Lecture 5: Examples of non-elementary amenable groups. The full topological group of Cantor minimal system.

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We begin with basic definitions. The Cantor space is denoted by \( C \), it is characterized up to a homeomorphism as a compact, metrizable, perfect and totally disconnected topological space. The group of all homeomorphisms of the Cantor space is denoted by \( \text{Homeo}(C) \). A Cantor dynamical system \((T, C)\) is the Cantor space together with its homeomorphism \( T \).

Let \( A \) be a finite set, we will call it an alphabet. A basic example of Cantor space is the set of all sequences in \( A \) indexed by integers, \( A^\mathbb{Z} \), and considered with product topology. A sequence \( \{\alpha_i\} \) converges to \( \alpha \) in this space if and only if for all \( n \) there exists \( i_0 \) such that for all \( i \geq i_0 \), we have that \( \alpha_i \) coincides with \( \alpha \) on the interval \([-n, n]\).

The basic example of a Cantor dynamical system is the shift on \( A^\mathbb{Z} \), i.e., the map \( s : A^\mathbb{Z} \to A^\mathbb{Z} \) is defined by

\[
s(x)(i) = x(i + 1)
\]

for all \( x \in A^\mathbb{Z} \).

The system \((T, C)\) is \textit{minimal} if there is no non-trivial closed \( T \)-invariant subset in \( C \). Equivalently, the closure of the orbit of \( T \) of any point \( p \) in \( C \) coincides with \( C \):

\[
\{T^ip : i \in \mathbb{Z}\} = C
\]

One of the basic examples of the Cantor minimal system is the \textit{odometer}, defined by the map \( \sigma : \{0,1\}^\mathbb{N} \to \{0,1\}^\mathbb{N} \):

\[
\sigma(x)(i) = \begin{cases} 
0, & \text{if } i < n, \\
1, & \text{if } i = n, \\
x(i), & \text{if } i > n 
\end{cases}
\]

where \( n \) is the smallest integer such that \( x(n) = 0 \), and \( \sigma(1) = 0 \). One can verify that the odometer is minimal homeomorphism.

While shift is not minimal, one can construct many Cantor subspaces of \( A^\mathbb{Z} \) on which the action of the shift is minimal. Closed and shift-invariant subsets of \( A^\mathbb{Z} \) are called \textit{subshifts}.

A sequence \( \alpha \in A^\mathbb{Z} \) is \textit{homogeneous}, if for every finite interval \( J \subset \mathbb{Z} \), there exists a constant \( k(J) \), such that the restriction of \( \alpha \) to any interval of the
size \(k(J)\) contains the restriction of \(\alpha\) to \(J\) as a subsequence. In other words, for any interval \(J'\) of the size \(k(J)\), there exist \(t \in \mathbb{Z}\) such that \(J + t \subset J'\) and \(\alpha(s + t) = \alpha(s)\) for every \(s \in J\).

**Theorem 0.0.1.** Let \(A\) be a finite set, \(T\) be the shift on \(A^\mathbb{Z}\), \(\alpha \in A^\mathbb{Z}\) and \(X = \text{Orb}_T(\alpha)\).

Then the system \((T, X)\) is minimal if and only if \(\alpha\) is homogeneous.

**Proof.** Assume that the sequence \(\alpha \in A^\mathbb{Z}\) is homogeneous. Let \(\beta \in \overline{\text{Orb}_T(\alpha)}\).

It is suffice to show that \(\alpha \in \overline{\text{Orb}_T(\beta)}\). Fix \(n > 0\), then there exist \(k(n, \alpha)\) such that the restriction of the sequence \(\alpha\) to any interval of the length \(k(n, \alpha)\) contains a copy of the restriction of \(\alpha\) to the interval \([-n, n]\). Thus, since \(\beta \in \overline{\text{Orb}_T(\alpha)}\) then the restriction of \(\beta\) to the interval \([-k(n, \alpha), k(n, \alpha)]\) contains a copy of \(\alpha\) restricted to \([-n, n]\). This implies that there exists a power \(i\) of \(T\) such that \(T^i(\beta)(j) = \alpha(j)\) for all \(j \in [-n, n]\). Since \(n\) is arbitrary large, we can find a sequence \(i_n\) of powers of \(T\), such that \(T^{i_n}(\beta)\) converges to \(\alpha\), therefore \((T, X)\) is minimal.

