Lecture 7: Minimal subshifts. Finite generation of the commutator subgroup.

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The aim of this section is to prove that every commutator subgroup of a Cantor minimal subshift is finitely generated, Theorem 0.0.3. This result is due to Hiroki Matui, [69]. Through this section we assume that $T$ is a minimal homeomorphism of the Cantor set.

Let $U$ be a clopen set in $C$, such that $U, T(U)$ and $T^{-1}(U)$ are pairwise disjoint. Define

$$f_U(x) = \begin{cases} 
T(x), & x \in T^{-1}(U) \cup U \\
T^{-2}(x), & x \in T(U) \\
x, & \text{otherwise}
\end{cases}$$

Obviously, $f_U$ is a homeomorphism of $C$ and $f_U \in [[T]]$. Moreover, we claim that $f_U$ is in the commutator subgroup $[[T]]'$. To verify the claim, define a symmetry in $[[T]]$:

$$g(x) = \begin{cases} 
T(x), & x \in T^{-1}(U) \\
T^{-1}(x), & x \in U.
\end{cases}$$

One verifies that $f_U = [g, f_U]$ by identifying $f_U$ with cycle $(123)$ and $g$ with cycle $(12)$.

Consider the following set of elements of $[[T]]'$

$$\mathcal{U} = \{f_U : U \text{ is clopen set and } U, T(U), T^{-1}(U) \text{ are pairwise disjoint}\}$$

**Lemma 0.0.1.** The commutator subgroup of the full topological group $[[T]]'$ is generated by $\mathcal{U}$.

**Proof.** Let $H$ be a subgroup of $[[T]]$ generated by $\mathcal{U}$. We start by showing that if $g \in [[T]]$ and $g^3 = e$, then $g$ is in $H$. Since each $f_U$ is of order 3, this will imply that $H$ is normal. Therefore because of simplicity of $[[T]]'$ we would be able to conclude that $H = [[T]]'$.

By Lemma ?? and Lemma ?? we can find a clopen set $A$ such that $A$, $g(A)$ and $g^2(A)$ are pairwise disjoint and $\text{supp}(g) = A \sqcup g(A) \sqcup g^2(A)$. Let now $B_i$ be a clopen partition of $C$ such that the restriction of $g$ to each $B_i$ coincides with a certain power of $T$. Consider the following partitions of $A$:

$$\mathcal{P}_0 = \{B_i \cap A\}_{1 \leq i \leq n},$$

$$\mathcal{P}_1 = g^{-1}\{B_i \cap g(A)\}_{1 \leq i \leq n},$$

$$\mathcal{P}_2 = g^{-2}\{B_i \cap g^2(A)\}_{1 \leq i \leq n}.$$
Denote the common refinement of $P_0$, $P_1$ and $P_2$ by $\{A_j\}_{1 \leq j \leq m}$. It has the property that for every $1 \leq j \leq m$ there are integers $k_j$, $l_j$ such that

$$
g|_{A_j} = T^{k_j}|_{A_j}, \quad g|_{g(A_j)} = T^{l_j}|_{g(A_j)}, \quad g|_{g^2(A_j)} = T^{-k_j-l_j}|_{g^2(A_j)}.\]

Now we can decompose $g = g_1 \ldots g_m$ as a product of commuting elements of $[[T]]$ defined by the restriction of $g$ onto $A_j \cup g(A_j) \cup g^2(A_j)$. This implies that it is sufficient to consider the case when $g$ is in $[[T]]$ of the order 3 and there exists a clopen set $A \subset C$ such that there are $k$ and $l$ with

$$
g|_{A} = T^k|_{A}, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)}.\]

Since for any $x \in A$ there exists a clopen neighborhood $U_x \subseteq A$ such that $\{T^i(U_x)\}_{1 \leq i \leq k+l}$ are pairwise disjoint and $A$ is compact, we can select a finite family $U_x$ that covers $A$. Let $C_1, \ldots, C_n$ be the partition of $A$ generated by these finite family. Let $g = g_1 \ldots g_m$ be the decomposition of $g$ into a product of commuting elements of $[[T]]$ defined by taking $g_i$ to be the restriction of $g$ to $C_i \cup g(C_i) \cup g^2(C_i)$.

