Lecture 17: Grigorchuk’s co-growth criteria

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0.1 Girgorchuk’s co-growth criteria

In 1978, [42], Grigorchuk introduced another amenability criteria related to the co-growth of finitely generated groups.

Let $\Gamma$ be a group generated by a finite set $S = \{x_1, \ldots, x_r\}$. Denote by $\gamma_n$ the number of all reduced words of length $n$ which are equal to identity in $\Gamma$ and by $\overline{\gamma}_n = \sum_0^n \gamma_n$ be the number of all reduced words of the length not greater then $n$ that are equal to identity in $\Gamma$. In other words, consider a presentation

$$\Gamma = \langle g_1, \ldots, g_k : r_1, r_2, \ldots \rangle,$$

then $\Gamma = \mathbb{F}_k/N$, where $N$ is a normal subgroup generated by $r_1, r_2, \ldots$ and the length of a word is defined by it’s length in the free group $\mathbb{F}_k$. Moreover, if we denote $E_n = \{w \in \mathbb{F}_k : |w| = n\}$, $N_n = E_n \cap N$, then $\gamma_n = |N_n|$.

**Theorem 0.1.1** (Grigorchuk’s co-growth criteria, ’78). A group $\Gamma$ generated by a finite set $S$ is amenable if and only if

$$\lim_{n \to \infty} \overline{\gamma}_n^{1/n} = |S|,$$

where $\overline{\gamma}_n$ is the number of all reduced words of the length not greater then $n$ that are equal to identity in $\Gamma$.

The key step in the proof is to apply Kesten’s criteria for the uniform on the self-adjoint set $S$ measure $\mu_S = \sum_{S} \delta_s$. Namely, we have $\Gamma$ is amenable if and only if

$$\lim_{n \to \infty} \mu_S^{2n}(e) = |S|.$$

If we develop the convolution on the left hand side of this equation we obtain that the value $\mu_S^{2n}(e)$ coincides with the coefficient next to identity in the set $S^{2n}$ considered with multiplicities. In other words, if we think of a free semigroup generated generated by $k$ elements, we can again consider the word length. In these notations $\mu_S^{2n}(e)$ is the number of all elements of the length $2n$ equal to identity in $\Gamma$. The proof will be based on a comparison of this quantity with $\overline{\gamma}_n$.

**Lemma 0.1.2.** For $n, m > 0$ we have

$$\begin{align*}
(2t - 2)(2t - 1)^{n-1}\gamma_m &\leq \gamma_{m+2n} \quad (1) \\
\gamma_n\gamma_m &\leq \gamma_{n+m+2} \quad (2)
\end{align*}$$
Proof. Let $\alpha \in F_r$ be a reduced word and define $\alpha(i) \in \{x_i, x_i^{-1} : 1 \leq i \leq r\}$ to the the $i$-th letter of $\alpha$. Assume $\alpha \in N_m$. Then for a word $w$ with $|w| = n$ which does not end with $\alpha(1)^{-1}$ or starts with $\alpha(m)$ we have $|waw^{-1}| = m + 2n$ and $waw^{-1} \in N$. Thus $waw^{-1} \in N_{m+2n}$. For each fixed $\alpha \in N_m$ there are $(2t - 2)(2t - 1)^{n-1}$ such $w$. Therefore, we have (1).

To prove the second inequality, take $\alpha \in N_m$, $\beta \in N_n$, then $\alpha u \beta u^{-1} \in N$, since $N$ is normal subgroup of $F_r$. Similarly to the first case, if 
\[
    u \in \{x_i, x_i^{-1} : 1 \leq i \leq r\} \setminus \{\alpha(m)^{-1}, \beta(1)^{-1}, \beta(n)\},
\]
then 
\[
    |\alpha u \beta u^{-1}| = m + n + 2.
\]
So we have $\gamma_m \gamma_n \leq \gamma_{m+n+2}$.

Lemma 0.1.3. For the sequence $\gamma_n$ one of the following conditions holds:

(i) $\gamma_n = 0$ for all $n \in \mathbb{N}$ if and only if $N = \{e\}$;

(ii) $\gamma_{2n+1} = 0$ for all $n \in \mathbb{N}$ and there exists $n_0$ such that for all $n \geq n_0$ we have $\gamma_{2n} \neq 0$;

(iii) There exists $n_0$ such that for all $n \geq n_0$ we have $\gamma_n \neq 0$.

