Appendix: First consider the conditions in Theorem 2.1. For a fixed $c > 0$, define the following extremal problem. Find $z \in H^1 = W_1^2[0,1]$ such that

$$\int_0^1 \dot{z}^2(s)\,ds = \min\left\{\int_0^1 \dot{x}^2(s)\,ds : x(0) = 1, x(1) = 0, x \geq 0, \int_0^1 x(s)\,ds = c\right\}.$$

We make the following observations about z. Define H to be the Hilbert space:

$$H = \{y \in H^1 : y(1) = 0\},$$

with inner product:

$$\langle x, y\rangle = \int_0^1 \dot{x}(s)\dot{y}(s)\,ds.$$

Then, K is a closed convex subset of H, where

$$K = \left\{y \in H : y(0) = 1,\ y \geq 0,\ \int_0^1 y(s)\,ds = c\right\}.$$

One sees that K is nonempty. If $0 < c \leq 1/2$, the function that is linear on $[0,a]$, and zero on $[a,1]$ is in K if the resulting triangle has area c (so that $a = 2c$). If $c > 1/2$, the quadratic function q, given in the paper, is in K :

$$q(t) = (1-t)(1-3t+6ct),\ 0 \leq t \leq 1.$$

It follows that K has a unique element z of minimal norm in H, and satisfies the inequality:

$$\langle z, y - z\rangle \geq 0,\ \forall y \in K. \tag{A.1}$$

Lemma A The function z, which is the unique element in K of minimal norm, has piecewise quadratic structure on sets where it is not identically zero. More precisely, if J is an open subinterval of $(0,1)$ on which $z(t) > 0$, $t \in J$, then

$$\frac{d^3 z}{dt^3}(t) = 0,\ t \in J.$$

Proof Let $\phi \in C_0^\infty(J)$ be an infinitely differentiable function with compact support in J, with mean value zero:

$$\int_J \phi(t)\,dt = 0.$$

In $(A.1)$, choose $y = y_\pm = z \pm \epsilon\phi$, where ϵ is small enough such that the minimum of z on the compact support of ϕ is at least as large as the maximum of $\epsilon\,|\phi|$ on this set. Note that z is continuous, which guarantees a positive minimum.

Also, note that $y_\pm \in K$. If we substitute y_+ in $(A.1)$, we obtain, after division by ϵ,

$$\langle z, \phi \rangle \geq 0.$$

Altogether, we have:

$$\int_J \dot{z}(s)\dot{\phi}(s)\, ds = 0.$$

Now suppose $\psi \in C_0^\infty$ is an arbitrary infinitely differentiable function with compact support in J. We observe that the derivative, $\phi = \dot{\psi}$, has mean value zero. We thus obtain:

$$\int_J \dot{z}(s)\ddot{\psi}(s)\, ds = 0,$$

for this arbitrary smooth compactly supported function ψ. Integration by parts yields:

$$\int_J z(s)\psi'''(s)\, ds = 0.$$

It follows, by definition, that z is a distribution solution of $D^3 z = 0$ on J. Thus, z is a classical solution (see I. Halperin, Theory of Distributions, 1952). This concludes the proof of Lemma A.

Now consider the conditions in Theorem 4.2. For a fixed $c > 0$, define a feasible set K in H^1 by:

$$K = \left\{ \omega \in H^1 : \omega(0) = 0,\ \omega(1) = \pi/2,\ |\omega| \leq \pi/2,\ \int_0^1 \cos\omega(s)\, ds = c \right\}.$$

This allows the definition of the following extremal problem. Find $\theta \in K$ such that:

$$\int_0^1 \dot{\theta}^2(s)\, ds = \min \left\{ \int_0^1 \dot{\omega}^2(s)\, ds : \omega \in K \right\}.$$

Existence of a minimizer has been demonstrated in Theorem 4.1. We now investigate the characterizations.

Lemma B Let θ be a minimizer and suppose $(a, b) \subset (0, 1)$ is an interval on which $0 < |\theta| < \pi/2$. Then θ satisfies the elastica equation on (a, b):

$$\ddot{\theta}(s) = C \sin\theta(s),\ a < s < b,$$

for some constant C.

