Compactness properties of operator multipliers

K. Juschenko, R. H. Levene, I. G. Todorov and L. Turowska

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Abstract

We continue the study of multidimensional operator multipliers initiated in [12]. We introduce the notion of the symbol of an operator multiplier. We characterise completely compact operator multipliers in terms of their symbol as well as in terms of approximation by finite rank multipliers. We give sufficient conditions for the sets of compact and completely compact multipliers to coincide and characterise the cases where an operator multiplier in the minimal tensor product of two C*-algebras is automatically compact. We give a description of multilinear modular completely compact completely bounded maps defined on the direct product of finitely many copies of the C*-algebra of compact operators in terms of tensor products, generalising results of Saar [21].

1 Introduction

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i, j)a_{ij})$ is the matrix of a bounded linear operator on $\ell_2$ whenever $(a_{ij})$ is such. The study of Schur multipliers was initiated by Schur in the early 20th century and since then has attracted considerable attention, much of which was inspired by A. Grothendieck’s characterisation of these objects in his Résumé [9]. Grothendieck showed that a function $\varphi$ is a Schur multiplier precisely when it has the form $\varphi(i, j) = \sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ satisfy the conditions $\sup_i \sum_{k=1}^{\infty} \left|a_k(i)\right|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} \left|b_k(j)\right|^2 < \infty$. In modern terminology, this characterisation can be expressed by saying that $\varphi$ is a

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Schur multiplier precisely when it belongs to the extended Haagerup tensor product $\ell_\infty \otimes_{eh} \ell_\infty$ of two copies of $\ell_\infty$.

Special classes of Schur multipliers, e.g. Toeplitz and Hankel Schur multipliers, have played an important role in analysis and have been studied extensively (see [19]). Compact Schur multipliers, that is, the functions $\varphi$ for which the mapping $(a_{ij}) \rightarrow (\varphi(i,j)a_{ij})$ on $B(\ell_2)$ is compact, were characterised by Hladnik [11], who identified them with the elements of the Haagerup tensor product $c_0 \otimes h c_0$.

A non-commutative version of Schur multipliers was introduced by Kissin and Shulman [14] as follows. Let $A$ and $B$ be $C^*$-algebras and let $\pi$ and $\rho$ be representations of $A$ and $B$ on Hilbert spaces $H$ and $K$, respectively. Identifying $H \otimes K$ with the Hilbert space $C_2(H^d,K)$ of all Hilbert-Schmidt operators from the dual space $H^d$ of $H$ into $K$, we obtain a representation $\sigma_{\pi,\rho}$ of the minimal tensor product $A \otimes B$ acting on $C_2(H^d,K)$. An element $\varphi \in A \otimes B$ is called a $\pi,\rho$-multiplier if $\sigma_{\pi,\rho}(\varphi)$ is bounded in the operator norm of $C_2(H^d,K)$. If $\varphi$ is a $\pi,\rho$-multiplier for any pair of representations $(\pi,\rho)$ then $\varphi$ is called a universal (operator) multiplier.

Multidimensional Schur multipliers and their non-commutative versions were introduced and studied in [12], where the authors gave, in particular, a characterisation of universal multipliers as certain weak limits of elements of the algebraic tensor product of the corresponding $C^*$-algebras, generalising the corresponding results of Grothendieck and Peller [9, 18] as previously conjectured by Kissin and Shulman in [14]. Let $A_1,\ldots,A_n$ be $C^*$-algebras. Like Schur multipliers, elements of the set $M(A_1,\ldots,A_n)$ of (multidimensional) universal multipliers give rise to completely bounded (multilinear) maps. Requiring these maps to be compact or completely compact, we define the sets of compact and completely compact operator multipliers denoted by $Mc(A_1,\ldots,A_n)$ and $Mcc(A_1,\ldots,A_n)$, respectively. The notion of complete compactness we use is an operator space version of compactness which was introduced by Saar [21] and subsequently studied by Oikhberg [15] and Webster [27]. Our results on operator multipliers rely on the main result of Section 3 where we prove a representation theorem for completely compact completely bounded multilinear maps. In [3] Christensen and Sinclair established a representation result for completely bounded multilinear maps which implies that every such map $\Phi : K(H_2,H_1) \otimes_h \cdots \otimes_h K(H_n,H_{n-1}) \rightarrow K(H_n,H_1)$ (where, for Hilbert spaces $H'$ and $H''$, we denote by $K(H',H'')$ the space of all compact operators from $H'$ into $H''$) has the form

$$\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \cdots (x_{n-1} \otimes 1)A_n,$$

(1)

for some index set $J$ and bounded block operator matrices $A_1 \in Mc(J(B(H_1)))$, $A_2 \in M_{J}(B(H_2)),\ldots,A_n \in M_{J,1}(B(H_n))$. In other words, $\Phi$ arises from an
element
\[ u = A_1 \odot \cdots \odot A_n \in \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n) \]
of the extended Haagerup tensor product of \( \mathcal{B}(H_1), \ldots, \mathcal{B}(H_n) \). Moreover, if \( \Phi \) is \( \mathcal{A}'_1, \ldots, \mathcal{A}'_n \)-modular for some von Neumann algebras \( \mathcal{A}_1, \ldots, \mathcal{A}_n \), then the entries of \( A_i \) can be chosen from \( \mathcal{A}_i \). We show in Section 3 that a map \( \Phi \) as above is completely compact precisely when it has a representation of the form (1) where
\[ u = A_1 \odot \cdots \odot A_n \in \mathcal{K}(H_1) \otimes_h (\mathcal{B}(H_2) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_{n-1})) \otimes_h \mathcal{K}(H_n). \]
This extends a result of Saar [21] in the two dimensional case. If, additionally, \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are von Neumann algebras and \( \Phi \) is \( \mathcal{A}'_1, \ldots, \mathcal{A}'_n \)-modular then \( u \) can be chosen from \( \mathcal{K}(\mathcal{A}_1) \otimes_h (\mathcal{B}_2 \otimes_{eh} \cdots \otimes_{eh} \mathcal{A}_{n-1}) \otimes_h \mathcal{K}(\mathcal{A}_n), \) where \( \mathcal{K}(\mathcal{A}) \) denotes the ideal of compact elements of a \( C^* \)-algebra \( \mathcal{A} \). As a consequence of this and a result of Effros and Kishimoto [4] we point out the completely isometric identifications
\[ CC(\mathcal{K}(H_2, H_1))^{**} \simeq (\mathcal{K}(H_1) \otimes_h \mathcal{K}(H_2))^{**} \simeq CB(\mathcal{B}(H_2, H_1)), \]
where \( CC(\mathcal{X}) \) and \( CB(\mathcal{X}) \) are the spaces of completely compact and completely bounded maps on an operator space \( \mathcal{X} \), respectively.

In Section 4 we pinpoint the connection between universal operator multipliers and completely bounded maps. This technical result is used in Section 5 to define the symbol \( u_\varphi \) of an operator multiplier \( \varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \) which, in the case \( n \) is even (resp. odd) is an element of \( \mathcal{A}_n \otimes_{eh} \mathcal{A}'_{n-1} \otimes_{eh} \cdots \otimes_{eh} \mathcal{A}_1^o \) (resp. \( \mathcal{A}_n \otimes_{eh} \mathcal{A}'_{n-1} \otimes_{eh} \cdots \otimes_{eh} \mathcal{A}_1^o \)). Here \( \mathcal{A}^o \) is the opposite \( C^* \)-algebra of \( \mathcal{A} \). This notion extends a similar notion that was given in the case of completely bounded masa-bimodule maps by Katavolos and Paulsen in [13]. We give a symbolic calculus for universal multipliers which is used to establish a universal property of the symbol related to the representation theory of the \( C^* \)-algebras under consideration.

The symbol of a universal multiplier is used in Section 6 to single out the completely compact multipliers within the set of all operator multipliers. In fact, we show that \( \varphi \in M_{\infty}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \) if and only if
\[ u_\varphi \in \begin{cases} \mathcal{K}(\mathcal{A}_n) \otimes_h (\mathcal{A}'_{n-1} \otimes_{eh} \cdots \otimes_{eh} \mathcal{A}_1) \otimes_h \mathcal{K}(\mathcal{A}_1^n) & \text{if } n \text{ is even} \\ \mathcal{K}(\mathcal{A}_n) \otimes_h (\mathcal{A}'_{n-1} \otimes_{eh} \cdots \otimes_{eh} \mathcal{A}_1^2) \otimes_h \mathcal{K}(\mathcal{A}_1) & \text{if } n \text{ is odd} \end{cases} \]
which is equivalent to the approximability of \( \varphi \) in the multiplier norm by operator multipliers of finite rank whose range consists of finite rank operators. It follows that a multidimensional Schur multiplier \( \varphi \in \ell_\infty(X_1 \times \cdots \times X_n) \) is compact if and only if \( \varphi \in c_0(X_1) \otimes_h (\ell_\infty(X_2) \otimes_{eh} \cdots \otimes_{eh} \ell_\infty(X_{n-1})) \otimes_h c_0(X_n). \)
In Section 7 we use Saar’s construction [21] of a completely bounded compact mapping which is not completely compact to show that the inclusion $M_{cc}(A_1, \ldots, A_n) \subseteq M_c(A_1, \ldots, A_n)$ is proper if both $K(A_1)$ and $K(A_n)$ contain full matrix algebras of arbitrarily large sizes. However, if both $K(A_1)$ and $K(A_n)$ are isomorphic to a $c_0$-sum of matrix algebras of uniformly bounded sizes then the sets of compact and completely compact multipliers coincide. The case when only one of $K(A_1)$ and $K(A_n)$ contains matrix algebras of arbitrary large size remains, however, unsettled. Finally, for $n = 2$, we characterise the cases where every universal multiplier is automatically compact: this happens precisely when one of the algebras $A_1$ and $A_2$ is finite dimensional and the other one coincides with its algebra of compact elements.

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2 Preliminaries

We start by recalling standard notation and notions from operator space theory. We refer the reader to [1], [6], [16] and [20] for more details.

If $H$ and $K$ are Hilbert spaces we let $\mathcal{B}(H, K)$ (resp. $\mathcal{K}(H, K)$) denote the set of all bounded linear (resp. compact) operators from $H$ into $K$. If $I$ is a set we let $H^I$ be the direct sum of $|I|$ copies of $H$ and set $H^\infty = H^\mathbb{N}$. An operator space $\mathcal{E}$ is a closed subspace of $\mathcal{B}(H, K)$, for some Hilbert spaces $H$ and $K$. The opposite operator space $\mathcal{E}^o$ associated with $\mathcal{E}$ is the space $\mathcal{E}^o = \{x^d : x \in \mathcal{E}\} \subseteq \mathcal{B}(K^d, H^d)$. Here, and in the sequel, $H^d = \{\xi^d : \xi \in H\}$ denotes the dual of the Hilbert space $H$, where $\xi^d(\eta) = (\eta, \xi)$ for $\eta \in H$. Note that $H^d$ is canonically conjugate-linearly isometric to $H$. We also adopt the notation $x^d \in \mathcal{B}(K^d, H^d)$ for the Banach space adjoint of $x \in \mathcal{B}(H, K)$, so that $x^d \xi^d = (x^* \xi)^d$ for $\xi \in K$. As usual, $\mathcal{E}^*$ will denote the operator space
dual of $\mathcal{E}$. If $n, m \in \mathbb{N}$, by $M_{n,m}(\mathcal{E})$ we denote the space of all $n$ by $m$ matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. The space $M_{n,m}(\mathcal{E})$ carries a natural norm arising from the embedding $M_{n,m}(\mathcal{E}) \subseteq \mathcal{B}(H^n, K^m)$. Let $I$ and $J$ be arbitrary index sets. If $v$ is a matrix with entries in $\mathcal{E}$ and indexed by $I \times J$, and $I_0 \subseteq I$ and $J_0 \subseteq J$ are finite sets, we let $v_{I_0,J_0} \in M_{I_0,J_0}(\mathcal{E})$ be the matrix obtained by restricting $v$ to the indices from $I_0 \times J_0$. We define $M_{I,J}(\mathcal{E})$ to be the space of all such $v$ for which

$$\|v\| \overset{\text{def}}{=} \sup\{\|v_{I_0,J_0}\| : I_0 \subseteq I, J_0 \subseteq J \text{ finite}\} < \infty.$$  

Then $M_{I,J}(\mathcal{E})$ is an operator space [6, §10.1]. Note that $M_{I,J}(\mathcal{B}(H, K))$ can be naturally identified with $\mathcal{B}(H^J, K^I)$ and every $v \in M_{I,J}(\mathcal{B}(H, K))$ is the weak limit of $\{v_{I_0,J_0}\}$ along the net $\{(I_0, J_0) : I_0 \subseteq I, J_0 \subseteq J \text{ finite}\}$. We set $M_I(\mathcal{E}) = M_{I,I}(\mathcal{E})$. For $A = (a_{ij}) \in M_I(\mathcal{E})$, we write $A^i = (a_{ij}^i) \in M_I(\mathcal{E}^{i})$.

### 2.1 Completely bounded maps and Haagerup tensor products

If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if the maps $\Phi^{(k)} : M_k(\mathcal{E}) \to M_k(\mathcal{F})$ given by $\Phi^{(k)}((a_{ij})) = (\Phi(a_{ij}))$ are bounded for every $k \in \mathbb{N}$ and $\|\Phi\|_{\text{cb}} \overset{\text{def}}{=} \sup_k \|\Phi^{(k)}\| < \infty$.

Given linear spaces $\mathcal{E}_1, \ldots, \mathcal{E}_n$, we denote by $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n$ their algebraic tensor product. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are operator spaces and $a^k = (a_{ij}^k) \in M_{n,m,k+1}(\mathcal{E}_k), m_k \in \mathbb{N}, k = 1, \ldots, n$, we define the multiplicative product

$$a^1 \circ \cdots \circ a^n \in M_{m_1,m_n+1}(\mathcal{E}_1 \circ \cdots \circ \mathcal{E}_n)$$

by letting its $(i, j)$-entry $(a^1 \circ \cdots \circ a^n)_{ij}$ be $\sum_{i_1, \ldots, i_n} a_{i_1,i_2}^1 \otimes a_{i_2,i_3}^2 \otimes \cdots \otimes a_{i_n,j}^n$. If $\mathcal{E}$ is another operator space and $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ is a multilinear map we let

$$\Phi^{(m)} : M_m(\mathcal{E}_1) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E})$$

be the map given by

$$(\Phi^{(m)}(a^1, \ldots, a^n))_{ij} = \sum_{i_2, \ldots, i_n} \Phi(a_{i_1,i_2}^1, a_{i_2,i_3}^2, \ldots, a_{i_n,j}^n),$$

where $a^k = (a_{s,t}^k) \in M_m(\mathcal{E}_k), k = 1, \ldots, n$. The multilinear map $\Phi$ is called completely bounded if there exists a constant $C > 0$ such that, for all $m \in \mathbb{N}$,

$$\|\Phi^{(m)}(a^1, \ldots, a^n)\| \leq C \|a^1\| \cdots \|a^n\|, \quad a^k \in M_m(\mathcal{E}_k), \quad k = 1, \ldots, n.$$
Set $\|\Phi\|_{cb} \overset{\text{def}}{=} \sup\{\|\Phi^{(m)}(a^1,\ldots,a^n)\| : m \in \mathbb{N}, \|a^1\|,\ldots,\|a^n\| \leq 1\}$. It is well-known (see [6, 17]) that a completely bounded multilinear map $\Phi$ gives rise to a completely bounded map on the Haagerup tensor product $E_1 \otimes_h \cdots \otimes_h E_n$ (see [6] and [20] for its definition the and basic properties).

The set of all completely bounded multilinear maps from $E_1 \times \cdots \times E_n$ into $E$ will be denoted by $CB(E_1 \times \cdots \times E_n)$. If $E_1,\ldots,E_n$ and $E$ are dual operator spaces we say that a map $\Phi \in CB(E_1 \times \cdots \times E_n,E)$ is normal [3] if it is weak* continuous in each variable. We write $CB^{\sigma}(E_1 \times \cdots \times E_n,E)$ for the space of all normal maps in $CB(E_1 \times \cdots \times E_n,E)$.

The extended Haagerup tensor product $E_1 \otimes_{eh} \cdots \otimes_{eh} E_n$ is defined [5] as the space of all normal completely bounded maps $u : E_1^* \times \cdots \times E_n^* \to \mathbb{C}$. It was shown in [5] that if $u \in E_1 \otimes_{eh} \cdots \otimes_{eh} E_n$ then there exist index sets $J_1,J_2,\ldots,J_{n-1}$ and matrices $a^1 = (a_1^{i_1}) \in M_1,J_1(E_1)$, $a^2 = (a_2^{i_2}) \in M_1,J_2(E_2),\ldots,a^n = (a_n^{i_n}) \in M_1,J_{n-1}(E_n)$ such that if $f_i \in E_i^*$, $i = 1,\ldots,n$, then

$$\langle u,f_1 \otimes \cdots \otimes f_n \rangle \overset{\text{def}}{=} u(f_1,\ldots,f_n) = \langle a^1,f_1 \rangle \cdots \langle a^n,f_n \rangle, \quad (2)$$

where $\langle a^k,f_k \rangle = (f_k(a_s^k))_{s,t}$ and the product of the (possibly infinite) matrices in (2) is defined to be the limit of the sums

$$\sum_{i_1 \in F_1,\ldots,i_{n-1} \in F_{n-1}} f_1(a_1^{i_1})f_2(a_2^{i_1,i_2}) \cdots f_n(a_n^{i_{n-1},i_n})$$

along the net $\{(F_1 \times \cdots \times F_{n-1}) : F_j \subseteq J_j \text{ finite}, 1 \leq j \leq n-1\}$. We may thus identify $u$ with the matrix product $a^1 \otimes \cdots \otimes a^n$; two elements $a^1 \otimes \cdots \otimes a^n$ and $\tilde{a}^1 \otimes \cdots \otimes \tilde{a}^n$ coincide if $\langle a^1,f_1 \rangle \cdots \langle a^n,f_n \rangle = \langle \tilde{a}^1,f_1 \rangle \cdots \langle \tilde{a}^n,f_n \rangle$ for all $f_i \in E_i^*$. Moreover,

$$\|u\|_{eh} = \inf\{\|a^1\| \cdots \|a^n\| : u = a^1 \otimes \cdots \otimes a^n\}.$$ 

The space $E_1 \otimes_{eh} \cdots \otimes_{eh} E_n$ has a natural operator space structure [5]. If $E_1,\ldots,E_n$ are dual operator spaces then by [5, Theorem 5.3] $E_1 \otimes_{eh} \cdots \otimes_{eh} E_n$ coincides with the weak* Haagerup tensor product $E_1 \otimes_{w^*} \cdots \otimes_{w^*} E_n$ of Blecher and Smith [2]. Given operator spaces $F_i$ and completely bounded maps $g_i : E_i \to F_i$, $i = 1,\ldots,n$, Effros and Ruan [5] define a completely bounded map

$$g = g_1 \otimes_{eh} \cdots \otimes_{eh} g_n : E_1 \otimes_{eh} \cdots \otimes_{eh} E_n \to F_1 \otimes_{eh} \cdots \otimes_{eh} F_n,$$

$$a^1 \otimes \cdots \otimes a^n \mapsto \langle a^1,g_1 \rangle \otimes \cdots \otimes \langle a^n,g_n \rangle$$

where $\langle a^k,g_k \rangle = (g_k(a_s^k))$. Thus

$$\langle g(u),f_1 \otimes \cdots \otimes f_n \rangle = \langle u, (f_1 \circ g_1) \otimes \cdots \otimes (f_n \circ g_n) \rangle \quad (3)$$
for $u \in \mathcal{E}_1 \otimes_{eh} \cdots \otimes_{eh} \mathcal{E}_n$ and $f_i \in \mathcal{F}^*_i$, $i = 1, \ldots, n$.

