Appendix: First consider the conditions in Theorem 2.1. For a fixed $c > 0$, define the following extremal problem. Find $z \in H^1 = W^1_2[0,1]$ such that

$$
\int_0^1 \dot{z}^2(s) \, ds = \min \left\{ \int_0^1 \dot{x}^2(s) \, ds : x(0) = 1, x(1) = 0, x \geq 0, \int_0^1 x(s) \, ds = c \right\}.
$$

We make the following observations about $z$. Define $H$ to be the Hilbert space:

$$
H = \left\{ y \in H^1 : y(1) = 0 \right\},
$$

with inner product:

$$
\langle x, y \rangle = \int_0^1 \dot{x}(s) \dot{y}(s) \, ds.
$$

Then, $K$ is a closed convex subset of $H$, where

$$
K = \left\{ y \in H : y(0) = 1, y \geq 0, \int_0^1 y(s) \, ds = c \right\}.
$$

One sees that $K$ is nonempty. If $0 < c \leq 1/2$, the function that is linear on $[0, a]$, and zero on $[a, 1]$ is in $K$ if the resulting triangle has area $c$ (so that $a = 2c$). If $c > 1/2$, the quadratic function $q$, given in the paper, is in $K$:

$$
q(t) = (1 - t)(1 - 3t + 6ct), \quad 0 \leq t \leq 1.
$$

It follows that $K$ has a unique element $z$ of minimal norm in $H$, and satisfies the inequality:

$$
\langle z, y - z \rangle \geq 0, \quad \forall y \in K. \tag{A.1}
$$

**Lemma A** The function $z$, which is the unique element in $K$ of minimal norm, has piecewise quadratic structure on sets where it is not identically zero. More precisely, if $J$ is an open subinterval of $(0,1)$ on which $z(t) > 0$, $t \in J$, then

$$
\frac{d^3 z}{dt^3}(t) = 0, \quad t \in J.
$$

**Proof** Let $\phi \in C_0^\infty(J)$ be an infinitely differentiable function with compact support in $J$, with mean value zero:

$$
\int_J \phi(t) \, dt = 0.
$$

In (A.1), choose $y = y_\pm = z \pm \epsilon \phi$, where $\epsilon$ is small enough such that the minimum of $z$ on the compact support of $\phi$ is at least as large as the maximum of $\epsilon |\phi|$ on this set. Note that $z$ is continuous, which guarantees a positive minimum.
Also, note that $y_\pm \in K$. If we substitute $y_+$ in (A.1), we obtain, after division by $\epsilon$,
\[ \langle z, \phi \rangle \geq 0. \]

Altogether, we have:
\[ \int_J z(s)\dot{\phi}(s) \, ds = 0. \]

Now suppose $\psi \in C^\infty_0$ is an arbitrary infinitely differentiable function with compact support in $J$. We observe that the derivative, $\phi = \psi$, has mean value zero. We thus obtain:
\[ \int_J z(s)\psi(s) \, ds = 0, \]

for this arbitrary smooth compactly supported function $\psi$. Integration by parts yields:
\[ \int_J z(s)\psi''(s) \, ds = 0. \]

It follows, by definition, that $z$ is a distribution solution of $D^3 z = 0$ on $J$. Thus, $z$ is a classical solution (see I. Halperin, Theory of Distributions, 1952). This concludes the proof of Lemma A.

Now consider the conditions in Theorem 4.2. For a fixed $c > 0$, define a feasible set $K$ in $H^1$ by:
\[ K = \left\{ \omega \in H^1 : \omega(0) = 0, \, \omega(1) = \pi/2, \, |\omega| \leq \pi/2, \, \int_0^1 \cos \omega(s) \, ds = c \right\}. \]

This allows the definition of the following extremal problem. Find $\theta \in K$ such that:
\[ \int_0^1 \theta^2(s) \, ds = \min \left\{ \int_0^1 \omega^2(s) \, ds : \omega \in K \right\}. \]

Existence of a minimizer has been demonstrated in Theorem 4.1. We now investigate the characterizations.

**Lemma B** Let $\theta$ be a minimizer and suppose $(a, b) \subset (0, 1)$ is an interval on which $0 < |\theta| < \pi/2$. Then $\theta$ satisfies the elastica equation on $(a, b)$:
\[ \ddot{\theta}(s) = C \sin \theta(s), \, a < s < b, \]

for some constant $C$.

