The Steady Boundary Value Problem for Charged Incompressible Fluids: PNP/Navier Stokes Systems

Joseph W. Jerome [a]

[a] *Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA*

Abstract

The initial-boundary value problem for the Poisson-Nernst-Planck/Navier-Stokes model was investigated in [Nonlinear Analysis 71 (2009), e2487–e2497], where an existence theory was demonstrated, based upon Rothe's method of horizontal lines. In this article, the steady case is considered, and the existence of a weak solution is established for the boundary-value problem. This solution satisfies a weak maximum principle for the concentrations relative to the boundary values. As noted in the above-mentioned citation, the model assumes significance because of its connection to the electrophysiology of the cell, including neuronal cell monitoring and microfluidic devices in biochip technology. The model has also been used in other applications, including electro-osmosis. The steady model is especially important in ion channel modeling, because the channel remains open for milliseconds, and the transients appear to decay on the scale of tens of nanoseconds.

Key words: Navier-Stokes, Poisson-Nernst-Planck, boundary-value problem for hybrid systems, existence, invariant region

1 Introduction

We consider the Poisson-Nernst-Planck/Navier-Stokes (PNP/NS) system of non-linear partial differential equations. The basic system was introduced by Rubinstein in [1]. Self-consistent charge transport is represented by the Poisson-Nernst-Planck system, and the fluid motion by a Navier-Stokes system with forcing terms. This model is capable of describing electro-chemical and fluid-mechanical transport throughout the cellular environment. This includes a range of spatial and temporal scales. For extensive applications, we refer the reader to [2–11], and, for appropriate numerical analysis, to [12,13].

A local smooth theory based upon these equations was derived for the Cauchy problem in [14], and a global theory of weak solutions for the initial-boundary-value problem in [15] (see also the paper of Schmuck [16]; this article is also interesting for its derivation, where ions of spherical shape are considered within the fluid). In this article, we analyze the *steady* boundary value problem in $\Omega \subset \mathbf{R}^m$, and demonstrate the existence of a weak solution (Theorem 6.1), satisfying a weak maximum principle for the concentrations. The steady model is especially important in ion channel modeling, because the channel remains open for milliseconds, and the transients appear to decay on the scale of tens of nanoseconds. This phenomenon is continually repeated with channel gating.

The model is described in the next section. The general approach to the Navier-Stokes subsystem is well documented in the mathematical literature, beginning with the seminal paper of Leray [17]. We adopt the definition of weak solution as presented in [18], and the use of Galerkin solutions as a tool of proof [19,20]. These approaches are discussed in detail in [21] and [22] (see also [23]). For the Fujita-Kato approach, see [24]. In this article, it has been found necessary to define a decomposition type map, with each of two component maps analyzed by the Galerkin method. Solutions are obtained via the Schauder theorem. Existence, but not uniqueness, is demonstrated. The hypotheses which we employ, particularly those which require certain inequalities among the viscosity coefficient, the domain diameter, the magnitude(s) of the boundary data, and some other parameters, are consistent with those in the mathematical literature cited above for the steady problem.

2 The Fluid/Transport System

2.1 The System Description

We recall the constitutive relations for the ionic current densities. They extend the usual relations, given in PNP theory, by the inclusion of velocity convection terms. If $\vec{v}$ is the velocity of the electrolyte, and the ionic concentrations are denoted by n, p, respectively, the current densities are:

$$\vec{J}_n = eD_n \nabla n - e\mu_n n \nabla \phi - e\vec{v}n, \tag{1}$$
$$\vec{J}_p = -eD_p \nabla p - e\mu_p p \nabla \phi + e\vec{v}p. \tag{2}$$

Here, J_n, J_p are the anion and cation current densities, with corresponding (constant) diffusion and mobility coefficients, D_n, D_p, μ_n, μ_p, respectively. The charge modulus is given by e, and ϕ is the electric potential. The Poisson equation, given shortly, describes the coupling. The enhanced PNP system is then given by, with ϵ the dielectric constant:

$$\frac{\partial n}{\partial t} - \frac{1}{e}\nabla\cdot \vec{J}_n = 0, \tag{3}$$

$$\frac{\partial p}{\partial t} + \frac{1}{e}\nabla\cdot \vec{J}_p = 0, \tag{4}$$

$$\vec{E} = -\nabla\phi, \tag{5}$$

$$\nabla\cdot(\epsilon\nabla\phi) = e(n-p) \quad \text{(Poisson equation)}. \tag{6}$$

The Einstein relations are employed: $D_n = (kT_0/e)\mu_n, D_p = (kT_0/e)\mu_p$. Here, T_0 is the ambient temperature; k denotes Boltzmann's constant.

The velocity of the electrolyte is determined by the Navier-Stokes equations:

$$\rho(\vec{v}_t + \vec{v}\cdot\nabla\vec{v}) - \eta\Delta\vec{v} = -\nabla P_f - e(p-n)\nabla\phi, \tag{7}$$

$$\nabla\cdot\vec{v} = 0, \tag{8}$$

where ρ is the constant (mass) density of the electrolyte, P_f denotes fluid pressure, and η is the constant dynamic viscosity. We shall make use of the kinematic viscosity, $\nu_* = \eta/\rho$, in the statement of the mathematical model. The above system is found essentially in [1]. As mentioned previously, we consider in this article the important case of the steady system. Its weak formulation is defined in a later subsection.

2.2 Boundary Conditions and Assumptions

We shall refer to the incompressibility condition (8) as a *divergence free* property. Projections onto such functions will be introduced subsequently. This follows the original idea of Leray, permitting a pressure-free formulation (see [22,21] for discussion). Scalar or vector functions with components in H^1 will be termed of *finite energy.* We now discuss the boundary conditions for the system. We distinguish between $\vec{v}, n, p$ and ϕ. For ϕ, the boundary of Ω decomposes into two components: Σ_1, which is a Dirichlet boundary component, and Σ_2, which serves as a zero flux boundary component. Dirichlet boundary values for $\vec{v}, n, p$ are imposed on all of $\partial\Omega$, with the *stipulation* that a nonnegative (outward) normal fluid velocity component condition $\vec{v}\cdot\vec{\nu} \geq 0$ is imposed on $\partial\Omega$. This stipulation is unnecessary if the Navier-Stokes system is replaced by the Stokes subsystem. More precisely, we assume the existence of functions $\vec{v}_B, \phi_B, n_B \geq 0, p_B \geq 0$, in appropriately smooth spaces (described in the following definition), such that the following requirements hold for the solution vector in the trace sense:

$$\vec{v}_{|\partial\Omega} = \vec{v}_{B|\partial\Omega},\ n_{|\partial\Omega} = n_{B|\partial\Omega},\ p_{|\partial\Omega} = p_{B|\partial\Omega},\ \phi_{|\Sigma_1} = \phi_{B|\Sigma_1}. \tag{9}$$

Here, we assume that $n_{B|\partial\Omega}$, $p_{B|\partial\Omega}$, are essentially bounded; also, that the boundary of Ω is sufficiently regular that the classical trace formulas and integration by parts formulas are valid. Finally, we assume that the Poisson solver is H^2 regularizing for L^2 data. This implies that the mixed boundary conditions imposed by ϕ_B are not completely arbitrary.