Proof. Suppose $N \neq \{e\}$, so (ii) does not hold. Then there exists $n_1$ such that $\gamma_{n_1} \neq 0$. Assume $n_1$ is odd, then by Lemma 0.1.2 we have 
\[
    0 < \gamma_{n_1} \gamma_{n_1} \leq \gamma_{2n_1+2}
\]
Again by Lemma 0.1.2 for all $k \geq 1$ we have 
\[
    0 < \gamma_{n_1} \leq \gamma_{n_1+2k} \text{ and } 0 < \gamma_{2n_1+2k+2}.
\]
So for all off and all $n$ starting with some $n_0$ we have $\gamma_n > 0$, which is the case (iii).

Assume that $\gamma_{2n+1} = 0$ for all $n$ and there exists $n_1$ such that $\gamma_{2n_1} \neq 0$. Then by Lemma 0.1.2 we have 
\[
    0 < \gamma_{2n_1} \leq \gamma_{2n_1+2k}
\]
for all $k > 0$. Thus the case (iii) follows. 

\[3\]
Proposition 0.1.4. For the sequence $\gamma_n$ we have the following

1. $\lim_{n \to \infty} (\gamma_{2n})^{\frac{1}{2n}}$ exists, if (iii) of Lemma 0.1.3 holds.

2. $\lim_{n \to \infty} (\gamma_n)^{\frac{1}{n}}$ exists, if (iii) of Lemma 0.1.3 holds.

Proof. Define $a_n = \log(\gamma_n - 2)$ for those $n$ such that $\gamma_n - 2 \neq 0$. Thus $a_n \geq 0$. By Lemma 0.1.2 we have

$$a_n + a_m = \log(\gamma_n - 2 \gamma_m - 2) \leq \log(\gamma_n + m - 2) = a_n + m.$$ 

Letting $b_n = -a_n$ we obtain that $b_n$ is sub additive ($b_{n+m} \leq b_n + b_m$). So if (iii) of Lemma 0.1.3 holds then $\lim_{n \to \infty} b_n/n$ and in the case when (ii) of Lemma 0.1.3 holds $\lim_{n \to \infty} b_{2n}/2n$ exists. So we have $\lim_{n \to \infty} a_n/n$ exists and therefore

$$\lim_{n \to \infty} \frac{a_n + 2}{n} = \lim_{n \to \infty} \log(\gamma_n)^{\frac{1}{n}}$$

exists and the statement follows.

Suppose that (iii) of Lemma 0.1.3 holds, then define

$$\gamma = \lim_{n \to \infty} (\gamma_n)^{\frac{1}{n}}$$

In the case if the condition (ii) of Lemma 0.1.3 holds, then define

$$\gamma = \lim_{n \to \infty} (\gamma_{2n})^{\frac{1}{2n}}$$

Proposition 0.1.5. Assume that either (ii) or (iii) of the Lemma 0.1.3 holds, then

$$\sqrt{2t-1} \leq \gamma \leq 2t - 1.$$ 

Proof. Since $\gamma_n = |N_n| \leq |E_n| = |\{w \in F : |w| = n\}| = 2t(2t - 1)^{n-1}$, we have

$$(\gamma_n)^{\frac{1}{n}} \leq (2t - 1)^{1-\frac{1}{n}}(2t)^{\frac{1}{n}},$$

so $\gamma \leq 2t - 1$.

Since either (ii) or (iii) of the Lemma 0.1.3, there exists $n$ with $\gamma_n \geq 1$. Then by Lemma 0.1.2

$$\gamma_{n+2k} \geq \gamma_n(2t-2)(2t-1)^{k-1}.$$
Hence we have
\[(\gamma_{n+2k})^{\frac{1}{n+2k}} \geq (\gamma_n)^{\frac{1}{n+2k}} (2t - 2)^{\frac{1}{n+2k}} (2t - 1)^{\frac{k+1}{2k+n}}.\]
Letting \(k \to \infty\) we obtain
\[\gamma \geq (2t - 1)^{\frac{1}{2}}.\]

**Proposition 0.1.6.** Let \(\tau_n = \gamma_0 + \ldots + \gamma_n\), then
\[\lim_{n \to \infty} (\tau_n)^{\frac{1}{n}} = \gamma.\]

**Proof.** We claim that
\[\gamma_{n-1} + \gamma_n \leq \tau_n \leq \frac{1}{n} (\tau_{n-1} + \tau_n).\]
The claim trivially implies the statement. The first inequality is obvious. By Lemma 0.1.2 we have
\[\gamma_m \leq \gamma_{m+2k},\]
for all \(m, k\). Thus
\[\gamma_{n-1} + \gamma_n \leq \gamma_{n-1} + \gamma_n\]
\[\gamma_{n-3} + \gamma_{n-2} \leq \gamma_{n-1} + \gamma_n\]
\[\ldots\]
\[\gamma_0 + \gamma_1 \leq \gamma_{n-1} + \gamma_n\]
Summing up these inequalities we obtain the claim. \(\Box\)

**Group ring of the free group \(\mathbb{F}_r\).**

Let \(X_n = \sum_{w \in E_n} w \in \mathbb{C}[\mathbb{F}_r]\), in particular, \(X_0 = 1, X_1 = g_1 + \ldots + g_k + g_i^{-1} + \ldots + g_k^{-1}\).