The following fact is a straightforward consequence of a well-known theorem due to Christensen and Sinclair [3], and it will be used throughout the exposition.

**Theorem 2.1.** Let $H_i$ be a Hilbert space and $\mathcal{R}_i \subseteq \mathcal{B}(H_i)$ be a von Neumann algebra, $i = 1, \ldots, n$. There exists an isometry $\gamma$ from $\mathcal{R}_1 \otimes_{eh} \cdots \otimes_{eh} \mathcal{R}_n$ onto the space of all $\mathcal{R}_1', \ldots , \mathcal{R}_n'$-modular maps in $C\mathcal{B}^*(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1))$, given as follows: if $u = A_1 \otimes \cdots \otimes A_n \in \mathcal{R}_1 \otimes_{eh} \cdots \otimes_{eh} \mathcal{R}_n$ then

$$\gamma(u)(T_1, \ldots, T_{n-1}) = A_1(T_1 \otimes I)A_2 \cdots A_{n-1}(T_{n-1} \otimes I)A_n,$$

for all $T_i \in \mathcal{B}(H_{i+1}, H_i)$, $i = 1, \ldots, n - 1$.

We now turn to the definition of slice maps which will play an important role in our proofs. Given $\omega_1 \in \mathcal{B}(H_1)^*$ we set $L_{\omega_1} = \omega_1 \otimes \mathrm{id}_{\mathcal{B}(H_2)}$. After identifying $\mathbb{C} \otimes \mathcal{B}(H_2)$ with $\mathcal{B}(H_2)$ we obtain a mapping called a left slice map $L_{\omega_1} : \mathcal{B}(H_1) \otimes_{eh} \mathcal{B}(H_2) \rightarrow \mathcal{B}(H_2)$. Similarly, for $\omega_2 \in \mathcal{B}(H_2)^*$ we obtain a right slice map $R_{\omega_2} : \mathcal{B}(H_1) \otimes_{eh} \mathcal{B}(H_2) \rightarrow \mathcal{B}(H_1)$. If $u = \sum_{i \in I} v_i \otimes w_i \in \mathcal{B}(H_1) \otimes_{eh} \mathcal{B}(H_2)$ where $v = (v_i)_{i \in I} \in M_{I,1}(\mathcal{B}(H_1))$ and $w = (w_i)_{i \in I} \in M_{I,1}(\mathcal{B}(H_2))$, then

$$L_{\omega_1}(u) = \sum_{i \in I} \omega_1(v_i) w_i$$

and

$$R_{\omega_2}(u) = \sum_{i \in I} \omega_2(w_i)v_i.$$  

Moreover,

$$\langle R_{\omega_2}(u), \omega_1 \rangle = \langle u, \omega_1 \otimes \omega_2 \rangle = \langle L_{\omega_1}(u), \omega_2 \rangle = \sum_{i \in I} \omega_1(v_i) \omega_2(w_i). \quad (4)$$

It was shown in [24] that if $\mathcal{E} \subseteq \mathcal{B}(H_1)$ and $\mathcal{F} \subseteq \mathcal{B}(H_2)$ are closed subspaces then, up to a complete isometry,

$$\mathcal{E} \otimes_{eh} \mathcal{F} = \{ u \in \mathcal{B}(H_1) \otimes_{eh} \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \text{ for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^* \} \quad (5)$$

$$= \{ u \in \mathcal{B}(H_1) \otimes_{eh} \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \text{ for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^* \}.$$  

Moreover [23],

$$\mathcal{E} \otimes_h \mathcal{F} = \{ u \in \mathcal{B}(H_1) \otimes_h \mathcal{B}(H_2) : L_{\omega_1}(u) \in \mathcal{F} \text{ and } R_{\omega_2}(u) \in \mathcal{E} \text{ for all } \omega_1 \in \mathcal{B}(H_1)^* \text{ and } \omega_2 \in \mathcal{B}(H_2)^* \}. \quad (6)$$

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Thus, \( \mathcal{E} \otimes_h \mathcal{F} \) can be canonically identified with a subspace of \( \mathcal{B}(H_1) \otimes_h \mathcal{B}(H_2) \) which, on the other hand, sits completely isometrically in \( \mathcal{B}(H_1) \otimes_c \mathcal{B}(H_2) \). These identifications are made in the statement of the following lemma which will be useful for us later.

**Lemma 2.2.** If \( H_1, H_2, H_3 \) are Hilbert spaces and \( \mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{B}(H_1), \mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{B}(H_2) \) are operator spaces, then

\[
(\mathcal{E}_1 \otimes_c \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_h \mathcal{F}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_h (\mathcal{F}_1 \cap \mathcal{F}_2)
\]

and

\[
(\mathcal{E}_1 \otimes_c \mathcal{F}_1 \otimes_c \mathcal{G}_1) \cap (\mathcal{E}_2 \otimes_h \mathcal{F}_2 \otimes_h \mathcal{G}_2) = (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_h (\mathcal{F}_1 \cap \mathcal{F}_2) \otimes_h (\mathcal{G}_1 \cap \mathcal{G}_2)
\]

are both associative, the second equation follows from the first. If \( u \in (\mathcal{E}_1 \otimes_c \mathcal{F}_1) \cap (\mathcal{E}_2 \otimes_h \mathcal{F}_2) \subseteq \mathcal{B}(H_1) \otimes_h \mathcal{B}(H_2) \) then

\[
L_\varphi(u) \in \mathcal{F}_1 \cap \mathcal{F}_2 \quad \text{and} \quad R_\psi(u) \in \mathcal{E}_1 \cap \mathcal{E}_2
\]

whenever \( \varphi \in \mathcal{B}(H_1)^* \) and \( \psi \in \mathcal{B}(H_2)^* \). By (6), \( u \in (\mathcal{E}_1 \cap \mathcal{E}_2) \otimes_h (\mathcal{F}_1 \cap \mathcal{F}_2) \). The converse inclusion follows immediately in light of the injectivity of the Haagerup tensor product.

\[ \tag*{\Box} \]

### 2.2 Operator multipliers

We now recall some definitions and results from [14] and [12] that will be needed later. Let \( H_1, \ldots, H_n \) be Hilbert spaces and \( H = H_1 \otimes \cdots \otimes H_n \) be their Hilbertian tensor product. Set \( HS(H_1, H_2) = C_2(H_1^d, H_2) \) and let \( \theta_{H_1, H_2} : H_1 \otimes H_2 \to HS(H_1, H_2) \) be the canonical isometry given by \( \theta(\xi_1 \otimes \xi_2) = (\xi_1, \eta)\xi_2 \) for \( \xi_1, \eta \in H_1 \) and \( \xi_2 \in H_2 \). When \( n \) is even, we inductively define

\[
HS(H_1, \ldots, H_n) \overset{\text{def}}{=} C_2(HS(H_2, H_3)^d, HS(H_1, H_4, \ldots, H_n)),
\]

and let \( \theta_{H_1, \ldots, H_n} : H \to HS(H_1, \ldots, H_n) \) be given by

\[
\theta_{H_1, \ldots, H_n}(\xi_{2,3} \otimes \xi) = \theta_{HS(H_2, H_3), HS(H_1, H_4, \ldots, H_n)}(\theta_{H_2, H_3}(\xi_{2,3}) \otimes \theta_{H_1, H_4, \ldots, H_n}(\xi)),
\]

where \( \xi_{2,3} \in H_2 \otimes H_3 \) and \( \xi \in H_1 \otimes H_4 \otimes \cdots \otimes H_n \). When \( n \) is odd, we let

\[
HS(H_1, \ldots, H_n) \overset{\text{def}}{=} HS(\mathbb{C}, H_1, \ldots, H_n).
\]

If \( K \) is a Hilbert space, we will identify \( C_2(\mathbb{C}^d, K) \) with \( K \) via the map \( S \to S(1^d) \). The isomorphism \( \theta_{H_1, \ldots, H_n} \) in the odd case is given by

\[
\theta_{H_1, \ldots, H_n}(\xi) = \theta_{\mathbb{C}, H_1, \ldots, H_n}(1 \otimes \xi).
\]

\[ \tag*{8} \]
We will omit the subscripts when they are clear from the context and simply write $\theta$.

If $\xi \in H_1 \otimes H_2$ we let $\|\xi\|_{op}$ denote the operator norm of $\theta(\xi)$. By $\| \cdot \|_2$ we will denote the Hilbert-Schmidt norm.

Let

$$
\Gamma(H_1, \ldots, H_n) = \begin{cases} 
(H_1 \otimes H_2) \circ (H_2 \otimes H_3)^d \circ \cdots \circ (H_{n-1} \otimes H_n) & \text{n even}, \\
(H_1 \otimes H_2)^d \circ (H_2 \otimes H_3) \circ \cdots \circ (H_{n-1} \otimes H_n) & \text{n odd}.
\end{cases}
$$

We equip $\Gamma(H_1, \ldots, H_n)$ with the Haagerup norm $\| \cdot \|_h$, where each of the terms of the algebraic tensor product is given the opposite operator space structure to the one arising from the embedding $H \otimes K \hookrightarrow (C_2(H^d, K), \| \cdot \|_{op})$. We denote by $\| \cdot \|_{2, \wedge}$ the projective norm on $\Gamma(H_1, \ldots, H_n)$ where each of the terms is given its Hilbert space norm.

Suppose $n$ is even. For each $\varphi \in \mathcal{B}(H)$ we let $S_\varphi : \Gamma(H_1, \ldots, H_n) \to \mathcal{B}(H_1^d, H_n)$ be the map given by

$$
S_\varphi(\xi) = \theta(\varphi(\xi_{1,2} \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n}))(\theta(\eta_{2,3}^d))(\theta(\eta_{1,5}^d)) \cdots (\theta(\eta_{n-2,n-1}^d))
$$

where $\xi = \xi_{1,2} \otimes \eta_{2,3}^d \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n)$ is an elementary tensor. In particular, if $A_i \in \mathcal{B}(H_i)$, $i = 1, \ldots, n$, and $\varphi = A_1 \otimes \cdots \otimes A_n$ then

$$
S_\varphi(\xi) = A_n \theta(\xi_{n-1,n}) \cdots A_3 \theta(\eta_{2,3}^d) A_2 \theta(\xi_{1,2}) A_1^d.
$$

Now suppose that $n$ is odd and let $\zeta \in \Gamma(H_1, \ldots, H_n)$ and $\xi_1 \in H_1$. Then

$$
\xi_1 \otimes \zeta \in H_1 \otimes \Gamma(H_1, \ldots, H_n) = \Gamma(\mathbb{C}, H_1, \ldots, H_n).
$$

For $\varphi \in \mathcal{B}(H)$ we let $S_\varphi(\zeta)$ be the operator defined on $H_1$ by

$$
S_\varphi(\zeta)(\xi_1) = S_{1 \otimes \varphi}(\xi_1 \otimes \zeta).
$$

Note that $S_{1 \otimes \varphi}(\xi_1 \otimes \zeta) \in C_2(\mathbb{C}^d, H_n)$; thus, $S_\varphi(\zeta)(\xi_1)$ can be viewed as an element of $H_n$. It was shown in [12] that $S_\varphi(\zeta) \in \mathcal{B}(H_1, H_n)$. If $\zeta = \eta_{1,2}^d \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}$ and $\varphi = A_1 \otimes \cdots \otimes A_n$ for $A_i \in \mathcal{B}(H_i)$, $i = 1, \ldots, n$ then

$$
S_\varphi(\zeta) = A_n \theta(\xi_{n-1,n}) \cdots A_3 \theta(\eta_{2,3}^d) A_2 \theta(\xi_{1,2}) A_1.
$$

As observed in [12, Remark 4.3], for any $\varphi \in \mathcal{B}(H)$ and $\zeta \in \Gamma(H_1, \ldots, H_n)$,

$$
\|S_\varphi(\zeta)\|_{op} \leq \|\varphi\| \|\zeta\|_{2, \wedge}. \tag{7}
$$

On the other hand, an element $\varphi \in \mathcal{B}(H)$ is called a concrete operator multiplier if there exists $C > 0$ such that $\|S_\varphi(\zeta)\|_{op} \leq C\|\zeta\|_h$ for each
\( \zeta \in \Gamma(H_1, \ldots, H_n) \). When \( n = 2 \), this is equivalent to \( \| S_\varphi(\zeta) \|_{op} \leq C \| \theta(\zeta) \|_{op} \) for each \( \zeta \in H_1 \otimes H_2 \). We call the smallest constant \( C \) with this property the concrete multiplier norm of \( \varphi \).

Now let \( A_i \) be a C*-algebra and \( \pi_i : A_i \to B(H_i) \) be a representation, \( i = 1, \ldots, n \). Set \( \pi = \pi_1 \otimes \cdots \otimes \pi_n : A_1 \otimes \cdots \otimes A_n \to B(H_1 \otimes \cdots \otimes H_n) \) (here, and in the sequel, by \( A \otimes B \) we will denote the minimal tensor product of the C*-algebras \( A \) and \( B \)). An element \( \varphi \in A_1 \otimes \cdots \otimes A_n \) is called a \( \pi_1, \ldots, \pi_n \)-multiplier if \( \pi(\varphi) \) is a concrete operator multiplier. We denote by \( \| \varphi \|_{\pi_1, \ldots, \pi_n} \) the concrete multiplier norm of \( \pi(\varphi) \). We call \( \varphi \) a universal multiplier if it is a \( \pi_1, \ldots, \pi_n \)-multiplier for all representations \( \pi_i \) of \( A_i \), \( i = 1, \ldots, n \). We denote the collection of all universal multipliers by \( M(A_1, \ldots, A_n) \); from this definition, it immediately follows that

\[
A_1 \otimes \cdots \otimes A_n \subseteq M(A_1, \ldots, A_n) \subseteq A_1 \otimes \cdots \otimes A_n.
\]

It was observed in [12] that if \( \varphi \in M(A_1, \ldots, A_n) \) then

\[
\| \varphi \|_m \overset{\text{def}}{=} \sup \{ \| \varphi \|_{\pi_1, \ldots, \pi_n} : \pi_i \text{ is a representation of } A_i, \ i = 1, \ldots, n \} < \infty.
\]

It is obvious that if \( A_i \) and \( B_i \) are C*-algebras and \( \rho_i : A_i \to B_i \) is a *-isomorphism, \( i = 1, \ldots, n \), then

\[
(\rho_1 \otimes \cdots \otimes \rho_n)(M(A_1, \ldots, A_n)) = M(B_1, \ldots, B_n).
\]

If \( \varphi \) is an operator, and \( \{ \varphi_\nu \} \) a net of operators, acting on \( H_1 \otimes \cdots \otimes H_n \), we say that \( \{ \varphi_\nu \} \) converges semi-stably to \( \varphi \) if \( (\varphi_\nu \xi, \eta) \to (\varphi \xi, \eta) \) for all \( \xi, \eta \in H_1 \otimes \cdots \otimes H_n \). The following characterisation of universal multipliers was established in [12] (see Theorem 6.5, the subsequent remark and the proof of Proposition 6.2) and will be used extensively in the sequel.

**Theorem 2.3.** Let \( A_i \subseteq B(H_i) \) be a C*-algebra, \( i = 1, \ldots, n \), and \( \varphi \in A_1 \otimes \cdots \otimes A_n \). Suppose that \( n \) is even. The following are equivalent:

(i) \( \varphi \in M(A_1, \ldots, A_n) \);

(ii) there exists a net \( \{ \varphi_\nu \} \) where \( \varphi_\nu = A_1^\nu \odot A_2^\nu \odot \cdots \odot A_n^\nu \) and \( A_i^\nu \) is a finite block operator matrix with entries in \( A_i \) such that \( \varphi_\nu \to \varphi \) semi-stably, \( \| \varphi_\nu \|_m \leq M \| A_2^\nu \| \| A_3^\nu \| \| A_4^\nu \| \cdots \| A_n^\nu \| \) and the operator norms \( \| A_i^\nu \| \) for \( i \) even and \( \| A_i^\nu \| \) for \( i \) odd, are bounded by a constant depending only on \( n \).

For every net \( \{ \varphi_\nu \} \) satisfying (ii) we have that \( S_{\varphi_\nu}(\zeta) \to S_{\varphi}(\zeta) \) weakly for all \( \zeta = \xi_1 \otimes \cdots \otimes \xi_{n-1} \in \Gamma(H_1, \ldots, H_n) \) and \( \sup_\nu \| \varphi_\nu \|_m \) is finite.

Moreover, the net \( \varphi_\nu \) can be chosen in (ii) so that \( A_i^\nu \to A_i \) (resp. \( A_i^\nu \to A_i^d \)) strongly for \( i \) even (resp. for \( i \) odd) for some bounded block operator matrix \( A_i \) with entries in \( A_i^e \) (resp. \( A_i^d \)) such that

\[
S_{id \odot \cdots \odot id(\varphi)}(\zeta) = A_n(\theta(\xi_{n-1}, n) \otimes I) \cdots (\theta(\xi_1, 2) \otimes I) A_1^d,
\]
for all $\zeta = \xi_1 \otimes \cdots \otimes \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n)$.

A similar statement holds if $n$ is odd.

Finally, recall that an element $a$ of a C*-algebra $\mathcal{A}$ is called compact if the operator $x \mapsto axa$ on $\mathcal{A}$ is compact. Let $\mathcal{K}(\mathcal{A})$ be the collection of all compact elements of $\mathcal{A}$. It is well known [7, 29] that $a \in \mathcal{K}(\mathcal{A})$ if and only if there exists a faithful representation $\pi$ of $\mathcal{A}$ such that $\pi(a)$ is a compact operator. Moreover, $\pi$ can be taken to be the reduced atomic representation of $\mathcal{A}$. The notion of a compact element of a C*-algebra will play a central role in Sections 6 and 7 of the paper.

3 Completely compact maps

We start by recalling the notion of a completely compact map introduced in [21] and studied further in [27] and [15]. By way of motivation, recall that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces then a bounded linear map $\Phi : \mathcal{X} \to \mathcal{Y}$ is compact if and only if for every $\varepsilon > 0$, there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that $\text{dist}(\Phi(x), F) < \varepsilon$ for every $x$ in the unit ball of $\mathcal{X}$.