2.3 Weak Solution Characterization

Definition 2.1 *Let $s \geq m/2$ be prescribed. We denote by $\mathcal{H}$ the divergence free functions in the m-fold Cartesian product of $H^1(\Omega)$, by $\mathcal{H}^s$ the intersection of $\mathcal{H}$ with the m-fold Cartesian product of $H^s(\Omega)$, and by $\mathcal{H}^s_0$ the zero trace subspace of $\mathcal{H}^s$. $(\mathcal{H}^s)^*$ is the dual of the latter space. The boundary data functions are assumed to be in L^∞ and, for $s^* = \max(2, s)$:*

$$\vec{v}_B \in \mathcal{H}^s, n_B \in H^s, p_B \in H^s, \phi_B \in H^{s^*}; \nabla\cdot \vec{v}_B = 0.$$

We define the regularity class $\mathcal{C} = \{\vec{u} = (\vec{v}, \phi, n, p)\}$ by the explicit conditions:

$$\vec{v} \in \mathcal{H},\ n \text{ and } p \in H^1,\ \phi \in H^2.$$

A weak solution of the PNP/Navier-Stokes system is a vector $\vec{u} \in \mathcal{C}$ such that (9) holds, such that ϕ is related to n, p via the weak formulation of (6), and, for $a(\vec{v}, \vec{v}, \vec{\psi}) := \int_\Omega \vec{v}\cdot \nabla\vec{v}\vec{\psi}\, d\xi$, and functions $\vec{\psi}, \omega_n, \omega_p$, in $\mathcal{H}^s_0 \times H^s_0 \times H^s_0$, we have:

$$\begin{aligned}
&\int_\Omega [\nu_* \nabla\vec{v}\cdot \nabla\vec{\psi}]d\xi \ + a(\vec{v}, \vec{v}, \vec{\psi}) + (e/\rho)\int_\Omega (p-n)\nabla\phi\cdot \vec{\psi}\, d\xi = 0 \\
&\int_\Omega [D_n \nabla n\cdot \nabla\omega_n - \left(\frac{eD_n}{kT_0}\right) n\nabla\phi\cdot \nabla\omega_n - \vec{v}n\cdot \nabla\omega_n] d\xi = 0 \\
&\int_\Omega [D_p \nabla p\cdot \nabla\omega_p + \left(\frac{eD_p}{kT_0}\right) p\nabla\phi\cdot \nabla\omega_p - \vec{v}p\cdot \nabla\omega_p] d\xi = 0. \qquad \square
\end{aligned}$$

If the second term of the first equation is suppressed, the system reduces to the PNP/Stokes system. For convenience, the first term of the first equation uses a dot product notation to express the tensor action of row by row summation of dot products. This action is often expressed as $A : B$ for matrices A, B.

3 Framework: The Stationary Problem and the Fixed Point Map

The framework for the analysis is motivated by the results described in the monograph [23] and refined in [15]. It is based upon finite-dimensional approximations, combined with an appropriate passage to the limit. The method has its origin in earlier studies of [19,20]. It is described fully in [21] and [22].

3.1 The Framework and Abstract Stationary Problem

Let X, Y be separable, reflexive real Banach spaces, with Y a subspace, densely and continuously embedded in X. We also suppose that X is compactly embedded in the real reflexive Banach space W. In addition, we consider a mapping, $A : X \mapsto Y^*$, given explicitly by

$$A(u) = Lu + a(u, u, \cdot\,) + F(u, \cdot\,), \tag{10}$$

where $L : X \mapsto X^*$ is an isomorphism, The structure of L is induced by a continuous, coercive bilinear form $B(\cdot\,, \cdot\,)$ on $X \times X$:

$$\langle Lu, v\rangle = B(u, v),\ B(u, u) \geq c\|u\|_X^2.$$

We outline the general assumptions now. In addition to the stated hypotheses, we assume:

(1) a is continuous on $X \times X \times Y$ and F is continuous on $X \times Y$.
(2) For each $u \in X$, $a(u, u, \cdot\,), F(u, \cdot\,)$ are continuous *linear* functionals on Y.
(3) The coerciveness property,

$$\langle A(u), u\rangle/\|u\|_X \to \infty,\ \text{as } \|u\|_X \to \infty, u \in Y,$$

holds in the norm on X for elements in Y.
(4) If $u_k \rightharpoonup u$ (weakly) in X and $u_k \to u$ in W, then

$$a(u_k, u_k, v) \to a(u, u, v),\ \forall v \in Y,$$

$$F(u_k, v) \to F(u, v),\ \forall v \in Y.$$

The following theorem was proven in [15]. It will be required in this article.

Theorem 3.1 *Under the stated hypotheses, there is an element $u \in X$ satisfying $A(u) = f_{|Y}$ for a prescribed $f \in X^*$. The estimate, $\|u\|_X \leq \|f\|_{X^*}$ holds.*

3.2 The Fixed Point Mapping: Summary Statement

The idea of the proof of existence for the system defined in Definition 2.1 is to define a decoupling map T, whose fixed point(s) serve as solutions. Theorem 3.1 is used to analyze each of the subsystems involved in the decoupling. Thus, set

$$\mathcal{K} = \{(\tilde{n}, \tilde{p}) \in L^2(\Omega) \times L^2(\Omega) : 0 \leq \tilde{n} \leq \alpha, 0 \leq \tilde{p} \leq \alpha\}, \tag{11}$$

where α is defined by:

$$\alpha = \max\{\sup_{\partial\Omega} n_B, \sup_{\partial\Omega} p_B\}.$$

The fixed point map $T : \mathcal{K} \to \mathcal{K}$ is defined by decoupling. One begins with $(\tilde{n}, \tilde{p}) \in \mathcal{K}$, and obtains ϕ from the mixed boundary value problem associated with the Poisson equation. One uses ϕ, as well as $(\tilde{n}, \tilde{p})$, in the Navier-Stokes system to obtain a (weak-Leray) solution $\vec{v}$, which is guaranteed to be unique by the assumptions. One designates this map by V. The function $\vec{v}$ is used in the PNP system to solve uniquely for (n, p). In the process, ϕ is determined implicitly (see (18) below). This map is designated by U. Next, one verifies that $T := U \circ V$ has range in $\mathcal{K}$. Finally, the hypotheses of the Schauder fixed point theorem are verified. Theorem 6.1 expresses the principal result of the paper: the existence of a weak solution of the system of Definition 2.1. All of the hypotheses are assembled there. The theorem contains a refinement, covering the case when the boundary values of the concentrations are bounded from below by a positive constant δ. In this case, the definition of $\mathcal{K}$ remains unchanged: however, an 'a posteriori' argument established in Corollary 5.1 implies the further property asserted in Theorem 6.1.

