1. The geometry of localization and completion

To motivate the construction of tmf, let us start with a classical situation for a commutative ring $R$. One way to describe a ring is to describe ring maps into it, i.e., to describe the functor that the ring represents. So given some other ring (or Hopf algebroid) $A$ defined by generators and relations, then ring maps from $A$ into $R$ can be thought of as $R$-valued solutions to the equations defining $A$. Since $R$ may be complicated, it can be easier to produce the set of maps to some simplified version of $R$, rather than to $R$ itself. Starting with the ideal generated by $p$, there is an evident simplification, that of completing $R$ with respect to $p$. The $p$-completion of $R$, denoted $R_p$, is a particularly natural object to map into since it is defined as an inverse limit: $R_p := \lim R/p^n$. Thus, to give a map to $A \to R$ one can start with a map to a simpler ring, $A \to R/p$ and pick a compatible lift to $R/p^n$ for each $n$. If one were only interested in maps into $R$ for which $p$ is the target of a nilpotent element, then we would be completely finished. In order to understand all maps into $R$, we must take account of when $p$ is the target of a non-nilpotent element. This is accomplished by looking at maps into the localization, $R[p^{-1}]$, since it has the opposite property that it only accepts ring maps for map $p$ is the target of a unit, as opposed to a nilpotent.

It turns out that this is sufficient. That is, the problem of understanding $R$-valued ring maps may be reduced to that $R_p$-valued and $R[p^{-1}]$-valued ring maps, together with a description of whether and how one can reconstruct the former from a pair of the latter. The solution to this is encapsulated in saying that the following picture

\[
\begin{array}{ccc}
R & \longrightarrow & R_p \\
\downarrow & & \downarrow \\
R[p^{-1}] & \longrightarrow & R_p[p^{-1}] \\
\end{array}
\]

is a pullback square in the category of rings. Restating the universal property of a limit, we can say that giving an $R$-valued ring map $\phi$ is equivalent to giving a pair of maps $(\phi', \hat{\phi})$ whose images agree in the localization of the completion, $R_p[p^{-1}]$. Thus, one can completely reconstruct the ring $R$ from three pieces of data: its completion at $p$, its localization at $p$, and also just one of the connecting ring maps $R[p^{-1}] \to R_p[p^{-1}]$ (the other one comes for free). (The situation is even better, in fact, since $R_p[p^{-1}] \cong R_p \otimes_R R[p^{-1}]$, the above diagram is also a pushout square of commutative rings, and so we have simultaneously described how to give a ring map out of $R_p[p^{-1}]$).

This situation has a very nice geometric description. Dually, let us now imagine we want to understand a scheme $X$ by looking at maps out of it, i.e., the functor $X$ corepresents. Let $x$ now denote a point (or any closed subscheme) of $X$. An potentially easier first step in constructing a map out of $X$ to some scheme (or stack) $Z$ is to specify where a formal neighborhood of the point $x$ is sent, $X_x \to Z$. If we were serious about constructing a map out of $X$, we now also have to produce a map from the complement of $x$, $X - x \to Z$. Of course, these maps must be compatible in that they agree on the formal punctured neighborhood of $x$, $X_x - x$. So we have completely determined what is necessary to give a map out of $X$, and this is again encapsulated in saying that

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is a pushout square in the category of schemes (and it also a pullback, incidentally). Thus, one can completely reconstruct $X$ from the upper left of the diagram. More precisely, $X$ is determined by exactly three pieces of data: the formal scheme $X_x$, the scheme $X - x$, and the attaching map $X_x - x \to X - x$. The situation with rings, above, can recovered from the geometry by taking global functions. In this case: $\mathcal{O}_X$ will be $R$; $\mathcal{O}_{X_x}$ will be $R_p$, where $x$ is Spec $R/p$; and $\mathcal{O}_{X - x}$ is $R[p^{-1}]$.

Going in the other direction, the affine formal geometry obtains from the algebra by applying $\text{Spf}$, the formal spectrum (see the glossary).