Thus this reduces the argument to the case when $g \in [[T]]$ has the following property: $g^3 = id$ and there exists a clopen set $A \subset C$ such that there are $k$ and $l$ with

$$
g|_{A} = T^k|_{A}, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)};\]

and $T^i(A) \cap T^j(A) = \emptyset$ for all $1 \leq i, j \leq k + l$. This element can be considered as a cycle $(k \, l \, k + l)$ of the permutation group $S_{k+l+1}$ and each cycle $(i-1 \, i \, i+1)$ is given by $f_{T^i(A)}$. Moreover, $g$ is in the alternating group $A_{k+l+1}$, which contains all 3-cycles. Thus $g$ is a product of elements of $U$, which finishes the lemma. \hfill \square

Note that the proof of lemma shows slightly more. Namely, for every prime number $p$ and an element of order $p$ in $[[T]]$, this element belongs to the commutator subgroup.

**Lemma 0.0.2.** Let $U$ and $V$ be clopen subsets of $C$, then the following holds

(i) If $T^2(V)$, $T(V)$, $V$, $T^{-1}(V)$, $T^{-1}(V)$ are pairwise disjoint and $U \subseteq V$, then for $\tau_U = f_{T^{-1}(U)} f_{T(U)}$ we have

$$
\tau_V f_U \tau_V^{-1} = f_{T(U)} \\
\tau_V^{-1} f_U \tau_V = f_{T^{-1}(U)}
$$

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(ii) If $V, U, T^{-1}(U), T(U) \cup T^{-1}(V), T(V)$ are pairwise disjoint then

$$[f_V, f_U^{-1}] = f_{T(U) \cap T^{-1}(V)}.$$ 

**Proof.** The proof of the lemma boils down to the identification of the elements involved in the statement with permutations.

(i). The support of $\tau_{V \setminus U}$ is disjoint from supports of other homomorphism, thus

$$\tau_V f_U \tau_V^{-1} = \tau_U f_U \tau_U^{-1} = f_{T(U)};$$

where the last identity is the consequence of the identity in the permutation group $(01234)(123)(04321) = (012)$.

(ii). Let $C = T(U) \cap T^{-1}(V)$. We can decompose $f_U = f_{T^{-1}(C)} f_{U \setminus T^{-1}(C)}$ and $f_V = f_{T(C)} f_{V \setminus T(C)}$. Thus

$$[f_V, f_U^{-1}] = [f_{T(C)}, f_{T^{-1}(C)}] = f_{T(C)} f_{T^{-1}(C)} f_{T(C)} f_{T^{-1}(C)} = f_C,$$

where the last identity is equivalent to the identity in the permutation group:


\[\square\]

**Theorem 0.0.3.** Let $T \in \text{Homeo}(C)$ is minimal homeomorphism. The commutator subgroup $[[T]]'$ is finitely generated if and only if $T$ is conjugate to a minimal subshift.

**Proof.** Assume that $T \in \text{Homeo}(C)$ is a minimal subshift, i.e., $T$ acts as a shift on the Cantor set $A^\mathbb{Z}$ for some finite alphabet $A$ and there exists a clopen $T$-invariant subset $X \subset A^\mathbb{Z}$ such that the action of $T$ on $X$ is minimal. Moreover, enlarging the alphabet and using the characterization of the minimal subshifts in terms of homogeneous sequences, we can assume that $x(i) \neq x(j)$ for every $|i - j| < 4$ and $x \in X$.

For every $n, m \in \mathbb{N}$ and $a_i \in A$, $-m \leq i \leq n$, define the cylinder sets in $X$ by

$$\langle\langle a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_n \rangle\rangle = \{x \in X : x(i) = a_i, -m \leq i \leq n\},$$
here the underlining of \(\alpha_0\) means that \(\alpha_0\) is in the 0's coordinate of \(\mathbb{Z}\)-enumeration. Since \(x(i) \neq x(j)\) for every \(|i - j| < 4\) we have that for every cylinder set \(U\) the sets \(T^{-2}(U), T^{-1}(U), U, T(U), T^2(U)\) are pairwise disjoint.