Assume \((T, X)\) is a minimal system. To reach a contradiction assume that \(\alpha\) is not homogeneous. Then there exists an interval \([-n, n]\), such that for any \(k\) there exists a subinterval of length \(k\) in \(\alpha\), which does not contain the restriction of \(\alpha\) to \([-n, n]\). Thus there exists a sequence \(m_k\) such that the interval \([-k, k]\) of \(T^{m_k}(\alpha)\) does not contain the restriction of \(\alpha\) to \([-n, n]\). Since the space is compact we can find a convergent subsequence in \(T^{m_k}(\alpha)\). Let \(\beta\) be a limit point. Then \(\alpha \notin \overline{\text{Orb}_T(\beta)}\), which gives a contradiction. Hence \(\alpha\) is homogeneous. \(\square\)

**The full topological groups.** The central object of this Chapter is the full topological group of a Cantor minimal system.

The full topological group of \((T, C)\), denoted by \([[T]]\), is the group of all \(\phi \in \text{Homeo}(C)\) for which there exists a continuous function \(n : C \to \mathbb{Z}\) such that

\[
\phi(x) = T^n(x)x \quad \text{for all } x \in C.
\]

Since \(C\) is compact, the function \(n(\cdot)\) takes only finitely many values. Moreover, for every its value \(k\), the set \(n^{-1}(k)\) is clopen. Thus, there exists a finite partition of \(C\) into clopen subsets such that \(n(\cdot)\) is constant on each piece of the partition.
Kakutani-Rokhlin partitions. Let $T$ be a minimal homeomorphism of the Cantor space $C$, we can associate a partition of $C$ as follows.

Let $D$ be a non-empty clopen subset of $C$. It is easy to check that for every point $p \in C$ the minimality of $T$ implies that the forward orbit $\{T^k p : k \in \mathbb{N}\}$ is dense in $C$. Define the first return function $t_D : D \to \mathbb{N}$:

$$t_D(x) = \min(n \in \mathbb{N} : T^n(x) \in D).$$

Since $t_D^{-1}[0, n] = T^{-n}(D)$, it follows that $t_D$ is continuous. Thus we can find natural numbers $k_1, \ldots, k_N$ and a partition $D = D_1 \sqcup D_2 \sqcup \ldots \sqcup D_N$ into clopen subsets, such that $t_D$ restricted to $D_i$ is equal to $k_i$ for all $1 \leq i \leq N$.

This gives a decomposition of $C$, called Kakutani-Rokhlin partition:

$$C = (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N)).$$

The family $D_i \sqcup T(D_i) \sqcup \ldots \sqcup T^{k_i}(D_i)$ is called a tower over $D_i$. The base of the tower is defined to be $D_i$ and the top of the tower is $T^{k_i}(D_i)$.

Refining of the Kakutani-Rokhlin partitions. Let $\mathcal{P}$ be a finite clopen partition of $C$ and let

$$C = (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N))$$

be the Kakutani-Rokhlin partition over a clopen set $D$ in $C$. There exist a refinement of the partition of $D_i = \bigsqcup_{j=1}^{j_i} D_{i,j}$ such that the partition

$$(D_{1,1} \sqcup T(D_{1,1}) \sqcup \ldots \sqcup T^{k_1}(D_{1,1})) \sqcup \ldots \sqcup (D_{1,j_1} \sqcup T(D_{1,j_1}) \sqcup \ldots \sqcup T^{k_1}(D_{1,j_1})) \ldots \ldots \ldots \ldots \ldots (D_{N,1} \sqcup T(D_{N,1}) \sqcup \ldots \sqcup T^{k_N}(D_{N,1})) \sqcup \ldots \sqcup (D_{N,j_N} \sqcup T(D_{N,j_N}) \sqcup \ldots \sqcup T^{k_N}(D_{N,j_N}))$$
of $C$ is a refinement of $\mathcal{P}$. Indeed, this can be obtained as follows. Assume there exists a clopen set $A \in \mathcal{P}$ such that $A \cap T^i(D_j) \neq \emptyset$ and $A \Delta T^i(D_j) \neq \emptyset$ for some $i,j$. Then we refine the partition $\mathcal{P}$ by the sets $T^s(T^{-i}(A) \cap D_j)$, $0 \leq s \leq k_j$. Since $\mathcal{P}$ is finite partition this operation is exhaustive.
Bibliography


[34] Elek, G., Monod, N., *On the topological full group of minimal $\mathbb{Z}^2$-systems*, to appear in Proc. AMS.