The preceding works just as well if one were working with stacks or derived stacks instead of schemes: that is, we might stratify a stack by a sequence of closed substacks, and the stack could then be reconstructed from its strata by gluing their formal neighborhoods. Furthermore, our stack might as well be equipped with a structure sheaf of $E_\infty$-ring spectra instead of just a sheaf of rings (i.e., it is a derived stack), and this would not interfere with our gluing procedure. The example we are most interested in is where $X$ is the moduli stack of elliptic curves, $\mathcal{M}_{\text{ell}}$, or better yet the compactification $\mathcal{M}_{\text{ell}}$ formed by including curves with nodal singularities. Our goal is to construct $\text{TMF}$, namely, a sheaf $\mathcal{O}_{\text{top}}$ of $E_\infty$-ring spectra on the étale site of $\mathcal{M}_{\text{ell}}$ satisfying the following local condition: for any affine étale open $\text{Spec } R \to \mathcal{M}_{\text{ell}}$ classifying an elliptic curve $C$ over Spec $R$, the value $\mathcal{O}_{\text{top}}(R)$ should be an $E_\infty$-ring spectrum whose underlying cohomology theory is given by the formal group of the elliptic curve $C \to \text{Spec } R$. To spell this out a little, every elliptic curve has an associated formal group formed by completion at the identity, and this leads to a map of stacks $\mathcal{M}_{\text{ell}} \to \mathcal{M}_{\text{FG}}$. The composition produces a flat map $\text{Spec } R \to \mathcal{M}_{\text{FG}}$ that, by the Landweber exact functor theorem, defines a cohomology theory. To recap, the condition for $\mathcal{O}_{\text{top}}$ is that this Landweber exact cohomology theory should be that defined by the $E_\infty$-ring spectrum $\mathcal{O}_{\text{top}}(C/R)$.

Of course, $\text{TMF}$ is then the global sections of $\mathcal{O}_{\text{top}}$, i.e., the homotopy limit of the diagram of $E_\infty$-ring spectra indexed by the étale site $(\mathcal{M}_{\text{ell}})_{\text{et}}$.

So, this just leaves us to show that this sheaf $\mathcal{O}_{\text{top}}$ of $E_\infty$-ring spectra actually exists (and that it is unique up to homotopy). The strategy for doing this is just the geometric picture above, appropriately armed with obstruction theory. Henceforth, we pick a prime $p$ to work at, and $\mathcal{M}_{\text{ell}}$ will denote the stack of elliptic curves over $\text{Spec } \mathbb{F}_p$. As noted in Andre and Jacob’s talks in this volume [4], there is an evident stratification of the moduli stack $\mathcal{M}_{\text{ell}}$ by the height of associated formal groups, which is either 1 or 2 (ordinary or supersingular, respectively). For each prime $p$, there are finitely many supersingular curves depending linearly on $p$, and so in this example our point $x$ becomes the substack $\mathcal{M}_{\text{ell}}^{ss} \hookrightarrow \mathcal{M}_{\text{ell}}$ just consists of finitely many (stacky) points. The open complement $\mathcal{M}_{\text{ell}} - \mathcal{M}_{\text{ell}}^{ss}$ consists of the open substack of ordinary curves, $\mathcal{M}_{\text{ell}}^{ord}$. The picture is as follows.

The reader can see why the prime $p$ has been suppressed from the notation. To further unencumber the notation, only the sheaves will be written. So the previous diagram (since the maps of sheaves go in the opposite way) becomes:

\[
\begin{align*}
(\mathcal{M}_{\text{ell}}^{ss} - \mathcal{M}_{\text{ell}}^{ss}, \mathcal{O}_{\text{top}}) &\longrightarrow (\mathcal{M}_{\text{ell}}^{ss}, \mathcal{O}_{\text{top}}^{ss}) \\
\downarrow &\downarrow \\
(\mathcal{M}_{\text{ell}}^{ord}, \mathcal{O}_{\text{top}}^{ord}) &\longrightarrow (\mathcal{M}_{\text{ell}}, \mathcal{O}_{\text{top}}^{ord})
\end{align*}
\]
The striking fact is that this corresponds exactly to localizing with respect to the Morava $K$-theories, $K(n)$. Loosely speaking, at the prime $p$, $K(n)$ is equipped with the unique formal group of height $n$, and so localizing with respect to $K(n)$ corresponds to restricting to the stratum of points whose associated formal group has height $n$. More precisely, if $M$ is a sheaf of $E_\infty$-ring spectra on . The diagram above can thereby be rewritten again as:

Passing to global sections recovers the usual Hasse square for $tmf$.

Remark 1.1. A technical point: in this particular case, passing to global sections commutes with the localizations. This is a nontrivial fact since the global sections functor is a homotopy limit, while localization is a left Quillen functor and thus does not necessarily preserve homotopy limits. However, it certain special cases localization does preserve homotopy limits. This is a consequence of the nilpotence theorem. More precisely, the particular case of $K(n)$-localizing an $E(n)$-local spectrum can be described as a homotopy limit. Here [6] Hovey-Strickland.

