

ON RESONANCES GENERATED BY CONIC DIFFRACTION

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ABSTRACT. We describe the resonances closest to the real axis generated by diffraction of waves among cone points on a manifold with Euclidean ends. These resonances lie asymptotically evenly spaced along a curve of the form

$$\frac{\operatorname{Im} \lambda}{\log |\operatorname{Re} \lambda|} = -\nu;$$

here $\nu = (n-1)/2L_0$ where n is the dimension and L_0 is the length of the longest geodesic connecting two cone points. Moreover there are asymptotically no resonances below this curve and above the curve

$$\frac{\operatorname{Im} \lambda}{\log |\operatorname{Re} \lambda|} = -\Lambda$$

for a fixed $\Lambda > \nu$.

1. Introduction

Let X^n be a manifold with cone points Y_1, \dots, Y_N and with Euclidean ends. We make the geometric assumption that there are no trapped geodesics that do not hit the cone points, and that there are a finite number of geodesics γ_{ij}^k connecting cone points Y_i and Y_j for each i, j (with possibly more than one geodesic connecting a given pair, hence the index k). We further assume that no two endpoints of any pair of these geodesics y, y' are π -related; loosely speaking, this means that no three cone points are collinear (see §2 below for precise definitions).

(*) The authors are grateful to Colin Guillarmou, Nicolas Burq, and Maciej Zworski for helpful conversations, and to an anonymous referee for comments on the manuscript. The second author thanks Institut Henri Poincaré and Université de Paris Sud for their hospitality. The work of L.H. is partially supported by the ANR program ANR-13-BS01-0007-01 GERASIC and J.W. acknowledges support from NSF grants DMS-1265568 and DMS-1600023.

Let L_0 denote the longest geodesic connecting two (not a priori distinct) cone points. Assume that no two distinct oriented geodesics between cone points of maximal length L_0 end at a common cone point⁽¹⁾ (e.g., one could assume that there is a unique geodesic of length L_0 and it connects two distinct cone points). Thus the longest path(s) along two successive conic geodesics is (are) obtained by traversing back and forth along a single geodesic, resulting in length $2L_0$. Let $2L'$ denote the length of the next longest path traversing two successive geodesics connecting cone points.

The main result of this paper is:

THEOREM 1.1. — *Let λ_j be a sequence of resonances of the Laplacian on a conic manifold X subject to the geometric hypotheses above, with*

$$-\frac{\operatorname{Im} \lambda_j}{\log |\operatorname{Re} \lambda_j|} \rightarrow \nu.$$

Then either $\nu = (n - 1)/2L_0$ or $\nu \geq \Lambda$ where

$$\Lambda = \min\{n/(2L_0), (n - 1)/(2L')\}.$$

More precisely, for a sequence of resonances satisfying

$$\limsup \left(-\frac{\operatorname{Im} \lambda_j}{\log |\operatorname{Re} \lambda_j|} \right) < \Lambda$$

there exist $\delta > 0$ and a constant C_{Im} such that, up to extracting a subsequence,

$$\operatorname{Im} \lambda_j = -\frac{(n - 1)}{2L_0} \log |\operatorname{Re} \lambda_j| + C_{\operatorname{Im}} + O(|\operatorname{Re} \lambda_j|^{-\delta});$$

also there exists C_{Re} such that the following quantization condition holds

$$(1.1) \quad \operatorname{Re} \lambda_j \in C_{\operatorname{Re}} + \frac{\pi}{L_0} \mathbb{Z} + O(|\operatorname{Re} \lambda_j|^{-\delta}).$$

This latter condition should be interpreted as saying the resonances have real parts in the union of the intervals $B(C_{\operatorname{Re}} + \pi k/L_0, C|k|^{-\delta})$ for $k \in \mathbb{Z}$ and for some fixed C .

Moreover, there is only a finite number of pairs $(C_{\operatorname{Re}}, C_{\operatorname{Im}})$: one for each geodesic of length L_0 .

The constants C_{Re} and C_{Im} are geometrically meaningful: they are related to the imaginary and real parts respectively of the logarithm of the product of diffraction coefficients along the corresponding maximal length closed diffractive geodesic undergoing two diffractions. It also follows from

⁽¹⁾Note that this hypothesis rules out a geodesic loop of length L_0 from a cone point to itself, since the geodesic and its reversal end at the same point.

this description that if the diffraction coefficient along every maximal geodesic vanishes then, for any sequence of resonances

$$\limsup \left(-\frac{\operatorname{Im} \lambda_j}{\log |\operatorname{Re} \lambda_j|} \right) \geq \Lambda.$$

The theorem applies via the method of images to the Dirichlet or Neumann problem on the exterior of one or more polygons in the plane, via a “doubling” in which two copies of the exterior domain are sewn together along their common edges to make a manifold with cone points—see e.g. [15, Section 1], [3, Section 1]. As long as no three vertices are collinear, the collinearity assumption is satisfied; at most one geodesic connects two cone points; nontrapping is an open condition (and always satisfied for the exterior of a single convex polygon); and the longest path condition certainly holds if, say, no two pairs of vertices are the same distance apart. Our geometric hypotheses are thus generically satisfied in the polygonal case (once nontrapping is stipulated); we expect them to be generic in the more general nontrapping conic setting as well.

Our main theorem consists of an upper bound for the locations of resonances of X lying in a log neighborhood of the real axis (albeit without multiplicity bounds), implying that resonances in this region can only concentrate on a log curve. We recall that previous work of Baskin–Wunsch [3] Galkowski [14] showed on the one hand [3] that *some* region of the form

$$-\frac{\operatorname{Im} \lambda_j}{\log |\operatorname{Re} \lambda_j|} > \nu_0 > 0, \quad |\lambda_j| > R$$

contains no resonances (subject to some genericity conditions on the relationship among the conic singularities); it was more precisely observed by Galkowski [14] that ν_0 could be taken to be $(n-1)/2L_0 + \epsilon$. On the other hand, the authors and Galkowski showed⁽²⁾ the following *existence* theorem for resonances using a trace formula:

THEOREM 1.2. — *Assume there is a single maximal orbit of length $2L_0$ undergoing two diffractions with nonvanishing diffraction coefficients, whose iterates are all isolated in the length spectrum. Then for every $\epsilon > 0$,*

$$\#\left\{ \lambda_j : -\frac{\operatorname{Im} \lambda_j}{\log |\operatorname{Re} \lambda_j|} < \frac{n-1}{2L_0} + \epsilon \right\} \cap B(0, r) > C_\epsilon r^{1-\epsilon}.$$

⁽²⁾This theorem was proved by the authors in the odd dimensional case and announced by the second author, with a sketch of the proof, at the Berkeley/Bonn/Paris Nord/Zürich PDE Video Seminar in September 2015. Galkowski subsequently published a proof covering both even and odd dimensional cases (citing the authors) as Theorem 5 of [14].

The proof employs a trace formula of Ford–Wunsch [12] (previously proved by the first author [15] in the case of flat surfaces) describing the singularities of the wave trace induced by diffractive closed orbits, together with a theorem relating resonances to the renormalized wave trace due to Bardos–Guillot–Ralston [1], Melrose [19], and Sjöstrand–Zworski [25] (as well as [30] for the even dimensional case). It also uses a Tauberian theorem of Sjöstrand–Zworski [24]. As a proof has appeared in [14] (see also [29]) we do not give one here.

Thus, previous results implied that infinitely many resonances lie in any logarithmic “strip”

$$\left| \frac{\operatorname{Im} \lambda}{\log |\operatorname{Re} \lambda|} + \frac{n-1}{2L_0} \right| < \epsilon.$$

The results at hand sharpen this result by pushing down further into the complex plane: we now know that below this first (approximate) logarithmic curve of resonances there is a gap region.

All this is in marked contrast with the case of a non-trapping smooth manifold with Euclidean ends, where classic results of Lax–Phillips [18] and Vainberg [27, 28] show that for every $\nu > 0$, there exists $R > 0$ so that the region

$$\operatorname{Im} \lambda > -\nu \log |\operatorname{Re} \lambda|, \quad |\lambda| > R$$

contains no resonances at all. The existence of resonances along log curves is thus a consequence of the *weak trapping* effects of repeated diffraction at the cone points (see discussion below). Our results therefore occupy a middle ground between the smooth nontrapping case and the case of a smooth manifold with trapped geodesics, where no matter how unstable the structure of the trapped set, there seem to be resonances closer to the real axis than those studied here: for instance there are now numerous results about the existence of resonances lying near lines parallel to the real axis, generated by normally hyperbolic trapping—cf. [10]. (That sequences of resonances should *always* exist in some strip near the real axis in cases of trapping on a smooth manifold with Euclidean ends is the content of the modified Lax–Phillips conjecture.)

Previous results on strings of resonances on log curves as in Theorem 1.1 include the much more precise study in [5] of the related special case of one orbit bouncing back and forth between an analytic corner and a wall. The treatise [4] contains similar (and highly refined) results in the setting of resonances generated by homoclinic orbits; such resonances are somewhat closer to the real axis than those discussed here, but the variable-order propagation of singularities techniques used below also appear in [4].

The appearance of the factor $(n - 1)/(2L_0)$ in our main theorem is quite natural from a dynamical point of view; after a semiclassical rescaling of the problem, we show that it represents the minimal “rate of smoothing” enjoyed by a solution to the semiclassical Schrödinger equation owing to its diffraction by cone points. More precisely, consider a putative semiclassical resonant state u_h —this would be a solution to the Schrödinger equation with a complex spectral parameter, here with imaginary part approximately $-2\nu h \log(1/h)$. As we discuss below, the semiclassical wavefront set for such a solution to the stationary Schrödinger equation propagates along geodesics that are permitted to branch at cone points. In evaluating the regularity of u_h , we show (Proposition 4.2) that it *loses* regularity along the forward bicharacteristic flow at a constant rate proportional to ν . At each diffraction, by contrast, u_h generically *gains* regularity by approximately the factor $h^{(n-1)/2}$. These gains and losses of regularity must be in balance along any closed branched geodesic in the wave-front set of u_h . The smallest ν will thus be obtained when the diffractive gain in semiclassical regularity along the branched orbit is as small as possible per unit length. We thus show that the optimal scenario is that of concentration along the closed branching orbit that diffracts as infrequently as possible: this is the orbit traveling back and forth between the two maximally separated cone points. Correspondingly there is a long-living resonant state concentrated along this orbit that loses energy to infinity via diffraction as infrequently as possible. It is an instance of the “weak trapping” phenomenon referred to above and yields the value $\nu = (n - 1)/(2L_0)$.

It is a natural conjecture that (at least generically), *all* resonances in any log neighborhood of the real axis lie on quantized log curves $\text{Im } \lambda \sim -\nu_j \log |\text{Re } \lambda|$ for some family of ν_j . We have been unable to gain sufficient control on error terms to verify this, however.

2. Conic geometry

We now specify our geometric hypotheses, which are much the same as those employed in [3]: we assume that our manifold has conic singularities and Euclidean ends, as follows.

Let X be a noncompact manifold with boundary, K a compact subset of X , and let g be a Riemannian metric on X° such that $X \setminus K$ is isometric to a union of finitely many exteriors of Euclidean balls $\bigsqcup_j (\mathbb{R}_{z_j}^n \setminus B^n(0, R_0))$ and such that g has conic singularities at the boundary of X :

$$g = dr^2 + r^2 h(r, dr, y, dy);$$

here g is assumed to be nondegenerate over X° and $h|_{\partial X}$ induces a metric on ∂X , while r is a boundary defining function. We let Y_α , $\alpha = 1, \dots, N$ denote the components of ∂X ; we will refer to these components in what follows as *cone points*, as each boundary component is a single point when viewed in terms of metric geometry.