3.3 The Boundary Condition Reductions

We make an explanatory comment here regarding the handling of boundary conditions. In order to maintain congruence with the existence theory, we shall write:

$$\vec{v} = \vec{v}_B + \vec{\sigma}, n = n_B + \nu, \ p = p_B + \pi,$$

where $\vec{\sigma}, \nu, \pi$ have zero boundary trace. However, in the actual verification of the continuity and coerciveness properties to follow, it is sometimes logically equivalent to consider $\vec{v}, n, p$, so that we adhere to this notationally simpler approach whenever possible.

4 The Mapping V

The mapping has been briefly discussed at the end of the previous section. Here, we give a precise analysis based upon Theorem 3.1.

4.1 Definition of V and Fundamental Lemma

Given $(\tilde{n}, \tilde{p}) \in \mathcal{K}$, and ϕ, satisfying the mixed boundary value problem for the equation, $\nabla \cdot (\epsilon \nabla \phi) = e(\tilde{n} - \tilde{p})$, define $\vec{v}$ to be a 'Leray' solution of the boundary value problem:

$$\int_\Omega [\nu_* \nabla \vec{v} \cdot \nabla \vec{\psi}] d\xi \ + a(\vec{v}, \vec{v}, \vec{\psi}) + (e/\rho) \int_\Omega (\tilde{p} - \tilde{n}) \nabla \phi \cdot \vec{\psi} \, d\xi = 0, \forall \vec{\psi} \in \mathcal{H}_0^s. \quad (12)$$

Lemma 4.1 *There is a solution $\vec{v}$ of the boundary value problem (12) if the product of diam(Ω) and the boundary value expression, $\|\vec{v}_B\|_{L^\infty}$, is sufficiently small with respect to ν_*. This is made precise in (15) below.*

We now link this mapping to the framework of Theorem 3.1.

4.1.1 Identification of Function Spaces and Mappings

Specifically, we require the finite energy spaces to be constrained further by zero boundary trace on $\partial\Omega$: $X = \mathcal{H}_0$. Similarly, the space Y is defined by $Y = \mathcal{H}_0^s$ and W is defined by $W = \prod_1^m L^2$. It will be convenient to use the equivalent norm on $\mathcal{H}_0$, defined by the standard $\prod H^1$ seminorm.

- We note that the assumptions of §3.1 hold for these spaces.

We now make the identifications with the mappings of §3.1.

- $B(\vec{\sigma}, \vec{\tau}) = \nu_* (\nabla \vec{\sigma}, \nabla \vec{\tau})_{L^2}$.
- The use of the form a requires the domain of the mapping to include two components for the fluid system. Theorem 3.1 does not require a to be trilinear, however. The form a is defined by

  $$a(\vec{\sigma}, \vec{\tau}, \cdot\,) = ((\vec{v}_B + \vec{\sigma}) \cdot \nabla (\vec{v}_B + \vec{\tau}), \cdot\,)_{L^2} = (\vec{v} \cdot \nabla \vec{w}, \cdot\,)_{L^2}.$$

  For later reference, we define: $b(\vec{v}, \vec{w}, \cdot\,) := a(\vec{\sigma}, \vec{\tau}, \cdot\,)$.
- We set $F = 0$, and

  $$f(\vec{\psi}) = (e/\rho)((\tilde{n} - \tilde{p}) \nabla \phi, \vec{\psi})_{L^2} - \nu_* (\nabla \vec{v}_B, \nabla \vec{\psi})_{L^2}.$$

4.2 Hypothesis Verification

We now proceed to verify the hypotheses of Theorem 3.1.

4.2.1 Continuity Properties

The analysis of the bilinear form B is standard. The properties of the functional a are now discussed. The earlier arguments of [21] essentially apply here, with minor adaptations. Joint continuity in the argument $(\vec{\sigma}, \vec{\tau}, \vec{\psi})$ (i. e., on $X \times X \times Y$) is equivalent to continuity of b in the argument $(\vec{v}, \vec{w}, \vec{\psi})$; continuity in the latter argument is a consequence of the following inequality:

$$|a(\vec{\sigma}, \vec{\tau}, \vec{\psi})| = |b(\vec{v}, \vec{w}, \vec{\psi})| \leq C\|\vec{v}\|_{L^2}\|\vec{w}\|_{H^1}\|\vec{\psi}\|_{H^s},$$

for $s \geq m/2$. This estimate first uses $b(\vec{v}, \vec{w}, \vec{\psi}) = -b(\vec{v}, \vec{\psi}, \vec{w})$, followed by the Hölder and Sobolev inequalities (see [21,23]). For the former inequality, the reciprocal indices are $1/2, 1/m, (m-2)/(2m)$, resp. Note that $s \geq m/2$ implies that H^{s-1} is continuously embedded into L^m. Continuity is established via the triangle inequality, after the standard addition and subtraction of like terms. Here, C is a constant obtained from the Sobolev embedding theorem [25].

4.2.2 Coerciveness

The bilinear form B is coercive by definition. It will be used in the subsequent analysis to absorb certain terms. We consider the functional a. First, notice that

$$a(\vec{\sigma}, \vec{\sigma}, \vec{\sigma}) = b(\vec{v}, \vec{v}, \vec{v} - \vec{v}_B) = b(\vec{v}, \vec{v}, \vec{v}) - b(\vec{v}, \vec{v}, \vec{v}_B).$$

We estimate these terms separately. The assumption on the normal boundary component implies that

$$b(\vec{v}, \vec{v}, \vec{v}) \geq 0,$$

since

$$b(\vec{v}, \vec{v}, \vec{v}) = \frac{1}{2}\int_\Omega \vec{v} \cdot \nabla|\vec{v}|^2 \, dx = \frac{1}{2}\int_\Omega \nabla \cdot (\vec{v}|\vec{v}|^2) \, dx,$$

which is then integrated by parts, after use of the divergence free property of $\vec{v}$. For the second term, we begin with the estimate:

$$-b(\vec{v}, \vec{v}, \vec{v}_B) \geq -\|\vec{v}\|_{L^2}\|\nabla\vec{v}\|_{L^2}\|\vec{v}_B\|_{L^\infty}. \quad (13)$$

