for \( P' = f^{-1}N_i \) and \( E' = N_i \). With this assumption, we can find an embedding \( s : M \to N_i \) arbitrarily close to the zero section \( z \) and for which \( f \) and \( s \) are transverse in \( N_i \). Composing the map \( s \) with the embedding of \( N_i \) into \( E \), we thus obtain a map \( \hat{s} : M \to E \) that is transverse to \( f : P \to E \).

Thus, it suffices to prove the transversality theorem under the assumption that \( E \) is a vector bundle over \( M \). We will first consider the case where the vector bundle is trivial, which we will then make use of in the case of a general vector bundle.

**First case:** \( E \) a trivial vector bundle.

Let \( E \) be a trivial \( k \)-dimensional vector bundle over \( M \), \( E \cong M \times \mathbb{R}^k \), and let \( f : P \to E \) be any smooth map, as before. Given a point \( x : * \to \mathbb{R}^k \), consider the following commuting diagram:

\[
P \xrightarrow{f} M \times \mathbb{R}^k \\
\downarrow \pi \hspace{1cm} \downarrow \text{id} \times x \\
\mathbb{R}^k \xleftarrow{x} M \times \{x\}
\]

Observe that \( x \) is a regular value of composite map \( \pi \circ f \) if and only if \( f \) is transverse to \( \text{id} \times x \). To see this, first assume that the derivative map \( d(\pi \circ f)|_p \) is a surjection onto the tangent space \( T_x \mathbb{R}^k \), for \( p \) a point in the inverse image \( f^{-1}(M \times \{x\}) \). Then \( d(\pi \circ f)|_p \oplus d(\text{id} \times x) \) is a surjection onto \( T_{f(p)}M \times \mathbb{R}^k \), since this tangent space \( T_{f(p)} \) can be split as a direct sum \( T_p \mathbb{R}^k \oplus T_p P \), where each of these summands is surjected upon by one of the two derivative maps. The converse, that the transversality of \( f \) and \( \text{id} \times x \) implies that \( x \) is a regular value of \( \pi \circ f \), obtains by the reverse bit of linear algebra. The Brown–Sard–Morse theorem now implies that the collection of \( x \in \mathbb{R}^k \) that regular values of \( \pi \circ f \) forms a dense subspace of \( \mathbb{R}^k \). Thus, a value of \( x \) for which \( f \cap \text{id} \times \{x\} \), is dense in \( \mathbb{R}^k \). We may therefore select a regular value \( x \) arbitrarily close to the origin \( 0 \in \mathbb{R}^k \), and \( f \) will be transverse to \( \text{id} \times x \). This proves the transversality theorem in the case of \( E \) a trivial bundle.

**Second case:** \( E \) a general vector bundle.

Let \( E \) be any vector bundle over \( M \), and let \( f : P \to E \) be a smooth map, as before. We can choose \( E^\perp \) such that the direct sum of vector bundles \( E \oplus E^\perp \) is a trivial bundle. Choosing a trivialization \( E \oplus E^\perp \cong M \times \mathbb{R}^k \), our situation is summarized in the following diagram:
Let us consider a class \([\gamma_k] \in \pi_n MO\) defined via the Pontryagin-Thom collapse map of the tubular neighborhood of an \(n\)-manifold \(M\) as a pullback of the following diagram:

\[
\begin{array}{ccc}
  f^{-1}(E \oplus E^\perp) & \xrightarrow{f} & E \oplus E^\perp \\
  \downarrow \pi_E & & \downarrow \text{id} \times x \\
  P & \xrightarrow{f} & E \times M
\end{array}
\]

By forming the pullback \(f^{-1}(E \oplus E^\perp)\) (which is manifold, since it fibers smoothly over \(P\)), we put ourselves in the situation of the first case: For the smooth map \(\tilde{f}: f^{-1}(E \oplus E^\perp) \to M \times \mathbb{R}^k\), valued in a trivial vector bundle, there exists a a point \(x: \ast \to \mathbb{R}^k\) such that the map \(\text{id} \times x: M \to M \times \mathbb{R}^k\) is transverse to \(\tilde{f}\).