For simplicity of notation, we will retain the notation $B^n(0, R)$ (with $R \gg 0$) for the union of K and the intersection(s) of this large ball with the Euclidean end(s) $X \setminus K$.

Theorem 1.2 of [20] implies that we may choose local coordinates (r, y) in a collar neighborhood of each Y_α such that the metric takes the form

$$(2.1) \quad g = dr^2 + r^2 h(r, y, dy),$$

where h is now a family in r (which is the distance function to the boundary) of smooth metrics on Y_α . The curves $\{r = r_0 \pm t, y = y_0\}$ are now unit-speed geodesics hitting the boundary, and indeed are the *only* such geodesics.

We will say that the conic manifold X is of *product type* if locally near ∂X the metric can be written in the form

$$(2.2) \quad g = dr^2 + r^2 h(y, dy)$$

in some product coordinates in a collar neighborhood of ∂X , where r is a defining function and where h is a *fixed* (i.e., r -independent) metric on ∂X .

We say that the concatenation of a geodesic entering the boundary at $y = y_0 \in Y_\alpha$ and another leaving at $y = y_1 \in Y_\alpha$ *at the same time* is a *geometric* geodesic if y_0, y_1 are connected by a geodesic in Y_α (with respect to the metric $h|_{r=0}$) of length π . Such concatenations of geodesics turn out to be exactly those which are locally approximable by families of geodesics in X° (see [20]). In the special case of a surface with conic singularity there are locally just two of these, corresponding to limits of families of geodesics that brush past the cone point on either side; more generally, there is a (locally) codimension-two family of such geodesics through any cone point.

By contrast, we say that concatenation of a geodesic entering the boundary at $y = y_0 \in Y_\alpha$ and another leaving at $y = y_1 \in Y_\alpha$ *at the same time* is a *diffractive* geodesic if there is no restriction on y_0, y_1 besides lying in the same boundary component Y_α . We say that a diffractive geodesic is *strictly diffractive* if it is not geometric. We say that two points in the cotangent bundle of $\mathbb{R} \times X$ are *diffractively related* (resp. strictly diffractively, geometrically) if they are connected by a diffractive geodesic (resp. strictly diffractive, geometric).

The principal results of [20] (see also [6, 7]) are that singularities for solutions of the wave equation propagate along diffractive geodesics, with

the singularities arising at *strictly* diffractive geodesics being generically weaker than the main singularities. More precisely, if q is a point with coordinates (r_0, y_0) lying close to a cone point Y_α , the solution

$$u \equiv e^{-it\sqrt{\Delta}}\delta_q$$

is shown to have a *conormal* singularity at the diffracted wavefront $r = t - r_0$ (for $t > r_0$) lying in $H^{-1/2-\epsilon}$ away from the geometric continuations of the geodesic from q to Y_α (whereas the main singularity is in $H^{-n/2-\epsilon}$). The symbol (and also precise order) of this singularity was analyzed in [12], based on computations in the product case by Cheeger–Taylor [6, 7], yielding the following:

PROPOSITION 2.1 ([12]). — *Let $p = (r, y)$ and $p' = (r', y')$ be strictly diffractively related points near cone point Y_α . Then near (t, p, p') , the Schwartz kernel of the half-wave propagator $e^{-it\sqrt{\Delta}}$ (acting on half-densities) has an oscillatory integral representation*

$$(2.3) \quad e^{-it\sqrt{\Delta}} = \int_{\mathbb{R}_\xi} e^{i(r+r'-t)\cdot\xi} a_D(t, r, y; r', y'; \xi) d\xi$$

whose amplitude $a_D \in S^0$ is

$$(2.4) \quad \frac{(rr')^{-\frac{n-1}{2}}}{2\pi i} \chi(\xi) \cdot \mathbf{D}_\alpha(y, y') \cdot \Theta^{-\frac{1}{2}}(Y_\alpha \rightarrow y) \Theta^{-\frac{1}{2}}(y' \rightarrow Y_\alpha) \omega_g(r, y) \omega_g(r', y')$$

modulo elements of $S^{-\frac{1}{2}+0}$. Here, $\chi \in \mathcal{C}^\infty(\mathbb{R}_\xi)$ is a smooth function satisfying $\chi \equiv 1$ for $\xi > 2$ and $\chi \equiv 0$ for $\xi < 1$. The factor $\mathbf{D}_\alpha(y, y')$ is the Schwartz kernel of the operator $e^{-i\pi\sqrt{\Delta_{Y_\alpha} + (n-2)^2/4}}$, while the factors $\Theta^{-\frac{1}{2}}$ are given by nonvanishing determinants of Jacobi fields (cf. [12] for details).

In the case where X is of product type near Y_α , the amplitude a_D admits an asymptotic expansion in powers of $|\xi|^{1/2}$.

3. Analytic preliminaries

We begin by making a semiclassical rescaling of our problem. Existence of a resonance λ implies the existence of a certain kind of “outgoing” solution u (an associated *resonant state*) of the equation

$$(\Delta - \lambda^2)u = 0.$$

Setting $\operatorname{Re} \lambda = h^{-1}$ and $\operatorname{Im} \lambda = -\nu \log \operatorname{Re} \lambda = -\nu \log h^{-1}$ gives

$$(3.1) \quad (h^2\Delta - z_h)u = 0$$

where

$$z_h = (1 - i\nu h \log(1/h))^2 = 1 + O((h \log(1/h))^2) - 2i\nu h \log(1/h).$$

Our aim is to understand the resonances of the family $h^2\Delta - z_h$ in a logarithmic neighbourhood of the real axis. More precisely, we look for resonances in the set of z_h with

$$\sqrt{z_h} \in \Omega_\epsilon \equiv \{(-\Lambda + \epsilon)h \log(1/h) < \operatorname{Im} \sqrt{z_h} < 0, \operatorname{Re} \sqrt{z_h} \in [1 - \epsilon, 1 + \epsilon]\}$$

Now we recall (cf. [23] or Chapter 4 of [11]) that the resonances of the Laplacian with argument having magnitude less than a fixed θ agree with poles of *complex-scaled* operator Δ_θ which coincides with the original Laplacian on a large compact set but which, near infinity, is deformed into the complex domain. Existence of a resonance at z_h is then equivalent to existence of an L^2 eigenfunction of the non-self-adjoint complex scaled problem.

As in [4, Section 2], [26] we will scale only to an angle $O(h \log(1/h))$; this restriction on the scaling has the virtue (albeit an inessential one here) that the overall propagation of singularities near the scaling is still bi-directional: while propagation into the scaling region loses powers of h , it only loses a finite number of such powers (see Proposition 4.2 below). Scaling to fixed angle, by contrast, would break the propagation of semiclassical singularities at the boundary of the scaled region. Thus we fix $M \gg 0$ and set θ so that

$$(3.2) \quad \tan \theta = Mh \log(1/h).$$

We let $D_x = -i\partial_x$ and we consider an operator given by the Laplace-Beltrami operator Δ_g on the compact part $K \subset X$ and, on the ends (with Euclidean coordinate $x \in \mathbb{R}^n$), given by the expression

$$\Delta_\theta = ((I + iF_\theta''(x))^{-1}D_x) \cdot ((I + iF_\theta''(x))^{-1}D_x),$$

where

$$F_\theta(x) = (\tan \theta) \cdot g(|x|)$$

for a function g chosen so that

$$g(t) = 0, \quad t \leq R_1, \quad g(t) = \frac{1}{2}t^2, \quad t > 2R_1, \quad g''(t) \geq 0.$$

We now set

$$P_\theta = h^2\Delta_\theta,$$

By construction, we have the following decomposition of X

- (1) The interior region $B^n(0, R_1)$ in which $P_\theta = h^2\Delta$.

- (2) The scaling region $R_1 \leq R \leq 2R_1$. In this region, we have $\text{Id} + iF''_\theta(x) = \text{Id} + \tan \theta G(x) = \text{Id} + iMh \log(1/h) \cdot G(x)$, for some $n \times n$ matrix G with smooth entries. We then obtain the asymptotic expansion

$$P_\theta \sim h^2 \Delta + \sum_{k \geq 1} (\tan \theta)^k Q_k$$

for some $Q_k \in \text{Diff}_h^2$.

- (3) The deep scaling region ($R \geq 2R_1$) in which

$$P_\theta = (1 + i \tan \theta)^{-2} h^2 \Delta.$$

This decomposition implies that there exists a constant C , independent of h , such that, for any v

$$(3.3) \quad \|(P_\theta - h^2 \Delta)v\|_{L^2} \leq Ch \log(1/h) \|(h^2 \Delta + 1)v\|_{L^2}$$

Let us assume as above, that, for a sequence of resonances, there exists $\nu, E \in (0, \infty)$ for which

$$z_h = E + o(1) - i(2\nu + o(1))h \log(1/h).$$

Since existence of resonances is equivalent to existence of eigenvalues of the complex-scaled operator P_θ , there exists a sequence of solutions to

$$(3.4) \quad (P_\theta - z_h)u_h = 0,$$

that is normalized in L^2 .

There is a slight ambiguity in the nomenclature since the eigenfunction u_h in the latter equation differs from the resonant state in (3.1), although the z_h are the same. We will say that this u_h is a resonant state.

The semiclassical principal symbol of $P_\theta - z_h$ is then given by

$$\sigma_h(h^2 \Delta_\theta) - E + o(1) + i(2\nu + o(1))h \log(1/h).$$

We easily see as in [23], [11] that for h sufficiently small,

$$-C_1 \theta |\xi|^2 \leq \text{Im } \sigma_h(h^2 \Delta_\theta) \leq 0$$

and

$$\text{Im } \sigma_h(h^2 \Delta_\theta) \leq -C_2 \theta |\xi|^2, \quad |x| > 2R_1.$$

Hence if M is sufficiently large (relative to ν), $P_\theta - z_h$ enjoys a kind of semiclassical hypoellipticity in $|x| > 2R_1$ in the sense that as $h \downarrow 0$ the imaginary part of its symbol is a nonvanishing multiple of $h \log(1/h)$ in this region. This will have consequences for regularity of solutions to $(P_\theta - z_h)u_h = O(h^\infty)$ that we will derive in the following sections.

4. Semiclassical wavefront set and propagation of singularities

Let $(u_h)_{h \geq 0}$ be a bounded sequence in $L^2(X)$. For a positive sequence ϵ_h , we will use the notation $u_h = O_{L^2}(\epsilon_h)$ to say that $\epsilon_h^{-1}u_h$ is a bounded sequence in $L^2(X)$.

For $(x, \xi) \in T^*X^\circ$ we define (cf. [31, Chapter 8])

$$(x, \xi) \notin \text{WF}_h^s u_h \iff \text{there exists } A \in \Psi_h(X) \text{ elliptic at } (x, \xi) \text{ and } Au_h = O_{L^2}(h^s).$$

Likewise

$$(x, \xi) \notin \text{WF}_h u_h \iff \text{there exists } A \in \Psi_h(X) \text{ elliptic at } (x, \xi) \text{ and } Au_h = O_{L^2}(h^\infty).$$

As usual, we have

$$\text{WF}_h u_h = \overline{\bigcup_s \text{WF}_h^s u_h}.$$

(Note that, for the moment, we are only dealing with wavefront set away from cone points)

By standard elliptic regularity in the semiclassical calculus, we have the following result.