By use of the decomposition, $\vec{v} = \vec{\sigma} + \vec{v}_B$, together with the triangle inequality, the coerciveness analysis of the term in (13) reduces to the analysis of the (numerator) term,

$$-\|\vec{\sigma}\|_{L^2}\|\vec{\sigma}\|_{\mathcal{H}_0}\|\vec{v}_B\|_{L^\infty}.$$

This expression, in turn, is absorbed into the corresponding part of B if $\|\vec{v}_B\|_{L^\infty}$ is sufficiently small, via Poincare's inequality [25, Theorem 12.17], which is stated as follows:

$$\|\vec{\sigma}\|_{L^2} \leq d_\Omega \|\nabla \vec{\sigma}\|_{L^2}, \ d_\Omega = \text{diam}(\Omega)/\sqrt{2}.$$

Notice that d_Ω has the units of length. Since we have defined the H_0^1 norm of $\vec{\sigma}$ as $\|\nabla \vec{\sigma}\|_{L^2}$, the inequality applies to give:

$$-\|\vec{\sigma}\|_{L^2}\|\vec{\sigma}\|_{\mathcal{H}_0}\|\vec{v}_B\|_{L^\infty} \geq -d_\Omega \|\vec{\sigma}\|^2_{\mathcal{H}_0}\|\vec{v}_B\|_{L^\infty}, \tag{14}$$

and this can be absorbed into B if

$$\nu_* > d_\Omega \|\vec{v}_B\|_{L^\infty}. \tag{15}$$

4.2.3 Sequential Convergence

Assumption (4) of §3.1 is almost immediate. The convergence result for a has been discussed in [21]. It is straightforward to see that f, as defined, is a continuous linear functional on $\mathcal{H}_0$. This completes the hypothesis verification. □

4.3 Summary Statement for the Stationary Problem for V

We have established that there is an element in the range of $V(\tilde{n}, \tilde{p})$. Uniqueness is known to hold if ν_* is sufficiently large [22, Theorem 4.2]. This will be assumed throughout this paper. Inequality (15) is consistent with the hypotheses of [22, Theorem 4.2].

Remark 4.1 *For later reference we recall the bound for solutions of (12). More precisely, this is a bound in $\mathcal{H}_0$ for the σ-component of $\vec{v}$, when expressed as $\vec{v} = \vec{v}_B + \sigma$. The bound of Theorem 3.1 translates here to a bound for:* $\|f\| \leq \|f_1\| + \|f_2\|$, *where*

$$\|f_1\| \leq \left(\frac{\alpha e}{\rho}\right) d_\Omega \|\nabla \phi\|_{L^2}, \|f_2\| \leq \nu_* \|\nabla \vec{v}_B\|_{L^2}.$$

It is possible to estimate to estimate $\|\nabla \phi\|_{L^2}$ by using the decomposition, $\phi = \phi_B + \phi_0$, where $\phi_0 = (-\Delta_0)^{-1}[(e/\epsilon)(\tilde{p} - \tilde{n}) + \Delta \phi_B]$ is the solution of the stated Poisson equation with mixed homogeneous boundary values. Note that $|\tilde{p} - \tilde{n}| \leq \alpha$.

5 The Mapping U

We give a precise analysis, based on Theorem 3.1.

5.1 *Definition of U and Fundamental Lemma*

Given a finite energy, divergence free, function $\vec{v}$, we define

$$U(\vec{v}) = (n, p),$$

where n, p satisfy the weak system, for $\omega_n, \omega_p \in H_0^s(\Omega)$:

$$\int_\Omega [D_n \nabla n \cdot \nabla \omega_n - \left(\frac{eD_n}{kT_0}\right) n \nabla \phi \cdot \nabla \omega_n - \vec{v} n \cdot \nabla \omega_n] d\xi = 0, \tag{16}$$

$$\int_\Omega [D_p \nabla p \cdot \nabla \omega_p + \left(\frac{eD_p}{kT_0}\right) p \nabla \phi \cdot \nabla \omega_p - \vec{v} p \cdot \nabla \omega_p] d\xi = 0. \tag{17}$$

In these equations, ϕ is defined *implicitly*:

$$\nabla \cdot (\epsilon \nabla \phi) = e(\tau(n) - \tau(p)), \tag{18}$$

where τ is the truncation operator:

$$\tau(u) = \begin{cases} u, \text{ for } 0 \leq u \leq \alpha, \\ 0, \quad \text{for } u < 0, \\ \alpha, \quad \text{for } u > \alpha. \end{cases}$$

Note that $\tau(n(x)) = n^+(x) - (n(x) - \alpha)^+$ follows from the definition of τ.

Lemma 5.1 *There is a unique solution pair (n, p) of the boundary value problem (16,17,18) if the product of α and d_Ω is sufficiently small (cf.(27)). The function pair (n, p) satisfies the invariant region property: $(n, p) \in \mathcal{K}$. In particular, $\tau(n) = n, \tau(p) = p$ in (18).*

We use Theorem 3.1 to establish existence. Uniqueness and the invariant region property are proved directly.

5.1.1 Identification of Function Spaces and Mappings

Specifically, we require the finite energy spaces to be constrained further by zero boundary trace on $\partial\Omega$:

$$X = H^1_0 \times H^1_0.$$

Similarly, the space Y is defined by

$$Y = H^s_0 \times H^s_0,$$

and W is defined by $W = \prod_1^2 L^2$. We retain the convention of the previous section for the norm in X; we use only the gradient semi-norm.

- We note that the assumptions of §3.1 hold for these spaces.

We now make the identifications with the mappings of §3.1. Accordingly, we write:

$$\vec{u} = (n, p) = (n_B + \nu, p_B + \pi),$$

and formulate the definitions in terms of $\vec{\zeta} = (\nu, \pi)$. Test functions are denoted by ω_n, ω_p.

- We begin with B, which depends upon $\vec{\zeta}, \omega_n, \omega_p$:

$$B(\vec{\zeta}; \omega_n, \omega_p) = D_n(\nabla\nu, \nabla\omega_n)_{L^2} + D_p(\nabla\pi, \nabla\omega_p)_{L^2}.$$

- We turn to the identification of F. Thus, define, in terms of $\vec{v}, \phi, n, p$:

$$F(\vec{v}; \phi, n, p; \omega_n, \omega_p) = -\mu_n(n\nabla\phi, \nabla\omega_n)_{L^2} - (n\vec{v}, \nabla\omega_n)_{L^2}$$

$$+\mu_p(p\nabla\phi, \nabla\omega_p)_{L^2} - (p\vec{v}, \nabla\omega_p)_{L^2}.$$

  Here, we use the relations: $\mu_n = \frac{eD_n}{kT_0}, \mu_p = \frac{eD_p}{kT_0}$.

