Chapter 15

Amenability criteria via actions

In this chapter we present one more amenability criteria, which found a lot of applications to classes of groups defined via actions on certain spaces. Firstly we give a group-theoretical version of criteria due to Stefaan Vaes, and then we deduce a topological version as a corollary due to Nekrashevych, de la Salle and the author. The topological version will be used to deduce amenability of groups acting on Bratteli diagrams in the later chapter.
15.1 Group theoretical version of the criteria

**Theorem 15.1.1.** Let $G$ act on two countable sets $Y$, $X$ and the action of $G$ on $X$ has amenable lamps. Assume that there exists a $G$-map $\pi : Y \to X$ with section $t : X \to Y$ which satisfy the following properties:

1. For every $g \in G$, the set \( \{ x \in X : t(gx) \neq gt(x) \} \) is finite;
2. For every $x \in X$, the quotient $G_x/N_x$ is amenable, where \( N_x = \{ g \in G_x : \text{for all } y \in x^{-1} : gy = y \} \leq G_x; \)
3. The group $H = \{ g \in G_x : t(gx) = gt(x) \text{ for all } x \in X \}$ is amenable;

Then $G$ is amenable.

**Proof.** The aim is to apply Theorem 4.5.5 to the following space:

\[ Z := \{ \phi : X \to Y \mid \phi \text{ is a section map of } \pi, \phi(x) = t(x) \text{ for all but finitely many } x \in X \} \]

Indeed, the group $G$ acts on $Z$ by $(g\phi)(x) = g\phi(g^{-1}x)$. Since $t$ is almost $G$-map we have that $g\phi$ is in $Z$. The proof will be decided in two parts. Both claims together with Lemma 4.5.5 imply that $G$ is amenable.

**Claim 1:** The action of $G$ on $Z$ has amenable stabilizers.

Note that, if $G$ has a non-amenable subgroup $K$, then for all $x \in X$ we have that $K \cap N_x$ is not amenable. Indeed, by Lemma 13.1.2 and Lemma 4.5.5 the action of $K$ on $K.x$ has an invariant mean. Thus by Lemma the group $K \cap G_x$ can not be amenable. Then $K \cap N_x$ is not amenable, since it is a kernel of the homomorphism $K \cap G_x \to G_x/N_x$ and $G_x/N_x$ is not amenable.

To reach a contradiction, assume that $G_\phi$ is not amenable for some $\phi$. Let $x_1, \ldots, x_n$ be the finite set for which $\phi(x) \neq t(x)$. Then by induction we see that $K = G_\phi \cap N_{x_1} \cap \ldots \cap N_{x_n}$ is not amenable. In order to reach a contradiction it is sufficient to show that $K$ is a subgroup of amenable group $H$. To prove the later we need to show that for all $x \in X$ and $g \in K$ we have $t(gx) = gt(x)$. Note that $g$ fixes all elements of $\{ x_1, \ldots, x_n \}$. Therefore
if \( x \) is in \( \{ x_1, \ldots, x_n \} \) we have \( gt(x) = t(x) = t(gx) \). Now assume that \( x \) is not in \( \{ x_1, \ldots, x_n \} \), then \( gx \) is not in \( \{ x_1, \ldots, x_n \} \). Thus \( \phi(x) = t(x) \) and \( \phi(gx) = t(gx) \). Moreover, since \( g \in G \phi \) we have \( \phi(gx) = g\phi(x) \). All together give us \( t(gx) = \phi(gx) = g\phi(x) = gt(x) \). Therefore \( K \) is a subgroup of \( H \) which gives a contradiction.