Now let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces. A completely bounded map $\Phi : \mathcal{X} \to \mathcal{Y}$ is called completely compact if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that $\text{dist}(\Phi^{(m)}(x), M_m(F)) < \varepsilon$, for every $x \in M_m(\mathcal{X})$ with $\|x\| \leq 1$ and every $m \in \mathbb{N}$. We extend this definition to multilinear maps: if $\mathcal{Y}, \mathcal{X}_1, \ldots, \mathcal{X}_n$ are operator spaces and $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is a completely bounded multilinear map, we call $\Phi$ completely compact if for each $\varepsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that $\text{dist}(\Phi^{(m)}(x_1, \ldots, x_n), M_m(F)) < \varepsilon$, for all $x_i \in M_m(\mathcal{X}_i)$, $\|x_i\| \leq 1$, $i = 1, \ldots, n$, and all $m \in \mathbb{N}$. We denote by $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ the space of all completely bounded completely compact multilinear maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ into $\mathcal{Y}$. A straightforward verification shows the following:

Remark 3.1. A completely bounded map $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is completely compact if and only if its linearisation $\tilde{\Phi} : \mathcal{X}_1 \otimes_h \cdots \otimes_h \mathcal{X}_n \to \mathcal{Y}$ is completely compact.

In view of this remark, we frequently identify the spaces $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ and $CC(\mathcal{X}_1 \otimes_h \cdots \otimes_h \mathcal{X}_n, \mathcal{Y})$. The next result is essentially due to Saar (see Lemmas 1 and 2 of [21]).
Proposition 3.2. (i) The space $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ is closed in $CB(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$.

(ii) Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be operator spaces. If $\Phi \in CC(\mathcal{E}, \mathcal{F})$ and $\Psi \in CB(\mathcal{F}, \mathcal{G})$ then $\Psi \circ \Phi \in CC(\mathcal{E}, \mathcal{G})$. If $\Phi \in CC(\mathcal{F}, \mathcal{G})$ and $\Psi \in CB(\mathcal{E}, \mathcal{F})$ then $\Phi \circ \Psi \in CC(\mathcal{E}, \mathcal{G})$.

Let $H_1, \ldots, H_n$ be Hilbert spaces. Recall the isometry
\[
\gamma : \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n) \to CB^a(\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}), \mathcal{B}(H_n, H_1))
\]
from Theorem 2.1. Let us identify a completely bounded map defined on $\mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1})$ with the corresponding completely bounded map defined on
\[
\mathcal{B}_h \overset{\text{def}}{=} \mathcal{B}(H_2, H_1) \otimes_h \cdots \otimes_h \mathcal{B}(H_n, H_{n-1}).
\]

For $u \in \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n)$ we let $\gamma_0(u)$ be the restriction of $\gamma(u)$ to
\[
\mathcal{K}_h \overset{\text{def}}{=} \mathcal{K}(H_2, H_1) \otimes_h \cdots \otimes_h \mathcal{K}(H_n, H_{n-1}).
\]

Proposition 3.3. The map $\gamma_0$ is an isometry from $\mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n)$ onto $CB(\mathcal{K}_h, \mathcal{B}(H_n, H_1))$.

Proof. Let $\Phi \in CB(\mathcal{K}_h, \mathcal{B}(H_n, H_1))$. Since $\Phi$ is completely bounded, its second dual
\[
\Phi^{**} : \mathcal{B}(H_2, H_1) \otimes_{\sigma_{eh}} \cdots \otimes_{\sigma_{eh}} \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)^{**}
\]
is completely bounded (here $\otimes_{\sigma_{eh}}$ denotes the normal Haagerup tensor product [5]). Let $Q : \mathcal{B}(H_n, H_1)^{**} \to \mathcal{B}(H_n, H_1)$ be the canonical projection. The multilinear map
\[
\tilde{\Phi} : \mathcal{B}(H_2, H_1) \times \cdots \times \mathcal{B}(H_n, H_{n-1}) \to \mathcal{B}(H_n, H_1)
\]
corresponding to $Q \circ \Phi^{**}$ is completely bounded and, by (5.22) of [5], weak* continuous in each variable. By Theorem 2.1, there exists an element $u \in \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n)$ such that $\tilde{\Phi} = \gamma(u)$. Hence $\gamma_0(u) = \gamma(u)|_{\mathcal{K}_h} = \tilde{\Phi}|_{\mathcal{K}_h} = \Phi$. Thus $\gamma_0$ is surjective.

Fix $u \in \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n)$. From the definition of $\gamma_0$ we have
\[
\|\gamma_0(u)\|_{cb} \leq \|\gamma(u)\|_{cb} = \|u\|_{eh}.
\]
On the other hand, the restrictions of the maps $Q \circ \gamma_0(u)^{**}$ and $\gamma(u)$ to $\mathcal{K}_h$ coincide, and since both maps are weak* continuous, $\gamma(u) = Q \circ \gamma_0(u)^{**}|_{\mathcal{B}_h}$. Hence,
\[
\|u\|_{eh} \leq \|Q \circ \gamma_0(u)^{**}\|_{cb} \leq \|\gamma_0(u)^{**}\|_{cb} = \|\gamma_0(u)\|_{cb}.
\]
Thus, $\gamma_0$ is an isometry. □
Theorem 3.4. Let $H_1, \ldots, H_n$ be Hilbert spaces. The image under $\gamma_0$ of the operator space $E \overset{\text{def}}{=} \mathcal{K}(H_1) \otimes_h (\mathcal{B}(H_2) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_{n-1})) \otimes_h \mathcal{K}(H_n)$ is $\mathcal{F} \overset{\text{def}}{=} \mathcal{C}(\mathcal{K}_h, \mathcal{K}(H_n, H_1))$.

Proof. We first establish the inclusion $\gamma_0(E) \subseteq \mathcal{F}$. If $\Phi = \gamma_0(u)$ where $u \in E$ then, by Proposition 3.3, $\Phi$ is the limit in the cb norm of maps of the form 

$$v = a \otimes B \otimes b \in \mathcal{K}(H_1) \otimes (\mathcal{B}(H_2) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_{n-1})) \otimes \mathcal{K}(H_n),$$

$a$ and $b$ have finite rank and $B$ is a finite matrix with entries in $\mathcal{B}(H_2) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_{n-1})$. But each such map has finite rank and hence is completely compact. Moreover, every operator in the image of $\gamma_0(v)$ has range contained in the range of $a$, which is finite dimensional. It follows that $\Phi$ takes compact values; it is completely compact by Proposition 3.2.

To see that $\mathcal{F} \subseteq \gamma_0(E)$, let $\Phi \in \mathcal{F}$. We will assume for technical simplicity that $H_1, \ldots, H_n$ are separable. Let $\{p_k\}_k$ (resp. $\{q_k\}_k$) be a sequence of projections of finite rank on $H_1$ (resp. $H_n$) such that $p_k \to I$ (resp. $q_k \to I$) in the strong operator topology. Let $\Psi_k : \mathcal{K}(H_n, H_1) \to \mathcal{K}(H_n, H_1)$ be the complete contraction given by $\Psi_k(x) = p_k x q_k$.

Let $\varepsilon > 0$. Since $\Phi$ is completely compact there exists a subspace $F \subseteq \mathcal{K}(H_n, H_1)$ of dimension $\ell < \infty$ such that $\text{dist}(\Phi^{(m)}(x), M_m(F)) < \varepsilon$ whenever $x \in M_m(\mathcal{K}_h)$ has norm at most one. Denote the restriction of $\Psi_k$ to $F$ by $\Psi_{k,F}$ and let $\iota$ be the inclusion map $\iota : F \hookrightarrow \mathcal{K}(H_n, H_1)$. By [6, Corollary 2.2.4], $\|\Psi_{k,F} - \iota\|_{cb} \leq \ell \|\Psi_{k,F} - \iota\|$. Since $F \subseteq \mathcal{K}(H_n, H_1)$, we have that $\Psi_{k,F}(x) \to x$ in norm for each $x \in F$. It follows easily that there exists $k_0$ such that $\|\Psi_{k,F} - \iota\|_{cb} < \varepsilon$ whenever $k \geq k_0$.

Let $x \in M_m(\mathcal{K}_h)$ be of norm at most one. Then there exists $y \in M_m(F)$ such that $\|\Phi^{(m)}(x) - y\| < \varepsilon$. Note that

$$\|y\| \leq \|\Phi^{(m)}(x) - y\| + \|\Phi^{(m)}(x)\| \leq \varepsilon + \|\Phi\|_{cb}.$$ 

Let $\Phi_k = \Psi_k \circ \Phi$. If $k \geq k_0$ then

$$\|\Phi_k^{(m)}(x) - x\| \leq \|\Phi_k^{(m)}(x) - \Psi_k^{(m)}(y)\| + \|\Psi_k^{(m)}(y) - y\| + \|y - \Phi^{(m)}(x)\|$$

$$= \|\Psi_k^{(m)}(\Phi^{(m)}(x) - y)\| + \|\Psi_{k,F} - \iota\|^{(m)}(y)\| + \|y - \Phi^{(m)}(x)\|$$

$$\leq 2\varepsilon + \varepsilon (\varepsilon + \|\Phi\|_{cb}).$$

This shows that $\|\Phi_k - \Phi\|_{cb} \to 0$.

By Proposition 3.2, it only remains to prove that each $\Phi_k$ lies in $\gamma_0(E)$. By Proposition 3.3, there exists an element

$$u = A_1 \otimes A_2 \otimes \cdots \otimes A_{n-1} \otimes A_n \in \mathcal{B}(H_1) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_n)$$

such that $\Phi_k = \gamma_0(u)$. Hence $\Phi_k \in \gamma_0(E)$. This completes the proof.
where $A_1 : H_1^\infty \to H_1$, $A_i : H_i^\infty \to H_i^\infty$, $i = 2, \ldots, n - 1$ and $A_n : H_n \to H_n^\infty$ are bounded operators, such that $\Phi = \gamma_0(u)$. Observe that $\Phi_k = \gamma_0(u_k)$ where $u_k = (p_k A_1) \otimes A_2 \otimes \cdots \otimes A_{n-1} \otimes (A_n q_k)$. It therefore suffices to show that $u_k \in \mathcal{E}$ for each $k$. Fix $k$ and let $p = p_k$, $q = q_k$. The operator $pA_1 : H_1^\infty \to H_1$ has finite dimensional range and is hence compact. For $i = 1, \ldots, n$, let $Q_{i,r} : H_i^\infty \to H_i^\infty$ be a projection with block matrix whose first $r$ diagonal entries are equal to the identity operator while the rest are zero. Then by compactness, $(pA_1)Q_{1,r} \to pA_1$ and $Q_{n,r}(A_n q) \to A_n q$ in norm as $r \to \infty$. Let $B = A_2 \otimes \cdots \otimes A_{n-1}, C_r = (pA_1)Q_{1,r} \otimes B \otimes Q_{n,r}(A_n q)$, $r \in \mathbb{N}$, and $C = (pA_1) \otimes B \otimes (A_n q)$. Then

$$
\|C_r - C\|_{\text{eh}} \leq \|C_r - (pA_1)Q_{1,r} \otimes B \otimes (A_n q)\|_{\text{eh}} + \|(pA_1)Q_{1,r} \otimes B \otimes (A_n q) - C\|_{\text{eh}} \\
\leq \|(pA_1)Q_{1,r}\| \cdot \|B\| \cdot \|Q_{n,r}(A_n q) - A_n q\| + \|(pA_1)Q_{1,r} - pA_1\| \cdot \|B\| \cdot \|A_n q\|.
$$

It follows that $\|C_r - C\|_{\text{eh}} \to 0$ as $r \to \infty$. Our claim will follow if we show that $C_r \in \mathcal{E}$. To this end, it suffices to show that if $A_1 = [a_1, \ldots, a_r, 0, \ldots]$ and $A_n = [b_1, \ldots, b_r, 0, \ldots]^t$, where $a_i, b_i$ are operators of finite rank, then $A_1 \otimes B \otimes A_n \in \mathcal{E}$. Let $A_1$ and $A_n$ be as stated and let $B' = (Q_{2,r} A_2) \otimes A_3 \otimes \cdots \otimes A_{n-2} \otimes (A_{n-1} Q_{n,r})$. Then $A_1 \otimes B \otimes A_n = A_1 \otimes B' \otimes A_{n+1}$ belongs to the algebraic tensor product $\mathcal{K}(H_1) \otimes (\mathcal{B}(H_2) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{B}(H_{n-1})) \otimes \mathcal{K}(H_n)$ and hence to $\mathcal{E} = \mathcal{K}(H_1) \otimes_h (\mathcal{B}(H_2) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{B}(H_{n-1})) \otimes_h \mathcal{K}(H_n)$.

**Remarks 3.5.** (i) It follows from Theorem 3.4 that if $\Phi : \mathcal{K}_n \to \mathcal{K}(H_n, H_1)$ is a mapping of finite rank whose image consists of finite rank operators then there exist finite rank projections $p$ and $q$ on $H_1$ and $H_n$, respectively, and $u \in (p\mathcal{K}(H_1)) \otimes_h (\mathcal{B}(H_2) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{B}(H_{n-1})) \otimes_h (\mathcal{K}(H_n) q)$ such that $\Phi = \gamma_0(u)$.

(ii) The identity $\mathcal{E}_1 \otimes_h (\mathcal{E}_2 \otimes_{\text{eh}} \mathcal{E}_3) = (\mathcal{E}_1 \otimes_{\text{eh}} \mathcal{E}_2) \otimes_{\text{eh}} \mathcal{E}_3$ does not hold in general; for an example, take $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{B}(H)$ and $\mathcal{E}_2 = \mathbb{C}$.

(iii) For every $\Phi \in CC(\mathcal{K}_n, \mathcal{K}(H_n, H_1))$ there exist $A_1 \in \mathcal{K}(H_i^h, H_1)$, $A_i \in \mathcal{B}(H_i^h, H_i^{j_i-1})$, $i = 2, \ldots, n - 1$ and $A_n \in \mathcal{K}(H_n, H_n^{j_n-1})$ such that

$$
\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \cdots (x_{n-1} \otimes 1)A_n,
$$

whenever $x_i \in \mathcal{K}(H_{i+1}, H_1)$, $i = 1, \ldots, n - 1$. Indeed, by Proposition 3.4, $\Phi(x_1 \otimes \cdots \otimes x_{n-1}) = A_1(x_1 \otimes 1)A_2 \cdots (x_{n-1} \otimes 1)A_n$ for some $A_1 \otimes A_2 \otimes \cdots \otimes A_n \in \mathcal{K}(H_1) \otimes_h (\mathcal{B}(H_2) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{B}(H_{n-1})) \otimes_h \mathcal{K}(H_n)$. Using an idea of Blecher and Smith [2, Theorem 3.1], we can choose $A_1 = [t_{ij}] \in J_i \subseteq \mathcal{M}_{j_1,1}(\mathcal{K}(H_1)) \subseteq \mathcal{B}(H_1^h, H_1)$ and $A_n = [s_{ij}] \in J_{n-1} \subseteq \mathcal{M}_{j_{n-1},1}(\mathcal{K}(H_n)) \subseteq \mathcal{B}(H_n, H_n^{j_n-1})$ such that
the sums $\sum_i s_i s_i^*$ and $\sum_j t_j t_j^*$ converge uniformly. Then $A_1$ is the norm limit of $A_1^F = [t_j^F]_{j \in J_1}$, where $F$ is a finite subset of $J_1$ and $t_j^F = t_j$ if $j \in F$ and $t_j^F = 0$ otherwise. Therefore $A_1 \in \mathcal{K}(H_1^{11}, H)$.

Similarly, $A_n \in \mathcal{K}(H_n, H_n^{1n-1})$.

In the case $n = 2$, Theorem 3.4 reduces to the following result which was established by Saar (Satz 6 of [21]) using the fact that every completely compact completely bounded map on $\mathcal{K}(H_1, H_2)$ is a linear combination of completely compact completely positive maps.

**Corollary 3.6.** A completely bounded map $\Phi : \mathcal{K}(H_1, H_2) \to \mathcal{K}(H_1, H_2)$ is completely compact if and only if there exist an index set $I$ and families $\{a_i\}_{i \in I} \subseteq \mathcal{K}(H_1)$ and $\{b_i\}_{i \in I} \subseteq \mathcal{K}(H_2)$ such that the series $\sum_{i \in I} b_i b_i^*$ and $\sum_{i \in I} a_i^* a_i$ converge uniformly and

$$\Phi(x) = \sum_{i \in I} b_i x a_i, \quad x \in \mathcal{K}(H_1, H_2).$$

We note in passing that Theorem 3.4 together with a result of Effros and Kishimoto [4] yields the following completely isometric identification:

**Corollary 3.7.** $CC(\mathcal{K}(H_2, H_1))^{**} \simeq (\mathcal{K}(H_1) \otimes \mathcal{K}(H_2))^{**} \simeq CB(\mathcal{B}(H_2, H_1))$.

Saar [21] constructed an example of a compact map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ which is not completely compact (see Section 7 where we give a detailed account of this construction). We note that a compact completely positive map $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ is automatically completely compact. Indeed, the Stinespring Theorem implies that there exist an index set $J$ and a row operator $A = [a_i]_{i \in J} \in \mathcal{B}(H^J, H)$ such that $\Phi(x) = \sum_{i \in J} a_i x a_i^*$, $x \in \mathcal{K}(H)$. The second dual $\Phi^{**} : \mathcal{B}(H) \to \mathcal{B}(H)$ of $\Phi$ is a compact map given by the same formula. A standard Banach space argument shows that $\Phi$ takes values in $\mathcal{K}(H)$, and hence $\Phi^{**}(I) \in \mathcal{K}(H)$. This means that $A A^* \in \mathcal{K}(H)$ and so $A \in \mathcal{K}(H^\infty, H)$ which easily implies that $\Phi$ is completely compact.

The previous paragraph shows that there exists a compact completely bounded map on $\mathcal{K}(H)$ which cannot be written as a linear combination of compact completely positive maps.

We finish this section with a modular version of Theorem 3.4. Given von Neumann algebras $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, we let $CC_{\mathcal{A}_1', \ldots, \mathcal{A}_n'}(K_h, \mathcal{K}(H_n, H_1))$ denote the space of $\mathcal{A}_1', \ldots, \mathcal{A}_n'$-modular completely compact maps from $K_h$ into $\mathcal{K}(H_n, H_1)$.