- Finally, the right hand side element f is given by

$$f(\omega_n, \omega_p) = -D_n(\nabla n_B, \nabla\omega_n)_{L^2} - D_p(\nabla p_B, \nabla\omega_p)_{L^2}.$$

5.2 Hypothesis Verification

5.2.1 Continuity Properties

The analysis of the bilinear form B is standard. We now consider the functional F, and estimate the individual terms for continuity.

(1) The terms, $-\mu_n(n\nabla\phi, \nabla\omega_n)_{L^2}$ and $\mu_p(p\nabla\phi, \nabla\omega_p)_{L^2}$, make use of the es-

timate:

$$|\mu_n(n\nabla\phi, \nabla\omega_n)_{L^2}| \leq C\|n\|_{L^2}\|\phi\|_{H^2}\|\omega_n\|_{H^s}, \tag{19}$$

and a similar estimate with n replaced by p.

(2) The terms, $-(n\vec{v}, \nabla\omega_n)_{L^2}$ and $-(p\vec{v}, \nabla\omega_p)_{L^2}$, make use of

$$|(n\vec{v}, \nabla\omega_n)_{L^2}| \leq C\|n\|_{H^1}\|\vec{v}\|_{L^2}\|\omega_n\|_{H^s}, \tag{20}$$

and a similar estimate with n replaced by p.

In (19) and (20), C is obtained from the Sobolev embedding theorem. This is sufficient to prove that F is continuous on $X \times Y$.

5.2.2 Coerciveness

The bilinear form B is coercive by definition. It will be used in the subsequent analysis to absorb certain terms.

- We now turn to F.

(1) Each of the terms,

$$-(n\vec{v}, \nabla\nu)_{L^2},\ -(p\vec{v}, \nabla\pi)_{L^2},$$

is handled similarly. For the first, we write $n = n_B + \nu$, and consider the resulting pair of terms:

$$-(n_B\vec{v}, \nabla\nu)_{L^2},\ -(\nu\vec{v}, \nabla\nu)_{L^2}.$$

The first of these two terms is estimated by a constant times the H_0^1 norm of ν; the constant results from the fact that the product, $n_B\vec{v}$, is in L^2. Thus, this term, when divided by the H_0^1 norm of ν, is estimated by a constant. We claim that the second term is zero. Since $\nu \in H_0^s$, we write:

$$\nu\vec{v}\cdot\nabla\nu = \frac{1}{2}\nabla\cdot(\vec{v}\nu^2).$$

The integral of this expression is zero by the divergence theorem.

(2) Estimation of

$$-\mu_n(n\nabla\phi, \nabla\nu)_{L^2},\ \mu_p(p\nabla\phi, \nabla\pi)_{L^2},$$

is handled as follows. We consider the expression for n. The expression for p follows the same logic. One begins with the decomposition for n and the term splitting. The fixed term involving n_B is treated as follows.

$$|\mu_n(n_B\nabla\phi, \nabla\nu)_{L^2}| \leq \mu_n\|n_B\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\nabla\nu\|_{L^2}.$$

Remark 4.1 permits the estimation of $\|\nabla\phi\|_{L^2}$ in terms of a fixed constant.

When divided by the appropriate norm, the product is thus estimated by a fixed constant. For the second term in the splitting, we write:

$$-\mu_n(\nu\nabla\phi, \nabla\nu)_{L^2} = -(\mu_n/2)(\nabla\phi, \nabla\nu^2)_{L^2}.$$

This is integrated by parts, and the definition of ϕ as the solution of the Poisson equation, based on $\tau(n), \tau(p)$, is utilized. One obtains the following relation:

$$-\mu_n(\nu\nabla\phi, \nabla\nu)_{L^2} = \frac{e\mu_n}{2\epsilon}\left[\int_\Omega \tau(n)\nu^2\,dx - \int_\Omega \tau(p)\nu^2\,dx\right].$$

If αd_Ω is sufficiently small, this can be absorbed into B by use of the Poincare inequality. More precisely, this holds if

$$\frac{e^2\alpha d_\Omega^2}{2\epsilon k T_0} < 1, \tag{21}$$

by use of the Einstein relations. The analysis of the p-equation is parallel.

Altogether, the coerciveness follows.

5.2.3 Sequential Convergence and Remarks

Assumption (4) of §3.1 is verified as follows. The terms involving μ_n, μ_p use the analysis of inequality (19), following the use of the Rellich theorem. The two terms involving $\vec{v}$ use the Rellich theorem and the fact that $\vec{v}\cdot\nabla\omega_n$ and $\vec{v}\cdot\nabla\omega_p$ are both square integrable. Again, f is continuous on X as defined. This completes the hypothesis verification. □

Remark 5.1 *Theorem 3.1 now applies to yield a solution of the system. In addition, any solution satisfies the norm gradient bound for (ν, π) induced by the linear functional f. In this case, the bound is written,*

$$\|f\| \leq D_n\|\nabla n_B\|_{L^2} + D_p\|\nabla p_B\|_{L^2}.$$

Remark 5.2 *The following sections treat the remaining logical issues required for a complete definition of U: the invariant region property and uniqueness. There is an important technical consideration which is briefly discussed now. In order to prove the results to follow, it will be necessary to select test functions ω_n, ω_p from a less restrictive function space than H_0^s. It is possible to show, via limits of C_0^∞ functions, that a function in H_0^1 can be selected, provided each inner product involves a triple pointwise product which is an integrable function. We will typically employ functions in $H_0^1 \cap L^\infty$.*

We establish upper and lower bounds. We begin with the former.

Lemma 5.2 *If* (n, p) *is a solution of the system: (16, 17, 18), then*

$$n \leq \alpha, \;\; p \leq \alpha, \;\; a.e..$$

Proof Fix an arbitrary positive real number β. In order to satisfy the restriction discussed in the previous remark, one uses $\omega_n = \sigma(n)$ as a test function in (16), where we define

$$\sigma(u) = \begin{cases} u - \alpha, \text{ for } \alpha \leq u \leq \alpha + \beta, \\ 0, \qquad\quad \text{for } u < \alpha, \\ \beta, \qquad\quad \text{for } u > \alpha + \beta. \end{cases}$$

In fact, $\sigma(n) = (n - \alpha)^+ - (n - \alpha - \beta)^+$ represents $\sigma(n)$ as the difference of functions in H_0^1. The following identities are essential for the analysis:

$$\nabla n \cdot \nabla \sigma(n) = \nabla \sigma(n) \cdot \nabla \sigma(n),$$

$$(n - \alpha)\nabla \sigma(n) = (1/2)\nabla[\sigma(n)]^2,$$
$$\sigma(n)\nabla \cdot (n\vec{v}) = \sigma(n)\vec{v} \cdot \nabla(n - \alpha)^+.$$