**Claim 2:** The action of \( G \) on \( Z \) is amenable.

Firstly we will show that there are probability measures \( \nu_{x,n} \in l_1(Y) \) with finite support in \( \pi^{-1}(x) \), such that

\[
\lim_n \| \nu_{gx,n} - g\nu_{x,n} \|_1 = 0 \quad \text{for all} \quad x \in X, g \in G. \tag{15.1}
\]

Let \( T \subset X \) be a \( G \)-orbit transversal of the action on \( X \), i.e. the set which contains one point from each orbit of \( G \). For every \( x \in T \) choose \( L_x \subset G \) such that the map from \( L_x \) into \( G \) defined by \( g \mapsto gx \) is bijective.

In general, if \( H_1 < H_2 < \Gamma \) then taking the push-forward of the map \( \Gamma/H_1 \to \Gamma/H_2 \), we obtain that the amenability of the action of \( \Gamma \) on \( \Gamma/H_1 \) implies amenability of the action of \( \Gamma \) on \( \Gamma/H_2 \). Therefore, since \( G_x/N_x \) is amenable and \( N_y \) stabilizers of all points of \( \pi^{-1}(x) \), the action of \( G_x \) on \( \pi^{-1}(x) \) is amenable. Thus we can find probability measures \( \nu_{x,n} \in l_1(\pi^{-1}(x)) \) with finite support such that

\[
\lim \| \nu_{x,n} - g\nu_{x,n} \|_1 = 0 \quad \text{for all} \quad x \in T, g \in G_x.
\]

Define the probability measure on \( Y \) for the \( G \)-orbits of \( x \) by \( \nu_{gx,n} := g\nu_{x,n} \) for \( x \in T \) and \( g \in L_x \). Fix now \( g \in G \) and \( x' \in X \). We can express \( x' = hx \) for some \( x \in T \) and \( h \in L_x \). Choose \( k \in L_x \) such that \( (gh)x = kx \). Then \( k^{-1}gh \) belongs to \( G_x \), thus applying (15.1) to this element we exactly what we need:

\[
0 = \lim_n \| \nu_{x,n} - k^{-1}gh\nu_{x,n} \|_1 = \lim_n \| k\nu_{x,n} - gh\nu_{x,n} \|_1
= \lim_n \| \nu_{hx,n} - g\nu_{hx,n} \|_1 = \lim_n \| \nu_{gx',n} - g\nu_{x',n} \|_1
\]

Let now \( X' \) be a finite subset of \( X \) on the complement of which the map \( \phi \) is \( S \)-map, i.e. for all \( x \in X \setminus X' \) and \( s \in S \) we have \( t(sx) = st(x) \). By Theorem 4.5.2 and remarks following it, for every \( \varepsilon > 0 \) we can find a finitely supported measure \( \mu \in l_1(P_f(X)) \) that satisfies

\[
\| s\mu - \mu \|_1 < \varepsilon \quad \text{for all} \quad s \in S
\]
and such that the support of it is in the the set of all finite subsets of $X$
which contain $X'$.

Let $X_0$ be the union of all finite sets in the support of $\mu$. Choose $n$ large
enough so that

$$\|\nu_{gx,n} - g\nu_{x,n}\|_1 \leq \varepsilon$$

for all $x \in X_0$ and $s \in S$.

Consider the following identification

$$Z = \{(w_x)_{x \in X} \in \prod_{x \in X} \pi^{-1}(x) : w_x = t(x) \text{ for all but finitely many } x \in X\}.$$ 

For every finite $F \subset X$ define a finitely supported probability measure
on $Z$ by

$$\mu_F = \prod_{x \in F} \nu_{x,n} \times \prod_{x \in X \setminus F} \delta_{t(x)}.$$ 

Consider a finitely supported probability measure on $Z$ which is defined
as a convex combination of $\mu_F$:

$$m = \sum_{F \in \mathcal{P}_f(X)} \mu(F)\mu_F.$$ 

Changing the summation indexes we have

$$m = \sum_{F \in \mathcal{P}_f(X)} \mu(gF)\mu_{gF},$$

thus from the approximation property of $\mu$ we obtain that $\|m - m'\|_1 < \varepsilon$, where

$$m' = \sum_{F \in \mathcal{P}_f(X)} \mu(F)\mu_{gF} = \sum_{F \in \mathcal{P}_f(X)} \mu(F) \left( \prod_{x \in F} \nu_{gx,n} \times \prod_{x \in X \setminus F} \delta_{gt(x)} \right).$$