**Corollary 3.8.** Let $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$, be von Neumann algebras. Set $\mathcal{K}'(\mathcal{A}_i) = \mathcal{K}(H_i) \cap \mathcal{A}_i$, for $i = 1$ and $i = n$. Then

$$\gamma_0(\mathcal{K}'(\mathcal{A}_1) \otimes_h (\mathcal{A}_2 \otimes_h \cdots \otimes_h \mathcal{A}_{n-1}) \otimes_h \mathcal{K}'(\mathcal{A}_n)) = CC_{\mathcal{A}_1', \ldots, \mathcal{A}_n'}(K_h, \mathcal{K}(H_n, H_1)).$$
Proof. By Theorems 2.1 and 3.4, the image of \( \mathcal{K}'(A_1) \otimes_h (A_2 \otimes_{eh} \cdots \otimes_{eh} A_{n-1}) \otimes_h \mathcal{K}'(A_n) \) under \( \gamma_0 \) is contained in \( CC_{A_1',\ldots,A_n'}(K_h,K(H_n,H_1)) \). For the converse, fix an element \( \Phi \in CC_{A_1',\ldots,A_n'}(K_h,K(H_1,H_n)) \). By Theorem 3.4, there exists a unique \( u \in K(H_1) \otimes_h (\mathcal{B}(H_2) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_{n-1})) \otimes_h K(H_n) \) such that \( \gamma_0(u) = \Phi \). By Theorem 2.1, \( u \in A_1 \otimes_{eh} \cdots \otimes_{eh} A_n \). Lemma 2.2 now shows that \( u \in \mathcal{K}'(A_1) \otimes_h (A_2 \otimes_{eh} \cdots \otimes_{eh} A_{n-1}) \otimes_h \mathcal{K}'(A_n) \). \( \square \)

4 Complete boundedness of multipliers

Our aim in this section is to clarify the relationship between universal operator multipliers and completely bounded maps, extending results of [12]. We begin with an observation which will allow us to deal with the cases of even and odd numbers of variables in the same manner. We use the notation established in Section 2.

Proposition 4.1. Let \( A_1, \ldots, A_n \) be \( C^* \)-algebras and \( \varphi \in M(A_1, \ldots, A_n) \). Let \( \pi_i \) be a representation of \( A_i \) on a Hilbert space \( H_i \), \( i = 1, \ldots, n \), and \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \). The map \( S_{\pi(\varphi)} \) takes values in \( K(H_1,H_n) \) if \( n \) is odd, and in \( K(H_1^d,H_n) \) if \( n \) is even.

Proof. For even \( n \), this is immediate as observed in [12]. Let \( n \) be odd. Assume without loss of generality that \( A_i \subseteq \mathcal{B}(H_i) \) and \( \pi_i \) is the identity representation. We call an element \( \zeta \in \Gamma(H_1, \ldots, H_n) \) thoroughly elementary if

\[
\zeta = \eta^d_{1,2} \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}
\]

where all \( \eta^d_{j,j+1} = \eta^d_j \otimes \eta^d_{j+1} \) and \( \xi_{j-1,j} = \xi_{j-1} \otimes \xi_j \) are elementary tensors. The linear span of the thoroughly elementary tensors is dense in the completion of \( \Gamma(H_1, \ldots, H_n) \) in \( \| \cdot \|_{2,A} \). Moreover, the linear span of the elementary tensors \( \varphi = \varphi_1 \otimes \cdots \otimes \varphi_n \) is dense in \( \mathcal{B}(H_1) \otimes \cdots \otimes \mathcal{B}(H_n) \). By (7) and since \( S_{\varphi}(\zeta) \) is linear in both \( \varphi \) and \( \zeta \), it suffices to show that \( S_{\varphi}(\zeta) \) is compact when \( \varphi \) is an elementary tensor and \( \zeta \) is a thoroughly elementary tensor. However, in this case \( S_{\varphi}(\zeta) \) has rank at most 1, since for every \( \xi_1 \in H_1 \),

\[
S_{\varphi}(\zeta)\xi_1 = \varphi_n\theta(\xi_{n-1,n}) \cdots \varphi_2\theta(\eta^d_{1,2})\varphi_1\xi_1 = \prod_{j=1}^{n-1} (\varphi_j\xi_j, \eta_j) \varphi_n\xi_n. \quad \square
\]

We now establish some notation. Let \( A_1, \ldots, A_n \) be \( C^* \)-algebras and \( \varphi \in A_1 \otimes \cdots \otimes A_n \). Assume that \( n \) is even and let \( \pi_1, \ldots, \pi_n \) be representations of \( A_1, \ldots, A_n \) on \( H_1, \ldots, H_n \), respectively. Set \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \). Using the
natural identifications, we consider the map $S_{\pi(\varphi)} : \Gamma(H_1, \ldots, H_n) \to H_1 \otimes H_n$ as a map (denoted in the same way)

$$S_{\pi(\varphi)} : C_2(H_1^d, H_2) \ast \cdots \ast C_2(H_{n-1}^d, H_n) \to C_2(H_1^d, H_n).$$

We let

$$\Phi = \Phi_{\pi(\varphi)} : C_2(H_1^d, H_2) \ast \cdots \ast C_2(H_{n-1}^d, H_n) \to C_2(H_1^d, H_n)$$

be the map given on elementary tensors by

$$\Phi_{\pi(\varphi)}(T_{n-1} \otimes \cdots \otimes T_1) = S_{\pi(\varphi)}(T_1 \otimes \cdots \otimes T_{n-1}).$$

Note that if $\varphi \in M(A_1, \ldots, A_n)$ then $\Phi_{\pi(\varphi)}$ is bounded when the domain is equipped with the Haagerup norm and the range with the operator norm. In this case, $\Phi_{\pi(\varphi)}$ has a unique extension (which will be denoted in the same way)

$$\Phi_{\pi(\varphi)} : (\mathcal{K}(H_{n-1}^d, H_n) \otimes_h \cdots \otimes_h \mathcal{K}(H_1^d, H_2), \| \cdot \|_h) \to (\mathcal{K}(H_1^d, H_n), \| \cdot \|_{op}).$$

If $n$ is odd then the map $\Phi_{\pi(\varphi)}$ is defined in a similar way. The map $\Phi_{\pi(\varphi)}$ will be used extensively hereafter.

The main result of this section is Theorem 4.3 where we show the relation between the complete boundedness of the mappings $\Phi_{\pi(\varphi)}$ and the property of $\varphi$ of being a multiplier. We will need the following lemma.

**Lemma 4.2.** Let $A_i \subseteq B(H_i)$ be a $C^\ast$-algebra, $i = 1, \ldots, n$, $\varphi \in A_1 \otimes \cdots \otimes A_n$ and $\psi = (\text{id}^{(2)} \otimes \cdots \otimes \text{id}^{(k)})(\varphi)$. Suppose that $n$ is even. If $T_i \in M_k(C_2(H_i^{(1)}, H_{i+1}^{(1)}))$ for even $i$ and $T_i \in M_k(C_2(H_i^{(1)}, H_{i+1}^{(1)}))$ for odd $i$ then

$$\Phi_{\psi^{(1)}}(T_{n-1} \otimes \cdots \otimes T_1) = \Phi_{\varphi}(T_{n-1} \otimes \cdots \otimes T_1),$$

where we identify $M_k(C_2(H_i^{(1)}, H_{i+1}^{(1)}))$ with $C_2((H_i^{(1)})^{(k)}, (H_{i+1}^{(1)})^{(k)})$ for even $i$, and $M_k(C_2(H_i^{(1)}, H_{i+1}^{(1)}))$ with $C_2(H_i^{(k)}(1), (H_{i+1}^{(1)})^{(k)})$ for odd $i$. A similar statement holds for odd $n$.

**Proof.** To simplify notation, we give the proof for $n = 2$; the general proof is similar. If $\varphi = a_1 \otimes a_2$ is an elementary tensor then $\Phi_{\varphi}(T) = a_2 T a_1^\ast$ and it is easily checked that the statement holds. By linearity, it holds for each $\varphi \in A_1 \otimes A_2$. Assume that $\varphi \in A_1 \otimes A_2$ is arbitrary. Let $\{ \varphi_m \} \subseteq A_1 \otimes A_2$ be a sequence converging in the operator norm to $\varphi$ and $\psi_m = (\text{id}^{(k)} \otimes \text{id}^{(k)})(\varphi_m)$. By (7), $\Phi_{\psi_m}(T) \to \Phi_{\varphi}(T)$ in the operator norm, for all $T \in C_2(H_1^{(1)}, H_2)$. This implies that if $S \in M_k(C_2(H_1^{(1)}, H_2))$, then $\Phi_{\psi_m}^{(k)}(S) \to \Phi_{\varphi}^{(k)}(S)$ in the operator norm of $M_k(C_2(H_1^{(1)}, H_2))$. Since $\psi_m \to \psi$ in the operator norm, we conclude that $\Phi_{\psi_m}(S) \to \Phi_{\varphi}(S)$ in the operator norm of $C_2((H_1^{(1)})^{(k)}, H_2^{(k)})$. It follows that $\Phi_{\psi}(S) = \Phi_{\varphi}^{(k)}(S)$. \qed
Theorem 4.3. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras and $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. The following are equivalent:

(i) $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$;

(ii) if $\pi_i$ is a representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, and $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded;

(iii) there exist faithful representations $\pi_i$ of $\mathcal{A}_i$, $i = 1, \ldots, n$, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_{\pi(\varphi)}$ is completely bounded.

Moreover, if the above conditions hold and $\pi$ is as in (iii) then $\|\varphi\|_m = \|\Phi_{\pi(\varphi)}\|_{cb}$.

Proof. For technical simplicity we take $n = 3$.

(i) $\Rightarrow$ (ii) Let $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation, $i = 1, 2, 3$. Then $\pi(\varphi) \in M(\pi_1(\mathcal{A}_1), \pi(\mathcal{A}_2), \pi_3(\mathcal{A}_3))$; thus, it suffices to assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ are concrete C*-algebras and that $\pi_i$ is the identity representation, $i = 1, 2, 3$.

Fix $k \in \mathbb{N}$ and let $\psi = (\text{id}^{(k)} \otimes \text{id}^{(k)} \otimes \text{id}^{(k)})(\varphi)$. Since $\varphi \in M(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, the map

$$\Phi_{\psi} : \mathcal{K}(H_2^{d(k)}, H_3^{k}) \otimes \mathcal{K}(H_1^{(k)}, H_2^{d(k)}) \to \mathcal{K}(H_1^{(k)}, H_3^{k})$$

is bounded with norm not exceeding $\|\varphi\|_m$. By Lemma 4.2, $\|\Phi_{\psi}\| \leq \|\varphi\|_m$. Since this inequality holds for every $k \in \mathbb{N}$, the map $\Phi_{\psi}$ is completely bounded.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$ and that $\pi_i$ is the identity representation, $i = 1, 2, 3$. Let $\lambda$ be a cardinal number, $\rho_i = \text{id}^{(\lambda)}(\varphi)$ be the amplification of the identity representation of multiplicity $\lambda$, $\psi = (\rho_1 \otimes \rho_2 \otimes \rho_3)(\varphi)$, and $\tilde{H}_i = H_i^{\lambda}$, $i = 1, 2, 3$. Fix $\varepsilon > 0$ and $\zeta \in \Gamma(H_1, H_2, H_3)$. Let

$$\tilde{T} = \tilde{T}_2 \otimes \tilde{T}_1 \in C_2(\tilde{H}_2^{\lambda}, \tilde{H}_3^{\lambda}) \otimes C_2(\tilde{H}_1, \tilde{H}_2^{\lambda})$$

be the element canonically corresponding to $\zeta$. Then there exist $k \in \mathbb{N}$ and canonical projections $P_i$ from $\tilde{H}_i$ onto the direct sum of $k$ copies of $H_i$ such that if $T_0 = (P_3 \tilde{T}_2 (P_2 \otimes I)) \otimes ((P_2 \otimes I) \tilde{T}_1 P_1)$ and if $\zeta_0$ is the element of $\Gamma(H_1^{(k)}, H_2^{(k)}, H_3^{(k)})$ corresponding to $T_0$ then $\|\zeta - \zeta_0\|_{2,\lambda} \leq \varepsilon$.

Set $\psi_0 = (\text{id}^{(k)} \otimes \text{id}^{(k)} \otimes \text{id}^{(k)})(\varphi)$. Arguing as in Lemma 4.2, we see that
\[ \| \Phi_\psi(T_0) \|_{op} = \| \Phi_\psi(T_0) \|_{op}. \] Using (7) and Lemma 4.2 we obtain
\[ \| S_\psi(\zeta) \|_{op} \leq \| S_\psi(\zeta - \zeta_0) \|_{op} + \| S_\psi(\zeta_0) \|_{op} \leq \| S_\psi(\zeta - \zeta_0) \|_{op} + \| \Phi_\psi(T_0) \|_{op} \]
\[ \leq \| \psi \| \| \zeta - \zeta_0 \|_{op} + \| \Phi_\psi(T_0) \|_{op} \leq \varepsilon \| \| \varphi \| + \| \Phi_\varphi^{(k)}(T_0) \|_{op} \]
\[ \leq \varepsilon \| \| \varphi \| + \| \Phi_\varphi \|_{ch} \| T_0 \|_{h} \]
\[ \leq \varepsilon \| \varphi \| + \| \Phi_\varphi \|_{ch} \| P_3 T_2 (P_2^d \otimes I) \|_{op} \| (P_2^d \otimes I) T_1 P_1 \|_{op} \]
\[ \leq \varepsilon \| \varphi \| + \| \Phi_\varphi \|_{ch} \| T_2 \|_{op} \| T_1 \|_{op}. \]

It follows that \( \| \varphi \|_{id^{(\lambda)}id^{(\lambda)}id^{(\lambda)}} \leq \| \Phi_\varphi \|_{ch}. \)

Now let \( \rho_1, \rho_2, \rho_3 \) be arbitrary representations of \( A_1, A_2, A_3 \), respectively. Then there exists a cardinal number \( \lambda \) such that each of the representations \( \rho_i \) is approximately subordinate to the representation \( id^{(\lambda)} \) (see [26] for the definition of approximate subordination and [10, Theorem 5.1]). By Theorem 5.1 of [12], \( \| \varphi \|_{\rho_1, \rho_2, \rho_3} \leq \| \varphi \|_{id^{(\lambda)}id^{(\lambda)}id^{(\lambda)}} \); now the previous paragraph implies that \( \| \varphi \|_{\rho_1, \rho_2, \rho_3} \leq \| \Phi_\varphi \|_{ch} \). It follows that \( \varphi \in \mathcal{M}(A_1, A_2, A_3) \) and \( \| \varphi \|_m \leq \| \Phi_\varphi \|_{ch} \). As the reversed inequality was already established, we conclude that \( \| \varphi \|_m = \| \Phi_\varphi \|_{ch} \).

\section{The symbol of a universal multiplier}

Our aim in this section is to generalise the natural correspondence between a function \( \varphi \in \ell^\infty \otimes_{eh} \ell^\infty \) and the Schur multiplier \( S_\varphi \) on \( B(\ell^2(N)) \) given by \( S_\varphi((a_{ij})) = (\varphi(i, j)a_{ij}) \). To each universal operator multiplier we will associate an element of an extended Haagerup tensor product which we call its symbol. This will be used in the subsequent sections to identify certain classes of operator multipliers.

Recall that if \( A \) is a C*-algebra, its opposite C*-algebra \( A^o \) is defined to be the C*-algebra whose underlying set, norm, involution and linear structure coincide with those of \( A \) and whose multiplication \( \cdot \) is given by \( a \cdot b = ba \). If \( a \in A \) we denote by \( a^o \) the element of \( A^o \) corresponding to \( a \). If \( \pi : A \to B(H) \) is a representation of \( A \) then the map \( \pi^d : a^o \to \pi(a)^d \) from \( A^o \) into \( B(H^d) \) is a representation of \( A^o \). Clearly, \( \pi \) is faithful if and only if \( \pi^d \) is faithful. If \( \pi_i : A_i \to B(H_i) \) are faithful representations, \( i = 1, \ldots, n \) (\( n \) even), then by [5, Lemma 5.4] there exists a complete isometry \( \pi_n \otimes_{eh} \pi_{n-1} \otimes_{eh} \cdots \otimes_{eh} \pi_1^d \) from \( A_n \otimes_{eh} A_{n-1} \otimes_{eh} \cdots \otimes_{eh} A_1 \) into \( B(H_n) \otimes_{eh} B(H_{n-1}) \otimes_{eh} \cdots \otimes_{eh} B(H_1^d) \) which sends \( a_n \otimes a_{n-1} \otimes \cdots \otimes a_1^d \) to \( \pi_n(a_n) \otimes \pi_{n-1}(a_{n-1})^d \otimes \cdots \otimes \pi_1(a_1)^d \).

Henceforth, we will consistently write \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \) and
\[
\pi' = \begin{cases} 
\pi_n \otimes_{eh} \pi_{n-1}^d \otimes_{eh} \cdots \otimes_{eh} \pi_1^d & \text{if } n \text{ is even}, \\
\pi_n \otimes_{eh} \pi_{n-1}^d \otimes_{eh} \cdots \otimes_{eh} \pi_1 & \text{if } n \text{ is odd}.
\end{cases}
\]
Let \( n \in \mathbb{N}, A_1, \ldots, A_n \) be \( \mathbb{C}^* \)-algebras, \( \pi_i \) be a representation of \( A_i, i = 1, \ldots, n \), and \( \varphi \in M(A_1, \ldots, A_n) \). Assume that \( n \) is even. By Theorem 4.3, the map
\[
\Phi_{\pi(\varphi)} : \mathcal{K}(H_{n-1}^d, H_n) \otimes \cdots \otimes \mathcal{K}(H_1^d, H_2) \to \mathcal{K}(H_1^d, H_n)
\]
is completely bounded. By Proposition 3.3, there exists a unique element \( u_{\varphi}^\pi \in \mathcal{B}(H_n) \otimes \cdots \otimes \mathcal{B}(H_1^d) \) such that \( \gamma_0(u_{\varphi}^\pi) = \Phi_{\pi(\varphi)} \). For example, if each \( A_i \) is a concrete \( \mathbb{C}^* \)-algebra and \( a_i \in A_i, i = 1, \ldots, n \), then
\[
u_{a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n} = a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_2 \otimes a_1^o.
\]
If \( n \) is odd then we define \( u_{\varphi}^\pi \) similarly.

The main result of this section is the following.

**Theorem 5.1.** Let \( A_1, \ldots, A_n \) be \( \mathbb{C}^* \)-algebras and \( \varphi \in M(A_1, \ldots, A_n) \). There exists a unique element
\[
u_{\varphi} \in \begin{cases} A_n \otimes \cdots \otimes \mathcal{A}_{n-1}^o \otimes \cdots \otimes \mathcal{A}_2 \otimes \mathcal{A}_1^o & \text{if } n \text{ is even}, \\ A_n \otimes \cdots \otimes \mathcal{A}_{n-1}^o \otimes \cdots \otimes \mathcal{A}_2^o \otimes \mathcal{A}_1 & \text{if } n \text{ is odd} \end{cases}
\]
with the property that if \( \pi_i \) is a representation of \( A_i \), for \( i = 1, \ldots, n \) then
\[
u_{\varphi}^\pi = \pi'(\nu_{\varphi}).
\]
The map \( \varphi \to \nu_{\varphi} \) is linear and if \( a_i \in A_i, i = 1, \ldots, n \), then
\[
u_{a_1 \otimes \cdots \otimes a_n} = \begin{cases} a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1 & \text{if } n \text{ is even}, \\ a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o & \text{if } n \text{ is odd}. \end{cases}
\]
Moreover, \( \|\varphi\|_m = \|\nu_{\varphi}\|_{\text{eh}} \).

**Definition 5.2.** The element \( \nu_{\varphi} \) defined in Theorem 5.1 will be called the symbol of the universal multiplier \( \varphi \).