LEMMA 4.1. — *Let $(P_\theta - z_h)u_h = O_{L^2}(h^\infty)$ with $\text{Re } z_h = E + o(1)$.*

We have

$$\text{WF}_h u_h \cap T^*X^\circ \subset \{|\xi|_g^2 = E\}.$$

This lemma implies that the wave-front set of u_h does not intersect the 0–section in T^*X_0 . Thus, we may test u_h against standard (non-semiclassical) pseudodifferential or Fourier Integral operators in order to understand its semiclassical wave-front. To understand the propagation of singularities, it thus suffices to study $U(t)u_h$, where $U(t)$ denotes the half-wave propagator $U(t) = \exp(-it\sqrt{\Delta})$.

The rest of this section is devoted to the proof of the following propagation of singularities result. What is novel here is the variable semiclassical order (in addition to the presence of complex scaling). Note also that this proposition only governs propagation in X° , away from cone points.

PROPOSITION 4.2. — *Let*

$$(P_\theta - z_h)u_h = O_{L^2}(h^\infty)$$

with

$$z_h = E + o(1) - i(2\nu + o(1))h \log(1/h).$$

- (1) For $t > 0$ assume that $\{\exp_{t'H}(q), t' \in [0, t]\} \subset T^*(X^\circ \cap B(0, R_1))$.

Then

$$q \notin \text{WF}_h^s u_h \implies \exp_{tH}(q) \notin \text{WF}_h^{s'} u_h$$

for all $s' < s - 2\nu t$.

- (2) For $t > 0$, assume that $\{\exp_{t'H}(q), t' \in [0, t]\} \subset T^*(X^\circ \cap B(0, R_1))$.

Then

$$\exp_{tH}(q) \notin \text{WF}_h^{s'} u_h \implies q \notin \text{WF}_h^s u_h$$

for all $s < s' + 2\nu t$,

- (3) There exists some $M_0 > 0$ such that for any $q \in T^*X^\circ$ and $t \in \mathbb{R}$ (not necessarily positive)

$$q \notin \text{WF}_h^s u_h \implies \exp_{tH}(q) \notin \text{WF}_h^{s'} u_h,$$

for all $s' < s - M_0|t|$ provided $\{\exp_{t'H}(q) : t' \text{ between } 0 \text{ and } t\} \subset T^*X^\circ$. In particular this implies

$$(4.1) \quad q \in \text{WF}_h u_h \iff \exp_{tH}(q) \in \text{WF}_h u_h,$$

provided the flow does not reach a cone point.

- (4) There exists $C > 0$ such that for $t > 0$ and $\{\exp_{t'H}(q), t' \in [0, t]\} \subset T^*(X \setminus B(0, 2R_1))$,

$$q \notin \text{WF}_h^s u_h \implies \exp_{tH}(q) \notin \text{WF}_h^{s'} u_h$$

for all $s' < s + Ct$.

Remark 4.3. — The main content of the proposition is that, in the interior region and under forward propagation, semiclassical regularity drops at a rate $2\nu t$ on the t -parametrized flow along H , the Hamilton vector field of the symbol of P . Note, though, that the vector field $H/2\sqrt{E}$ induces unit speed geodesic flow, hence the rate of regularity loss along unit-speed geodesics is $\nu t/\sqrt{E}$, where E is the real part of the spectral parameter and $-2i\nu h \log(1/h)$ the imaginary part.

The situation is more complicated near the boundary of the scaling region, where there are gains or losses in regularity owing to the scaling which compete with the $2\nu t$ loss rate, but these gains and losses are also of finite order. The last part illustrates that deep within the scaling region, forward propagation *gains* regularity; this will be our substitute for elliptic regularity in the scaling region, since we do not have full semiclassical elliptic regularity with the angle $h \log(1/h)$ scaling employed here: semiclassical singularities instead propagate but decay. (Cf. [4, Lemma 8.4] for related results.)

Remark 4.4. — This proposition can be proved using commutator arguments involving operators of variable semiclassical order. We refer the reader to [13, Section 2.3] for a thorough treatment of the symbols of these operators. The main difference with ordinary commutator arguments is the presence of $\log(1/h)$ losses in the computation of the Poisson bracket (see also [2, Appendix A] for analogous discussion in the homogeneous setting). We have chosen a different approach using the half-wave propagator since we will need to understand $U(t)u_h$ anyway to go through the conical points.

In the next subsection, we prove parts (1) and (2) of Proposition 4.2 and in the following one we will turn to the parts (3) and (4).

4.1. Propagation in the interior region

Consider a resonant state u_h , i.e., a solution to

$$(P_\theta - z_h)u_h = 0.$$

Then of course locally in $B(0, R_1)$ we have simply

$$(h^2\Delta - z_h)u_h = 0$$

so that, by finite speed of propagation,

$$\frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}u_h = \frac{h \sin(t\frac{\sqrt{z_h}}{h})}{\sqrt{z_h}}u_h,$$

in $B(0, R_1 - A)$ for any $|t| < A$.

We need a little more work to understand $U(t)u_h$.

LEMMA 4.5. — *Let u_h be a resonant state associated with the resonance z_h with $\sqrt{z_h} \in \Omega_e$. For any $|t| < A$ and any open set V such that $\bar{V} \subset B(0, R_1 - A) \cap X^\circ$.*

$$(4.2) \quad U(t)u_h = e^{-it\sqrt{z_h}/h}u_h + O(h^\infty)$$

in $L^2(V)$.

Proof. — Since $(P_\theta - z_h)u_h = 0$, we have $(h\sqrt{\Delta} + \sqrt{z_h})(h\sqrt{\Delta} - \sqrt{z_h})u_h = 0$ in V . Using the results of Appendix A, we know that, in V , $\sqrt{\Delta}$ is a pseudodifferential operator with real symbol so that, by ellipticity of the first factor, we obtain $(h\sqrt{\Delta} - \sqrt{z_h})u_h = O(h^\infty)$ in $L^2(V)$. We now set

$$v_h(t, \cdot) = \frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}}u_h.$$

We have

$$\begin{aligned} U(t)u_h &= \partial_t v_h(t, \cdot) - i\sqrt{\Delta}v_h \\ &= \cos\left(t\frac{\sqrt{z_h}}{h}\right)u_h - i\frac{h\sin\left(t\frac{\sqrt{z_h}}{h}\right)}{\sqrt{z_h}}\sqrt{\Delta}u_h. \end{aligned}$$

The claim follows since $h\sqrt{\Delta}u_h = \sqrt{z_h}u_h + O(h^\infty)$. \square

Proof of (1) and (2) of Proposition 4.2. — Let (x_0, ξ_0) in $T^*X^\circ \cap B(0, R_1)$ and t such that the geodesic of length t emanating from (x_0, ξ_0) stays in $T^*X^\circ \cap B(0, R_1)$. We write ϕ_t for \exp_{tH} .

If $(x_0, \xi_0) \notin \text{WF}_h^s(u_h)$ then we can find microlocal cutoffs Π_0 near (x_0, ξ_0) and Π near $\phi_t(x_0, \xi_0)$ such that $\Pi_0 u_h = O_{L^2}(h^s)$ and $\Pi U(t)(I - \Pi_0)$ is smoothing. It follows that

$$\begin{aligned} \Pi u_h &= e^{it\sqrt{z_h}/h}\Pi U(t)u_h + O_{L^2}(h^\infty) \\ &= e^{it\sqrt{z_h}/h}\Pi U(t)\Pi_0 u_h + O_{L^2}(h^\infty) \end{aligned}$$

By unitarity of $U(t)$,

$$\|e^{it\sqrt{z_h}/h}\Pi U(t)\Pi_0 u_h\| \leq h^{-2(\nu t+0)}\|\Pi_0 u_h\|,$$

hence

$$(4.3) \quad \Pi u_h = O_{L^2}(h^{s-2\nu t-0}). \quad \square$$

4.2. Propagation in the scaling region

We now prove a propagation result which is valid everywhere in T^*X° , including the scaling region.

Let $(x_0, \xi_0) \notin \text{WF}_h^s(u_h)$. Assume that $\phi_t(x_0, \xi_0)$ lies over X° for all $t \in [0, T]$, for some $T > 0$. We will show that $\phi_t(x_0, \xi_0) \notin \text{WF}_h^{s-M_0 t-\epsilon} u_h$ for all $t \in [0, T]$ and $\epsilon > 0$; the case of negative values of t will follow by the exact same argument, which is not sensitive to choices of sign. The constant M_0 will be defined during the proof.

We prove by (descending) induction that for any $N \in \mathbb{N}$ the following holds true: for any $\epsilon > 0$, we have $\phi_t(x_0, \xi_0) \notin \text{WF}_h^{s-N-M_0 t-\epsilon}$ for $t \in [0, T]$.

This assertion certainly holds true if N is large enough (since $u_h \in L^2$). We now assume that it holds true for some N , set $s' = s - N$, and choose $\epsilon > 0$.

There exists $\Pi_0 \in \Psi_h(X^\circ)$, elliptic at (x_0, ξ_0) , with $\Pi_0 u = O_{L^2}(h^s)$. Let

$$\Pi(t) = U(t)\Pi_0 U(-t);$$

by the results of Appendix A this is in fact a pseudodifferential operator elliptic at $\phi_t(x_0, \xi_0)$. By our inductive hypothesis, we may shrink Π_0 if necessary (but still include a *fixed* open neighborhood of (x_0, ξ_0) in its elliptic set) and then additionally find $\tilde{\Pi}(t)$ elliptic on $\text{WF}' \Pi(t)$ with $\tilde{\Pi}(t)u = O_{L^2}(h^{s'-M_0t-\frac{\epsilon}{2}})$.

Microlocally in T^*X° , the operator $h\sqrt{\Delta} + \sqrt{z_h}$ is an elliptic pseudodifferential operator so that

$$\sqrt{\Delta}u_h - \frac{\sqrt{z_h}}{h}u_h = Q_h u_h,$$

where we have set

$$Q_h := h^{-1}(h\sqrt{\Delta} + \sqrt{z_h})^{-1}(h^2\Delta - P_\theta).$$

Thus $Q_h \in \log(1/h)\Psi_h(X^\circ)$ in the scaling region (and vanishes where the scaling vanishes).

Let $f(t) = \|\Pi(t)u_h\|$. Then we compute:

$$\begin{aligned} 2ff' &= \frac{d}{dt}f^2 = 2\text{Im} \left\langle [\sqrt{\Delta}, \Pi(t)]u_h, \Pi(t)u_h \right\rangle \\ &= -2\text{Im} \left\langle \Pi(t)\sqrt{\Delta}u_h, \Pi(t)u_h \right\rangle \\ &= -2\text{Im} \left\langle \Pi(t)(\sqrt{z_h}/h + Q_h)u_h, \Pi(t)u_h \right\rangle \\ &= -2\text{Im} \sqrt{z_h}/h \|\Pi(t)u_h\|^2 - 2\text{Im} \langle \Pi(t)Q_h u_h, \Pi(t)u_h \rangle \\ &= -2\text{Im} \sqrt{z_h}/h \|\Pi(t)u_h\|^2 - 2\text{Im} \langle Q_h \Pi(t)u_h, \Pi(t)u_h \rangle \\ &\quad - 2\text{Im} \langle [\Pi(t), Q_h]u_h, \Pi(t)u_h \rangle. \end{aligned}$$

There exists a constant M_0 (which is independent of s') such that

$$M_0 > \frac{(-\text{Im} \sqrt{z_h}/h) + \|Q_h\|}{\log(1/h)},$$

and another constant C such that $\|[\Pi(t), Q_h]u_h\| \leq Ch \log(1/h) \|\tilde{\Pi}(t)u_h\|$. Hence we obtain

$$\begin{aligned} 2ff' &\leq 2M_0 \log(1/h) \|\Pi(t)u_h\|^2 + Ch \log(1/h) \|\tilde{\Pi}(t)u_h\| \|\Pi(t)u_h\| \\ &\leq 2M_0 \log(1/h) f^2 + Ch^{1+s'-M_0t-\frac{\epsilon}{2}} \log(1/h) f. \end{aligned}$$

An application of the Gronwall inequality now yields

$$f(t) \leq f(0)h^{-M_0t} + Ch^{1+s'-M_0t-\epsilon} h^{\frac{\epsilon}{2}} \log(1/h),$$

hence our assumption on Π_0 yields

$$f(t) \leq Ch^{1+s'-M_0t-\epsilon},$$

which, since $\Pi(t)$ is elliptic on $\phi_t(x_0, \xi_0)$, completes the proof of the induction step. Part (3) of the proposition follows by setting $N = 0$.