Moreover

$$gm = \sum_{F \in \mathcal{P}_f(X)} \mu(F) \left( \prod_{x \in F} g\nu_{x,n} \times \prod_{x \in X \setminus F} \delta_{gt(x)} \right).$$
Let $F$ be such that $\mu(F) \neq 0$ then $X_0 \subset F$ and $t(sx) = st(x)$ for all $x \in X \setminus F$ and $s \in S$. Thus

$$\left\| \prod_{x \in F} \nu_{x,n} \times \prod_{x \in X \setminus F} \delta_{t(x)} - \prod_{x \in F} \nu_{x,n} \times \prod_{x \in X \setminus F} \delta_{t(gx)} \right\|_1 \leq \sum_{x \in X} \| \nu_{g,x,n} - g\nu_{x,n} \|_1 \leq \varepsilon |X'|$$

Therefore we get

$$\| gm - m \|_1 \leq \| gm - m' \|_1 + \varepsilon \leq \varepsilon (|X'| + 1)$$

which proves the second claim. □
15.2 Actions by homeomorphisms. Preliminary definitions

Let $G$ act by homeomorphisms on a topological space $\mathcal{X}$.

**Definition 15.2.1. The full topological group** of the action of $G$, denoted by $[[G]]$, is the group of all homeomorphisms $h$ on $\mathcal{X}$ such that for any point $x \in \mathcal{X}$ there exists a neighborhood of it where $h$ acts as an element of $G$.

The notion of the full topological group of an action is defined using a local structure of action. This suggests that we can eventually study a more general notion using instead of the group $G$ a set which is "closed under local multiplication". For this purpose, we introduce a groupoid structure, i.e., the structure where the notion of the full topological group will still make sense.

**Definition 15.2.2. A groupoid** is a set $\mathcal{G}$ such that the following holds. For each $g \in \mathcal{G}$ the inverse of $g$ is defined, i.e., there is a map $^{-1} : \mathcal{G} \to \mathcal{G}$ with $g \mapsto g^{-1}$ and there is a partially defined multiplication $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ which satisfy the following axioms:

- **Associativity.** If both $a \cdot b$ and $b \cdot c$ are defined then $a \cdot (b \cdot c)$ and $(a \cdot b) \cdot c$ are defined and equal;

- **Inverse.** Both $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are defined;

- **Identity.** If $a \cdot b$ is defined, then $a \cdot b \cdot b^{-1} = a$ and $a^{-1} \cdot a \cdot b = b$.

To emphasize the structure of groupoid we will denote all groupoids by calligraphic letters.

We will define a groupoid structure on a set of homeomorphisms between open subsets of $\mathcal{X}$ as follows. A germ of homeomorphism $g$ of $\mathcal{X}$ is an equivalence class of pairs $(g, x)$ where $x \in \mathcal{X}$ and $g$ is a homeomorphism between a neighborhood of $x$ and a neighborhood of $g(x)$. Two germs $(g_1, x_1)$ and $(g_2, x_2)$ are equal if and only if $x_1 = x_2$ and both $g_1$ and $g_2$ coincide on a neighborhood of $x_1$. Each germ is considered up to equivalence. A composition of two germs $(g_1, x_1)(g_2, x_2)$ is defined to be the germ $(g_1g_2, x_2)$, if $g_2(x_2) = x_1$. The inverse of a germ $(g, x)$ is the germ $(g, x)^{-1} = (g^{-1}, g(x))$.