We now define the embedding \(s: M \to E\) to be the composite \(\pi_E \circ (\text{id} \times x)\). We now show that \(f\) is indeed transverse to \(s\), and this will complete the proof of the transversality theorem.

The transversality \(\tilde{f} \cap \text{id} \times \{x\}\) implies that for any \(e \in E \oplus E^\perp\) in the image of \(\tilde{f}\) and \(\text{id} \times \{x\}\) and \(\hat{e} \in f^{-1}(e)\) the following diagram commutes

\[
\begin{array}{ccc}
  T\hat{e}f^{-1}(E \oplus E^\perp) \oplus T\hat{e}M & \xrightarrow{T\hat{e}(E \oplus E^\perp)} & T\hat{e}(E \oplus E^\perp) \\
  \downarrow \pi_{\hat{e}} & & \downarrow \pi_{\hat{e}} \\
  T\pi(\hat{e})E \oplus T\pi M & \xrightarrow{\cdots} & T\pi\pi(\hat{e})E
\end{array}
\]

The surjectivity of the dotted arrow in the above diagram is forced by the surjectivity of all the other maps in this diagram, so we conclude the transversality of \(f \cap \pi_E \circ (\text{id} \times x)\). I.e., \(f\) is transverse to \(s\).

We now make immediate use of the transversality theorem to finish Thom’s proof of the equivalence \(\Omega^\text{un}_{n} \cong \pi_n MO\).

**Theorem 2.1.** \(\Theta\) is an isomorphism.

**Proof.** Let us consider a class \([f] \in \pi_n MO\) and choose a representative

\[f: (S^{n+k}, \ast) \to (\text{Th}(\gamma^k_k), \ast)\]

for \(k\) sufficiently large. By smooth approximation, we may select \(f\) so that its restriction \(f'\) to the inverse image of complement of a small neighborhood of the basepoint of \(\text{Th}(\gamma^k_k)\),

\[f': S^{n+k} - f^{-1}(\ast) \to \text{Th}(\gamma^k_k) - \{\ast\} \cong \text{Disk}^n(\gamma^k_k)\]

is a smooth map of manifolds. (For convenience, I assume that this neighborhood is just the point, itself.) Our goal is to define an \(n\)-dimensional manifold \(M\) which corresponds to this class \([f]\), so that \(\Theta([M]) = [f]\). This will, of course, imply the surjectivity of our map \(\Theta\).

We may apply Thom’s Transversality Theorem and choose an embedding \(s\) of \(\text{Gr}_k(\mathbb{R}^*) \hookrightarrow \text{Th}(\gamma^k_k) - \{\ast\}\) near the zero section, such that \(s\) is transverse to \(f'\). Define the desired \(n\)-manifold \(M\) as a pullback of the following diagram:

\[
\begin{array}{ccc}
  f'^{-1}(\text{Gr}_k(\mathbb{R}^*)) & \xrightarrow{\text{Gr}_k(\mathbb{R}^*)} & S^{n+k} - f^{-1}(\ast) \\
  \downarrow & & \downarrow \\
  \text{Gr}_k(\mathbb{R}^*) & \xrightarrow{\text{Th}(\gamma^k_k) - \{\ast\}} & \text{Th}(\gamma^k_k) - \{\ast\}
\end{array}
\]

2. **Completion of the proof of \(\Omega_n \cong \pi_n MO\)**

Recall from the last lecture the construction of well-defined homomorphism \(\Theta: \Omega^\text{un}_{n} \to \pi_n MO\) defined via the Pontryagin-Thom collapse map of the tubular neighborhood of an \(n\)-manifold \(M\) embedded into Euclidean space.

**Theorem 2.1.** \(\Theta\) is an isomorphism.

**Proof.** Let us consider a class \([f] \in \pi_n MO\) and choose a representative

\[f: (S^{n+k}, \ast) \to (\text{Th}(\gamma^k_k), \ast)\]

for \(k\) sufficiently large. By smooth approximation, we may select \(f\) so that its restriction \(f'\) to the inverse image of complement of a small neighborhood of the basepoint of \(\text{Th}(\gamma^k_k)\),

\[f': S^{n+k} - f^{-1}(\ast) \to \text{Th}(\gamma^k_k) - \{\ast\} \cong \text{Disk}^n(\gamma^k_k)\]

is a smooth map of manifolds. (For convenience, I assume that this neighborhood is just the point, itself.) Our goal is to define an \(n\)-dimensional manifold \(M\) which corresponds to this class \([f]\), so that \(\Theta([M]) = [f]\). This will, of course, imply the surjectivity of our map \(\Theta\).