Finally, in the deep scaling region, we follow the same argument as in the un-scaled region, using that

$$P_\theta u_h = z_h u_h \implies \sqrt{\Delta} u_h = (1 + ih \tan \theta) \frac{\sqrt{z_h}}{h} u_h$$

so that we obtain, in place of (4.3), the *gain* in regularity

$$\|\Pi u_h\| = O(h^{s+Ct}).$$

5. Microlocal concentration on the outgoing set

We now deduce from the preceding section the fact that the wavefront set of a resonant state only lives on the outgoing set, which we define below. Unless otherwise specified below, we will take the asymptotics (5.1) for z_h as a standing assumption from now on, so that we may apply the propagation theorems obtained above.

COROLLARY 5.1. — *Let*

$$(P_\theta - z_h)u_h = O(h^\infty)$$

with

$$(5.1) \quad z_h = E + o(1) - i(2\nu + o(1))h \log(1/h).$$

Let $q \in T^*X^\circ$ and assume that $\exp_{-tH}(q) \in T^*X^\circ$ for all $t > 0$. Then $q \notin \text{WF}_h u_h$.

Proof. — The assumption on the flowout of q means, given our standing hypothesis that geodesics in X° are non-trapped, that the backward flowout of q eventually escapes into the deep scaling region $X \setminus B(0, 2R_1)$. Thus there exists T_0 such that for $t \geq T_0$, $\pi(\exp_{-tH} q) \notin B(0, 2R_1)$. Since $u_h \in L^2$, part 4 of Proposition 4.2, applied to a neighborhood of the backward flowout of $\exp_{-T_0H}(q)$ for arbitrarily long times, gives $\exp_{-T_0H}(q) \notin \text{WF}_h u_h$. Part 3 then yields $q \notin \text{WF}_h u_h$. \square

This proposition tells us that a resonant state u_h can only have wavefront on rays that emanate from the conical points. This leads to the following definition.

DEFINITION 5.2. — *Let Γ_\pm denote the flowout/flowin from/to the union of cone points, i.e.*

$$\Gamma_\pm = \{q \in T^*X^\circ : \exp_{tH}(q) \rightarrow Y \text{ as } t \rightarrow t_0^\pm \text{ for some } t_0 \leq 0\}$$

Let $\Gamma \equiv \Gamma_+ \cap \Gamma_-$ denote the geodesics propagating among the cone points.

Locally near Y , in coordinates (r, y) from (2.1) with canonical dual coordinates ξ, η , we have $\Gamma_{\pm} = \{\xi \gtrless 0, \eta = 0\}$. The preceding corollary thus says that

$$\text{WF}_h(u_h) \subset \Gamma_+.$$

The set Γ is the trapped set that corresponds to our setting (cf. Chapter 6 in [11]). It consists in the geodesic rays that connect two (not necessarily distinct) conical points. We now proceed to show that a non-trivial resonant state must have some wavefront set on Γ . This will result from the composition with the half-wave propagator near a conical point and of the known structure of this operator.

6. Composition with the wave propagator

Recall that $U(t)$ denotes the half-wave propagator :

$$U(t) = e^{-it\sqrt{\Delta}}.$$

The propagation results of [20], [6, 7] translate immediately into statements on propagation of semiclassical wavefront set:

PROPOSITION 6.1. — *Let $A_h, B_h \in \Psi_h(X^\circ)$ be compactly supported semiclassical pseudodifferential operators with microsupports in a neighborhood of a cone point Y_α that are strictly diffractively related (i.e. no geometric geodesics through Y_α connect a point in $\text{WF}'_h A_h$ and $\text{WF}'_h B_h$) and with*

$$(\text{WF}'_h A_h \cup \text{WF}'_h B_h) \cap (0 - \text{section}) = \emptyset.$$

Then for any f_h with $(P_h - z_h)f_h = O(h^\infty)$ we have

$$\text{WF}_h(A_h U(t) B_h f_h) \subset D \circ \text{WF}_h f_h$$

where D denotes the canonical relation in canonical coordinates (r, ξ, θ, η)

$$\{(r, \xi, y, \eta, r', \xi', y', \eta') : r + r' = t, \eta = \eta' = 0, \xi = -\xi'\}.$$

Moreover a quantitative version of this result holds: there exists N such that

$$\text{WF}_h^s(A_h U(t) B_h f_h) \subset D \circ \text{WF}_h^{s+N} f_h.$$

As a special case, this result tells us that if there is no wavefront set at all on geodesics arriving at Y_α then there is no wavefront set on geodesics leaving it.

Proof. — The main result of [20] is that near diffractively related points, for fixed t , the Schwartz kernel of $U(t)$ is a conormal distribution with respect to $r + r' = t$. Then (8.4.8) of [31] shows that in fact we locally have

$$\mathrm{WF}_h(U(t)) \setminus \{0\text{-section}\} = N^*\{r + r' = t\}.$$

The mapping property on WF_h then follows from the usual results on mapping properties of FIOs.

The quantitative version follows from the closed graph theorem. (We could of course get an explicit N but it is immaterial for our purposes.) \square

Remark 6.2. — The precise form of the principal symbol of the conormal distribution will not concern us so much as the mere fact of conormality (and the order of the distribution).

As a consequence of this lemma and of Proposition 6.1 together with our “free” propagation results above (Proposition 4.2), we may now draw the desired conclusion about the microsupport of a resonant state.

COROLLARY 6.3. — *Suppose $u_h \in L^2(X)$ and*

$$(P_\theta - z_h)u_h = 0.$$

Then $\mathrm{WF}_h u_h \subset \Gamma_+$. If $\mathrm{WF}_h u_h \cap \Gamma = \emptyset$ then $u_h = O(h^\infty)$.

Proof. — The containment in Γ_+ is simply Corollary 5.1 above.

Now suppose that there is no wavefront set along $\Gamma = \Gamma_+ \cap \Gamma_-$. Consider one cone point Y_α . It is possible to choose, for small ϵ , B_h that microlocalizes near the sphere of radius ϵ centered at Y_α in the incoming directions and A_h that microlocalizes near the same sphere but in the outgoing directions, so that, setting $t = 2\epsilon$,

$$\mathrm{WF}_h A_h U(t) u_h = \mathrm{WF} A_h U(t) B_h u_h.$$

Applying Proposition 6.1, we find $\mathrm{WF}_h A_h U(t) B_h u_h = \emptyset$. The latter equality and equation (4.2) then implies that $\mathrm{WF}_h A_h u_h = \emptyset$. Since this argument works for any α , we see from our propagation results above that $u_h = O(h^\infty)$ globally. \square

It is thus sufficient to understand u_h near the rays in Γ in order to understand its global behavior. In the following section, we encode the behaviour of u_h near such a ray by restricting u_h to a transversal cross-section. This is reminiscent of the construction of the quantum monodromy matrix of Nonnenmacher–Sjöstrand–Zworski (see [21, 22]).

7. Restriction and extension

Let $S \subset X^\circ$ be an open oriented hypersurface and let (x, y) denote normal coordinates near S , i.e., x denotes the signed distance from the nearest point on S , which then has coordinate $y \in S$. We let ξ, η denote canonical dual variables to (x, y) in T^*X .

In the following proposition, we construct an extension operator \mathcal{E} that builds a microlocal solution to $(P_h - z_h) = O(h^\infty)$ given data on S (Cauchy data).

PROPOSITION 7.1. — *There exists an amplitude a and a phase*

$$\phi = ((y - y') \cdot \eta + x\sqrt{1 - |\eta|_{g(0,y)}^2} + O(x^2))$$

having phase variable $\eta \in \mathbb{R}^{n-1}$ such that the operator \mathcal{E} with kernel defined by

$$\mathcal{E}(x, y, y') \equiv \left(\frac{h}{\sqrt{z_h}}\right)^{-(n-1)} \int a(x, y, y', \eta; h) e^{i\phi\sqrt{z_h}/h} d\eta$$

solves

$$\begin{aligned} (P_h - z_h)\mathcal{E}f_h &= O(h^\infty), \\ \mathcal{E}f_h|_{x=0} &= f_h + O(h^\infty), \end{aligned}$$

for any f_h such that $\text{WF}(f_h) \subset \{|\eta| < \frac{1}{2}\}$. The amplitude a enjoys an asymptotic expansion in nonnegative powers of $h/\sqrt{z_h}$, with coefficients that are smooth functions of y', p .

Proof. — We employ the Ansatz

$$\left(\frac{\sqrt{z_h}}{h}\right)^{n-1} \int a(x, y, y', \eta; h/\sqrt{z_h}) e^{i\phi\sqrt{z_h}/h} f(y') d\eta dy',$$

where a is assumed to have an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_j(x, y, y', \eta) \left(\frac{h}{\sqrt{z_h}}\right)^j.$$

We find that the Cauchy data is reproduced (modulo $O(h^\infty)$) so long as

$$(7.1) \quad a|_{x=0} \equiv 1$$

and $\phi|_{x=0} = (y - y') \cdot \eta$. On the other hand, applying $P_h - z_h$ to this expression yields, first, the eikonal equation

$$z_h((\partial_x \phi)^2 + |\nabla_y \phi|_g^2) - z_h = 0,$$

which as usual can locally be solved in the form

$$\phi = ((y - y') \cdot \eta + x\sqrt{1 - |\eta|_{x,y}^2} + O(x^2)),$$

by parametrizing the Lagrangian given by flowout along the Hamilton flow in (x, ξ, y, η) of the set

$$\{(x = 0, \xi = \sqrt{1 - |\eta|_{g(0,y)}^2}, y = y', \eta = \eta')\}.$$

Next, the leading transport equation reads

$$2ih\sqrt{z_h}\nabla\phi \cdot \nabla a_0 + ih\sqrt{z_h}\Delta(\phi)a_0 = 0.$$

This can be solved by integrating from $x = 0$ to give a smooth solution with Cauchy data (7.1).

The next transport equation picks out the term $(h/\sqrt{z_h})a_1$ and now reads

$$2ih\left(\frac{h}{\sqrt{z_h}}\right)\sqrt{z_h}\nabla\phi \cdot \nabla a_1 + ih\sqrt{z_h}\left(\frac{h}{\sqrt{z_h}}\right)\Delta(\phi)a_1 - h^2\Delta(a_0) = 0;$$

this likewise has a smooth solution a_1 . Subsequent transport equations take the same form.

We may now Borel sum the results of solving the transport equations to find that the operator \mathcal{E} with kernel

$$\mathcal{E}(x, y, y') \equiv \left(\frac{h}{\sqrt{z_h}}\right)^{-(n-1)} \int a(x, y, y', \eta; h) e^{i\phi\sqrt{z_h}/h} d\eta$$

solves

$$\begin{aligned} (P_h - z_h)\mathcal{E}f_h &= O(h^\infty), \\ \mathcal{E}f_h|_{x=0} &= f_h + O(h^\infty). \quad \square \end{aligned}$$

This extension operator gives us a way to parametrize (microlocally) any solution to $(P_h - z_h)u_h = O(h^\infty)$.