In order to prove the theorem we have to establish a number of auxiliary results.

If \( \omega \in \mathcal{B}(H)^* \) we let \( \bar{\omega} \in \mathcal{B}(H^d)^* \) be the functional given by \( \bar{\omega}(a^d) = \omega(a) \). Note that if \( \omega = \omega_{\xi, \eta} \) is the vector functional \( a \mapsto (a \xi, \eta) \) then \( \bar{\omega} = \omega_{\eta^d, \xi^d} \).

**Lemma 5.3.** Let \( A_i \subseteq \mathcal{B}(H_i) \) be a \( \mathbb{C}^* \)-algebra, \( \xi_i, \eta_i \in H_i \) and \( \omega_i = \omega_{\xi_i, \eta_i} \), \( i = 1, \ldots, n \). Suppose that \( \varphi \in M(A_1, \ldots, A_n) \). Then
\[
\langle \varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n \rangle = \begin{cases} \langle u_{\varphi}^id, \omega_n \otimes \bar{\omega}_{n-1} \otimes \cdots \otimes \bar{\omega}_1 \rangle & \text{n even}, \\ \langle u_{\varphi}^id, \omega_n \otimes \bar{\omega}_{n-1} \otimes \cdots \otimes \omega_1 \rangle & \text{n odd}. \end{cases}
\]
Lemma 5.4. Let \( u,v \) respectively and let \( \mathcal{H} \) be a Hilbert space. The proof is complete.

Proof. We only consider the case \( n \) is even since the proof for odd \( n \) is similar. Suppose that \( \varphi \) is an elementary tensor, say \( \varphi = a_1 \otimes \cdots \otimes a_n \). Then \( u^\varphi = a_n \otimes a^d_{n-1} \otimes \cdots \otimes a^d_1 \) and thus
\[
(\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n) = \prod_{i=1}^n (a_i \xi_i, \eta_i) = \langle u^\varphi, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle.
\]
By linearity, (9) holds for each \( \varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \).

Now let \( \varphi \) be an arbitrary element of \( M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \). By Theorem 2.3, there exists a net \( \{\varphi_\nu\} \subseteq \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \) and representations \( u^\varphi = A_n \otimes \cdots \otimes A_1 \) and \( u^\varphi_\nu = A^\nu_n \otimes \cdots \otimes A^\nu_1 \), where \( A^\nu_i \) are finite matrices with entries in \( \mathcal{A}_i \) if \( i \) is even and in \( \mathcal{A}^d_i \) if \( i \) is odd, such that \( \varphi_\nu \to \varphi \) semi-strongly, \( A^\nu_i \to A_i \) strongly and all norms \( \|A_i\|, \|A^\nu_i\| \) are bounded by a constant depending only on \( n \). As in (2), we have
\[
\langle u^\varphi_\nu, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle = \langle A_n, \omega_n \rangle \langle A_{n-1}, \omega_{n-1} \rangle \ldots \langle A_1, \tilde{\omega}_1 \rangle.
\]
Moreover, all norms \( \|\langle A^\nu_i, \omega_i \rangle\| \) (for even \( i \)) and \( \|\langle A^\nu_i, \tilde{\omega}_i \rangle\| \) (for odd \( i \)) are bounded by a constant depending only on \( n \), and the strong convergence of \( A^\nu_i \) to \( A_i \) implies that \( \langle A^\nu_i, \omega_i \rangle \) converges strongly to \( \langle A_i, \omega_i \rangle \). Indeed, it is easy to check that if \( \xi, \eta \in H, A \in M_1(\mathcal{B}(H)) \equiv \mathcal{B}(H \otimes \ell_2(I)) \) and \( \zeta \in \ell_2(I) \) for some index set \( I \) then
\[
\langle A, \omega_{\xi,\eta} \rangle \zeta^2 = (A(\xi \otimes \xi), \eta \otimes \langle A, \omega_{\xi,\eta} \rangle \zeta).
\]
This implies that \( \|\langle A_i - A^\nu_i, \omega_i \rangle \eta\| \leq C\|\langle A_i - A^\nu_i \rangle(\xi \otimes \eta)\| \) for some constant \( C > 0 \), and the strong convergence follows.

Since operator multiplication is jointly strongly continuous on bounded sets, it now follows from (10) that
\[
\langle u^\varphi_\nu, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle \to \langle u^\varphi, \omega_n \otimes \tilde{\omega}_{n-1} \otimes \cdots \otimes \tilde{\omega}_1 \rangle.
\]
On the other hand, since \( \varphi_\nu \to \varphi \) semi-strongly,
\[
(\varphi_\nu(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n) \to (\varphi(\xi_1 \otimes \cdots \otimes \xi_n), \eta_1 \otimes \cdots \otimes \eta_n).
\]
The proof is complete.

Lemma 5.4. Let \( H_i \) be a Hilbert space and \( \mathcal{E}_i \subseteq \mathcal{B}(H_i) \) be an operator space, \( i = 1, \ldots, n \). Suppose that \( \mathcal{X} \) and \( \mathcal{Y} \) are closed subspaces of \( \mathcal{E}_1 \) and \( \mathcal{E}_n \), respectively and let \( u, v \in \mathcal{E}_1 \otimes_h \cdots \otimes_h \mathcal{E}_n \). If
\[
R_\omega(u) \in \mathcal{X} \quad \text{and} \quad L_{\omega'}(v) \in \mathcal{Y}
\]
whenever \( \omega = \omega_2 \otimes \cdots \otimes \omega_n \) and \( \omega' = \omega'_1 \otimes \cdots \otimes \omega'_{n-1} \) where every \( \omega_i, \omega'_i \in \mathcal{B}(H_i) \) is a vector functional, then
\[
u \in \mathcal{E}_1 \otimes_h \mathcal{E}_2 \otimes_h \cdots \otimes_h \mathcal{E}_n \quad \text{and} \quad v \in \mathcal{E}_1 \otimes_h \mathcal{E}_2 \otimes_h \cdots \otimes_h \mathcal{E}_n \otimes \mathcal{Y}.
\]
Proof. Let $F_i$ be the span of the vector functionals on $B(H_i)$. By linearity, $R_\omega(u) \in X$ for each $\omega \in F_2 \odot \cdots \odot F_n$. Now suppose that
\[ \omega \in (B(H_2) \otimes_{eh} \cdots \otimes_{eh} B(H_n))_* = C_1(H_2) \otimes_h \cdots \otimes_h C_n(H_n). \]
There exists a sequence $(\omega_m) \subseteq F_2 \odot \cdots \odot F_n$ such that $\omega_m \to \omega$ in norm. Hence
\[ \|R_\omega(u) - R_{\omega_m}(u)\|_{B(H_i)} \leq \|\omega - \omega_m\| \|u\|_{eh} \to 0, \]
whence $R_\omega(u) = \lim_m R_{\omega_m}(u) \in X$. Spronk’s formula (5) now implies that $u \in X \otimes_{eh} E_2 \otimes_{eh} \cdots \otimes_{eh} E_n$. The assertion concerning $v$ has a similar proof. \qed

We will use slice maps defined on the minimal tensor product of several $C^*$-algebras as follows. Assume that $A_i \subseteq B(H_i)$ and $\omega_i \in B(H_i)^*$, $i = 1, \ldots, n$, and let $\varphi \in A_1 \otimes \cdots \otimes A_n$. If $1 \leq i_1 < \cdots < i_k \leq n$ and $\{\ell_1, \ldots, \ell_{n-k}\}$ is the complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$, let
\[ \Lambda_{\omega_{i_1}, \ldots, \omega_{i_k}} : A_1 \otimes \cdots \otimes A_n \to A_{\ell_1} \otimes \cdots \otimes A_{\ell_{n-k}} \]
be the unique norm continuous linear mapping given on elementary tensors by
\[ \Lambda_{\omega_{i_1}, \ldots, \omega_{i_k}}(a_1 \otimes \cdots \otimes a_n) = \omega_{i_1}(a_{i_1}) \cdots \omega_{i_k}(a_{i_k}) a_{\ell_1} \otimes \cdots \otimes a_{\ell_{n-k}}. \]

**Proposition 5.5.** Let $A_i \subseteq B(H_i)$, $i = 1, \ldots, n$, be $C^*$-algebras and let $\varphi \in M(A_1, \ldots, A_n)$. Then
\[ u_{\varphi}^{id} \in \begin{cases} A_n \otimes_{eh} A_{n-1}^d \otimes_{eh} \cdots \otimes_{eh} A_1^d & \text{if } n \text{ is even,} \\ A_n \otimes_{eh} A_{n-1}^d \otimes_{eh} \cdots \otimes_{eh} A_1 & \text{if } n \text{ is odd.} \end{cases} \]

**Proof.** We only consider the case $n = 3$. Let $u = u_{\varphi}^{id}$; by definition, $u \in B(H_3) \otimes_{eh} B(H_3) \otimes_{eh} B(H_1)$. Let $\xi, \eta \in H_i$ and $\omega_i = \omega_{\xi, \eta}$, $i = 1, 2, 3$. Then by (4) and Lemma 5.3,
\[ (R_{\omega_2 \otimes \omega_1}(u))_{\xi_3, \eta_3} = (R_{\omega_2 \otimes \omega_1}(u), \omega_3) = (u, \omega_3 \otimes \bar{\omega}_2 \otimes \omega_1) = (\varphi(\xi_1 \otimes \xi_2 \otimes \xi_3), \eta_1 \otimes \eta_2 \otimes \eta_3) = (\Lambda_{\omega_1, \omega_2}(\varphi))_{\xi_3, \eta_3}. \]
Thus
\[ R_{\omega_2 \otimes \omega_1}(u) = \Lambda_{\omega_1, \omega_2}(\varphi) \in A_3. \]
Lemma 5.4 now implies that $u \in A_3 \otimes_{eh} B(H_2^d) \otimes_{eh} B(H_1)$.

Let $w = R_{\omega_1}(u)$. By the previous paragraph, $w \in A_3 \otimes_{eh} B(H_2^d)$. By (4) and Lemma 5.3,
\[ (L_{\omega_3}(w))_{\eta_2^d, \xi_2^d} = (L_{\omega_3}(w), \bar{\omega}_2) = (R_{\omega_1}(u), \omega_3 \otimes \bar{\omega}_2) = (u, \omega_3 \otimes \bar{\omega}_2 \otimes \omega_1) = (\Lambda_{\omega_1, \omega_3}(\varphi))_{\xi_2, \eta_2} = (\Lambda_{\omega_1, \omega_3}(\varphi))_{\eta_2^d, \xi_2^d}. \]
Hence \( L_{\omega}(w) = \Lambda_{\omega_1, \omega_3}(\varphi)^d \in A_2^d \) and, by Lemma 5.4, \( w \in A_3 \otimes_{eh} A_2^d \). Applying this lemma again shows that \( u \in A_3 \otimes_{eh} A_2^d \otimes_{eh} B(H_1) \). Continuing in this fashion we see that \( u \in A_3 \otimes_{eh} A_2^d \otimes_{eh} A_1 \). □

**Lemma 5.6.** Let \( A_1, \ldots, A_n \) be \( C^* \)-algebras and let

\[
\rho_i : A_i \to \mathcal{B}(K_i), \quad \theta_i : \rho_i(A_i) \to \mathcal{B}(H_i)
\]

be representations, \( i = 1, \ldots, n \). Suppose that

(i) for any cardinal number \( \kappa \), the representations \( \theta_i^{(\kappa)} : \rho_i(A_i) \to \mathcal{B}(H_i^{\kappa}) \) are strongly continuous, and

(ii) whenever \( \varphi \in M(A_1, \ldots, A_n) \) and \( \{ \varphi_\nu \} \) is a net in \( A_1 \otimes \cdots \otimes A_n \) such that \( \rho(\varphi_\nu) \to \rho(\varphi) \) semi-weakly and \( \sup_\nu \| \varphi_\nu \|_m < \infty \) then \( \Phi_{\theta_i \rho_i(\varphi_\nu)} \to \Phi_{\theta_i \rho_i(\varphi)} \) pointwise weakly.

Then \( u_\varphi^{\theta_0} = \theta'(u_\varphi^\rho) \), for each \( \varphi \in M(A_1, \ldots, A_n) \).

**Proof.** If \( \varphi = a_1 \otimes \cdots \otimes a_n \) is an elementary tensor, then \( u_\varphi^\rho = \rho'(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1^\rho) \), so

\[
u_\varphi^{\theta_0} = (\theta \circ \rho)'(a_n \otimes a_{n-1} \otimes \cdots \otimes a_1^\rho) = \theta'(u_\varphi^\rho).
\]

By linearity, the claim also holds for \( \varphi \in A_1 \otimes \cdots \otimes A_n \).

If \( \varphi \in M(A_1, \ldots, A_n) \) is arbitrary then \( \rho(\varphi) \in M(\rho(A_1), \ldots, \rho(A_n)) \) and by Theorem 2.3 and Proposition 5.5, there exist a net \( \{ \varphi_\nu \} \subseteq A_1 \otimes \cdots \otimes A_n \) such that \( \rho(\varphi_\nu) \to \rho(\varphi) \) semi-weakly, a representation \( u_\varphi^\rho = A_n \otimes \cdots \otimes A_1 \), where \( A_i \in M_n(\rho_i(A_i)) \subseteq \mathcal{B}(K_i^{\kappa}) \) if \( i \) is even and \( A_i \in M_\kappa(\rho_i^d(A_i^{\kappa})) \subseteq \mathcal{B}(K_i^{\kappa}) \) if \( i \) is odd (\( \kappa \) being a suitable index set), whose operator matrix entries belong to \( \rho_i(A_i) \) if \( i \) is even and to \( \rho_i^d(A_i^{\kappa}) \) if \( i \) is odd, and representations \( u_\varphi^\rho = A_n^\nu \otimes \cdots \otimes A_1^\nu \) where the \( A_i^\nu \) are finite matrices with operator entries in \( \rho_i(A_i) \) if \( i \) is even and \( \rho_i^d(A_i^{\kappa}) \) if \( i \) is odd such that \( A_i^\nu \to A_i \) strongly and all norms \( \| A_i^\nu \|, \| A_i \| \) are bounded.

Now \( \theta'(u_\varphi^\rho) = \tilde{A}_n \otimes \cdots \otimes \tilde{A}_1 \) and \( \theta'(u_\varphi^\rho) = \tilde{A}_n^\nu \otimes \cdots \otimes \tilde{A}_1^\nu \) where \( \tilde{A}_i \) and \( \tilde{A}_i^\nu \) are the images of \( A_i \) and \( A_i^\nu \) under \( \theta_i^{(\kappa)} \) or \( (\theta_i^{(d)})^{(\kappa)} \) according to whether \( i \) is even or odd. By assumption (i),

\[
\gamma_0(\theta'(u_\varphi^\rho))(T_{n-1} \otimes \cdots \otimes T_1) \to \gamma_0(\theta'(u_\varphi^\rho))(T_{n-1} \otimes \cdots \otimes T_1) \quad (11)
\]

weakly for all \( T_{n-1} \in C_2(H_{n-1}^{d-1}, H_n), \ldots, T_1 \in C_2(H_1^{d-1}, H_2) \). On the other hand, assumption (ii) and the first paragraph of the proof show that

\[
\gamma_0(\theta'(u_\varphi^\rho)) = \gamma_0(u_\varphi^{\theta_0 \rho}) = \Phi_{\theta_0 \rho_i(\varphi_\nu)} \to \Phi_{\theta_0 \rho_i(\varphi)} = \gamma_0(u_\varphi^{\theta_0 \rho})
\]

pointwise weakly. Using (11) we conclude that \( \gamma_0(u_\varphi^{\theta_0 \rho}) = \gamma_0(\theta'(u_\varphi^\rho)) \); since \( \gamma_0 \) is injective we have that \( u_\varphi^{\theta_0 \rho} = \theta'(u_\varphi^\rho) \). □
Proof of Theorem 5.1. We will only consider the case \( n \) is even. Let \( \rho_i : \mathcal{A}_i \to \mathcal{B}(K_i) \) be the universal representation of \( \mathcal{A}_i \), \( i = 1, \ldots, n \). Set \( \rho = \rho_1 \otimes \cdots \otimes \rho_n \) and \( \rho^\prime = \rho_n \otimes \rho_{n-1}^1 \otimes \cdots \otimes \rho_1^1 \). By Proposition 5.5, \( u_\varphi^\rho \) lies in the image of \( \rho^\prime \); we define \( u_\varphi = (\rho^\prime)^{-1}(u_\varphi^\rho) \).

Let \( \kappa \) be a nonzero cardinal number and let \( \sigma_i = \rho_i(\kappa) \). If \( \theta_i = \text{id}_{\rho_i(\kappa)} = \sigma_i \circ \rho_i^{-1} \) then it follows from the proof of Proposition 6.2 of [12] that the hypotheses of Lemma 5.6 are satisfied, so
\[
(u_\varphi^{\sigma}) = \theta'(u_\varphi^\rho) = (\theta' \circ \rho')(u_\varphi) = \sigma'(u_\varphi).
\]

Now let \( \pi_i \) be an arbitrary representation of \( \mathcal{A}_i \). It is well known (see e.g. [25]) that \( \pi_i \) is unitarily equivalent to a subrepresentation of \( \sigma_i = \rho_i(\kappa) \) for some \( \kappa \). Hence there exist unitary operators \( v_i, i = 1, \ldots, n \) (acting between appropriate Hilbert spaces) and subspaces \( H_i \subset K_i^\kappa \), such that if \( \tau_i(x) = v_i x v_i^* \vert_{H_i} \) then \( \pi_i = \tau_i \circ \sigma_i \). Examining the proof of Proposition 6.2 of [12], we see that \( \tau \) satisfies the hypotheses of Lemma 5.6, so
\[
(u_\varphi^{\pi}) = \tau'(u_\varphi^\sigma) = (\tau \circ \sigma)'(u_\varphi) = \pi'(u_\varphi).
\]

The uniqueness of \( u_\varphi \) follows from the injectivity of \( \gamma_0 \). The linearity of the map \( \varphi \to u_\varphi \) and its values on elementary tensors are straightforward. The fact that \( \| \varphi \|_m = \| u_\varphi \|_{eh} \) follows from Proposition 3.3 and Theorem 4.3. \( \square \)

Remarks. (i) Let \( \mathcal{A}_i \subset \mathcal{B}(H_i), i = 1, \ldots, n \) be concrete C*-algebras of operators. Taking \( \pi_i \) to be the identity representation for \( i = 1, \ldots, n \) and writing \( \text{id} = \pi_1 \otimes \cdots \otimes \pi_n \) gives \( u_\varphi = u_{\text{id}} \) if we identify \( \mathcal{A}_i^o \) with \( \mathcal{A}_i^d \).

(ii) Theorem 5.1 implies that if \( \mathcal{A}_i, i = 1, \ldots, n, \) are concrete C*-algebras then the entries of the block operator matrices \( \mathcal{A}_i \) appearing in the representation of \( \varphi \) in Theorem 2.3 can be chosen from \( \mathcal{A}_i, i = 1, \ldots, n. \)

6 Completely compact multipliers

In this section we introduce the class of completely compact multipliers and characterise them within the class of all universal multipliers using the notion of the symbol introduced in Section 5. We will need the following lemma.