The use of the third identity arises after integration by parts of the final term in (16). Note that the right hand side of this identity has a zero integral, which follows from the following divergence expression:

$$\sigma(n)\vec{v} \cdot \nabla(n - \alpha)^+ = \begin{cases} \nabla \cdot (\vec{v}(1/2)[\sigma(n)]^2), \text{ for } n \leq \alpha + \beta, \\ \beta \, \nabla \cdot (\vec{v}(n - \alpha)^+), \;\text{ for } n > \alpha + \beta. \end{cases}$$

The application of the three identities gives:

$$D_n \int_\Omega |\nabla \sigma(n)|^2 \, dx = \mu_n \int_\Omega \nabla \phi \cdot \nabla (1/2)[\sigma(n)]^2 \, dx + \alpha \mu_n \int_\Omega \nabla \phi \cdot \nabla \sigma(n) \, dx. \tag{22}$$

If the first term on the right hand side of (22) is integrated by parts, one obtains for this term:

$$\mu_n \int_\Omega \nabla \phi \cdot \nabla (1/2)[\sigma(n)]^2 \, dx = \frac{e\mu_n}{\epsilon} \int_\Omega [(\tau(p) - \alpha) - (\tau(n) - \alpha)](1/2)[\sigma(n)]^2 \, dx.$$

Since

$$(\tau(n) - \alpha)[\sigma(n)]^2 = 0, \ (\tau(p) - \alpha)[\sigma(n)]^2 \leq 0,$$

one concludes that

$$\mu_n \int_\Omega \nabla\phi \cdot \nabla(1/2)[\sigma(n)]^2 \, dx \leq 0.$$

In a similar manner, the second right hand side term is estimated as:

$$\alpha\mu_n \int_\Omega \nabla\phi \cdot \nabla\sigma(n) \, dx = \frac{\alpha e\mu_n}{\epsilon} \int_\Omega [(\tau(p) - \alpha) - (\tau(n) - \alpha)]\sigma(n) \, dx \leq 0.$$

One concludes from these inequalities:

$$D_n \int_\Omega [\nabla\sigma(n)]^2 \, dx \leq 0.$$

This inequality implies that $\sigma(n) = 0$. In particular, for each $\beta > 0$, the set $\{x \in [\alpha, \alpha + \beta] : n(x) > \alpha\}$ has measure zero. This completes the proof that $n \leq \alpha$. A parallel proof works for p. □ We derive (nonnegativity) lower bounds in the following lemma. Following the lemma, we present a corollary which provides a sharper lower bound if the boundary data are bounded away from zero. The corollary uses the result of the lemma, but requires a slightly stronger version of (21). We mention here that the sharper bounds do not require a modification of the set $\mathcal{K}$.

Lemma 5.3 *If inequality (21) holds, one has*

$$n \geq 0, p \geq 0, \ a.e.,$$

for any solution (n, p) satisfying (16, 17, 18).

Proof One uses $\omega_n = n^-$ as a test function in (16). Here, n^- is the negative part of n, defined in the standard manner. According to Remark 5.2, this is an admissible choice. One obtains, after the use of identities analogous to those used in the proof of the previous lemma,

$$D_n \int_\Omega [\nabla n^-]^2 \, dx = \mu_n \int_\Omega \nabla\phi \cdot \nabla(1/2)[n^-]^2 \, dx. \tag{23}$$

Integration by parts yields

$$\mu_n \int_\Omega \nabla\phi \cdot \nabla(1/2)[n^-]^2 \, dx = \frac{e\mu_n}{\epsilon} \int_\Omega [\tau(p) - \tau(n)](1/2)[n^-]^2 \, dx.$$

By use of the identity, $\tau(n)[n^-]^2 = 0$, one has

$$D_n \int_\Omega [\nabla n^-]^2 \, dx \leq \frac{e\mu_n}{\epsilon} \int_\Omega \alpha(1/2)[n^-]^2 \, dx.$$

If inequality (21) holds, then the right hand side can be absorbed into the left hand side, and one concludes that $n^- = 0$. The proof for p is parallel. $\Box$

Corollary 5.1 *Suppose that δ is defined by:*

$$\delta = \min\{\inf_{\partial\Omega} n_B, \inf_{\partial\Omega} p_B\}, \tag{24}$$

and that $\delta > 0$. Suppose the strengthened inequality,

$$\frac{e^2 d_\Omega^2}{\epsilon k T_0}\left(\frac{\alpha}{2} + \delta\right) < 1, \tag{25}$$

holds. Then

$$n \geq \delta, p \geq \delta, \text{ a.e.},$$

for any solution (n, p) satisfying (16, 17, 18).

Proof The proof requires both of the equations, for n and p, resp. For $\sigma(\cdot) = (\cdot - \delta)^-$, these are written as the following system:

$$\begin{aligned}
D_n \int_\Omega |\nabla\sigma(n)|^2 \, dx &= \mu_n \int_\Omega \nabla\phi \cdot \nabla(1/2)[\sigma(n)]^2 \, dx + \delta\mu_n \int_\Omega \nabla\phi \cdot \nabla\sigma(n) \, dx, \\
D_p \int_\Omega |\nabla\sigma(p)|^2 \, dx &= -\mu_p \int_\Omega \nabla\phi \cdot \nabla(1/2)[\sigma(p)]^2 \, dx - \delta\mu_p \int_\Omega \nabla\phi \cdot \nabla\sigma(p) \, dx.
\end{aligned}$$

Notice that the choice of test functions leads to the vanishing of the velocity terms, as in each of the two previous lemmas. The first terms on the r.h.s. of the system equations are estimated as in the previous lemma, via similar manipulations:

$$\mu_n \int_\Omega \nabla\phi \cdot \nabla(1/2)[\sigma(n)]^2 \, dx \leq \frac{e\mu_n\alpha}{2\epsilon} \int_\Omega [\sigma(n)]^2 \, dx,$$

$$-\mu_p \int_\Omega \nabla\phi \cdot \nabla(1/2)[\sigma(p)]^2 \, dx \leq \frac{e\mu_p\alpha}{2\epsilon} \int_\Omega [\sigma(p)]^2 \, dx.$$

Each of these terms is absorbed into the respective l.h.s. by inequality (25). The second r.h.s. terms of the system equations must be added and estimated. One obtains, after integration by parts:

$$\delta\mu_n \int_\Omega \nabla\phi \cdot \nabla\sigma(n) \, dx - \delta\mu_p \int_\Omega \nabla\phi \cdot \nabla\sigma(p) \, dx = (e\delta/\epsilon) \int_\Omega (n-p)(\mu_n\sigma(n) - \mu_p\sigma(p)) \, dx. \tag{26}$$