PROPOSITION 7.2. — *Let u_h be a solution to $(P_h - z_h)u_h = O(h^\infty)$ such that $\text{WF}_h u_h \subset \{\xi > 0\}$. Set $f_h = u_h|_{x=0}$. Then microlocally near $(p_0, \dot{\gamma}_{ij}^k)$,*

$$u_h - \mathcal{E}f_h = O(h^\infty).$$

Proof. — The proof relies on the microlocal energy estimates of [8, Section 3.2] (based, in turn on [17, Sections 23.1-23.1] in the homogeneous setting). According to these estimates, for any solution to $(P_h - z_h)w_h = O(h^\infty)$ with $\text{WF}' w_h \subset \{\xi > 0\}$ we have the bound

$$(7.2) \quad \|w_h|_{x=x_1}\| \lesssim h^{-C|x_1|} \|w_h|_{x=0}\|;$$

the only real change needed from the treatment in [8] stems from the fact that operators A_\pm used there are no longer self-adjoint, with $A_\pm^* - A_\pm = O(h \log(1/h))$, leading to the growth in norms in the equation above: the LHS of equation (3.10) of [8] now has a factor of h^{Cy} inside the supremum arising from the non-self-adjointness.

Equation (7.2) suffices to show that our solution

$$w_h = u_h - \mathcal{E}f_h$$

to

$$\begin{aligned} (P_h - z_h)w_h &= O(h^\infty), \\ w_h|_{x=0} &= O(h^\infty) \end{aligned}$$

must itself be $O(h^\infty)$, as desired. \square

We can now use standard FIO methods first to change the phase function to the Riemannian distance along the geodesic γ and then to extend \mathcal{E} in a microlocal neighbourhood of γ even past conjugate points.

Changing the phase function means that, for (x, y) strictly away from S but still in a small neighborhood, we may also write u_h in the form

$$(7.3) \quad u_h(p) = \int \left(\frac{h}{\sqrt{z_h}} \right)^{-(n-1)/2} \tilde{a}(p, y'; h/\sqrt{z_h}) e^{i\sqrt{z_h} \operatorname{dist}(p, y')/h} f_h(y'; h) dy' + O(h^\infty)$$

where dist denotes the Riemannian distance. Again the amplitude has an expansion in integer powers of $h/\sqrt{z_h}$.

To see that this is possible, it suffices to observe that we may do a stationary phase expansion in the y', η integrals (Proposition B.3 below), reducing the number of phase variables from $(n-1)$ to zero. In this case, by construction the value of $d_{x,y}\phi$ where $d_\eta\phi = 0$ is just the tangent to the geodesic flowout from $(x = 0, \xi, y, \eta)$ where $\xi^2 + |\eta|^2 = 1$, so that it agrees with $d_{x,y} \operatorname{dist}((x, y), (0, y'))$, hence the new oscillatory term becomes exactly $e^{i \operatorname{dist} \sqrt{z_h}/h}$. The amplitude has an expansion in powers of $h/\sqrt{z_h}$, with an overall factor of $(h/\sqrt{z_h})^{(n-1)/2}$, since the η integral was over \mathbb{R}^{n-1} .

We now proceed to extend the structure theorem for $\mathcal{E} : f_h \rightarrow u_h$ globally along a given geodesic, even past conjugate points.

PROPOSITION 7.3. — *Fix a geodesic γ intersecting S orthogonally at $p_0 \in S$. Let $p \in \gamma$ not be conjugate to p_0 . Subject to the assumptions of Proposition 7.1, microlocally near $(p, \dot{\gamma})$ we have $u_h = \mathcal{E}(f_h)$ with*

$$(7.4) \quad \mathcal{E}f_h = \int \left(\frac{h}{\sqrt{z_h}} \right)^{-(n-1)/2} a(p, y', \eta; z_h, h) e^{i \operatorname{dist}(p, y') \sqrt{z_h}/h} f_h(y') d\eta dy'.$$

The function dist should be interpreted here (potentially far beyond the injectivity radius) as the smooth function given by distance along the family

of geodesics remaining microlocally close to γ , i.e. specified by the locally-defined smooth inverse of the exponential map.

Proof. — Using the preceding construction, for any x_0 small enough we can construct \mathcal{E}_{x_0} starting from the surface $x = x_0$. Denoting by \mathcal{R}_{x_0} the restriction to the surface $x = x_0$ we then have, by construction the following semigroup property :

$$(7.5) \quad \mathcal{E}_x \mathcal{R}_x \mathcal{E} = \mathcal{E}.$$

The proof then follows by decomposing the geodesic into small enough steps $[x_i, x_{i+1}]$ and applying stationary phase. \square

DEFINITION 7.4. — *Since we will often be referring to symbols that have a half-step polyhomogeneous expansion in $h/\sqrt{z_h}$ in the product type case (i.e., with metric of the form (2.2) near the boundary), but only have a leading order asymptotics modulo $O((h/\sqrt{z_h})^{1/2-0})$ times the leading order power in the general case, we will simply say that the function in question has adapted half-step asymptotics to cover both cases.*

This distinction will not be of great importance for the results presented here, but we will maintain it in the hope of future applications in which the product type case may offer stronger results.

7.1. Undergoing one diffraction

Now we study the composition of the microlocalized extension operator and the microlocalized wave propagator when one diffraction occurs.

Let γ be a geodesic orthogonal to S at $p_0 \in S$ and terminating at cone point Y_j , with p_0 not conjugate to Y_j (see [3] for a definition of conjugacy in this context). Away from the conjugate locus of p_0 let dist_S be the distance function from S measured along geodesics near γ . Let dist_j denote the distance function from the cone point Y_j , and let $\text{dist}_{S,j}$ denote the restriction of dist_j to S , near p_0 .

PROPOSITION 7.5. — *Let $\text{WF}'_h A$ and $\text{WF}'_h B$ contain only diffractively related points, and let $\sqrt{z_h} \in \Omega_\epsilon$. There exists a symbol $c \in S_0$ with adapted half-step asymptotics such that for*

$$(7.6) \quad \text{dist}_j(x) < t < \text{dist}_{S,j}(p_0) + \text{dist}_j(x),$$

we have

$$AU(t)B\mathcal{E}(x, y') = c(x, y'; h/\sqrt{z_h})e^{i(\text{dist}_{S,j}(y') + \text{dist}_j(x) - t)\sqrt{z_h}/h}.$$

Proof. — We will employ stationary phase to compose the two oscillatory integral representations (2.3) and (7.3). In particular, we must evaluate an integral of the form

$$\left(\frac{h}{\sqrt{z_h}}\right)^{-(n-1)/2} \int \int_0^\infty a_D e^{i(\text{dist}_j(x) + \text{dist}_j(x'') - t)\xi} a_S e^{i \text{dist}_S(x'', y') \sqrt{z_h}/h} d\xi dx'',$$

where a_S has a complete asymptotic expansion in $\sqrt{z_h}/h$ while a_D has adapted half-step asymptotics. We would like to formally make the change of variables $\xi = \xi' \sqrt{z_h}/h$ and then do stationary phase (as in Appendix B) in the small parameter $h/\sqrt{z_h} \downarrow 0$; justifying this deformation into the complex in fact proceeds as follows. To begin, we let $\eta = h\xi$ so that we are trying to evaluate

$$h^{-1} \left(\frac{h}{\sqrt{z_h}}\right)^{-(n-1)/2} \int \int_0^\infty a_D e^{i(\text{dist}_j(x) + \text{dist}_j(x'') - t)\eta/h} a_S e^{i \text{dist}_S(x'', y') \sqrt{z_h}/h} d\eta dx''.$$

By the usual method of nonstationary phase, we then find that the integral is unchanged modulo $O(h^\infty)$ if we insert a compactly supported cutoff $\chi(\eta)$, equal to 1 for $|\eta| < 2$. Finally, we replace $\chi(\eta)a_D(\dots, \eta)$ with an almost-analytic extension in η , and set $\xi' = \eta/\sqrt{z_h}$, justifying the resulting contour deformation exactly as in the proof of Lemma B.2 in the appendix. This finally yields

$$\left(\frac{h}{\sqrt{z_h}}\right)^{-(n+1)/2} \int \int_0^\infty a_D e^{i(\text{dist}_j(x) + \text{dist}_j(x'') - t)\xi' \sqrt{z_h}/h} a_S e^{i \text{dist}_S(x'', y') \sqrt{z_h}/h} d\xi' dx''.$$

Finally we apply Proposition B.3 to justify a formal stationary phase expansion of this expression. Stationary points are where

$$(7.7) \quad \text{dist}_j(x) + \text{dist}_j(x'') - t = 0,$$

$$(7.8) \quad \nabla_{x''} \text{dist}_j(x'') + \xi' \nabla_{x''} \text{dist}_S(x'', y') = 0.$$

The latter equation implies that $\xi' = 1$ and that x'' must lie on the unique geodesic near γ connecting y' to Y_j , hence at the stationary point, $\text{dist}_j(x'') + \text{dist}_S(x'', y') = \text{dist}_{S,j}(y')$. Thus stationary phase yields a result of the form

$$\text{symbol} \cdot e^{i(\text{dist}_{S,j}(y') + \text{dist}_j(x) - t)\sqrt{z_h}/h}.$$

Since the stationary phase is in the $n+1$ variables (x'', ξ) we gain an overall factor $(h/\sqrt{z_h})^{(n+1)/2}$, reducing the overall power of $h/\sqrt{z_h}$ to zero. \square

We can extend this formula for larger times by precomposing and post-composing with $U(t)$ (see [12, Section 5]). Using the semigroup property for $U(t)$, and stationary phase to compute the compositions, we see that this formula continues to hold, correctly interpreted for x far from Y_j as well,

as long as x is not conjugate to Y_j and we microlocalize near the geodesic connecting Y_j to x .

PROPOSITION 7.6. — *Let A be microsupported sufficiently close to a geodesic γ coming from Y_j , in a small neighborhood of a point not conjugate to Y_j along γ . There exists a symbol $c \in S_0$ with adapted half-step asymptotics such that for*

$$\text{dist}_j(x) < t < \text{dist}_{S,j}(p_0) + \text{dist}_j(x),$$

$$AU(t)\mathcal{E}(x, y') = c(x, y'; h/\sqrt{z_h})e^{i(\text{dist}_{S,j}(y') + \text{dist}_j(x) - t)\sqrt{z_h}/h},$$

where the distance is interpreted as distance along geodesics microlocally close to γ .

Finally, we examine what happens when we again restrict to a hypersurface. Let the geometric setup be as above, with S' a hypersurface orthogonal to a geodesic from Y_j . Let \mathcal{R} denote the operation of restriction to S' , and let $\text{dist}_{j,S'}$ denote the distance function from the cone point Y_j to S' .

PROPOSITION 7.7. — *There exists a symbol $c \in S_0$ such that for*

$$\text{dist}_j(y) < t < \text{dist}_j(y') + \text{dist}_j(y),$$

$$\mathcal{R}U(t)\mathcal{E}(y, y') = c(y, y'; h/\sqrt{z_h})e^{i(\text{dist}_{S,j}(y') + \text{dist}_{j,S'}(y) - t)\sqrt{z_h}/h}.$$

8. Monodromy data

In this section we examine relations that hold among the restrictions of a resonant state to cross sections of geodesics in Γ .

Consider the directed multigraph whose vertices are the cone points and whose edges are the oriented geodesics connecting pairs of cone points. Let E be the edge set of this graph and for $e, f \in E$ write $e \rightarrow f$ if e and f are adjacent in the sense of digraphs, i.e. if e terminates at some vertex Y_i while f emanates from Y_i . For any edge e let \bar{e} denote the edge corresponding to the same geodesic but with opposite orientation.