We may apply Thom’s Transversality Theorem and choose an embedding \(s\) of \(\text{Gr}_k(\mathbb{R}^*) \hookrightarrow \text{Th}(\gamma^k_k) - \{\ast\}\) near the zero section, such that \(s\) is transverse to \(f'\). Define the desired \(n\)-manifold \(M\) as a pullback of the following diagram:

\[
\begin{array}{ccc}
  f'^{-1}(\text{Gr}_k(\mathbb{R}^*)) & \xrightarrow{\text{Gr}_k(\mathbb{R}^*)} & S^{n+k} - f^{-1}(\ast) \\
  \downarrow & & \downarrow \\
  \text{Gr}_k(\mathbb{R}^*) & \xrightarrow{\text{Th}(\gamma^k_k) - \{\ast\}} & \text{Th}(\gamma^k_k) - \{\ast\}
\end{array}
\]
I.e., $M$ is the transverse intersection of $S^{n+k}$ and $Gr_k(\mathbb{R}^n)$ inside $Th(\gamma^k_s)$.

Note that $M$ comes with an embedding into $S^{n+k}$, and the basepoint of $S^{n+k}$ is, by construction, not in the image of this embedding. By identifying $S^{n+k} - \{\ast\} \cong \mathbb{R}^{n+k}$, we obtain an embedding of $M$ into the Euclidean space $\mathbb{R}^{n+k}$. Applying the Pontryagin-Thom collapse to the normal bundle of this embedding, as in the previous lecture, we obtain a pointed map $\Theta(M) : S^{n+k} \rightarrow MO(n+k)$.

Let us now construct a homotopy between the map $\Theta(M)$ and our original map $f$. $f$ can be chosen so that its restriction to $M$ exactly classifies the normal bundle of $M$, and its restriction to the tubular neighborhood of $M$ in $S^{n+k}$ agrees with the Pontryagin-Thom collapse map. By contracting whatever the map $f$ does outside of $M$'s tubular neighborhood to zero, we obtain a homotopy between $f$ and $\Theta(M)$, and thus we have shown the surjectivity of $\Theta$.

Injectivity of $\Theta$: Since $\Theta$ is a homomorphism, to demonstrate the injectivity of $\Theta$ it suffices to assume that for an $n$-manifold $M$ for which $\Theta[M] = 0$, that it is therefore the case that $M \simeq \partial W^{n+1}$. I.e., $M$ is a boundary of an $(n+1)$-manifold. Since we are assuming the map $\Theta(M)$ is null-homotopic, let us choose a regular homotopy $[0,1] \times S^{n+k} \rightarrow Th(\gamma^k_s)$ from the map $\Theta(M)$ to the constant map $\{1\} \times S^{n+k} \rightarrow * \rightarrow Th(\gamma^k_s)$. Note that the above transverse inverse image construction applied to the constant map produces an empty $n$-manifold. Thus, applying this the transversality construction to the map $[0,1] \times S^{n+k} \hookrightarrow [0,1] \times S^{n+k} \rightarrow Th(\gamma^k_s)$ produces an $(n+1)$-dimensional manifold $W^{n+1}$ with boundary $M$.

Remark 2.2. The above proof can repeated essentially verbatim to prove the more general isomorphism $\Omega^B_n \cong \pi_*MB$. Here $B$ consists of a sequence $\ldots B_n \rightarrow B_{n+1}$ with compatible fibrations $\alpha_n : B_n \rightarrow BO(n)$, $\Omega^B_n$ consists of cobordism classes of $n$-manifolds with structure $B$ on their normal bundles, and $MB$ is the (Thom) spectrum with $MB(n) = Th(\alpha^n_s \gamma^n)$. The proof is the same, one need only verify at each step that the $B$ structure can be carried along through each of constructions.

References