Proof The feasible set K is not convex, but nonetheless we consider the energy functional

$$E(\omega) = \int_0^1 \dot{\omega}^2(s)\, ds,$$

for functions ω in K, and attempt to exploit a derivative characterization, taken on trajectories in K. Suppose $\phi \in C_0^\infty(a,b)$ is of mean value zero, i.e.,

$$\int_a^b \phi(t)\,dt = 0.$$

Now θ has a particular sign on (a,b). For concreteness, we assume that $0 < \theta_0 \leq \theta(t) \leq \theta_1 < \pi/2$, for t in the compact support of ϕ. For $|\epsilon| > 0$ sufficiently small, we consider the functions:

$$\theta_\epsilon(t) = \arccos(\cos\theta(t) + \epsilon\phi(t)), \; a < t < b.$$

Here, arccos is the standard branch, and we choose $|\epsilon| > 0$ sufficiently small so that the argument of arccos is strictly constrained to $(0,1)$ for $t \in (a,b)$. One extends the definition of $\theta_\epsilon(t)$ to all of $[0,1]$ via $\theta_\epsilon(t) = \theta(t)$ on the complement of (a,b). We have the differentiation formula:

$$\frac{d \arccos t}{dt} = \frac{-1}{\sqrt{1-t^2}}, \; a < t < b.$$

Moreover, $\theta_\epsilon \in K$ and $\theta_0 = \theta$. Now define the differentiable function of ϵ :

$$\Phi(\epsilon) := E(\theta_\epsilon), \; 0 \leq \; |\epsilon| \; < \; \epsilon_0,$$

for sufficiently small ϵ_0. We will show that the derivative $\Phi'(0)$, exists and

$$\Phi'(0) = 0$$

since

$$E(\theta_\epsilon) \geq E(\theta), \; \forall\, 0 < \; |\epsilon| \; < \; \epsilon_0.$$

The computation of $\Phi'(0)$ proceeds in two stages. The explicit representation of $\Phi(\epsilon)$. The subsequent calculation of the derivative. By direct calculation,

$$\theta'_\epsilon = \frac{\sin\theta\theta' - \epsilon\phi'}{\sqrt{1-(\cos\theta + \epsilon\phi)^2}}, \text{on (a, b)}.$$

The difference quotient may be represented as:

$$\frac{\Phi(\epsilon) - \Phi(0)}{\epsilon} := \int_a^b \left(\frac{(\theta'_\epsilon)^2 - (\theta')^2}{\epsilon} \right) dt,$$

since θ_ϵ agrees with θ on the complement of (a,b). In computing the limit of the difference quotient involving Φ, Lebesgue's dominated convergence theorem can be used to validate the interchange of $\lim_{\epsilon \to 0}$ with the operation $\int_a^b$. Moreover, the resulting limit inside the integral can be evaluated by direct differentiation of $(\theta'_\epsilon)^2$ with respect to ϵ, followed by evaluation at $\epsilon = 0$. In order to facilitate this calculation, fix $t \in (a, b)$ and set

$$G(\epsilon) := \left[\frac{(\sin\theta\theta' - \epsilon\phi')^2}{1 - (\cos\theta + \epsilon\phi)^2} \right],$$

where $[\,\cdot\,]$ is evaluated at fixed t. After some simplification, one obtains:

$$G'(0) = 2\frac{(\theta')^2 \cos\theta\phi}{\sin^2\theta} - 2\frac{\theta'\phi'}{\sin\theta}.$$

Now suppose that an arbitrary function $\psi \in C_0^\infty(a, b)$ is given. Set $\phi = \psi'$ to obtain the mean value zero property for ϕ. We have,

$$\Phi'(0) = 0 = \int_a^b \left\{ \frac{(\theta')^2 \cos\theta\psi'}{\sin^2\theta} - \frac{\theta'\psi''}{\sin\theta} \right\} dt.$$

If the first term is integrated by parts, so that ψ'' becomes a multiplier of both terms, and if

$$h(t) = \int_a^t \left\{ \frac{(\theta')^2 \cos\theta\phi}{\sin^2\theta} \right\} ds,$$

then standard distribution theory yields the result that $h(t) + \theta'/\sin\theta$ is a linear function, hence differentiable. Since h is differentiable, the second term must also be differentiable. We conclude that θ' has a derivative. This permits integration by parts in the second integrand term in the expression $\Phi'(0) = 0$, so that ψ' becomes a multiplier of both terms. In particular, we conclude that there is a constant C such that, on the interval (a, b),

$$(\theta')^2 \frac{\cos\theta}{\sin^2\theta} + \left(\frac{\theta'}{\sin\theta} \right)' = C.$$

Upon simplification, this gives the pendulum equation stated in the lemma. If (a, b) corresponds to the case where θ is negative, a nonstandard branch of arccos is selected, yielding a differentiation formula without a minus sign. All other details are the same, and one is led to the same equation. This concludes the proof of Lemma B.