To each directed edge e we fix a patch of oriented hypersurface S_e intersecting it orthogonally at a point not conjugate to either of the cone points at which e originates and terminates. There is no particular need to require that S_e and $S_{\bar{e}}$ be identical as unoriented surfaces (but of course that is one option). We arrange for the sake of simplicity that each S_e intersects only the edges e, \bar{e} .

For each $e \in E$ let d_e^\pm denote the functions on S_e given by the distances to the cone points at the end point (+) and starting point (−) of e . Let ℓ_e denote the length of the edge e (hence of course $\ell_e = \ell_{\bar{e}}$).

For each edge $e \in E$, let \mathcal{E}_e resp. \mathcal{R}_e respectively denote the parametriz for the extension operator (7.4) and the restriction operators from/to the oriented surface S_e orthogonal to this edge (as discussed in §7). Given $f, e \in E$ with $f \rightarrow e$, with incidence at the cone point Y_j let t_{fe} denote a number exceeding $\text{dist}(S_f, Y_j) + \text{dist}(Y_j, S_e)$ by a small fixed quantity $\epsilon_1 > 0$.

LEMMA 8.1. — *Consider a sequence of solutions to*

$$(P_\theta - z_h)u_h = 0, \quad h \downarrow 0.$$

For each $e \in E$,

$$(8.1) \quad \mathcal{R}_e u_h - \sum_{f \rightarrow e} e^{it_{fe}\sqrt{z_h}/h} \mathcal{R}_e U(t_{fe}) \mathcal{E}_f \mathcal{R}_f u_h = O(h^\infty).$$

Proof. — We first recall that, for any t ,

$$\mathcal{R}_e u_h = \mathcal{R}_e e^{it\sqrt{z_h}/h} U(t) u_h + O(h^\infty);$$

choose $t = d + \varepsilon$ where d is the distance from S_e to the cone point Y_j from which the edge e emanate and ε is small enough so that that $t < t_{fe} - \varepsilon$ for all $f \rightarrow e$. By propagation of singularities (Proposition 6.1) $\mathcal{R}_e u_h$ is determined, modulo $O(h^\infty)$, by u_h on the sphere of radius ε centered at Y_j . Denote by \mathcal{V} the $\frac{\varepsilon}{2}$ neighborhood of this sphere. Since $\text{WF}_h u_h$ is a subset of the geodesics represented by the edges in E , we may let A_f denote microlocalizers along all edges $f \rightarrow e$, supported in \mathcal{V} with $\text{WF}'_h(I - A_f)$ disjoint from all points near f that are diffractively related to points in S_e in time t . By the semiclassical Egorov theorem [31, Theorem 11.1] for $e^{-is\sqrt{\Delta}} = e^{-is\sqrt{h^2\Delta}/h}$ (with $s = t_{fe} - t$), we may write

$$A_f U(t_{fe} - t) = U(t_{fe} - t) A'_f$$

where A'_f is microsupported near the intersection of f with S_f with $\text{WF}'_h(I - A'_f)$ disjoint from a smaller neighborhood of this intersection.

By Proposition 6.1, then,

$$(8.2) \quad \mathcal{R}_e u_h = \sum_{f \rightarrow e} \mathcal{R}_e e^{it\sqrt{z_h}/h} U(t) A_f u_h + O(h^\infty).$$

and by (4.2) and a further application of Proposition 6.1,

$$\begin{aligned} \mathcal{R}_e e^{it\sqrt{z_h}/h} U(t) A_f u_h &= \mathcal{R}_e e^{it_{f_e}\sqrt{z_h}/h} U(t) A_f U(t_{f_e} - t) u_h + O(h^\infty) \\ &= \mathcal{R}_e e^{it_{f_e}\sqrt{z_h}/h} U(t_{f_e}) A'_f u_h + O(h^\infty) \\ &= \mathcal{R}_e e^{it_{f_e}\sqrt{z_h}/h} U(t_{f_e}) \mathcal{E}_f \mathcal{R}_f u_h + O(h^\infty). \quad \square \end{aligned}$$

Thus we conclude that the functions $\mathcal{R}_e u_h$ satisfy a set of relations which we can now employ to deduce constraints on $\text{Im } z_h$. By Proposition 7.7 we find the following:

PROPOSITION 8.2. — *There exist symbols c_{f_e} of order zero with adapted half-step asymptotics such that for each e ,*

$$(8.3) \quad \mathcal{R}_e u_h = \sum_{f \rightarrow e} A_{f_e} \mathcal{R}_f u_h + O(h^\infty)$$

where A_{f_e} has Schwartz kernel

$$(8.4) \quad A_{f_e}(y, y') = c_{f_e}(y, y'; h/\sqrt{z_h}) e^{i(d_e^-(y) + d_f^+(y'))\sqrt{z_h}/h}.$$

Proof. — We insert the representation of the wave propagator from Proposition 7.7 into (8.1). \square

Thus we have a matrix equation with operator-valued entries for the restriction to the hypersurfaces.

PROPOSITION 8.3. — *Let M be the constant used in defining the scaling region and L_0 the longest geodesic between two cone points. For each e , there exists a smooth amplitude $s_e(y; h) \in h^{-1-ML_0} S(1)$ such that*

$$\mathcal{R}_e u_h = e^{id_e^-(y)\sqrt{z_h}/h} s_e(y; h) + O(h^\infty).$$

Remark 8.4. — No claim is made here about polyhomogeneity of s_e in h .

Proof. — We want to prove that $y \mapsto s_e(y; h)$ is smooth and that any seminorm $\|\partial^\beta s(\cdot; h)\|_\infty$ is $O(h^{-1-ML_0})$. The fact that s is smooth follows from the fact that u_h is smooth by ellipticity. Using the eigenvalue equation we also have $\|u_h\|_{H_{\text{loc}}^k} = O(h^{-k})$, so that, roughly speaking, we lose one power of h by differentiating. The content of the proposition is thus that this loss actually does not occur.

We apply the operator given in (8.4) to u_h , noting that we can pull the factor $e^{id_e^-(y)\sqrt{z_h}/h}$ out of the integral. What remains is to show that the remaining factors of the form

$$(8.5) \quad \tilde{s}_{f_e}(y; h) = \int c_{f_e}(y, y'; h/\sqrt{z_h}) e^{id_f^+(y')\sqrt{z_h}/h} (\mathcal{R}_f u_h)(y') dy'$$

are in fact smooth amplitudes.

Using Sobolev embedding, and the fact that u_h is a solution to the eigenvalue equation, we have

$$\|\mathcal{R}_f u_h\|_{L^2(y)} \leq Ch^{-1}.$$

Replacing in (8.5) and using the Cauchy-Schwarz inequality we obtain

$$|\tilde{s}_{f_e}(y, h)| \leq Ch^{-1}h^{-ML_0} \left(\int |c_{f_e}(y, y'; \frac{h}{\sqrt{z_h}})|^2 dy' \right)^{\frac{1}{2}}.$$

The integral is uniformly bounded (in h) owing to the fact that c_{f_e} is a symbol. Moreover, \tilde{s}_{f_e} enjoys iterated regularity under differentiation in y , as y derivatives of (8.5) only hit the factor c_{f_e} , hence all y -derivatives are $O(h^{-1-ML_0})$. \square

It is convenient to rewrite (8.3) in terms of the amplitudes s_e :

$$(8.6) \quad s_e = \sum_{f \rightarrow e} e^{i\ell_f \frac{\sqrt{z_h}}{h}} \mathcal{M}_{f_e} s_f,$$

where \mathcal{M}_{f_e} has Schwartz kernel

$$(8.7) \quad M_{f_e}(y, y') = c_{f_e}(y, y'; \frac{h}{\sqrt{z_h}}) e^{i\frac{\sqrt{z_h}}{h} \phi_f(y, y')}$$

with $\phi_f(y, y') = d_f^+(y') + d_f^-(y') - \ell_f$.

Observe that the phase ϕ_f has a unique non-degenerate critical point at $y' = 0$ and satisfies $\phi_f(0) = 0$.

Equation (8.6) can be rewritten as

$$(\text{Id} - \mathcal{M}) \mathbf{s} = O(h^\infty),$$

which is typical of a monodromy operator in such settings. In our setting, we can pass to a discrete set of restriction data, given by the jets of the restriction to S_e at e . (Compare, e.g., Proposition 4.3 of [5] and see also [21, 22]). This will reduce the monodromy equation to a finite dimensional system.

For each $\alpha \in \mathbb{N}^{n-1}$, we thus let

$$s_e^\alpha = \partial_y^\alpha s_e,$$

so that

$$s_e(y) = \sum_{|\alpha| < N} s_e^\alpha \frac{y^\alpha}{\alpha!} + R_{e,N}, \quad R_{e,N} = O(|y|^N),$$

Applying the stationary phase computation in Appendix B gives the asymptotic expansion

$$(8.8) \quad \mathcal{M}_{fe}(y'^{\alpha}) \sim \left(\frac{h}{\sqrt{z_h}}\right)^{\frac{n-1}{2}} \sum_{k \geq \frac{|\alpha|}{2}} m_{ef\alpha k}(y; \frac{h}{\sqrt{z_h}}) \left(\frac{h}{\sqrt{z_h}}\right)^k,$$

where $m_{ef\alpha k}$ is a symbol.

Remark 8.5. — If $|\alpha|$ is odd, a closer inspection shows that this contribution is $O(h^{\infty})$.

PROPOSITION 8.6. —

- (1) For each $f \rightarrow e$, α , β there exist $C(\alpha, \beta, e, f; h)$, bounded in $h \downarrow 0$, such that

$$(8.9) \quad s_e^{\alpha} = \sum_{f \rightarrow e} \sum_{j < N} \sum_{|\beta| \leq 2j} C(\alpha, \beta, j, e, f; \frac{h}{\sqrt{z_h}}) \left(\frac{h}{\sqrt{z_h}}\right)^{(n-1)/2+j} e^{i\ell_f \frac{\sqrt{z_h}}{h}} s_f^{\beta} + O\left(\left(\frac{h}{\sqrt{z_h}}\right)^{(n-1)/2+N} e^{i\ell_f \frac{\sqrt{z_h}}{h}}\right).$$

The coefficients $C(\alpha, \beta, j, e, f; \frac{h}{\sqrt{z_h}})$ enjoy adapted half-step asymptotics, and in particular

$$C(\alpha, 0, 0, e, f; \frac{h}{\sqrt{z_h}}) \equiv C(\alpha, e, f) + O\left(\left(\frac{h}{\sqrt{z_h}}\right)^{1/2-0}\right),$$

with $C(\alpha, e, f)$ independent of $\frac{h}{\sqrt{z_h}}$.

- (2) For all $m_0 > 0$ there exist $m_1 > 0$ and $N_1 \in \mathbb{N}$ such that whenever $s_e^{\alpha} = O(h^{m_1})$ for all $e \in E$, $|\alpha| \leq N_1$, we have

$$s_e(y) = O(h^{m_0}) \text{ for all } e \in E.$$

Proof. — We truncate the asymptotic expansion (8.8) at $k = N$. We plug it into (8.6) and then extract the coefficient s_e^{α} from this expansion. This gives (8.9). The assertions on the coefficients follow by inspection. For the last assertion, we first observe that

$$|e^{i\ell_f \frac{\sqrt{z_h}}{h}}| = O(h^{-ML_0}),$$

where M is the constant used to define the scaling region and L_0 the longest geodesic between cone points. We then choose N so that $\frac{n-1}{2} + N - ML_0 \geq m_0$. We then set $N_1 = 2N$, and $m_1 = m_0 - \frac{n-1}{2}$, so that in (8.9) all the terms are $O(h^{m_0})$. \square

9. Proof of Theorem 1.1

We now prove Theorem 1.1.