**Proof** The feasible set $K$ is not convex, but nonetheless we consider the energy functional
\[ E(\omega) = \int_0^1 \dot{\omega}^2(s) \, ds, \]
for functions $\omega$ in $K$, and attempt to exploit a derivative characterization, taken
on trajectories in $K$. Suppose $\phi \in C^0_0(a, b)$ is of mean value zero, i.e.,

$$\int_a^b \phi(t) \, dt = 0.$$ 

Now $\theta$ has a particular sign on $(a, b)$. For concreteness, we assume that $0 < 
\theta_0 \leq \theta(t) \leq \theta_1 < \pi/2$, for $t$ in the compact support of $\phi$. For $|\epsilon| > 0$ sufficiently
small, we consider the functions:

$$\theta_\epsilon(t) = \text{arccos}(\cos \theta(t) + \epsilon \phi(t)), \quad a < t < b.$$ 

Here, arccos is the standard branch, and we choose $|\epsilon| > 0$ sufficiently small so that
the argument of arccos is strictly constrained to $(0, 1)$ for $t \in (a, b)$. One
extends the definition of $\theta_\epsilon(t)$ to all of $[0, 1]$ via $\theta_\epsilon(t) = \theta(t)$ on the complement
of $(a, b)$. We have the differentiation formula:

$$\frac{d \text{arccos} t}{dt} = \frac{-1}{\sqrt{1 - t^2}}, \quad a < t < b.$$ 

Moreover, $\theta_\epsilon \in K$ and $\theta_0 = \theta$. Now define the differentiable function of $\epsilon$ :

$$\Phi(\epsilon) := E(\theta_\epsilon), \quad 0 \leq |\epsilon| < \epsilon_0,$$

for sufficiently small $\epsilon_0$. We will show that the derivative $\Phi'(0)$, exists and

$$\Phi'(0) = 0$$

since

$$E(\theta_\epsilon) \geq E(\theta), \quad \forall 0 < |\epsilon| < \epsilon_0.$$ 

The computation of $\Phi'(0)$ proceeds in two stages. The explicit representation
of $\Phi(\epsilon)$. The subsequent calculation of the derivative. By direct calculation,

$$\theta'_\epsilon = \frac{\sin \theta \phi' - \epsilon \phi'}{\sqrt{1 - (\cos \theta + \epsilon \phi')^2}}, \text{ on } (a, b).$$

The difference quotient may be represented as:

$$\frac{\Phi(\epsilon) - \Phi(0)}{\epsilon} := \int_a^b \left( \frac{(\theta'_\epsilon)^2 - (\theta')^2}{\epsilon} \right) dt,$$

since $\theta_\epsilon$ agrees with $\theta$ on the complement of $(a, b)$. In computing the limit of the
difference quotient involving $\Phi$, Lebesgue’s dominated convergence theorem can
be used to validate the interchange of $\lim_{\epsilon \to 0}$ with the operation $\int_a^b$. Moreover,
the resulting limit inside the integral can be evaluated by direct differentiation
of \((\theta')^2\) with respect to \(\epsilon\), followed by evaluation at \(\epsilon = 0\). In order to facilitate this calculation, fix \(t \in (a, b)\) and set

\[
G(\epsilon) := \left[ \frac{(\sin \theta \theta' - \epsilon \phi')^2}{1 - (\cos \theta + \epsilon \phi)^2} \right],
\]

where \([\cdot]\) is evaluated at fixed \(t\). After some simplification, one obtains:

\[
G'(0) = 2 \frac{(\theta')^2 \cos \theta \phi'}{\sin^2 \theta} - 2 \frac{\theta' \phi'}{\sin \theta}.
\]

Now suppose that an arbitrary function \(\psi \in C_0^\infty(a, b)\) is given. Set \(\phi = \psi'\) to obtain the mean value zero property for \(\phi\). We have,

\[
\Phi'(0) = 0 = \int_a^b \left\{ \frac{(\theta')^2 \cos \theta \psi'}{\sin^2 \theta} - \frac{\theta' \psi''}{\sin \theta} \right\} dt.
\]

If the first term is integrated by parts, so that \(\psi''\) becomes a multiplier of both terms, and if

\[
h(t) = \int_a^t \left\{ \frac{(\theta')^2 \cos \theta \phi}{\sin^2 \theta} \right\} ds,
\]

then standard distribution theory yields the result that \(h(t) + \theta' / \sin \theta\) is a linear function, hence differentiable. Since \(h\) is differentiable, the second term must also be differentiable. We conclude that \(\theta'\) has a derivative. This permits integration by parts in the second integrand term in the expression \(\Phi'(0) = 0\), so that \(\psi'\) becomes a multiplier of both terms. In particular, we conclude that there is a constant \(C\) such that, on the interval \((a, b)\),

\[
(\theta')^2 \frac{\cos \theta}{\sin^2 \theta} + \left( \frac{\theta'}{\sin \theta} \right)' = C.
\]

Upon simplification, this gives the pendulum equation stated in the lemma. If \((a, b)\) corresponds to the case where \(\theta\) is negative, a nonstandard branch of \(\text{arccos}\) is selected, yielding a differentiation formula without a minus sign. All other details are the same, and one is led to the same equation. This concludes the proof of Lemma B.