**Lemma 0.1.7.** There are integers \(a_{i,n}\) such that \((X_1)^n = \sum_{i=0}^{n} a_{i,n} X_i\) that satisfy
(i) $a_{i,n} = \delta_i(n)$ for $n = 0, 1$;
(ii) $a_{0,n+1} = 2ta_{1,n}$;
(iii) $a_{i,n+1} = a_{i-1,n} + (2t - 1)a_{i+1,n}$.

Proof. The property (i) is obvious by the definition of $X_i$. We will show the rest of statement by the induction on $n$. For $n = 0$ we have $(X_1)^0 = e = a_{0,0}X_0$ with $a_{0,0} = 1$. Assume $(X_1)^n = \sum_{i=0}^{n} a_{i,n}X_i$. Then

$$(X_1)^{n+1} = \left(\sum_{i=0}^{n} a_{i,n}X_i\right)X_1$$

The only way to obtain identity in this product is for each $u = x_i^{\pm 1}$ in $X_1$ find $u^{-1}$ in $a_{1,n}X_1$. There are $2t \cdot a_{1,n}$ ways to do this, so $a_{0,n+1} = 2ta_{1,n}$.

Let $|g| = i > 0$, $g = wu$ with $|w| = i - 1$, then there are two ways to obtain $g$ in $(X_1)^{n+1}$:

1. find $w$ in $a_{i-1,n}X_{i-1}$ and there are $a_{i-1,n}$ possibilities of doing this;
2. find $gv$ in $a_{i+1,n}X_{i+1}$, $v \neq u^{-1}$ and there are $a_{i+1,n}(2t - 1)$ possibilities of doing this.

So there are $a_{i-1,n} + (2t - 1)a_{i+1,n}$ appearances of $g$ in $(X_1)^{n+1}$. Since this number does not depend on the choice of $g$ we have

$$a_{i,n+1} = a_{i-1,n} + (2t - 1)a_{i+1,n}. \quad \Box$$

Define $g_i(z) = \sum_{n \geq i} a_{i,n}z^n$. Then from (iii) and (iii) of Lemma 0.1.7 above, we have $g_0 = 1 + 2tzg_1$ and

$$g_i = zg_{i-1} + (2t - 1)zg_{i+1}. \quad (3)$$

Lemma 0.1.8.

$$g_0(z) = (1 - \sum_{n \geq 0} C_n 2t(2t - 1)^n z^{2n+2})^{-1},$$

where $C_n = \frac{1}{n+1} = \binom{2n}{n}$ is the Catalan number.
Proof. We need to show that

\[ g_0(z) = 1 + \sum_{n \geq 0} C_n 2t(2t - 1)^n z^{2n+2} g_0. \]

We start with \( g_0 = 1_t z g_1 \) and successively apply to \( g_i \) the formula

\[ g_i = z g_{i-1} + (2t - 1) z g_{i+1}. \]

Stopping each time we arrive at \( g_0 \). We can look at this process as walking on \( \mathbb{N} \) such that each step to the left gives multiplication by \( z \) and each step to the right gives multiplication by \( (2t - 1) z \). Thus we have

\[
\begin{align*}
 g_0 &= 1 + 2t z \sum_{\text{walks from 1 ending at 0}} z^{\#\{\text{left steps}\}} [(2t - 1) z]^{\#\{\text{right steps}\}} g_0 \\
 &= 1 + 2t z \sum_{n \geq 0} C_n z^{n+1} [(2t - 1) z]^n g_0 \\
 &= 1 + \sum_{n \geq 0} C_n 2t(2t - 1)^n z^{2n+2} g_0,
\end{align*}
\]

where \( C_n \) is the number of walks from 1 to 0 of length \( 2n + 1 \), which end when one arrives at 0, thus, \( C_n \) is the Catalan number. \( \square \)

Corollary 0.1.9. Let \( \rho_1 \) be the radius of convergence of \( g_0(z) \), then

\[ \rho_1 = (2\sqrt{2t - 1})^{-1}. \]

Proof. By Lemma 0.1.8 we have

\[ g_0(z) = (1 - \sum_{n \geq 0} C_n 2t(2t - 1)^n z^{2n+2})^{-1}, \]

The radius of the convergence of \( 1 - \sum_{n \geq 0} C_n 2t(2t - 1)^n z^{2n+2} \) is equal to

\[ \lim_{n \to \infty} [2t(2t - 1)^n C_n]^{\frac{1}{2n+2}} = \sqrt{2t - 1} \lim_{n \to \infty} C_n^{\frac{1}{n}} = 2\sqrt{2t - 1}, \]

since by the Stirling formula \( C_n \sim 4^n \). Hence, the statement follows. \( \square \)
Group ring of the group $\Gamma$.