Lemma 6.1. Let \( \mathcal{A}_i \subset \mathcal{B}(H_i), i = 1, \ldots, n \) be a C*-algebra, \( a \in \mathcal{A}_1, b \in \mathcal{A}_n \) and \( \varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \). Let \( \psi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \) be given by
\[
\psi = \begin{cases} 
(a \otimes I \otimes \cdots \otimes I \otimes b)\varphi & \text{if } n \text{ is even,} \\
(a \otimes I \otimes \cdots \otimes I \otimes I)\varphi(b \otimes \cdots \otimes I \otimes b) & \text{if } n \text{ is odd.}
\end{cases}
\]
Then \( \psi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \) and
\[
\Phi_\psi(x) = \begin{cases} 
  b\Phi_\varphi(x) a^d & \text{if } n \text{ is even}, \\
  b\Phi_\varphi(x) a & \text{if } n \text{ is odd}.
\end{cases}
\tag{12}
\]

**Proof.** For technical simplicity, we will only consider the case \( n = 2 \). Let \( a_i \in \mathcal{A}_i, \ i = 1, 2, \) and \( \varphi = a_1 \otimes a_2 \). In this case \( \psi = (aa_1) \otimes (ba_2) \) so
\[
\Phi_\psi(T) = ba_2 T(aa_1)^d = ba_2 T a_1^d a^d = b \Phi_\varphi(T) a^d.
\]

By linearity, (12) holds whenever \( \varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2 \).

Assume that \( \varphi \in M(\mathcal{A}_1, \mathcal{A}_2) \) is arbitrary. Fix an operator \( T \in C_2(H_1^d, H_2) \).

By Theorem 2.3, there exists a net \( \{ \varphi_\nu \} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2 \) such that \( \varphi_\nu \to \varphi \) semi-weakly, \( \sup_\nu \| \varphi_\nu \|_m < \infty \) and \( \Phi_\varphi(T) \to \Phi_\varphi(T) \) weakly.

Let \( \psi_\nu = (a \otimes b) \varphi_\nu \); then \( \psi_\nu \to \psi \) semi-weakly. Clearly, \( \psi_\nu \in \mathcal{A}_1 \otimes \mathcal{A}_2 \); in particular \( \psi_\nu \in M(\mathcal{A}_1, \mathcal{A}_2) \). By the previous paragraph, \( \Phi_\psi_\nu(\cdot) = b\Phi_\varphi(\cdot)a^d \) and hence \( \Phi_\psi_\nu(T) \to b\Phi_\varphi(T) a^d \) weakly. If \( \varphi_\nu = B_1^\nu \otimes B_2^\nu \) then \( \psi_\nu = (aB_1^\nu) \otimes ((b \otimes I)B_2^\nu) \). It follows from Theorem 2.3 that \( \psi \in M(\mathcal{A}_1, \mathcal{A}_2) \) and that \( \Phi_\psi(T) \to \Phi_\psi(T) \) weakly. Thus \( \Phi_\psi(T) = b\Phi_\varphi(T) a^d \).

Given faithful representations \( \pi_1, \ldots, \pi_n \) of the C*-algebras \( \mathcal{A}_1, \ldots, \mathcal{A}_n \), respectively, we define
\[
\begin{align*}
M_{cc}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n) &= \{ \varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) : \Phi_\pi(\varphi) \text{ is completely compact} \} \\
M_{ff}^\pi(\mathcal{A}_1, \ldots, \mathcal{A}_n) &= \{ \varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) : \text{the range of } \Phi_\pi(\varphi) \\
&\quad \text{is a finite dimensional space of finite-rank operators} \}. 
\end{align*}
\]

**Theorem 6.2.** Let \( \mathcal{A}_i \subseteq \mathcal{B}(H_i) \) be a C*-algebra, \( i = 1, \ldots, n \), and \( \varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n) \). The following are equivalent:

(i) \( \varphi \in M_{cc}^\text{id}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \);

(ii) \( u_\varphi^\text{id} \in \begin{cases} (\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_h (A_{n-1}^d \otimes \cdots \otimes \mathcal{A}_2) \otimes_h (\mathcal{K}(H_1^d) \cap \mathcal{A}_1^d) & n \text{ even}, \\
(\mathcal{K}(H_n) \cap \mathcal{A}_n) \otimes_h (A_{n-1}^d \otimes \cdots \otimes \mathcal{A}_2^d) \otimes_h (\mathcal{K}(H_1) \cap \mathcal{A}_1) & n \text{ odd}; 
\end{cases} \)

(iii) there exists a net \( \{ \varphi_\alpha \} \subseteq M_{ff}^\text{id}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \) such that \( \| \varphi_\alpha - \varphi \|_m \to 0 \).

**Proof.** We will only consider the case \( n \) is even.

(i) \( \Rightarrow \) (ii) Theorem 3.4 implies that
\[
u_\varphi^\text{id} \in \mathcal{K}(H_n) \otimes_h (\mathcal{B}(H_{n-1}^d) \otimes \cdots \otimes \mathcal{B}(H_2)) \otimes_h \mathcal{K}(H_1^d)
\]

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Lemma 6.3. If compact multipliers.

\[ q \text{ finite rank operators. By Remark 3.5 (i) there exist finite rank projections } p \text{ and } \rho \text{ representation of } A \text{ while, by Proposition 5.5, } \pi M \text{ and } \rho \text{.} \]

The conclusion now follows from Lemma 2.2.

(ii)⇒(i) By Theorem 3.4, \( \Phi_\varphi = \gamma_0(u^\text{id}_\varphi) \) is completely compact.

(ii)⇒(iii) Let \( p = B(H_1) \) (resp. \( q = B(H_n) \)) be the projection onto the span of all ranges of operators in \( K(H_1) \cap A_i \) (resp. \( K(H_n) \cap A_i \)), and let \( \{p_\alpha\} \subseteq K(H_1) \cap A_i \) (resp. \( \{q_\alpha\} \subseteq K(H_n) \cap A_i \)) be a net of finite rank projections which tends strongly to \( p \) (resp. \( q \)). It is easy to see that \( \Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = q\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1)p, \) for all \( T_1 \in K(H_1^d, H_2), \ldots, T_{n-1} \in K(H_{n-1}^d, H_n). \)

Now consider the sets \( \varphi_\alpha(\cdot) = q_\alpha \Phi_\varphi(\cdot)p_{\alpha}^d; \) hence \( \varphi_\alpha \in M_{ff}^d(A_1, \ldots, A_n). \) We have already seen that \( \Phi_\varphi \) is completely compact, and it follows from the proof of Theorem 3.4 that \( \Phi_\varphi \to \Phi_\varphi \) in the cb norm. By Theorem 4.3, \( \| \varphi - \varphi_\alpha \|_m \to 0. \)

(iii)⇒(i) is immediate from Proposition 3.2 and Theorem 4.3 and the fact that finite rank maps are completely compact. \( \square \)

Now consider the sets

\[ M_{cc}(A_1, \ldots, A_n) = \bigcup_{\pi} M_{cc}^\pi(A_1, \ldots, A_n) \]

\[ M_{ff}(A_1, \ldots, A_n) = \bigcup_{\pi} M_{ff}^\pi(A_1, \ldots, A_n) \]

where the unions are taken over all \( \pi = \pi_1 \otimes \cdots \otimes \pi_n, \) each \( \pi_i \) being a faithful representation of \( A_i. \) We refer to the first of these as the set of completely compact multipliers.

Lemma 6.3. If \( \rho_i \) is the reduced atomic representation of \( A_i, \) \( i = 1, \ldots, n, \) and \( \rho = \rho_1 \otimes \cdots \otimes \rho_n \) then \( M_{ff}(A_1, \ldots, A_n) = M_{ff}^\rho(A_1, \ldots, A_n). \)

Proof. Again, we give the proof for the even case only. We must show that \( M_{ff}^\rho(A_1, \ldots, A_n) \subseteq M_{ff}^\pi(A_1, \ldots, A_n) \) whenever \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \) where each \( \pi_i \) is a faithful representation of \( A_i. \) Without loss of generality, we may assume that each \( \pi_i \) is the identity representation of \( A_i \subseteq B(H_i). \) Let \( \varphi \in M_{ff}^\rho(A_1, \ldots, A_n) \) so that the range of \( \Phi_\varphi \) is finite dimensional and consists of finite rank operators. By Remark 3.5 (i) there exist finite rank projections \( p \) and \( q \) on \( H_1^d \) and \( H_n, \) respectively, such that \( u^\text{id}_\varphi \) lies in the intersection of

\[ (qK(H_n)) \otimes_h (B(H_{n-1}) \otimes \cdots \otimes \otimes_h B(H_2)) \otimes_h (K(H_1^d)p) \]

and \( A_n \otimes \otimes_h A_1. \) By Lemma 2.2, \( u^\text{id}_\varphi \) lies in

\[ (qK(H_n) \cap A_n) \otimes_h (B(H_{n-1}) \otimes \cdots \otimes \otimes_h B(H_2)) \otimes_h (K(H_1^d)p \cap A_1). \]
Hence there exists a representation \( u^\text{id}_\varphi = A_n \odot \cdots \odot A_1 \) of \( u^\text{id}_\varphi \) such that \( A_n = qA_n \) and \( A_1 = A_1 p \). Suppose that \( A_n = [b_1, b_2, \ldots] \), where \( b_j \in A_n \) for each \( j \), and let \( q_j \) be the orthogonal projection onto the range of \( b_j \). Setting \( Q_m = \bigvee_{j=1}^m q_j \) we see that \( \{Q_m\} \) is an increasing sequence of projections in \( A_n \) dominated by \( q \). It follows that \( \bigvee_{m=1}^\infty Q_m \in A_n \). We may thus assume that \( q \in A_n \). Similarly, we may assume that \( p \in A_1^d \). Now

\[
\rho'(u_\varphi) = (\rho_n(q)\rho_n(A_n)) \odot \cdots \odot (\rho_1(A_1)\rho_1(p)).
\]

By \([29]\), \( \rho_n(q) \) and \( \rho_1(p) \) have finite rank. By Lemma 6.1, \( \varphi \in M^\text{en}_1(A_1, \ldots, A_n) \).

\[\square\]

We are now ready to prove the main result of this section.

**Theorem 6.4.** Let \( A_1, \ldots, A_n \) be \( C^* \)-algebras and \( \varphi \in M(A_1, \ldots, A_n) \). The following are equivalent:

(i) \( \varphi \in M_\text{cc}(A_1, \ldots, A_n) \);

(ii) \( u_\varphi \in \begin{cases} K(A_n) \otimes_h (A_n^{a-1} \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} A_2) \otimes_h K(A_1) & \text{if } n \text{ is even,} \\ K(A_n) \otimes_h (A_n^{a-1} \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} A_2) \otimes_h K(A_1) & \text{if } n \text{ is odd}; \end{cases} \)

(iii) there exists a net \( \{\varphi_\alpha\} \subseteq M^\text{ff}(A_1, \ldots, A_n) \) such that \( \|\varphi_\alpha - \varphi\|_m \to 0 \).

**Proof.** We will only consider the case \( n \) is even.

(i)\(\Rightarrow\)(ii) Choose \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \) such that \( \varphi \in M^\text{en}_1(A_1, \ldots, A_n) \); after identifying \( A_i \) with its image under \( \pi_i \), we may assume that each \( \pi_i \) is the identity representation of a concrete \( C^* \)-algebra \( A_i \subseteq B(H_i) \). By Theorem 6.2, \( u^\text{id}_\varphi \) lies in

\[
(K(H_n) \cap A_n) \otimes_h (A_n^{a-1} \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} A_2) \otimes_h (K(H_1) \cap A_1^d).
\]

The conclusion follows from the fact that \( K(H_i) \cap A_i \subseteq K(A_i) \) for \( i = 1, n \).

(ii)\(\Rightarrow\)(i) Let \( \rho_i \) be the reduced atomic representation \( A_i \to B(H_i) \) for \( i = 1, \ldots, n \). Since \( \rho_\ell \) is an isometry, \( u^\ell_\varphi = \rho'(u_\varphi) \) lies in

\[
\rho_\ell(K(A_n)) \otimes_h (\rho^{a-1}_{n-1}(A_n^{a-1}) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \rho_2(A_2)) \otimes_h \rho_1^d(K(A_1^d)).
\]

By Theorem 7.5 of \([28]\), \( K(H_i) \cap \rho_i(A_i) = \rho_i(K(A_i)) \). By Theorem 6.2, \( \varphi \in M^\text{en}_1(A_1, \ldots, A_n) \).

(i)\(\Rightarrow\)(iii) is immediate from Theorem 6.2.

(iii)\(\Rightarrow\)(i) Suppose that \( \{\varphi_\alpha\} \subseteq M^\text{ff}(A_1, \ldots, A_n) \) is a net such that \( \|\varphi_\alpha - \varphi\|_m \to 0 \). By Lemma 6.3, \( \{\varphi_\alpha\} \subseteq M^\text{en}_1(A_1, \ldots, A_n) \), where \( \rho \) is the tensor product of the reduced atomic representations of \( A_1, \ldots, A_n \). By Theorem 6.2, \( \varphi \in M^\text{en}_1(A_1, \ldots, A_n) \subseteq M^\text{en}(A_1, \ldots, A_n) \).

\[\square\]
In the next theorem we show that in the case $n = 2$ one more equivalent condition can be added to those of Theorem 6.4.

**Theorem 6.5.** Let $A$ and $B$ be $C^*$-algebras and $\varphi \in M(A, B)$. The following are equivalent:

(i) $\varphi \in M_{cc}(A, B)$;

(ii) there is a sequence $\{\varphi_k\}_{k=1}^{\infty} \subseteq K(A) \otimes K(B)$ such that $\|\varphi_k - \varphi\|_m \to 0$ as $k \to \infty$.

**Proof.** (i)$\Rightarrow$(ii) By Theorem 6.4, $u_\varphi \in K(B) \otimes hK(A)$; thus $u_\varphi = \sum_{i=1}^{\infty} b_i \otimes a_i^o$ where $a_i^o \in K(A)$, $b_i \in \mathcal{K}(B)$, $i \in \mathbb{N}$, and the series $\sum_{i=1}^{\infty} b_i b_i^*$ and $\sum_{i=1}^{\infty} a_i^o a_i^o$ converge in norm. Let $\varphi_k = \sum_{i=1}^{k} a_i \otimes b_i \in A \otimes B$. By Theorem 5.1, $u_{\varphi_k} = \sum_{i=1}^{k} b_i \otimes a_i^o$ and $\|\varphi - \varphi_k\|_m = \|u_\varphi - u_{\varphi_k}\|_{eh} \to 0$ as $k \to \infty$.

(ii)$\Rightarrow$(i) Assume that $A$ and $B$ are represented concretely. It is clear that $\varphi_k \in M_{cc}(A, B)$. By Theorem 4.3, $\|\Phi_{id(\varphi)} - \Phi_{id(\varphi_k)}\|_{cb} = \|\varphi - \varphi_k\|_m$. Proposition 3.2 now implies that $\Phi_{id(\varphi)}$ is completely compact, in other words, $\varphi \in M_{cc}(A, B)$. \hfill $\square$

### 7 Compact multipliers

In this section we compare the set of completely compact multipliers with that of compact multipliers. We exhibit sufficient conditions for these two sets of multipliers to coincide, and show that in general they are distinct. Finally, we address the question of when any universal multiplier in the minimal tensor product of two $C^*$-algebras is automatically compact. We show that this happens precisely when one of the $C^*$-algebras is finite dimensional while the other coincides with the set of its compact elements.

#### 7.1 Automatic complete compactness

We will need the following result complementing Theorem 3.4. Notation is as in Section 3.

**Proposition 7.1.** If $\Phi : \mathcal{K}_h \to \mathcal{K}(H_n, H_1)$ is a compact completely bounded map then $\gamma_0^{-1}(\Phi) \in \mathcal{K}(H_1) \otimes_{eh} B(H_2) \otimes_{eh} \cdots \otimes_{eh} B(H_{n-1}) \otimes_{eh} \mathcal{K}(H_n)$.

**Proof.** Fix $\varepsilon > 0$. By compactness, there exist $y_1, \ldots, y_\ell \in \mathcal{K}(H_n, H_1)$ such that $\min_{1 \leq i \leq \ell} \|\Phi(x) - y_i\| < \varepsilon$ for each $x \in \mathcal{K}_h$ with $\|x\| \leq 1$.

Let $\{p_\alpha\}$ (resp. $\{q_\alpha\}$) be a net of finite rank projections finite rank in $\mathcal{K}(H_1)$ (resp. $\mathcal{K}(H_n)$) such that $p_\alpha \to I$ (resp. $q_\alpha \to I$) strongly and let $\Phi_\alpha : \mathcal{K}_h \to \mathcal{K}(H_n, H_1)$ be the map given by $\Phi_\alpha(x) = p_\alpha \Phi(x) q_\alpha$. Let $u =$
The converse of Proposition 7.1 does not hold, even for Remark. $\gamma M$ is compact. We call the elements of $\{p_o y_i a - y_i\}$ and the range of $\sum_{i=1}^p a_i y_i a - y_i$ so $B$ functional. Lemma 5.3 and a straightforward verification shows that if $u$ and $\omega$ and $u \xi_1 A_1 \otimes \cdots \otimes A_n$ and $\omega = \omega_2 \otimes \cdots \otimes \omega_n$ then

$$\langle R_\omega(v)\xi_1, \eta_1 \rangle = \langle v, \omega_1 \otimes \cdots \otimes \omega_n \rangle = \langle \gamma_0(v)(\zeta)\xi_n, \eta_1 \rangle,$$

(13)

where

$$\zeta = \left((\eta_2^* \otimes \xi_1) \otimes (\eta_3^* \otimes \xi_2) \otimes \cdots \otimes (\eta_{n-1}^* \otimes \xi_{n-2}) \otimes (\eta_n^* \otimes \xi_{n-1})\right) \in K_h$$

is an elementary tensor whose components are rank one operators.

Since $\gamma_0(u_o) \to \gamma_0(u)$ in norm, (13) implies that $R_\omega(u_o) \to R_\omega(u)$ in the operator norm of $K(H_1)$. Since $R_\omega(u_o) \in K(H_1)$, we obtain $R_\omega(u) \in K(H_1)$. By Lemma 5.4, $u \in K(H_1) \otimes K(H_2) \otimes K(H_n)$ and $\omega = \omega_2 \otimes \cdots \otimes \omega_n \in B(K_n)$, the conclusion now follows.

**Remark.** The converse of Proposition 7.1 does not hold, even for $n = 2$. Indeed, let $\{p_i\}_{i=1}^\infty$ be a family of pairwise orthogonal rank one projections on a Hilbert space $H$ and let $u = \sum_{i=1}^\infty p_i \otimes p_i$. Then $u \in K(H) \otimes K(H)$ and the range of $\gamma_0(u)$ consists of compact operators, but $\gamma_0(u)(p_i) = p_i$ for each $i$, so $\gamma_0(u)$ is not compact.

