Lecture 12: Weak containment of representations

by Kate Juschenko
0.1 Weak containment of representations

Let $\Gamma$ be a discrete group with two representations $\pi : \Gamma \to B(\mathcal{H})$ and $\rho : \Gamma \to B(\mathcal{K})$ by unitary operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ correspondingly. The representation $\pi$ is **weakly contained** in $\rho$, denoted by $\pi \prec \rho$, if for every $\xi \in \mathcal{H}$, finite set $E$ of $\Gamma$ and $\varepsilon > 0$ there are $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that for all $g \in E$ we have

$$\left| \langle \pi(g) \xi, \xi \rangle - \sum_{i=1}^{n} \langle \rho(g) \eta_i, \eta_i \rangle \right| < \varepsilon.$$

Denote by $1_\Gamma : \Gamma \to \mathbb{C}$ the trivial representation, i.e., $1_\Gamma(g) = 1$ for every $g \in \Gamma$.

**Theorem 0.1.1.** A discrete group $\Gamma$ is amenable if and only if $1_\Gamma \prec \lambda$.

**Proof.** We will show that $1_\Gamma \prec \lambda$ is equivalent to the existence of an almost invariant vector for $\lambda$, which by Theorem ?? is equivalent to the amenability of $\Gamma$.

Assume that $\lambda$ admits an almost invariant vector. It is sufficient to show that for every $\varepsilon > 0$ and a finite set $E$ of $\Gamma$ there are $\eta_1, \ldots, \eta_n \in l^2(\Gamma)$ such that

$$|1 - \sum_{i=1}^{n} \langle \lambda(t) \eta_i, \eta_i \rangle| < \varepsilon,$$

for every $t \in E$.

This follows if we take $n = 1$ and $\eta_1 = \xi$, where $\xi$ is almost invariant vector, i.e., $||\lambda(g) \xi - \xi|| < \varepsilon$ for every $g \in E$.

Conversely, assume that $1_\Gamma \prec \lambda$, we will deduce that $\lambda$ has an almost invariant vector. By the definition, for every $\varepsilon > 0$ and a finite subset $E$ of $\Gamma$ there are $\eta_1, \ldots, \eta_n \in l^2(\Gamma)$, such that

$$|1 - \sum_{i=1}^{n} \langle \lambda(t) \eta_i, \eta_i \rangle| < \varepsilon,$$

for every $t \in E$.  \hspace{1cm} (1)

Assuming that $E$ contains the identity element $e$, we obtain

$$|1 - \sum_{i=1}^{n} \| \eta_i \|^2 | < \varepsilon.$$

Rescaling the norm we may assume that $\sum_{i=1}^{n} \| \eta_i \|^2 = 1$ and (1) is still satisfied.
To reach a contradiction assume $\lambda$ does not have an almost invariant vector. Then there exists $C > 0$ and a finite set $S \subset \Gamma$ such that for every $\xi \in l^2(\Gamma)$ we have

$$\|\xi\|_2^2|S| - \sum_{\gamma \in S} \langle \lambda(\gamma)\xi, \xi \rangle > C\|\xi\|_2^2$$

Now applying this to the vectors $\eta_1, \ldots, \eta_n$ and summing up, we obtain

$$|S| - \sum_{a \in S} \sum_{i=1}^n \langle \lambda(\gamma)\eta_i, \eta_i \rangle > C$$

This implies that there exists $\gamma \in S$ such that

$$1 - \sum_{i=1}^n \langle \lambda(\gamma)\eta_i, \eta_i \rangle > C/|S| > 0,$$

which contradicts to (1), thus $\lambda$ admits an almost invariant vector. \qed

More on weak containment of representations can be found in the book [8].
Bibliography


[34] Elek, G., Monod, N., *On the topological full group of minimal $\mathbb{Z}^2$-systems*, to appear in Proc. AMS.