Fix any $\epsilon > 0$. We assume throughout that

$$\sqrt{z_h} \in \Omega_\epsilon \equiv \{(-\Lambda + \epsilon)h \log(1/h) < \operatorname{Im} \sqrt{z_h} < 0, \operatorname{Re} \sqrt{z_h} \in [1 - \epsilon, 1 + \epsilon]\}.$$

Our aim is to show that if z_h is such a sequence of resonances, with $\operatorname{Re} \sqrt{z_h} \rightarrow E$ and $\operatorname{Im} \sqrt{z_h} \sim -\nu h \log(1/h)$ then we must have $\nu = (n-1)/2L_0$, while $\operatorname{Re} \sqrt{z_h}$ satisfy a quantization condition. As discussed above, we fix M , the parameter in our complex scaling, with $M \gg \Lambda$ so that square roots of eigenvalues of P_θ in Ω_ϵ agree with resonances of P in that set. By Proposition 4.2 and Proposition 6.1, there exists some $m_0 > 0$ such that if $\operatorname{WF}^{m_0} u_h \cap \Gamma = \emptyset$, then $\operatorname{WF}^\epsilon u_h = \emptyset$, hence u_h could not possibly be an L^2 -normalized resonant state. It thus suffices to show that if $\sqrt{z_h} \in \Omega_\epsilon$ does not satisfy the quantization condition or the condition on the imaginary part, we must have $\operatorname{WF}_h^{m_0} u_h \cap \Gamma = \emptyset$ for this fixed, potentially large, $m_0 > 0$. Moreover, again by Proposition 4.2, it suffices to show absence of $\operatorname{WF}^{m_1} u_h$ at the intersection of Γ with the edges $e \in E$, for some potentially larger $m_1 > 0$. To show this, in turn, we see by the second part of Proposition 8.6 that it suffices to show that if the desired conditions on z_h are not met then each s_e^α is $O(h^{m_2})$ for all α with $|\alpha| \leq N_2$ for some large (but geometrically determined) N_2 .

Thus, we suppose that either the quantization condition or the imaginary part condition is *violated* and we aim to show that consequently $s_e^\alpha = O(h^{m_2})$ for this finite list of values of α .

For any $N > N_2$, we let A_N denote the weighted directed edge adjacency matrix for a multigraph with multiple edges (e, α) for each edge $e \in E$ as above, but now with $\alpha \in \mathbb{N}^{n-1}$ a multi-index, with $|\alpha| < N$ and with $(e, \alpha) \rightarrow (f, \beta)$ an adjacency iff $e \rightarrow f$ in our original multigraph of directed geodesics with edge set E . The (e, α, f, β) entry of A_N may thus be nonzero only if $f \rightarrow e$, and in that case is defined to be:

$$(A_N)_{e, \alpha, f, \beta} \equiv \sum_{j \in \mathbb{N}: |\beta| \leq 2j < 2N} C(\alpha, \beta, j, e, f; \frac{h}{\sqrt{z_h}}) \left(\frac{h}{\sqrt{z_h}}\right)^{(n-1)/2+j} e^{il_f \frac{\sqrt{z_h}}{h}},$$

with $C(\alpha, \beta, j, e, f; \frac{h}{\sqrt{z_h}})$ given by Proposition 8.6.

LEMMA 9.1. — *If $\sqrt{z_h} \in \Omega_\epsilon$, and*

$$(P_\theta - z_h)u_h = 0$$

then viewed as an h -dependent vector $\mathbf{s} = \{s_e^\alpha\}$, the Cauchy data s_e^α of u_h on the hypersurfaces S_e satisfies

$$(9.1) \quad \mathbf{s} = A_N \cdot \mathbf{s} + \mathbf{r}, \quad \mathbf{r} = O(h^{(n-1+N)/2-L_0(\Lambda-\epsilon)}).$$

Proof. — This is just a truncation of the result of Proposition 8.6, where we also use the fact that for $\sqrt{z_h} \in \Omega_\epsilon$, $z_h, z_h^{-1} = O(1)$ while $\text{Im} \sqrt{z_h} > (-\Lambda + \epsilon)h \log(1/h)$, hence

$$\left| e^{i\ell_f \sqrt{z_h}/h} \right| = O(h^{-\ell_f(\Lambda-\epsilon)}) = O(h^{-L_0(\Lambda-\epsilon)}),$$

while the s_e^α are all bounded as $h \downarrow 0$. \square

Since the entries in A_N are all seen to be $O(h^{(n-1)/2-L_0(\Lambda-\epsilon)})$ by the same reasoning as in the proof of Lemma 9.1, we find applying A_N to both sides of our relation that

$$(9.2) \quad \begin{aligned} A_N^2 \mathbf{s} &= A_N \mathbf{s} - A_N \mathbf{r} \\ &= \mathbf{s} - \mathbf{r} - A_N \mathbf{r} \\ &= \mathbf{s} + O(h^{(n-1+N)/2-L_0(\Lambda-\epsilon)}) + O(h^{(n-1)+N/2-2L_0(\Lambda-\epsilon)}) \\ &= \mathbf{s} + O(h^{(n-1+N)/2-L_0(\Lambda-\epsilon)}), \end{aligned}$$

since $\Lambda \leq (n-1)/2L_0$.

Now we examine the entries in A_N^2 . This matrix has diagonal entries, corresponding to two-cycles in our digraph. The largest such entries are the two with zero multi-index that correspond to traversing f, \bar{f} or \bar{f}, f , for f a maximal edge, i.e., $\ell_f = L_0$. We assume from now on that there are J such maximal edges. By assumption, no two geodesics with length L_0 are incident on the same cone point. It is thus impossible to have an off-diagonal entry in A_N^2 with a maximal contribution and the only way to obtain these largest entries is to traverse one maximal oriented geodesic f_j and then return whence we came on \bar{f}_j .

The remaining entries are all bounded by either

$$O(h^{(n-1)-(2L')(\Lambda-\epsilon)})$$

in the case in which at least one edge traversed is not maximal, or else by

$$O(h^{(n-1)+1-2L_0(\Lambda-\epsilon)})$$

in the case that one multi-index β (and hence one value of j) is nonzero. In either case, we find that the entry is bounded by $O(h^{\epsilon'})$ with

$$(9.3) \quad \epsilon' \equiv \min \left\{ (n-1) - 2L'(\Lambda-\epsilon), (n-1) + 1/2 - 2L_0(\Lambda-\epsilon) \right\}.$$

Note that this is a positive number, by definition of Λ ; the $1/2$ term in the second expression is not optimal in the argument above but will be

necessary below. Thus, if we write A_N in block form with the edges $(f, 0)$ and $(\bar{f}, 0)$ with f maximal listed first, we have

$$A_N^2 = Q_N^0 + O(h^{\epsilon'})$$

with

$$Q_N^0 = \begin{pmatrix} \tilde{B}_1 & & & & \\ & \ddots & & & \mathbf{0} \\ & & \tilde{B}_J & & \\ & & & 0 & \\ \mathbf{0} & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

where the 2×2 block \tilde{B}_j is given by

$$\tilde{B}_j = (A_N)_{(\bar{f}_j, 0, f_j, 0)} (A_N)_{(f_j, 0, \bar{f}_j, 0)} \text{Id}_{2 \times 2}$$

and where the number of these blocks equals the number of maximal geodesics. Replacing these matrix entries with their leading order approximation, we now get the improvement

$$A_N^2 = Q_N + O(h^{\epsilon'})$$

with

$$Q_N = \begin{pmatrix} B_1 & & & & \\ & \ddots & & & \mathbf{0} \\ & & B_J & & \\ & & & 0 & \\ \mathbf{0} & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

and where, for each j ⁽³⁾

$$B_j = C(0, \bar{f}_j, f_j) C(0, f_j, \bar{f}_j) h^{(n-1)} e^{2iL_0 \sqrt{z_n}/h} \text{Id}_{2 \times 2}.$$

Here we have used the fact that approximating $(A_N)_{(f, 0, \bar{f}, 0)}$ by $C(0, f, \bar{f}) h^{(n-1)/2} e^{iL_0 \sqrt{z_n}/h}$ incurs an error of $O(h^{(n-1)/2+1/2-0} e^{iL_0 \sqrt{z_n}/h})$. Hence for any N sufficiently large, for $\sqrt{z_n} \in \Omega_\epsilon$ we have

$$\begin{aligned} (A_N)_{(\bar{f}, 0, f, 0)} (A_N)_{(f, 0, \bar{f}, 0)} &= C(0, \bar{f}, f) C(0, f, \bar{f}) h^{(n-1)} e^{2iL_0 \sqrt{z_n}/h} \\ &= O(h^{(n-1)+1/2-0} e^{i2L_0 \sqrt{z_n}/h}) \\ &= O(h^{\epsilon'}), \end{aligned}$$

⁽³⁾ In the sequel, we will drop the index j for readability.

with ϵ' given by (9.3) above; this argument is where the $1/2$ gain in the second term in the minimum taken in (9.3) is now optimal.

Now choose N big enough that

$$\frac{(n-1+N)}{2} - L_0\Lambda \gg m_2 + \epsilon',$$

where we recall that m_2 is the decay rate required to show that u_h could not be an L^2 -normalized eigenfunction of P_θ . Then by (9.2) we have the simple equation

$$(\text{Id} - Q_N + O(h^{\epsilon'}))\mathbf{s} = O(h^{m_2+\epsilon'}).$$

Clearly, if $\text{Id} - Q_N$ is invertible with $(\text{Id} - Q_N)^{-1} = O(h^{-\epsilon'/2})$ we can then invert to obtain

$$\mathbf{s} = O(h^{m_2+\epsilon'/2}),$$

which is the desired estimate: if this holds, then u_h could not have been a normalized resonant state after all.

Thus in order for a resonance to exist, we must have, by contrast, a lower bound on the inverse of $(\text{Id} - Q_N)^{-1}$. In particular, for some maximal edge f , we must have

$$\left| C(0, \bar{f}, f)C(0, f, \bar{f})h^{(n-1)}e^{2iL_0\sqrt{z_h}/h} - 1 \right|^{-1} \geq Ch^{-\epsilon'/2},$$

i.e.,

$$C(0, \bar{f}, f)C(0, f, \bar{f})h^{(n-1)}e^{2iL_0\sqrt{z_h}/h} = 1 + O(h^{\epsilon'/2}).$$

Taking log, this yields

$$2iL_0\frac{\sqrt{z_h}}{h} + (n-1)\log h + \log C(0, \bar{f}, f)C(0, f, \bar{f}) \in 2\pi i\mathbb{Z} + O(h^{\epsilon'/2}).$$

The equality of imaginary parts yields for the constant

$$\begin{aligned} C_{\text{Re}} &= -\text{Im} \log C(0, \bar{f}, f)C(0, f, \bar{f})/2L_0, \\ \text{Re} \sqrt{z_h} &\in h(C_{\text{Re}} + \frac{\pi}{L_0}\mathbb{Z}) + O(h^{1+\epsilon'/2}) \end{aligned}$$

while taking real parts gives for the constant

$$\begin{aligned} C_{\text{Im}} &= \text{Re} \log C(0, \bar{f}, f)C(0, f, \bar{f})/2L_0 \\ \text{Im} \sqrt{z_h} &= -\frac{(n-1)}{2L_0}h \log(1/h) + C_{\text{Im}}h + O(h^{1+\epsilon'/2}). \end{aligned}$$

Recalling that semiclassical rescaling gave

$$\text{Re} \sqrt{z_h} = 1, \quad h = (\text{Re} \lambda)^{-1}, \quad \frac{\text{Im} \sqrt{z_h}}{h} = \text{Im} \lambda,$$

this yields the statements of the theorem.