The integrand of the latter expression is analyzed by decomposing Ω into four sets (up to sets of measure zero), depending on whether $n(x) < \delta, n(x) \geq \delta$, $p(x) < \delta, p(x) \geq \delta$. These inequalities present four possibilities. Since we know that n, p are nonnegative from the previous lemma, a study of the four cases yields

$$(n - p)(\mu_n\sigma(n) - \mu_p\sigma(p)) \leq \mu_n[\sigma(n)]^2 + \mu_p[\sigma(p)]^2,$$

in each case. Thus, the sum, (26), can be absorbed into the sum of the l.h.s. of the system equations, via (25). This completes the argument, since one can proceed as previously. □

5.4 Uniqueness for Lemma 5.1

Lemma 5.4 *If the product αd_Ω is sufficiently small, uniqueness holds in the sense of Lemma 5.1. Specifically, we require:*

$$\left(\frac{e^2\alpha d_\Omega}{\epsilon k T_0}\right)\left(\frac{2\max(D_n, D_p)}{\min(D_n, D_p)}\sqrt{\|(-\Delta_0)^{-1}\|} + \frac{d_\Omega}{2}\right) < 1. \tag{27}$$

In this inequality, $\|(-\Delta_0)^{-1}\|$ has the units of length squared.

Proof For a given finite energy, divergence free, function $\vec{v}$, we suppose that (n_i, p_i) are solutions of the system (16,17,18). We will use the notation:

$$B(\nu, \pi) = D_n(\nabla\nu, \nabla\nu)_{L^2} + D_p(\nabla\pi, \nabla\pi)_{L^2}.$$

The roles of ν, π will be assumed in this proof by $n_1 - n_2, p_1 - p_2$, resp. These are the choices of the test functions ω_n, ω_p also. These are valid choices as observed in Remark 5.2. We can now write, for the subtraction of the equations for n_1, n_2:

$$D_n(\nabla\nu, \nabla\nu)_{L^2} = \mu_n(\nu\nabla\phi_1, \nabla\nu)_{L^2} + \mu_n(n_2\nabla(\phi_1 - \phi_2), \nabla\nu)_{L^2} + (\vec{v}, \nu\nabla\nu)_{L^2}.$$

Similarly, the subtraction of the equations for p_1, p_2 yields:

$$D_p(\nabla\pi, \nabla\pi)_{L^2} = -\mu_p(\pi\nabla\phi_1, \nabla\pi)_{L^2} - \mu_p(n_2\nabla(\phi_1 - \phi_2), \nabla\pi)_{L^2} + (\vec{v}, \pi\nabla\pi)_{L^2}.$$

The third term on the right hand side of each of these equations is zero since $\vec{v}$ is divergence free and ν, π have zero boundary trace. The second (right hand side) terms are the most subtle; we temporarily defer their analysis. The first terms require only their respective equations. We write:

$$\mu_n(\nu\nabla\phi_1, \nabla\nu)_{L^2} = \frac{\mu_n e}{2\epsilon}(\tau(p_1) - \tau(n_1), \nu^2)_{L^2}$$

for the first equation, with a similar representation for the second. Here, we have integrated by parts, and applied the Poisson equation. Since $|\tau(p_1) - \tau(n_1)| \leq \alpha$, Inequality (27) permits this term to be absorbed into the left hand side. Now the inequality,

$$\frac{2\max(D_n, D_p)}{\min(D_n, D_p)}\sqrt{\|(-\Delta_0)^{-1}\|}\left(\frac{e^2\alpha d_\Omega}{\epsilon k T_0}\right) < 1, \tag{28}$$

guarantees that the sum of the second terms can be absorbed into the left hand side (cf. (27). To verify this, consider the estimate,

$$|\mu_n(n_2\nabla(\phi_1-\phi_2),\nabla\nu)_{L^2}|+|\mu_p(p_2\nabla(\phi_1-\phi_2),\nabla\pi)_{L^2}|$$

$$\leq \alpha(\mu_n\|\nabla\nu\|_{L^2}+\mu_p\|\nabla\pi\|_{L^2})\|\nabla(\phi_1-\phi_2)\|_{L^2}.$$

The term, $\|\nabla(\phi_1-\phi_2)\|_{L^2}$, is the central term to bound. If the square of this term is integrated by parts, and the Poisson equation is utilized, one obtains:

$$\|\nabla(\phi_1-\phi_2)\|_{L^2}^2=(e/\epsilon)^2((-\Delta_0)^{-1}(\pi-\nu),\pi-\nu))_{L^2}.$$

Upon further estimation, including the triangle inequality applied to the term, $\|\pi-\nu\|_{L^2}$, one sees that the sum

$$|\mu_n(n_2\nabla(\phi_1-\phi_2),\nabla\nu)_{L^2}|+|\mu_p(p_2\nabla(\phi_1-\phi_2),\nabla\pi)_{L^2}|$$

is bounded by the quadratic form,

$$c[A\xi^2+(A+B)\xi\eta+B\eta^2],$$

where $c=(e\alpha d_\Omega/\epsilon)\sqrt{\|(-\Delta_0)^{-1}\|}$, and where we have made the identifications:

$$\xi=\|\nabla\nu\|_{L^2},\eta=\|\nabla\pi\|_{L^2},A=\mu_n,B=\mu_p.$$

This quadratic form can be absorbed into $B(\nu,\pi)$ if (28) holds; this form, together with each first right hand side term, can be absorbed if the more comprehensive inequality (27) holds. In particular, both ν and π are zero. Uniqueness follows. □

5.5 Summary Statement

Lemma 5.1 validates the definition of the mapping U. By joining the results for V and U we have a well-defined mapping $T=U\circ V$ from $\mathcal{K}$ to $\mathcal{K}$. Furthermore, by the construction of the composite mappings, it is immediate that a fixed point of T is a solution in the sense of Definition 2.1, and conversely. However, this discussion does not imply the uniqueness of fixed points of T.

6 Existence via the Schauder Theorem

We verify the hypotheses of Schauder's fixed point theorem for T.

In the previous sections, we have demonstrated that the mapping,

$$T : \mathcal{K} \mapsto \mathcal{K},$$

is well defined. We prove, consecutively, that V and U are sequentially L^2-continuous. The continuity of T is implied by Lemmas 6.1 and 6.2.