Given C*-algebras $A_1, \ldots, A_n$, we let $M_c(A_1, \ldots, A_n)$ be the collection of all $\varphi \in M(A_1, \ldots, A_n)$ for which there exist faithful representations $\pi_1, \ldots, \pi_n$ of $A_1, \ldots, A_n$, respectively, such that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then the map $\Phi_\pi(\varphi)$ is compact. We call the elements of $M_c(A_1, \ldots, A_n)$ compact multipliers.

As a consequence of the previous result we obtain the following fact.

**Proposition 7.2.** Let $A_1, \ldots, A_n$ be C*-algebras and let $\varphi \in M(A_1, \ldots, A_n)$. Then

$$u_\varphi \in \begin{cases} K(A_n) \otimes A_{n-1} \otimes \cdots \otimes A_2 \otimes K(A_1) & \text{if } n \text{ is even}, \\ K(A_n) \otimes A_{n-1} \otimes \cdots \otimes A_2 \otimes K(A_1) & \text{if } n \text{ is odd}. \end{cases}$$
Proof. We only consider the case \( n \) is even. We may assume that \( A_i \subseteq \mathcal{B}(H_i) \) is a concrete non-degenerate \( C^* \)-algebra, \( i = 1, \ldots, n \), and that \( \Phi_\varphi \) is compact. By Propositions 5.5 and 7.1, \( v^\varphi_i \) belongs to
\[
(\mathcal{K}(H_n) \otimes_{eh} \mathcal{B}(H_{n-1}^d) \otimes_{eh} \cdots \otimes_{eh} \mathcal{B}(H_2) \otimes_{eh} \mathcal{K}(H_1^d)) \cap (A_n \otimes_{eh} \cdots \otimes_{eh} A_1^d).
\]
Since \( \mathcal{K}(H_n) \cap A_n \subseteq \mathcal{K}(A_n) \) and \( \mathcal{K}(H_1^d) \cap A_1^d \subseteq \mathcal{K}(A_1^d) \), an application of (5) shows that \( v^\varphi_i \in \mathcal{K}(A_n) \otimes_{eh} A_{n-1}^d \otimes_{eh} \cdots \otimes_{eh} A_2 \otimes_{eh} \mathcal{K}(A_1^d) \).

If \( \{A_j\}_{j \in J} \) is a family of \( C^* \)-algebras, we will denote by \( \bigoplus_{j \in J}^c A_j \) and \( \bigoplus_{j \in J}^\ell A_j \) their \( c_0 \)- and \( \ell^\infty \)-direct sums, respectively.

**Theorem 7.3.** Let \( A_1, \ldots, A_n \) be \( C^* \)-algebras, and suppose that \( \mathcal{K}(A_1) \) is isomorphic to \( \bigoplus_{j \in J}^c M_m \), and \( \mathcal{K}(A_3) \) is isomorphic to \( \bigoplus_{j \in J}^\ell M_n \), where \( J \) is some index set and \( \sup_{j \in J} m_j \) and \( \sup_{j \in J} n_j \) are finite. Then
\[
M_c(A_1, \ldots, A_n) = M_{c0}(A_1, \ldots, A_n).
\]

*Proof.* We give the proof for \( n = 3 \); the case of a general \( n \) is similar. Let \( m = \sup\{m_j, n_j : j \in J\} \). By hypothesis, \( \mathcal{K}(A_1) \) and \( \mathcal{K}(A_3) \) may both be embedded in the \( C^* \)-algebra \( \mathcal{C} \equiv \bigoplus_{j \in J}^c M_m \) for some \( m \in \mathbb{N} \); without loss of generality, we may assume that this embedding is an inclusion and that \( A_i \) is represented faithfully on some Hilbert space \( H_i \) such that \( H_1 \) and \( H_3 \) both contain the Hilbert space
\[
H = \bigoplus_{j \in J} \mathbb{C}^m.
\]
Given \( \varphi \in M_c(A_1, A_2, A_3) \), Proposition 7.2 implies that the symbol \( u_\varphi \) of \( \varphi \) can be written in the form \( u_\varphi = A_3 \circ A_2 \circ A_1 \), where the entries of \( A_3 \) and \( A_1 \) belong to \( \mathcal{C} \). Let \( \{e_{ij} : i, j = 1, \ldots, m\} \) be the canonical matrix unit system of \( M_m \) and let \( P_k = \bigoplus_{j \in J}^c e_{kk} \in \bigoplus_{j \in J}^{\ell \infty} M_m \), \( k = 1, \ldots, m \).

For \( k, \ell, s, t = 1, \ldots, m \), we set \( A_3^{k,\ell} = P_k A_3 (P_\ell \otimes I) \) and \( A_1^{s,t} = (P_s \otimes I) A_1 P_t \) and define
\[
u_{k,\ell,s,t} = A_3^{k,\ell} \circ A_2 \circ A_1^{s,t} \quad \text{and} \quad \Phi_{k,\ell,s,t} = \gamma_0(\nu_{k,\ell,s,t}).
\]
Then \( \gamma_0(u_\varphi) = \Phi = \sum_{k,\ell,s,t} \Phi_{k,\ell,s,t} \) so it suffices to show that each of the maps \( \Phi_{k,\ell,s,t} \) is completely compact. Now
\[
\Phi_{k,\ell,s,t}(T_2 \otimes T_1) = P_k \Phi(P_\ell T_2 \otimes T_1 P_s) P_t = A_3^{k,\ell} ((P_\ell T_2) \otimes I) A_2 ((T_1 P_s) \otimes I) A_1^{s,t}.
\]
Thus, \( \Phi_{k,\ell,s,t} \) can be considered as a completely bounded multilinear map from \( \mathcal{K}(H_2^d) \times \mathcal{K}(P_\ell H, H_2^d) \) into \( \mathcal{K}(P_\ell H, P_k H) \). Since \( \Phi \) is compact, it follows that \( \Phi_{k,\ell,s,t} \) is compact.

Take a basis \( \{e_i^j : i = 1, \ldots, m, j \in J\} \) of \( H = \bigoplus_{j \in J} \mathbb{C}^m \), where for each \( j \in J \), the standard basis of the \( j \)-th copy of \( \mathbb{C}^m \) is \( \{e_i^j : i = 1, \ldots, m\} \). Let
\( U_k : P_k H \to P_1 H \) be the unitary operator defined by \( U_k e^j_k = e^j_1 \). Consider the mapping \( \Psi : \mathcal{K}(H^2_1, P_1 H) \times \mathcal{K}(P_1 H, H^2_2) \to \mathcal{K}(P_1 H, P_1 H) \) given by
\[
\Psi(T_2 \otimes T_1) = U_k \Phi_k,\ell,s,t(U_\ell T_2 \otimes T_1 U_s) U_t.
\]
To show that \( \Phi_k,\ell,s,t \) is completely compact it suffices to show that \( \Psi \) is. Let \( C_0 = P_1^c P_1 \); then \( C_0 \) is isomorphic to \( c_0 \) and its commutant \( C'_0 \) has a cyclic vector. Moreover, \( \Psi \) is a \( C'_0 \)-modular multilinear map. Let \( \{ p_\alpha \} \) be a net of finite dimensional projections belonging to \( C_0 \), such that \( s-lim p_\alpha = I_{P_1 H} \). Consider the completely bounded multilinear maps \( \Psi_\alpha(x) = p_\alpha \Psi(x) p_\alpha \). Since the ranges of the \( p_\alpha \)'s are finite dimensional, \( \Psi_\alpha \) has finite rank and is hence completely compact. Since \( \Psi \) is compact, we may argue as in the proof of Proposition 7.1 to show that \( \| \Psi_\alpha - \Psi \| \to 0 \). Now the maps \( \Psi \) and \( \Psi_\alpha \) are \( C'_0 \)-modular and \( C'_0 \) has a cyclic vector, so by the generalisation [12, Lemma 3.3] of a result of Smith [23, Theorem 2.1],
\[
\| \Psi_\alpha - \Psi \|_{cb} = \| \Psi_\alpha - \Psi \| \to 0.
\]
Proposition 3.2 now implies that \( \Psi \) is completely compact. \( \square \)

The following corollary extends Proposition 5 of [11] to the case of multidimensional Schur multipliers. We recall from [12] that with every \( \varphi \in \ell_\infty(X_1 \times \cdots \times X_n) \) we associate a mapping \( S_\varphi : \ell_2(X_1 \times X_2) \otimes \cdots \otimes \ell_2(X_{n-1} \times X_n) \to \ell_2(X_1 \times X_n) \) which extends the usual Schur multiplication in the case \( n = 2 \). We equip the domain of \( S_\varphi \) with the Haagerup norm where each of the terms is given its operator space structure arising from its embedding into the corresponding space of Hilbert-Schmidt operators endowed with the operator norm.

**Corollary 7.4.** Let \( X_1, \ldots, X_n \) be sets and \( \varphi \in \ell_\infty(X_1 \times \cdots \times X_n) \). The following are equivalent:

(i) \( S_\varphi \) is compact;

(ii) \( \varphi \in c_0(X_1) \otimes_h (\ell_\infty(X_2) \otimes_{eh} \cdots \otimes_{eh} \ell_\infty(X_{n-1})) \otimes_h c_0(X_n) \).

**Proof.** Assume first that \( S_\varphi \) is compact. For notational simplicity we assume that \( X_i = \mathbb{N}, i = 1, \ldots, n \). It follows from [12, Section 3] that the map \( S_\varphi \) induces a completely bounded compact map
\[
\hat{S}_\varphi : \mathcal{C}_2 \times \cdots \times \mathcal{C}_2 \to \mathcal{C}_2
\]
defined by \( \hat{S}_\varphi(T_{f_1}, \ldots, T_{f_n}) = T_{S_\varphi(f_1, \ldots, f_n)} \), where \( T_f \) is the Hilbert-Schmidt operator with kernel \( f \). By Proposition 7.1, \( \varphi = \gamma^{-1}_0(\hat{S}_\varphi) \in \mathcal{K}(\ell_2) \otimes_{eh} B(\ell_2) \otimes_{eh} \cdots \otimes_{eh} B(\ell_2) \otimes_{eh} \mathcal{K}(\ell_2) \). Since \( S_\varphi \) is bounded, \( \varphi \) is a Schur multiplier and by [12,
Theorem 3.4, \( \varphi \in \ell_\infty \otimes_{eh} \cdots \otimes_{eh} \ell_\infty \). Hence \( \varphi \in c_0 \otimes_{eh} \ell_\infty \otimes_{eh} \cdots \otimes_{eh} \ell_\infty \otimes_{eh} c_0 \).

We may now argue as in the last paragraph of the preceding proof to show that \( \varphi \in c_0 \otimes_{eh} (\ell_\infty \otimes_{eh} \cdots \otimes_{eh} \ell_\infty) \otimes_{eh} c_0 \).

Our next aim is to show that if both \( K(A_1) \) and \( K(A_n) \) contain full matrix algebras of arbitrarily large sizes then the completely compact multipliers form a proper subset of the set of compact multipliers. Saar [21] has provided an example of a compact completely bounded map on \( K(H) \) (where \( H \) is a separable Hilbert space) which is not completely compact. It turns out that Saar’s example also shows that the sets of compact and completely compact multipliers are distinct, in the case under consideration.

We will need some preliminary results. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras. Recall that a linear map \( \Phi : \mathcal{A} \to \mathcal{B} \) is called symmetric (or hermitian) if \( \Phi = \Phi^* \) where \( \Phi^* : \mathcal{A} \to \mathcal{B} \) is the map given by \( \Phi^*(a) = (\Phi(a^*))^* \). By \( S_A \) we denote the unit ball of \( \mathcal{A} \) and set \( S^h_A = \{ a \in S \mathcal{A} : a = a^* \} \). The following lemma is a special case of Satz 6 of [21]. We include a direct proof for the convenience of the reader.

**Lemma 7.5.** Let \( H \) be a Hilbert space. If \( \Phi : \mathcal{A} \to K(H) \) is a symmetric, completely compact linear map with \( \| \Phi \|_{cb} \leq 1 \), then there exists a positive operator \( c \in K(H) \) such that \( \Phi(n)(a) \leq c \otimes 1_n \) for all \( a \in S^h_{M_n(\mathcal{A})} \) and all \( n \in \mathbb{N} \). Moreover, \( c \) can be chosen to have norm arbitrarily close to one.

**Proof.** We first show that for a given \( \varepsilon > 0 \) there exists a finite rank projection \( p \) on \( H \) such that

\[
\| \Phi(n)(a) - (p \otimes 1_n)\Phi(n)(a)(p \otimes 1_n) \| \leq \varepsilon \quad \text{for any} \quad a \in S_{M_n(\mathcal{A})}.
\]  

(14)

Since \( \Phi \) is completely compact, there exists a finite dimensional subspace \( F \subset K(H) \) such that \( \text{dist}(\Phi(n)(a), M_n(F)) \leq \varepsilon/3 \) for any \( a \in M_n(\mathcal{A}) \), \( \| a \| \leq 1 \) and any \( n \in \mathbb{N} \). Let \( S_{F,1+\varepsilon} = \{ x \in F : \| x \| \leq 1 + \varepsilon \} \) and let \( k = \dim F \). Choose a finite rank projection \( p \in K(H) \) such that

\[
\| x - pxp \| < \frac{\varepsilon}{k(3 + \varepsilon)} \quad \text{for all} \quad x \in S_{F,1+\varepsilon}
\]

and let \( \Psi : F \to K(H) \) be defined by \( \Psi(x) = x - pxp \). By [6, Corollary 2.2.4], \( \Psi \) is completely bounded and \( \| \Psi \|_{cb} \leq k\| \Psi \| \). This implies that

\[
\| \Psi(n)(y) \| \leq k\| \Psi \| \| y \| \leq \frac{\varepsilon}{3 + \varepsilon} \| y \| \leq \frac{\varepsilon}{3}
\]

for all \( y \in M_n(F) \) with \( \| y \| \leq 1 + \varepsilon/3 \).
Now for \( a \in S^h_{M_n(A)} \) let \( y \in M_n(F) \) be such that \( \|\Phi^n(a) - y\| \leq \varepsilon/3 \).
Then \( \|y\| \leq \|\Phi^n(a)\| + \varepsilon/3 \leq 1 + \varepsilon/3 \). Hence

\[
\|\Phi^n(a) - (p \otimes 1_n)\Phi^n(a)(p \otimes 1_n)\| \\
\leq \|\Phi^n(a) - y\| + \|\Psi^n(y)\| + \|(p \otimes 1_n)(y - \Phi^n(a))(p \otimes 1_n)\| \\
\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\]

proving (14). Next we fix \( \varepsilon > 0 \) and choose a finite rank projection \( q_1 \) on \( H \) such that

\[
\|\Phi^n(a) - (q_1 \otimes 1_n)\Phi^n(a)(q_1 \otimes 1_n)\| \leq \frac{\varepsilon}{2}, \quad a \in M_n(A), \quad \|a\| \leq 1, \quad n \in \mathbb{N}.
\]

Let \( r_1 : A \to \mathcal{K}(H) \) be the mapping given by \( r_1(a) = \Phi(a) - q_1\Phi(a)q_1, \ a \in A \). Then \( r_1 = \Psi \circ \Phi \), where \( \Psi : \mathcal{K}(H) \to \mathcal{K}(H) \) is the completely bounded map given by \( \Psi(x) = x - q_1xq_1 \). By Proposition 3.2, \( r_1 \) is completely compact. Moreover, \( \|r_1\|_{cb} \leq \varepsilon/2 \) and \( \Phi(a) = q_1\Phi(a)q_1 + r_1(a), \ a \in A \). Proceeding by induction, we can find sequences of finite rank projections \( q_i \) and completely compact symmetric mappings \( r_i \) such that \( \|r_i\|_{cb} \leq \varepsilon/2^i \) and

\[
\Phi(a) = q_1\Phi(a)q_1 + \sum_{i=1}^{\infty} q_{i+1}r_i(a)q_{i+1}, \quad a \in A.
\]

Let \( c = q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1} \). We have that \( \Phi^n \) and \( r_i^n \) are symmetric and

\[
\Phi^n(a) = (q_1 \otimes 1_n)\Phi^n(a)(q_1 \otimes 1_n) + \sum_{i=1}^{\infty} (q_{i+1} \otimes 1_n)r_i^n(a)(q_{i+1} \otimes 1_n),
\]

for each \( a \in A \). Now

\[
\Phi^n(a) \leq (q_1 \otimes 1_n)\|\Phi\|_{cb} + \sum_{i=1}^{\infty} (q_{i+1} \otimes 1_n)\|r_i\|_{cb} \leq (q_1 + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} q_{i+1}) \otimes 1_n = c \otimes 1_n
\]

for all \( a \in S^h_{M_n(A)} \). By construction, \( c \) is compact and \( \|c\| \leq 1 + \varepsilon \). \( \square \)

Let \( H \) be an infinite dimensional separable Hilbert space and \( \{q_k\}_{k \in \mathbb{N}} \) be a family of pairwise orthogonal projections in \( \mathcal{B}(H) \) with rank \( q_k = k \) and \( \sum_{k=1}^{\infty} q_k = I \). Set \( p_n = \sum_{k=1}^{n} q_k, \ n \in \mathbb{N} \). Let \( \Phi_k : \mathcal{B}(q_k H) \to \mathcal{B}(q_k H), \ k \in \mathbb{N} \), be symmetric linear maps such that

\[
\|\Phi_k\|_{cb} = 1, \quad \|\Phi_k\| \to 0 \text{ as } k \to \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \|\Phi_k\|^2 < \infty, \quad (15)
\]

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where \( \|\Phi_k\|_2 \) denotes the norm of the mapping \( \Phi_k \) when \( \mathcal{B}(q_k H) \simeq C_2(q_k H) \) is equipped with the Hilbert-Schmidt norm. Identifying \( \mathcal{B}(q_k H) \) with \( q_k \mathcal{B}(H) q_k \), let \( \Phi : \mathcal{K}(H) \to \mathcal{B}(H) \) be the map given by the norm-convergent sum

\[
\Phi(x) = \sum_{k=1}^{\infty} \oplus \Phi_k(q_kxq_k), \quad x \in \mathcal{K}(H).
\]  

(16)

An example of such a map is obtained by taking \( \Phi_k = k^{-1}\tau_k \) where \( \tau_k \) is the transposition map \( \mathcal{B}(q_k H) \simeq M_k \to M_k \simeq \mathcal{B}(q_k H) \), which is symmetric and an isometry for both the operator and the Hilbert-Schmidt norm. It is well known (see e.g. [20, p. 419]) that \( \|\tau_k\|_{cb} = k \) and hence conditions (15) are satisfied.

The next lemma is a straightforward extension of [21, pp. 32–34].

Lemma 7.6. If \( \Phi \) is a map satisfying (15) and (16) then the range of \( \Phi \) consists of compact operators. Moreover, \( \Phi \) is completely contractive and compact but not completely compact.