The purpose of the rest of this note is to outline one part of this construction: the supersingular locus, $L_{K(2)}\mathcal{O}_{\text{top}}$. Due to Serre-Tate, the formal neighborhood $\hat{M}_{\text{ell}}$ may be identified with its image in $\mathcal{M}_{FG}$. Thus, to define the sheaf $L_{K(2)}\mathcal{O}_{\text{top}}$, it is sufficient to produce an $E_\infty$-ring spectra associated to the formal group of each supersingular elliptic $C$, together with an action of the automorphisms of $C$ on the spectrum.

The Hopkins-Miller theorem does this, not just for height 2, but for all heights. The original motivation was in the opposite direction. Hopkins and Miller wanted to construct higher analogues of real $K$-theory.

2. The Hopkins-Miller theorem

Definition 2.1. The category $\mathcal{FG}$ has as objects height $n$ formal groups over a perfect field of characteristic $p$. The morphisms are

Definition 2.2. The category $E_{LT}^\infty$-rings of Lubin-Tate $E_\infty$-ring spectra is the full subcategory of $E_\infty$-rings of fibrant/cofibrant even-periodic $E(k,\Gamma)$ such that $\pi_E$ is a complete local ring whose reduction modulo $m$, its maximal ideal, is a perfect field $k$ of characteristic $p$ whose associated formal group $\Gamma$ has height $n$. Furthermore, the formal group over $\pi_*E \cong A(k,\Gamma)[u^{\pm 1}]$, where $A(k,\Gamma)$ is the universal deformation of $(k,\Gamma)$ and $u$ is an invertible element of degree 2.

Remark 2.3. Note that the definitions of $E_{LT}^\infty$-rings and $\mathcal{FG}$ both depend on the characteristic $p$, and the height $n$, but we will suppress them from the notation.

Theorem 2.4. For every $p$ and $n$ finite, there exists a functor from the category $\mathcal{FG}$ to the category of $E_\infty$-ring spectra. The underlying homology theory of the spectrum assigned to a formal group $(k,\Gamma)$ is the Landweber exact theory whose coefficient ring is $W(k)[[u_0,\ldots,u_{n-1}]]$ and with associated formal group the universal deformation of $\Gamma$. That is, there is a lift below:
Moreover, the space of lifts is contractible. As a consequence, the functor $\pi_0(-)/m$ is a weak equivalence of topological categories.

**Remark 2.5.** The statement of the theorem for $A_{\infty}$-ring spectra is due to Mike Hopkins and Haynes Miller. The extension to $E_{\infty}$ uses a more difficult version of the obstruction theory, and is due to Hopkins and Paul Goerss, [3].

**Proof.** The proof of the theorem has three parts, of varying difficulty:

(a) The formal group of universal deformation $A(k, \Gamma)$ is regular, and so defines a Landweber exact homology theory. Furthermore, morphisms in $FG$ define natural transformations of homology theories. This constructs the functor $FG \rightarrow \{\text{homology theories}\}$.

(b) For every $(k, \Gamma)$, there exists at least one $E_{\infty}$-ring spectrum $E(k, \Gamma)$.

(c) For any two $E, F$ in $E_{\infty}$-rings associated to $(k, \Gamma), (k', \Gamma')$, the map $\text{Map}_{E_{\infty}}(E, F) \rightarrow FG(\Gamma, \Gamma')$ is a weak equivalence of topological spaces.

We begin:

(a) Landweber exactness of $L \rightarrow A(k, \Gamma)$ is an easy consequence of the construction of the formal group on the universal deformation. The functoriality is an easy trick. See Rezk, [2]. That was easy.

(b) This uses the obstruction theory of Hopkins-Miller [5], Goerss-Hopkins [3]. Fix $(k, \Gamma)$ and set $E = E(k, \Gamma)$, which exists as a homotopy commutative spectrum because of (a). The idea is to show that the following space, is contractible.

**Definition 2.6.** Fix a homology theory $E$ and a commutative ring $A$ in the category of $E_{\infty}$-comodules. The realization space of $A$, $R(A)$, is defined to be the classifying space of the following topological category: the objects are $E_{\infty}$-ring spectra $X$ such that $E_*X \cong A$, and the morphisms are $E_{\infty}$-ring maps that induce an isomorphism on $E$-homology.

Goerss-Hopkins obstruction theory, as surveyed by Vigleik in [1], proves that the obstructions to existence of a point in $R(A)$ lie in certain André-Quillen cohomology groups of simplicial algebras over a simplicial operad.

(c) Here we set up a spectral sequence for mapping spaces of $E_{\infty}$-ring spectra.

This section is unfinished.

**References**