Remark 9.2. — As shown by Galkowski [14], if one of the diffraction coefficients $C(0, \bar{f}, f)$ does not vanish then the width of the resonance free logarithmic region depends on L_0 . Our argument shows when all these coefficients vanish, we indeed obtain a larger resonance-free logarithmic region, so that this condition is sharp.

Appendix A. (Micro)-locality of $\sqrt{\Delta}$

The aim of this appendix is to prove the two following facts for the Laplace operator Δ on a manifold with conical singularities.

PROPOSITION A.1. — *Let X be a manifold with conical singularities and Δ its self-adjoint Laplace operator (Friedrichs extension). Then*

- (1) *For any open sets, $U, V \in X$ such that $U \cap V = \emptyset$, and $V \subset X^\circ$, for any N , $\Delta^N \sqrt{\Delta}$ is continuous from $L^2(V)$ into $L^2(U)$.*
- (2) *For any open set U such that $\bar{U} \subset X^\circ$, $\sqrt{\Delta}$ seen as an operator from $H^1(U)$ into $L^2(U)$ is a (first order) pseudodifferential operator.*

Both results will follow from studying the heat kernel and using the transform :

$$\sqrt{\Delta} = \frac{\Delta}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-t\Delta} t^{-\frac{1}{2}} dt.$$

First, we take $\rho \in C_0^\infty([0, +\infty))$ that is identically 1 on $[0, 2t_0]$ for some $t_0 > 0$ and write $\psi = 1 - \rho$. Since

$$\int_0^\infty e^{-t\Delta} \psi(t) t^{-\frac{1}{2}} dt = e^{-t_0\Delta} \int_0^\infty e^{-(t-t_0)\Delta} \psi(t) t^{-\frac{1}{2}} dt,$$

we see that the operator that is defined by the former integral is smoothing.

Hence, both claims will follow from the same claims where $\sqrt{\Delta}$ is replaced by

$$\Delta \int_0^\infty \rho(t) e^{-t\Delta} t^{-\frac{1}{2}} dt.$$

Multiplying by Δ does not modify the statements so that it suffices to study the operator-valued integral :

$$(A.1) \quad \int_0^\infty e^{-t\Delta} \rho(t) t^{-\frac{1}{2}} dt$$

Proof of (1). — For any $a \in L^2(U)$, we define the distribution T_a on $\mathbb{R} \times V$ by

$$(T_a, \phi(t)b(y))_{\mathcal{D}' \times \mathcal{D}} = \int_0^\infty \langle a, e^{-t\Delta}b \rangle_{L^2} \phi(t) dt.$$

Since $\lim_{t \rightarrow 0^+} \langle a, e^{-t\Delta}b \rangle_{L^2} = 0$, a simple calculation shows that, in the distributional sense

$$(\partial_t + \Delta_y)T_a = 0, \text{ in } \mathcal{D}'(\mathbb{R} \times V).$$

By hypoellipticity in $\mathbb{R} \times V$, it follows that T_a is smooth.

Since T_a vanishes identically for $t < 0$

$$\forall (a, b) \in L^2(U) \times L^2(V), t \mapsto \langle e^{-t\Delta}a, b \rangle$$

is smooth on $[0, \infty)$ and vanishes to infinite order at 0.

In particular, for any N and k , the N -th derivative of the latter function vanishes to order k at 0. So the quantity

$$t^{-k} \langle \Delta^N e^{-t\Delta}a, b \rangle$$

is bounded on $(0, 1]$.

By the principle of uniform boundedness this implies that

$$\|\Delta^N e^{-t\Delta}\|_{L^2(V) \rightarrow L^2(U)} = O(t^k).$$

Plugging this bound into the integral (A.1) yields the result. \square

Proof of (2). — We begin by choosing \tilde{U} that is compactly embedded into X° and such that $U \subset \tilde{U}$. We denote by e the heat kernel on X and by \tilde{e} the heat kernel on a complete smooth Riemannian manifold \tilde{X} in which \tilde{U} is embedded.

We denote by r the distribution on $\mathbb{R} \times U \times U$ that is defined by

$$(r, \phi) = \int_0^\infty \int_{U \times U} (e(t, x, y) - \tilde{e}(t, x, y)) \phi(t, x, y) dx dy dt.$$

Observing that, for any ϕ

$$\int_{U \times U} (e(t, x, y) - \tilde{e}(t, x, y)) \phi(t, x, y) dx dy \xrightarrow[t \rightarrow 0]{} 0,$$

we obtain that, in $\mathcal{D}'(\mathbb{R} \times U \times U)$

$$(2\partial_t + \Delta_x + \Delta_y)r = 0.$$

So by hypoellipticity, r is smooth on $\mathbb{R} \times U \times U$.

Consequently, in (A.1), if we replace Δ by the Laplace operator on \tilde{X} , we make an error whose kernel is

$$\int_0^\infty \rho(t)r(t, x, y)t^{-\frac{1}{2}} dt.$$

Since r is smooth and vanishes to infinite order at $t = 0$ this integral is a smooth function of x and y .

It follows that $\sqrt{\Delta}$ in U coincides with $\sqrt{\tilde{\Delta}}$ up to a smoothing operator. \square

Appendix B. Stationary phase

In this appendix we discuss the method of stationary phase when the large parameter is allowed to be complex, with imaginary part comparable to the logarithm of the real part. We will parallel the treatment and notation in [31, Section 3.5]. The outcome will be that we may treat the factor of w_h below as a formal parameter, but we have been unable to find a justification for these manipulations in the published literature.

As before we write

$$\Omega_\epsilon \equiv \{w_h : (-\Lambda + \epsilon)h \log(1/h) < \operatorname{Im} w_h < 0, \operatorname{Re} w_h \in [1 - \epsilon, 1 + \epsilon]\}.$$

For $a \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ real valued and $w_h \in \Omega_\epsilon$, we define

$$I_h(a, \varphi; w_h) \equiv \int_{\mathbb{R}^n} e^{i\varphi w_h/h} a \, dx.$$

Note that the exponential term may be polynomially growing in h owing to the presence of the factor $w_h \in \mathbb{C}$. We will use throughout the fact that $h/w_h = O(h)$ for $w_h \in \Omega_\epsilon$.

LEMMA B.1. — *Let $w_h \in \Omega_\epsilon$. If $d\varphi \neq 0$ on $\operatorname{supp} a$, then $I_h(a, \varphi; w_h) = O(h^\infty)$.*

Proof. — As in Lemma 3.14 of [31], we simply integrate by parts using the operator

$$L = \frac{h}{iw_h} \frac{1}{|\partial\varphi|^2} \partial\varphi \cdot \partial,$$

chosen so that

$$L^k e^{i\varphi w_h/h} = e^{i\varphi w_h/h}.$$

The integration by parts then gains $(h/w_h)^k = O(h^k)$. \square

Thus as in the usual case, we may (decomposing a using a partition of unity) read off stationary phase asymptotics as a sum of asymptotics associated to each critical point, for a nondegenerate φ ; also we may use the Morse Lemma to convert φ into a diagonal quadratic form near each of those critical points. The only difficulty is then to compute quadratic stationary phase asymptotics, as in Theorem 3.13 of [31].

LEMMA B.2. — Let $w_h \in \Omega_\epsilon$. Let

$$\varphi(x) = \frac{1}{2} \langle Qx, x \rangle$$

be a quadratic phase, with Q a nonsingular, symmetric, real matrix. For all $N \in \mathbb{N}$,

$$I_h(a, \varphi; w_h) = \left(\frac{2\pi h}{w_h} \right)^{\frac{n}{2}} \frac{e^{\frac{i\pi}{4} \operatorname{sgn} Q}}{|\det Q|^{1/2}} \left(\sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{h}{w_h} \right)^k \left(\frac{\langle Q^{-1}D, D \rangle}{2i} \right)^k a(0) + O(h^N) \right).$$

Proof. — Let \tilde{a} denote an almost analytic extension of a with support in a small neighborhood (in \mathbb{C}^n) of $\operatorname{supp} a$ (see [16, Section 3.1], as well as, for instance, [9, Chapter 8]). Then

$$\begin{aligned} (B.1) \quad I_h(a, \varphi; w_h) &= \int_{\mathbb{R}^n} e^{i\varphi(x)w_h/h} a(x) dx_1 \wedge \cdots \wedge dx_n \\ &= \int_{\mathbb{R}^n} e^{i\varphi(x\sqrt{w_h})/h} a(x) dx_1 \wedge \cdots \wedge dx_n \\ &= \int_{\Gamma} e^{i\varphi(z)/h} \tilde{a}(z/\sqrt{w_h}) w_h^{-n/2} dz_1 \wedge \cdots \wedge dz_n \end{aligned}$$

where Γ is the complex contour $\{z_j = \sqrt{w_h}x_j, x_j \in \mathbb{R}\}$. Since \tilde{a} is compactly supported, we may apply Stokes's theorem on the domain

$$\Upsilon = \{z \in \mathbb{C}^n : z_j = ((1-s) + s\sqrt{w_h})x_j, x_j \in \mathbb{R}, s \in [0, 1]\}$$

to obtain

$$\begin{aligned} (B.2) \quad I_h(a, \varphi; w_h) &= \int_{\mathbb{R}^n} e^{i\varphi(x)/h} \tilde{a}(x/\sqrt{w_h}) w_h^{-n/2} dx_1 \wedge \cdots \wedge dx_n \\ &\quad + \iint_{\Upsilon} e^{i\varphi(z)/h} \bar{\partial}[\tilde{a}(z/\sqrt{w_h}) w_h^{-n/2} dz_1 \wedge \cdots \wedge dz_n]. \end{aligned}$$

By almost-analyticity of \tilde{a} , the latter integral is $O(h^\infty)$ since the support of the integrand is compact and over this compact set $\operatorname{Im}(z/\sqrt{w_h}) = O(h \log(1/h))$ for $z \in \Upsilon$. The former integral is then an ordinary stationary phase integral with quadratic phase to which we may directly apply Theorem 3.13 of [31]. Note of course that in applying the usual formula for the Gaussian integral, we are using the fact that

$$\begin{aligned} (B.3) \quad \frac{\partial^{|\alpha|}}{\partial x^\alpha} (\tilde{a}(x/\sqrt{w_h})) \Big|_{x=0} &= w_h^{-|\alpha|/2} \frac{\partial^{|\alpha|}}{\partial z^\alpha} \tilde{a}(0) \\ &= w_h^{-|\alpha|/2} \frac{\partial^{|\alpha|}}{\partial x^\alpha} a(0). \end{aligned}$$

□

Assembling the foregoing results, we finally arrive at the desired stationary phase expansion in general. (Cf. Theorem 3.16 of [31].)

PROPOSITION B.3. — *Let $w_h \in \Omega_\epsilon$, $a \in C_c^\infty(\mathbb{R}^n)$. Suppose x_0 is the unique point in $\text{supp } a$ at which $\partial\varphi = 0$ and that $\det \partial^2\varphi(x_0) \neq 0$. Then there exist differential operators $A_{2k}(x, D)$ of order $\leq 2k$ such that for all $N \in \mathbb{N}$*

$$I_h(a, \varphi; w_h) = \left(\sum_{k=0}^{N-1} A_{2k}(x, D)a(x_0) \left(\frac{h}{w_h} \right)^{k+n/2} \right) e^{\frac{i\varphi(x_0)w_h}{h}} + O(h^{N+n/2}).$$

In particular,

$$A_0 = (2\pi)^{n/2} |\det \partial^2\varphi(x_0)|^{-1/2} e^{\frac{i\pi}{4} \text{sgn } \partial^2\varphi(x_0)}.$$

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