Proof. Fix \( x \in \mathcal{K}(H) \). Since \( \|\Phi_k\| \to_{k \to \infty} 0 \) as we have \( p_n \Phi(x)p_n \to \Phi(x) \) in norm, so \( \Phi(x) \in \mathcal{K}(H) \). Each of the maps \( x \mapsto \Phi_k(q_kxq_k) \) is completely contractive, so \( \Phi \) is completely contractive.

Next, note that \( \Phi \) maps the unit ball of \( \mathcal{K}(H) \) into \( U \defeq U_1 \oplus U_2 \oplus \cdots \), where \( U_k \) is the ball of radius \( \|\Phi_k\| \) in \( q_k \mathcal{B}(H) q_k \). Since \( U \) is compact, the map \( \Phi \) is compact.

If \( \Phi \) were completely compact then by Lemma 7.5, there would exist a positive compact operator \( c \) on \( H \) such that

\[
\Phi^{(k)}(x) \leq c \otimes 1_k \text{ for all } x \in S_{M_k(\mathcal{K}(H))}^h \text{ and all } k \in \mathbb{N}.
\]

Hence for every \( k \in \mathbb{N} \) and \( x \in S_{M_k(\mathcal{B}(H))}^h \),

\[
\Phi^{(k)}(q_k \otimes 1_k)x(q_k \otimes 1_k)) = (q_k \otimes 1_k)\Phi^{(k)}(x)(q_k \otimes 1_k) \leq q_k c q_k \otimes 1_k.
\]

However, \( \|\Phi^{(k)}\| = \|\Phi_k\|_{cb} = 1 \) by [22] and \( \Phi^{(k)}_k \) is symmetric, so

\[
\|q_k c q_k\| = \|q_k c q_k \otimes 1_k\| \geq \sup\{\|\Phi^{(k)}_k(x)\| : x \in S_{M_k(q_k \mathcal{B}(H) q_k)}^h\} = 1,
\]

which is impossible since \( c \) is compact. \( \square \)

Lemma 7.7. Let \( C = \bigoplus_{k \in \mathbb{N}}^{c_0} \mathcal{B}(q_k H) \subseteq \mathcal{K}(H) \). Then there exists \( \varphi \in M(C, C) \) such that \( \Phi = \Phi_{\text{id}(\varphi)} \).
Proof. Let \( \varphi_k \in \mathcal{B}(q_k H)^d \otimes \mathcal{B}(q_k H) \) be such that \( \Phi_{id(\varphi_k)} = \Phi_k, \ k \in \mathbb{N} \), where the family \( \{ \Phi_k \}_{k=1}^{\infty} \) satisfies (15). Then \( \| \varphi_k \|_{\min} = \| \Phi_k \|_2 \). Let \( \psi_n = \sum_{k=1}^{n} \varphi_k \).

If \( n < m \) then \( \| \psi_m - \psi_n \|_{\min} = \| \sum_{k=n+1}^{m} \Phi_k \|_2 \) so

\[
\| \psi_m - \psi_n \|_{\min} \leq \left( \sum_{k=n+1}^{m} \| \Phi_k \|_2^2 \right)^{1/2}.
\]

By (15), the sequence \( \{ \psi_n \} \) converges to an element \( \varphi \in \mathcal{C}^d \otimes \mathcal{C} \). Moreover, for every \( x \in \mathcal{C}_2(H) \) we have

\[
\Phi_{id(\varphi)}(x) = \lim_{n \to \infty} p_n \Phi_{id(\varphi)}(x)p_n = \lim_{n \to \infty} \Phi_{id(\psi_n)}(x) = \Phi(x),
\]

where the limits are in the operator norm. So \( \Phi_{id(\varphi)} = \Phi \) which is completely contractive by Lemma 7.6, so \( \varphi \in \mathcal{M}(\mathcal{C}_d, \mathcal{C}) \) by Theorem 4.3.

Given \( \mathcal{C}^*-\)algebras \( \mathcal{A}_i \subseteq \mathcal{B}(H_i), \ i = 1, \ldots, n, \) and \( \psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{n-1} \), we may define a bounded linear map \( \mathcal{A}_1 \otimes \mathcal{A}_n \to \mathcal{B}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n \), where \( \mathcal{B}_1 = \mathcal{A}_1 \) if \( n \) is even and \( \mathcal{B}_1 = \mathcal{A}_1^d \) if \( n \) is odd, by

\[
a \otimes b \mapsto \begin{cases} a \otimes \psi \otimes b & \text{if } n \text{ is even,} \\ a^d \otimes \psi \otimes b & \text{if } n \text{ is odd.} \end{cases}
\]

We write \( \iota_\psi \) for the restriction of this map to \( \mathcal{M}(\mathcal{A}_1, \mathcal{A}_n) \).

**Lemma 7.8.** (i) The range of \( \iota_\psi \) is contained in \( \mathcal{M}(\mathcal{B}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \).

(ii) \( \iota_\psi(M_{c}^d(\mathcal{A}_1, \mathcal{A}_n)) \subseteq M_c^d(\mathcal{B}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \).

(iii) Suppose that \( n \) is even and \( \omega \in (\mathcal{B}(H_{n-1}^d) \otimes \cdots \otimes \mathcal{B}(H_{2}^d))_+ \). Writing

\[
M_\omega : \mathcal{B}(H_{n}) \otimes \cdots \otimes \mathcal{B}(H_{2}^d) = \mathcal{B}(H_{n-1}^d) \otimes \cdots \otimes \mathcal{B}(H_{2}^d) \to \mathcal{B}(H_{n}^d) \otimes \cdots \otimes \mathcal{B}(H_{2}^d)
\]

for the “middle slice map” \( M_\omega = R_\omega \otimes \cdots \otimes \mathcal{B}(H_{2}^d) \), we have

\[
M_\omega(u_{\iota_\psi(\varphi)}) = \omega(\tilde{\psi})u_\varphi
\]

where \( \tilde{\psi} = c_2^d \otimes \cdots \otimes c_2 \). The same is true, mutatis mutandis, if \( n \) is odd.

**Proof.** Let \( \varphi \in \mathcal{M}(\mathcal{A}_1, \mathcal{A}_n) \). By Theorem 2.3, there exist a net \( \{ \varphi_\nu \} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_n \) and representations \( u_{\varphi_\nu}^d = A_2^\nu \otimes \mathcal{A}_1^\nu \) and \( u_{\varphi}^d = A_2 \otimes \mathcal{A}_1 \), where \( \mathcal{A}_i^\nu \) are finite matrices with entries in \( \mathcal{A}_i^d \) if \( i = 1 \) and in \( \mathcal{A}_n \) if \( i = 2 \), such that \( \varphi_\nu \to \varphi \) semi-weakly, \( \mathcal{A}_i^\nu \to \mathcal{A}_i \) strongly and \( \sup_{\nu} \| A_i^\nu \| < \infty \).

(i) It is easy to see that \( \iota_\psi(\varphi_\nu) \) satisfies the boundedness conditions of Theorem 2.3 and converges semi-weakly to \( \iota_\psi(\varphi) \), which is therefore a universal multiplier.
(ii) Suppose that $n$ is even and let $\iota = \iota_\psi$. It is immediate to check that if
$$\phi \in \mathcal{A}_1 \otimes \mathcal{A}_n$$
and $T_1 \in \mathcal{K}(H^1_1, H_2), \ldots, T_{n-1} \in \mathcal{K}(H^d_{n-1}, H_n)$ then
$$\Phi_{\iota(\phi)}(T_{n-1} \otimes \ldots \otimes T_1) = \Phi_\phi(T_{n-1} c^1_{n-1} \ldots c_2 T_1).$$

Note that this equation holds for any $\phi \in M(\mathcal{A}_1, \mathcal{A}_n)$ since $\Phi_{\phi^\vee}(T) \to \Phi_\phi(T)$
and $\Phi_{\iota(\phi^\vee)}(T_{n-1} \otimes \ldots \otimes T_1) \to \Phi_{\iota(\phi)}(T_{n-1} \otimes \ldots \otimes T_1)$ weakly for any $T, T_1, \ldots, T_{n-1}$. Since $\Phi_{\iota(\phi)}$ is the composition of the bounded mapping $X_{n-1} \otimes \ldots \otimes X_1 \mapsto X_{n-1} c^1_{n-1} \ldots c_2 X_1$ with $\Phi_\phi$, it follows that if $\phi$ is a compact
operator multiplier then so is $\iota(\phi)$.

(iii) We have that
$$\Phi_{\iota(\phi^\vee)}(T_{n-1} \otimes \ldots \otimes T_1) = A^*_2(T_{n-1} \otimes 1)(c^1_{n-1} \otimes 1) \ldots (c_2 \otimes 1)(T_1 \otimes 1) A^*_2$$
$$\to A_2(T_{n-1} \otimes 1)(c^1_{n-1} \otimes 1) \ldots (c_2 \otimes 1)(T_1 \otimes 1) A_1$$
weakly. On the other hand, $\Phi_{\iota(\phi^\vee)}(T_{n-1} \otimes \ldots \otimes T_1) \to \Phi_{\iota(\phi)}(T_{n-1} \otimes \ldots \otimes T_1)$
which implies that $u_{\iota(\phi)} = A_2 \circ (c^1_{n-1} \otimes 1) \circ \ldots \circ (c_2 \otimes 1) \circ A_1$. It follows
that $M_\phi(u_{\iota(\phi)}) = \omega(\psi) u_\phi$.

\begin{theorem}
Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be $C^*$-algebras with the property that both
$\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ contain full matrix algebras of arbitrarily large sizes. Then
the inclusion $M_{\text{cc}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq M_\phi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is proper.
\end{theorem}

\begin{proof}
We may assume that $\mathcal{A}_i \subseteq \mathcal{B}(H_i)$, $i = 1, \ldots, n$ for some Hilbert spaces
$H_1, \ldots, H_n$. First suppose that $n = 2$ and let $H$ be an infinite dimensional
separable Hilbert space with $H^d \subseteq H_1$ and $H \subseteq H_2$. Let $\mathcal{C} \simeq \bigoplus_{k \in \mathbb{N}} M_k$
be the $C^*$-algebra from Lemma 7.7. Then $\mathcal{C}^d \subseteq \mathcal{A}_1$ and $\mathcal{C} \subseteq \mathcal{A}_2$. By the
injectivity of the minimal tensor product of $C^*$-algebras, $\mathcal{C}^d \otimes \mathcal{C} \subseteq \mathcal{A}_1 \otimes \mathcal{A}_2$.

Let $\phi \in \mathcal{C}^d \otimes \mathcal{C}$ be given by Lemma 7.7. It follows from Lemma 7.6 that
$\phi \in M_{\text{cc}}(\mathcal{A}_1, \mathcal{A}_2) \setminus M_{\text{cc}}^d(\mathcal{A}_1, \mathcal{A}_2)$. Since faithful representations of
$\mathcal{A}_1$ and $\mathcal{A}_2$ restrict to representations of $\mathcal{C}$ containing the identity subrepresentation up
to unitary equivalence, we have that $\phi \in M_\phi(\mathcal{A}_1, \mathcal{A}_2)$.

Suppose now that $n$ is even. Let $\phi \in M_{\text{cc}}(\mathcal{A}_1, \mathcal{A}_n) \setminus M_{\text{cc}}(\mathcal{A}_1, \mathcal{A}_n)$, fix any
non-zero $\psi = c_2 \otimes \cdots \otimes c_{n-1} \in \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{n-1}$ and let us write $\iota = \iota_\psi$.
Suppose that $\iota(\phi)$ is a completely compact multiplier. By Theorem 6.4, $u_{\iota(\phi)} \in \mathcal{K}(\mathcal{A}_n) \otimes_h (\mathcal{A}_{n-1} \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{A}_2) \otimes_h \mathcal{K}(\mathcal{A}_1^\vee)$.

Let $\tilde{\psi} = c^1_{n-1} \otimes \cdots \otimes c_2 \in \mathcal{A}_{n-1} \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{A}_2$ and fix $\omega \in (\mathcal{B}(H^d_{n-1}) \otimes_{\text{eh}} \cdots \otimes_{\text{eh}} \mathcal{B}(H_2))_*$ such that $\omega(\tilde{\psi}) \neq 0$. By Lemma 7.8 (iii), $M_\omega(u_{\iota(\phi)}) = \omega(\tilde{\psi}) u_\phi$
and hence $u_\phi \in \mathcal{K}(\mathcal{A}_n) \otimes_h \mathcal{K}(\mathcal{A}_1^\vee)$ which by Theorem 6.4 contradicts the
assumption that $\phi$ is not a completely compact multiplier.

If $n$ is odd then the same proof works with minor modifications.
\end{proof}
Remark 7.10. We do not know whether the sets $M_{cc}(A, B)$ and $M_c(A, B)$ are distinct if $K(A)$ contains matrix algebras of arbitrarily large sizes, while $K(B)$ does not (and vice versa). To show that the inclusion $M_{cc}(C, c_0) \subseteq M_c(C, c_0)$ is proper it would suffice to exhibit mappings $\Phi_k : M_k \to M_k$ which satisfy (15) and are left $D_k$-modular (where $D_k$ is the subalgebra of all diagonal matrices of $M_k$). This modularity condition would enable us to find $\varphi_k \in M_k^d \otimes D_k$ such that $\Phi_k = \Phi_{id(\varphi_k)}$ using the method of Lemma 7.7 and we could then conclude from Lemma 7.6 that $M_{cc}(C, c_0) \not\subseteq M_c(C, c_0)$. However, we do not know if such mappings $\Phi_k$ exist.

7.2 Automatic compactness

We now turn to the question of when every universal multiplier is automatically compact. We will restrict to the case $n = 2$ for the rest of the paper. We will first establish an auxiliary result in a different but related setting. Suppose that $A$ and $B$ are commutative $C^*$-algebras and assume that $A = C_0(X)$ and $B = C_0(Y)$ for some locally compact Hausdorff spaces $X$ and $Y$. The $C^*$-algebra $C_0(X) \otimes C_0(Y)$ will be identified with $C_0(X \times Y)$ and $M(A, B)$ with a subset of $C_0(X \times Y)$. Elements of the Haagerup tensor product $C_0(X) \otimes_h C_0(Y)$, as well as of the projective tensor product $C_0(X) \hat{\otimes} C_0(Y)$, will be identified with functions in $C_0(X \times Y)$ in the natural way. Note that, by Grothendieck’s inequality, $C_0(X) \otimes_h C_0(Y)$ and $C_0(X) \hat{\otimes} C_0(Y)$ coincide as sets of functions.

Proposition 7.11. Let $X$ and $Y$ be locally compact infinite Hausdorff spaces. Then $C_0(X) \otimes_h C_0(Y) \subseteq M(C_0(X), C_0(Y))$ and this inclusion is proper.

Proof. The inclusion $C_0(X) \otimes_h C_0(Y) \subseteq M(C_0(X), C_0(Y))$ follows from Corollary 6.7 of [14]. To show that this inclusion is proper, suppose first that $X$ and $Y$ are compact. By Theorem 11.9.1 of [8], there exists a sequence $(f_i)_{i=1}^\infty \subseteq C(X) \otimes_h C(Y)$ such that $\sup_{i \in \mathbb{N}} \|f_i\|_h < \infty$ converging uniformly to a function $f \in C(X \times Y) \setminus C(X) \otimes_h C(Y)$. By Corollary 6.7 of [14], $f \in M(C(X), C(Y))$. The conclusion now follows.

Now assume that both $X$ and $Y$ are locally compact but not compact (the case where one of the spaces is compact while the other is not is similar). Let $\tilde{X} = X \cup \{\infty\}$ and $\tilde{Y} = Y \cup \{\infty\}$ be the one point compactifications of $X$ and $Y$. Then $C(\tilde{X}) = C_0(X) + \mathbb{C}1$ and $C(\tilde{Y}) = C_0(Y) + \mathbb{C}1$, where 1 denotes the constant function taking the value one. Moreover, it is easy to see that

$$C(\tilde{X}) \otimes C(\tilde{Y}) = C_0(X \times Y) + C_0(X) + C_0(Y) + \mathbb{C}1$$
and
\[ C(\tilde{X}) \otimes C(\tilde{Y}) = C_0(X) \otimes C_0(Y) + C_0(X) + C_0(Y) + \mathbb{C}1. \]  
(17)

By the first part of the proof, there exists \( \varphi \in M(C(\tilde{X}), C(\tilde{Y})) \setminus C(\tilde{X}) \otimes_h C(\tilde{Y}) \). Write \( \varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \) where \( \varphi_1 \in C_0(X \times Y) \), \( \varphi_2 \in C_0(X) \), \( \varphi_3 \in C_0(Y) \) and \( \varphi_4 \in \mathbb{C}1 \). Suppose that \( \varphi_1 \in C_0(X) \otimes_h C_0(Y) \). By (17), \( \varphi \in C(\tilde{X}) \otimes C(\tilde{Y}) \), a contradiction.  
\[ \square \]

**Theorem 7.12.** Let \( A \) and \( B \) be \( C^\ast \)-algebras. The following are equivalent:

(i) either \( A \) is finite dimensional and \( K(B) = B \), or \( B \) is finite dimensional and \( K(A) = A \);

(ii) \( M_c(A, B) = M(A, B) \);

(iii) \( M_{cc}(A, B) = M(A, B) \).

**Proof.** (i)⇒(iii) Suppose that \( A \) is finite dimensional and \( K(B) = B \), and that \( A \subseteq B(H_1) \) and \( B \subseteq B(H_2) \) for some Hilbert spaces \( H_1 \) and \( H_2 \) where \( H_1 \) is finite dimensional. Fix \( \varphi \in M(A, B) \). Then \( \varphi \) is the sum of finitely many elements of the form \( a \otimes b \) where \( a \) has rank one and \( b \in K(H_2) \); such elements are completely compact multipliers by Theorem 6.4.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Assume that both \( A \) and \( B \) are infinite dimensional and are identified with their image under the reduced atomic representation. If either \( K(A) \) or \( K(B) \) is finite dimensional then there exists an elementary tensor \( a \otimes b \in (A \otimes B) \setminus (K(A) \otimes K(B)) \). By Proposition 7.2, \( a \otimes b \notin M_c(A, B) \).

We can therefore assume that both \( K(A) \) and \( K(B) \) are infinite dimensional. Then, up to a *-isomorphism, \( c_0 \) is contained in both \( K(A) \) and \( K(B) \). By Proposition 7.11, there exists \( \varphi \in M(c_0, c_0) \setminus (c_0 \otimes_h c_0) \). Then \( \varphi \in M(A, B) \) and \( \Phi_{\text{id}}(\varphi) \) is not compact by Hladnik’s result [11]. Since the restrictions to \( c_0 \) of any faithful representations of \( A, B \) contain representations unitarily equivalent to the identity representations, we see that \( \varphi \) is not a compact multiplier.

Thus at least one of the \( C^\ast \)-algebras \( A \) and \( B \) is finite dimensional; assume without loss of generality that this is \( A \). Suppose that \( B \neq K(B) \) and fix an element \( b \in B \setminus K(B) \). Let \( a \in A \) be a non-zero element. By Proposition 7.2, the elementary tensor \( a \otimes b \) is not a compact multiplier.  
\[ \square \]

**References**


