Lecture 11: Hulanicki’s criteria.

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0.1 Hulanicki’s criteria

Let \( \lambda : \Gamma \to B(l^2(\Gamma)) \) be the left regular representation of \( \Gamma \):

\[
\lambda(t)f = \delta_t * f \quad \text{for every } f \in l^2(\Gamma).
\]

Here the convolution of two functions is defined by

\[
(f * h)(t) = \sum_{s \in \Gamma} f(s)h(s^{-1}t).
\]

Equivalently, \((\lambda(t)h)(s) = h(t^{-1}s)\). For a function \( f \in l^1(\Gamma) \), we denote

\[
\lambda(f) = \sum_{t \in \Gamma} f(t)\lambda(t)
\]

**Theorem 0.1.1.** For a discrete group \( \Gamma \) the following is equivalent:

(i) \( \Gamma \) is amenable

(ii) There is constant \( C > 0 \) such that for every positive finitely supported function \( f \) on \( \Gamma \) we have

\[
\sum_{t \in \Gamma} f(t) \leq C \| \sum_{t \in \Gamma} f(t)\lambda(t) \|_{B(l^2(\Gamma))}
\]

(iii) The same as in (ii) with constant \( C = 1 \).

**Proof.** Assume that \( \Gamma \) is amenable and let \( f \geq 0 \) be a finitely supported function. Let \( F_n \) be a Følner sequence that \( 1/n \)-approximates the support of \( f \), i.e.,

\[
|gF_n \Delta F_n| \leq 1/n|F_n|, \quad \text{for every } g \in \text{supp}(f).
\]

Denote by \( \xi_n = \frac{1}{\sqrt{|F_n|}} \chi_{F_n} \in l^2(\Gamma) \). Note that \( \|\xi_n\|_2 = 1 \). Assume that

\[
\| \sum_{t \in \Gamma} f(t)\lambda(t) \| \leq 1.
\]
Then we have
\[
\langle \sum_{t \in \Gamma} f(t) \lambda(t) \xi_n, \xi_n \rangle = \sum_{t \in \Gamma} f(t) \langle \lambda(t) \xi_n, \xi_n \rangle = \sum_{s,t \in \Gamma} f(t) \xi_n(t^{-1}s) \overline{\xi_n(s)} \leq 1.
\]

But \(\|\lambda(t) \xi_n - \xi_n\| = \|\delta_t * \xi_n - \xi_n\| \to 0\), when \(n \to \infty\). Thus we have
\[
1 \geq \lim_{n \to \infty} \left| \sum_{s,t \in \Gamma} f(t) \xi_n(t^{-1}s) \overline{\xi_n(s)} \right| = \lim_{n \to \infty} \left| \sum_{s,t \in \Gamma} f(t) \xi_n(s) \overline{\xi_n(s)} \right| = \sum_{t \in \Gamma} f(t).
\]

This, by homogeneity of the inequality, implies that
\[
\sum_{t \in \Gamma} f(t) \leq \| \sum_{t \in \Gamma} f(t) \lambda(t) \|.
\]

Now assume (ii), we will deduce (iii). Note that
\[
\lambda(f * \ldots * f) = \lambda(f) \ldots \lambda(f).
\]
Applying (ii) to the convolution of \(f\) we get
\[
\left( \sum_{t \in \Gamma} f(t) \right)^n = \sum_{t \in \Gamma} f * \ldots * f(t) \leq C \| \lambda(f) \ldots \lambda(f) \| \leq C \| \lambda(f) \|^n.
\]

Thus we obtain
\[
\sum_{t \in \Gamma} f(t) \leq C^{1/n} \| \lambda(t) \|,
\]
which implies (iii).

Assume now (iii). Let \(E\) be a finite subset of \(\Gamma\), then applying (iii) to \(f = \chi_E\) we obtain
\[
\| \sum_{t \in E} \lambda(t) \| = |E|,
\]
which by Kesten's criteria implies amenability of \(\Gamma\).
Bibliography


[34] Elek, G., Monod, N., *On the topological full group of minimal Z²-systems*, to appear in Proc. AMS.