Lemma 6.1 *V is sequentially continuous in the L^2 topology on $\mathcal{K}$.*

Proof To prove that V is sequentially continuous, suppose that

$$(\tilde{n}_j, \tilde{p}_j) \to (\tilde{n}, \tilde{p}) \text{ in } L^2,$$

and $V(\tilde{n}_j, \tilde{p}_j) = \vec{v}_j$. By pointwise subsequential convergence, one sees that $(\tilde{n}, \tilde{p}) \in \mathcal{K}$. By Remark 4.1, one has an (energy) $\mathcal{H}$ bound on $\vec{v}_j$, and, by weak compactness, there is a weakly convergent subsequence, which is relabeled. Thus,

$$\vec{v}_j \rightharpoonup \vec{v}, \text{ in } \mathcal{H},$$
$$\vec{v}_j \to \vec{v}, \text{ in } L^2.$$

Rellich's theorem has been used for the L^2 convergence. We show that $\vec{v} = V(\tilde{n}, \tilde{p})$. This means that it suffices to show, by uniqueness, that

$$\int_\Omega [\nu_* \nabla \vec{v} \cdot \nabla \vec{\psi}] d\xi \; + a(\vec{v}, \vec{v}, \vec{\psi}) + (e/\rho) \int_\Omega (\tilde{p} - \tilde{n}) \nabla \phi \cdot \vec{\psi} \, d\xi = 0, \tag{29}$$

for all $\vec{\psi} \in \mathcal{H}_0^s$. One may here assume the relations, for $j \geq 1$:

$$\int_\Omega [\nu_* \nabla \vec{v}_j \cdot \nabla \vec{\psi}] d\xi \; + a(\vec{v}_j, \vec{v}_j, \vec{\psi}) + (e/\rho) \int_\Omega (\tilde{p}_j - \tilde{n}_j) \nabla \phi_j \cdot \vec{\psi} \, d\xi = 0,$$

for all $\vec{\psi} \in \mathcal{H}_0^s$. To conclude the proof, we verify that (29) follows by taking the limit of these relations. The limits of the first two terms use the same analysis employed in the continuity subsection for V. The final limit is standard; it follows from [26, Lemma 3.4], for example. This concludes the proof. □

The analysis of the companion result for U is presented now.

Lemma 6.2 *U is sequentially continuous, in the L^2 topology, on the range of V, and may be extended to the limit points of this set.*

Proof The proof follows the general template of the previous proof for V. Suppose that

$$\vec{v}_j \to \vec{v},$$

and $U(\vec{v}_j) = (n_j, p_j)$. By Remark 5.1, one has an (energy) $\mathcal{H}$ bound on (n_j, p_j), and, by weak compactness, there is a weakly convergent subsequence, which is relabeled. Thus, for the first components,

$$n_j \rightharpoonup n, \text{ in } H^1,$$

$$n_j \to n, \text{ in } L^2.$$

Rellich's theorem has been used for the L^2 convergence. A parallel set of convergence results holds for the p-components. We show that $(n, p) = U(\vec{v})$. This means that it suffices to show, by uniqueness, that

$$\int_\Omega [D_n \nabla n \cdot \nabla \omega_n - \left(\frac{eD_n}{kT_0}\right) n \nabla \phi \cdot \nabla \omega_n - \vec{v} n \cdot \nabla \omega_n] d\xi = 0,$$

for all $\omega_n \in H_0^s$, with the parallel result for p. One may assume that

$$\int_\Omega [D_n \nabla n_j \cdot \nabla \omega_n - \left(\frac{eD_n}{kT_0}\right) n_j \nabla \phi_j \cdot \nabla \omega_n - \vec{v}_j n_j \cdot \nabla \omega_n] d\xi = 0,$$

for all $\omega_n \in H_0^s$, with the parallel result for p. As in the previous proof, term-by-term limits are straightforward. Once again, a reference is [26, Lemma 3.4]. This concludes the proof. □

6.2 Relative Compactness and the Schauder Theorem

The bounds for the range of the individual mappings, expressed in terms of bounded energy, are expressed in Remark 4.1 and Remark 5.1. The latter remark provides the relative compactness of T via Rellich's theorem. Since $\mathcal{K}$ is closed and convex in the L^2 product space, this completes the verification of the hypotheses of Schauder's theorem for the mapping T [27]. A fixed point is a solution, satisfying the invariant region bounds.

6.3 The Principal Theorem

We first list the hypotheses separately. They are enumerated as H1–H4.

H1 The assumptions of Section 2 are required.
H2 The hypotheses for uniqueness of stationary Navier-Stokes systems are required (see [22, Theorem 4.2]). In particular, these hold if ν_* is sufficiently large in relation to the boundary values specified for $\vec{v}$ and in relation to the linear functional f discussed in Remark 4.1.
H3 Inequality (15) holds.
H4 Inequality (27) holds.

Theorem 6.1 *Assume that the set of hypotheses enumerated as H1–H4 hold. There exists a weak solution of the system of Definition 2.1. The components* n, p *satisfy the invariant region bounds:* $0 \leq n \leq \alpha, 0 \leq p \leq \alpha$. *If the initial data are bounded away from zero in the sense of (24), then the conclusion may be strengthened to* $\delta \leq n \leq \alpha, \delta \leq p \leq \alpha$, *provided (27) is strengthened to:*

$$\left(\frac{e^2 \alpha d_\Omega}{\epsilon k T_0}\right)\left(\frac{2\max(D_n, D_p)}{\min(D_n, D_p)}\sqrt{\|(-\Delta_0)^{-1}\|} + d_\Omega\left(\frac{1}{2} + \frac{\delta}{\alpha}\right)\right) < 1. \tag{30}$$

6.4 Summation

In his monograph [28], C. Koch highlights ion diffusion in Chapter 11, as part of the global picture of transmission, signaling, and bio-computation. It seems clear that ion channel currents are now seen as a fundamental topic of study within this context. The coupling of the PNP model to the Navier-Stokes model incorporates two elements: (i) effect of fluid fluctuations on current densities; (ii) microscopic size of carriers, extending beyond point size charge visualization. Moreover, the steady problem assumes importance because of scale: the duration of the open channel is several orders of magnitude greater than the transient scale. These were the motivating factors in our analysis of this model. Independently, in the articles cited in the bibliography, our computations suggest the impact of the coupled model; these computations appear to support the further study of the PNP/Navier-Stokes model.

The reader has no doubt noticed that we have idealized the mobilities in this article: we have assumed that they are (possibly different) constants, and that the Einstein relations hold with respect to the diffusion coefficients. Eventually, mobilities with electric field dependence need to be studied. This was done in [29] for the semiconductor problem, where the model allowed for mobilities which are Lipschitz continuous, strictly positive, functions of the electric field. Here, we discuss reducing the assumption of the Einstein relations to one comparing diffusion to low-field mobility, and allowing mobility to be field-dependent. A careful study of the proofs of this paper indicates the following. In those parts of the paper which employ only continuity, it appears that Lipschitz continuity of the mobility-electric field product suffices. However, in those estimates in which right hand side terms must have a divergence decomposition into a negatively signed part and a part which can be absorbed into the left hand side, a special structure for the mobility is required. This comes into play when coerciveness and uniqueness are studied, and even in the invariant region arguments. It appears that further study is required to identify a natural structure, and to ensure that this is compatible with the electrochemistry of fluid-assisted charge transport. We intend to study this topic in a future publication.

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