Lecture 9: Elementary subexponentially amenable groups

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0.1 Elementary subexponentially amenable groups

Recall that a group $\Gamma$ generated by a finite set $S$ is of subexponential growth if $\lim_{n \to \infty} |B(n)|^{1/n} = 1$, where $B(n)$ is the ball of radius $n$ in the Cayley graph $(\Gamma, S)$.

Let $SG_0$ be the class of groups whose all finitely generated subgroups have exponential growth. Let $\alpha > 0$ be an ordinal such that we have already defined $SG_\beta$ for all $\beta < \alpha$. Then if $\alpha$ is a limit ordinal, we set

$$SG_\alpha = \bigcup_{\beta < \alpha} SG_\beta$$

and if $\alpha$ is not a limit ordinal, we set $SG_\alpha$ to be the class of groups which can be obtained from $SG_{\alpha-1}$ by applying either extension or a direct union one time. Define

$$SG = \bigcup_\alpha SG_\alpha$$

**Theorem 0.1.1.** The class $SG$ is the smallest class of groups which contain all groups of sub-exponential growth and is closed under taking subgroups, quotients, extensions, and direct unions.

The class $SG$ will be called as the class of elementary subexponentially amenable groups. Obviously, the class $SG$ contains the class of elementary amenable groups $EG$.

In this section we will show that Basilica group is not elementary subexponentially amenable, which was proved in [48].

**Theorem 0.1.2.** The Basilica group is not elementary subexponentially amenable. In particular, it is not elementary amenable.

The proof will be subdivided into two lemmas. We will use the following notations for the lemmas. Since $G$ acts on the rooted a binary tree, we denote the stabilizer of the $n$-th level of the tree by $St_G(n)$.

Then there are naturally defined $2^n$ projections $p_k$, $1 \leq k \leq 2^n$, defined on $St_G(n)$ as a restriction to the $k$-th tree on the $n$-th level.

**Lemma 0.1.3.** For the Basilica group $G$ we have $G' \geq G' \times G'$. 
Proof. It is sufficient to show that the projections onto both coordinates of $\text{St}_G(1)$ coincide with $G$. Direct computation shows

$$
a^b = (a^{-1}, 1)(1, b)(1, a) = (a^{-1}ba, 1) = (b^a, 1) \in \text{St}_G(1)
$$

$$
b^2 = (a, a) \in \text{St}_G(1),
$$

and moreover $a = (1, b) \in \text{St}_G(1)$. Thus, $p_1(\text{St}_G(1)) = p_2(\text{St}_G(1)) = G$.

Note that

$$
[a, b^2] = (1, b^{-1})(a^{-1}, a^{-1})(1, b)(a, a) = (1, b^{-1}a^{-1}ba) = (1, [b, a]).
$$

Summarizing above, we obtain

$$
G' \geq \langle[a, b^2]^G \rangle \geq \{e\} \times \langle[b, a]^G \rangle = \{e\} \times G'.
$$

Moreover, $(\{e\} \times G')^b = G' \times \{e\}$, therefore we have $G' \geq G' \times G'$. \( \square \)

**Lemma 0.1.4.** For every normal subgroup $H$ of the Basilica group $G$ there exists a subgroup $K < H$ such that $K$ admits a quotient group containing $G$.

**Proof.** Let $g$ be a non-trivial element. Then let $n$ be a level of the tree which is fixed by $g$ and the next level is not fixed by $g$. Then for some $1 \leq k \leq 2^n$ $g$ has the form

$$
g = (g_1, \ldots, g_{k-1}, \overline{g}, g_{k+1}, \ldots, g_{2^n})
$$

where $\overline{g}$ is an automorphism of the tree which does not fix the first level. In other words, $\overline{g} = (g_1, g_2) \epsilon$, for some $g_1, g_2 \in G$.

For every $\overline{h} \in G'$ put $h = (1, \ldots, 1, \overline{h}, 1, \ldots, 1)$, where $\overline{h}$ is based on the $k$-th place. By Lemma 0.1.3, we obtain $h \in G'$ and thus

$$
[g, h] = (1, \ldots, 1, [\overline{g}, \overline{h}], 1, \ldots, 1) \in H.
$$

The same lemma implies that $\overline{h}$ can be chosen as $(w, 1)$, where $w$ is any element of $G'$. Hence

$$
[\overline{g}, \overline{h}] = (w, g_1^{-1}w^{-1}g_1).
$$

This implies that there exists a subgroup $K$ in $H$ such that $p_1(p_k(K))$ contains $G'$. Since we have the following expressions

$$
[b^{-1}, a] = (b^{-1}, b) \quad \text{and} \quad [a, b^2] = (1, b^{-1}a^{-1}ba)
$$

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we obtain that $b, b^a \in p_2(G')$. But

$$b^2 = (a, a) \text{ and } b^a = (b, b^{-1}a).$$

Thus $p_2(p_2(G'))$ contains $G$, which implies the desired statement. \hfill \square

Proof of the Theorem 0.1.2. Since, by the Corollary ??, the Basilica group has exponential growth we have that it is not in the class $SG_0$. To reach a contradiction, assume $G \in SG$ and let $\alpha$ be a minimal ordinal such that $G \in SG_\alpha$. Since $\alpha$ is minimal it is not limit ordinal. Thus $G$ is obtained from the class $SG_{\alpha-1}$ by taking either extension or a direct union. It can not be obtained as a direct union, since otherwise $G \in SG_{\alpha-1}$ which contradicts to minimality of $\alpha$. Hence $G$ is obtained as an extension of groups from the class $SG_{\alpha-1}$. Therefore we can find a normal subgroup $H$ of $G$, which is in the class $SG_{\alpha-1}$. Since all these classes are closed by taking subgroups and quotients, by Lemma 0.1.4 we obtain $G \in SG_{\alpha-1}$. \hfill \square
Bibliography


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