A groupoid of germs of homeomorphisms on $\mathcal{X}$ is a set of germs of homeomorphisms of $\mathcal{X}$ that is closed under composition (when it is possible
to compose two germs) and taking inverse and contains all germs of the form 
\((Id_{\mathcal{X}}, x)\) for \(x \in \mathcal{X}\). In particular, if a group \(G\) acts by homeomorphisms on \(\mathcal{X}\), the set of all its germs \((g, x)\) for \(g \in G\) and \(x \in \mathcal{X}\) forms the groupoid of germs, which we will call the groupoid of germs of the action of \(G\) on \(\mathcal{X}\).

The **topological full group of a groupoid of germs** \(\mathcal{G}\), denoted by \([[\mathcal{G}]]\), is the set of all homeomorphisms \(F : \mathcal{X} \rightarrow \mathcal{X}\) such that all germs of \(F\) belong to \(\mathcal{G}\). If \(\mathcal{G}\) is a group, then this coincides with usual notion of the full topological group \([[G]]\).

The groupoid of germs has a natural topology defined by the basis of open sets of the form \(\{(g, x) : x \in U\}\), where \(g \in G\) and \(U \subset \mathcal{X}\) is open.

Denote by \(o(g, x) = x\) and \(t(g, x) = g(x)\) the origin and the target of the germ \((g, x)\). The **isotropy group** or the group of germs at \(x \in \mathcal{X}\) is the group of all \(\gamma \in \mathcal{G}\) such that \(o(\gamma) = t(\gamma) = x\). In the case, when \(\mathcal{G}\) is the groupoid of germs of the action of a group \(G\) on \(\mathcal{X}\), then the group of germs \(\mathcal{G}_x\) is the quotient of the stabilizer \(G_x\) of \(x\) by the subgroup of all elements of \(G\) that act trivially on a neighborhood of \(x\).
15.3 Topological version of criteria

For $x \in \mathcal{X}$ the group of germs of $G$ at $x$ is the quotient of the stabilizer of $x$ by the subgroup of elements acting trivially on a neighborhood of $x$.

**Theorem 15.3.1.** Let $G$ be a finitely generated group of homeomorphisms of a topological space $\mathcal{X}$, and $\mathcal{G}$ be its groupoid of germs. Let $\mathcal{H}$ be a groupoid of germs of homeomorphisms of $\mathcal{X}$. Suppose that the following conditions hold.

(i) The group $[[\mathcal{H}]]$ is amenable.

(ii) For every generator $g$ of $G$ the set of points $x \in \mathcal{X}$ such that $(g, x) \notin \mathcal{H}$ is finite.

(iii) For every singular point $x \in \mathcal{X}$, the action of $G$ on the orbit of $x$ is recurrent.

(iv) The groups of germs $\mathcal{G}_x$ are amenable.

Then the group $G$ is amenable. Moreover, if the space $\mathcal{X}$ is compact then the topological full group $[[G]]$ is amenable.

**Proof.** We will deduce the theorem from its combinatorial version Theorem 15.1.1. Let $X$ be the set of all points $x \in \mathcal{X}$ such that there exists $g \in G$ with $(g, x) \notin \mathcal{H}$. Let $Y$ be the quotient space $\mathcal{G}/\mathcal{H}$, where two germs $\gamma$ and $\nu$ in $\mathcal{G}$ are equivalent modulo $\mathcal{H}$ if there exists $h \in \mathcal{H}$ such that $\nu = \gamma h$.

Define the $G$-map $\pi : Y \rightarrow X$ to be the target map and the section map $t : X \rightarrow Y$ defined by $t(x) = (e, x)$ for all $x \in X$. In these settings we have that $\mathcal{G}_x = G_x/N_x$ is amenable and $t(gx) = gt(x)$ if and only if $(g, x) \in \mathcal{H}$. Thus we can take $H$ to be $G \cap [[\mathcal{H}]]$, which is amenable by assumptions. Therefore all conditions of Theorem 15.1.1 are satisfied, and thus the group $G$ is amenable.

Since the conditions of the theorem are local, we have that $[[G]]$ is amenable. 

\[\square\]
Bibliography


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