Let \(H\) be a subgroup of \([[T]]'\) generated by the finite set of cylinders:

\[
\{ f_U : U = \langle \langle \alpha_0 \rangle \rangle, \alpha, b, c \in A \}. 
\]

We will show that \(H = [[T]]'\). By Lemma 0.0.1 it is sufficient to show that for every cylinder sets \(U \in X\), we have \(f_U \in H\).

Since

\[
f_{T(\langle \langle \alpha \rangle \rangle)} = \prod_{b \in A} f_{\langle \langle \alpha b \rangle \rangle}, \quad f_{T^{-1}(\langle \langle \alpha \rangle \rangle)} = \prod_{b \in A} f_{\langle \langle b \rangle \rangle},
\]

we immediately have \(f_{T(\langle \langle \alpha \rangle \rangle)}, f_{T^{-1}(\langle \langle \alpha \rangle \rangle)}\), thus \(\tau_{\langle \langle \alpha \rangle \rangle}\) is in \(H\). Applying Lemma 0.0.2 to the sets

\[
U = \langle \langle a_m \ldots a_{-1}a_a1 \ldots a_n \rangle \rangle \subseteq \langle \langle a_0 \rangle \rangle = V
\]

we obtain:

\[
\tau_{\langle \langle a_0 \rangle \rangle} f_U \tau_{\langle \langle a_0 \rangle \rangle}^{-1} = f_T(U), \quad \tau_{\langle \langle a_0 \rangle \rangle}^{-1} f_U \tau_{\langle \langle a_0 \rangle \rangle} = f_{T^{-1}(U)}.
\]

Hence we conclude the statement of the lemma by induction on \(m + n\) and applying Lemma 0.0.2 (ii) to the sets \(V = \langle \langle a_m \ldots a_{-1}a_a1 \rangle \rangle\) and \(U = \langle \langle a_1 a_2 \rangle \rangle\).

To prove the converse, assume that \(T \in \text{Homeo}(C)\) is minimal and \([[T]]'\) is finitely generated. Let \(g_1, \ldots, g_n\) be the generating set of \([[T]]'\) and \(n_i : C \to \mathbb{Z}\) be continuous maps that satisfy:

\[
g_i(x) = T_{n_i}(x), \quad x \in C.
\]

Let \(P = \{ P_1, \ldots, P_n \} \) be the common refinement of the partition \(\{ n_i^{-1}(k) \}_{k \in \mathbb{Z}}\).

We will consider \(P\) as a finite alphabet together with shift map \(s : P \to P\).

Define a continuous map \(S : X \to P\) by the property that \(S(x)(k) = P_s\), if \(T^k(x) \in P_s\). It is easy to verify that \(S\) is a factor map. Define a homeomorphism \(f_i \in \text{Homeo}(C)\) by \(f_i(z) = S^k(z)\) when \(z(0) \subseteq n_i^{-1}(k)\). It is easy to see that \(f_i \in [[s]]\) and \(Sg_i = f_i S\). It remains to show that \(S\) is injective.

Suppose \(x, y \in C\) are distinct and \(S(x) = S(y)\). Let \(g \in [[T]]'\) such that \(g(x) \neq x\) and \(g(y) = y\). By assumptions \([[T]]'\) is finitely generated, thus we can write \(g\) as a word on the generators \(w(g_1, \ldots, g_n)\).
\[ S_g(x) = S_w(g_1, \ldots, g_n)(x) \]
\[ = w(f_1, \ldots, f_n)S(x) \]
\[ = w(f_1, \ldots, f_n)S(y) \]
\[ = S_w(g_1, \ldots, g_n)(y) \]
\[ = S_g(y) = S(x). \]

Hence, for some \( k \) we have \( s^kS(x) = S(T(x)) = S(x) \), which contradicts to the minimality of \( s \) and thus of \( T \). \( \square \)
Bibliography


[34] Elek, G., Monod, N., *On the topological full group of minimal \( \mathbb{Z}^2 \)-systems*, to appear in Proc. AMS.


[67] Lebesgue, H., Sur l’intégration et la recherche des fonctions primitive, professées au au Collège de France (1904)