Recall $\gamma = F_r/N$, $F_r = \langle x_1, \ldots, x_r \rangle$. Let $\pi : \mathbb{C}[F_r] \to \mathbb{C}[\Gamma]$ be the linear extension to the group rings of the quotient map. Denote by $\alpha = \xi_1 = \pi(X_1), \ldots, \xi_n = \pi(X_n)$. By Lemma 0.1.7 we have

$$\alpha^n = \sum_{i=0}^{n} a_{i,n} \xi_i.$$

Note that

$$\tau(\xi_i) = \tau(\sum_{|g|=i} \pi(g))$$
$$= |\{g \in F : |g| = i, \pi(g) = e\}|$$
$$= |\{g \in N : |g| = i\}|$$
$$= \gamma_i.$$

Thus we have

$$\tau(\alpha^n) = \sum_{i=0}^{n} a_{i,n} \gamma_i.$$

**Theorem 0.1.10.** If $\gamma > 0$, then $\|\alpha\| = \gamma + (2t - 1)/\gamma$.

**Proof.** Recall

$$\|\alpha\| = \lim_{n \to \infty} (\tau(\alpha^n))^{1/n} = \lim_{n \to \infty} \left(\sum_{i=0}^{n} a_{i,n} \gamma_i\right)^{1/n}.$$

Since $\gamma_i^{1/n}$ converges to $\gamma$, it is easy to show that

$$\lim_{n \to \infty} \left(\sum_{i=0}^{n} a_{i,n} \gamma_i\right)^{1/n} = \lim_{n \to \infty} \left(\sum_{i=0}^{n} a_{i,n} \gamma^i\right)^{1/n}.$$

Then the function $h(s) = \lim_{n \to \infty} \left(\sum_{i=0}^{n} a_{i,n} s^i\right)^{1/n}$ is continuous.

Consider the power series $f(z) = \sum_{i,n} a_{i,n} \gamma^i z^n$. Its radius of convergence is equal to

$$\rho = \left(\lim_{n \to \infty} \left(\sum_{i,n} a_{i,n} \gamma^i\right)^{1/n}\right)^{-1} = 1/\|\alpha\|.$$
Let us now commute it

\[ f(z) = \sum_{i,n} a_{i,n} \gamma^i z^n = \sum_{i \geq 0} \gamma^i z^n = \sum_{i \geq 0} \gamma^i g_i(z) \]

\[ = g_0(z) + \sum_{i \geq 1} \gamma^i [zg_{i-1}(z) + (2t - 1) zg_{i+1}(z)] \]

\[ = g_0(z) + \gamma z f(z) + (2t - 1) \gamma \sum_{i \geq 1} \gamma^{i+1} g_{i+1}(z) \]

\[ = g_0(z) + \gamma z f(z) + (2t - 1) \gamma \left[ f(z) - \gamma g_1(z) - g_0(z) \right]. \]

Hence

\[ f(z) = \frac{g_0(z)[1 - (2t - 1)z/\gamma - (2t - 1)/(2t)] + (2t - 1)/(2t)}{1 - z[\gamma + (2t - 1)/z]} \]

Thus the radius of convergence of \( f(z) \) is the minimum of \( \rho_2 = (\gamma + (2t - 1)/\gamma)^{-1} \) and \( \rho_1 = (2\sqrt{2t - 1})^{-1} \), by Corollary 0.1.9. Note that

\[ \gamma + (2t - 1)/\gamma \geq 2\sqrt{2t - 1}, \]

for all \( \gamma > 0 \). So we have \( \rho_2 \leq \rho_1 \) and therefore the radius of convergence of \( f(z) \) is equal to \( (\gamma + (2t - 1)/\gamma)^{-1} \).

So \( \| \alpha \| = \gamma + (2t - 1)/\gamma \).

By Kesten’s criteria, \( \Gamma \) is amenable if and only if

\[ \| \alpha \|_{B(\alpha(\Gamma))} = \| \alpha \|_1 = 2t. \]

Thus \( \Gamma \) is amenable if and only if \( \gamma = 2t - 1 \), which completes the proof of the Theorem 0.1.1.
Bibliography


Elek, G., Monod, N., *On the topological full group of minimal $\mathbb{Z}^2$-systems*, to appear in Proc. AMS.


