A Nisnevich Square in Motivic Homotopy Theory

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Abstract

Here we present basic calculations in motivic homotopy theory and what it means for a “space” to be contractible.

1 Elementary Distinguished Squares

Let’s unpack a little the idea of an EDS. First of all the category $Sm/k$ of smooth schemes over a base field $k$ cannot be a model category because it is not complete. So to make it complete we can use the Yoneda embedding. But this is bad because for any scheme $X$ such that $X = U \cup V$ where $U \to X \leftarrow V$ are open embeddings, we lose the pushout squares:

\[
\begin{array}{ccc}
U \cap V & \rightarrow & V \\
\downarrow & & \downarrow \\
U & \rightarrow & U \cup V
\end{array}
\]

Furthermore, we want not just any homotopy theory on $Sm/k$ but one parametrized by $A^1$, the affine line. So we want to make sure the canonical projection $X \times_k A^1 \to X$ is an equivalence. And that’s pretty much the only thing we need.

Now the idea of an EDS is to capture the above property — the representable corresponding to an EDS must be a push-out in the category of sheaves (valued on sets). So recall:

**Definition 1.1.** An elementary distinguished square (EDS) is a cartesian square:

\[
\begin{array}{ccc}
U \times_X V & \rightarrow & V \\
\downarrow & & \downarrow \\
U & \rightarrow & X
\end{array}
\]

where $i$ is an open embedding, $p$ is an etale morphism and (excisive) $p : p^{-1}(X - U)_{red} \to (X - U)_{red}$ is an isomorphism.

Every open embedding is an etale morphism — so that we can think of the examples where $V \to X$ is an open embedding. In particular the fact that the complement of $U$ is an isomorphism gives us this “decomposition” of $X$ in terms of $U$ and $V$.

**Theorem 1.2** (MV99, 3.1.4). A pre-sheaf $F$ on $Sm/k$ is a sheaf if and only if $F$ takes an EDS to a pullback diagram in $Set$.

Of course the sheaf here is with respect to the Nisnevich topology but we don’t really need to get into details. Here’s the crucial property for a Nisnevich topology:

**Theorem 1.3.** Every representable presheaf is a Nisnevich sheaf.
The proof of the above uses the fact that this is true in the etale topology and uses faithfully flat descent. Which is nice.

**Corollary 1.4.** Every EDS is a push out diagram in $Sm/k$.

**Proof.** To show that a diagram: 

$$
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow \quad & \quad \downarrow p \\
U & \longrightarrow & X
\end{array}
$$

is a pushout, it suffices to show that the diagram under applying $Hom(-, Z)$ for all $Z$ is a pullback in Sets. But this is true by the theorem above. In particular we see that $Hom(X, Z) \cong Hom(V, Z) \times_{Hom(U \times_X V, Z)} Hom(U, Z)$ which unpacks to the fact that: the data of a map $X \to Z$ is the same data as a map $f : V \to Z, g : U \to Z$ that agrees on $U \times_X V$.

At this point, we should really think of an EDS as the algebro-geometric analog of the square we see in topology.

**Theorem 1.5.** Every EDS is a pushout diagram in $Shv((Sm/K)_{Nis})$. In particular, $V / (U \times_X V) \cong X/U$ canonically in $Shv((Sm/k)_{Nis})$

**Proof.** Let $F$ be a Nisnevich sheaf and suppose that we are looking at the following test diagram where the schemes are being thought of as representables:

$$
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow \quad & \quad \downarrow p \\
U & \longrightarrow & X
\end{array}
$$

A map of sheaves $V \to F$ and $U \to F$, corresponds via the Yoneda lemma to an element of $F(U)$ and $F(V)$ respectively. Let us call these elements $\alpha, \beta$ respectively. Now the theorem above gives us that $F(X) \cong F(V) \times_{F(U \times_X V)} F(U)$ and $(\alpha, \beta)$ does indeed live in the right hand side which canonically determines an element in the left hand side and reversing the Yoneda gives us a map of sheaves: $X(= Hom_{Sch/k}(-, X)) \to F$.

We denote by $Spc(k)$ the homotopy category of $k$-spaces which is the category obtained from $Sm/k$ by the above shenanigans. Here’s a paragraph (paraphrased) from Asok’s lecture that sort of clarifies things:

The $A_1$-homotopy category can be defined my means of a universal property. The $A_1$-homotopy category is the universal category constructed from $Sm/K$ in which (0) we adjoin all colimits and then (1) $X$ can be recovered up to homotopy from its Nisnevich cover and (2) if $X$ is a space, then the projection morphism: $A_1 \times X \to X$ is an equivalence.

An immediate upshot of this is that we can do the constructions that we know and love in topology:

1. wedge product is a pushout of 2 pointed spaces over their identified base point
2. the smash product is defined as $X \times Y/(X \wedge Y)$
3. multiple smash products
2 Some Basic Functors

Definition 2.1. Fix a space $X \in \text{Spc}(k)$, we may think of $U \in \text{Sm}/k$ as the representable sheaf. Then the sheaf of $A^1$ connected components is the Nisnevich sheaf on $\text{Sm}_k$ associated to the presheaf $U \mapsto [U,X]$ where $[-,-]$ is the morphism set in the homotopy category of $k$-spaces.

Definition 2.2. Given a pointed space $(X,x)$, the $i$-th homotopy sheaf is the Nisnevich sheaf on $\text{Sm}/k$ associated to $U \mapsto [S^i_{x} \wedge U_+, (X,x)]$. We write this as $\pi_i(X,x) : \text{Sm}/k \to \text{Set}$.

Of course we formally show that the homotopy sheaves are actually landing in groups and abelian groups for

Proposition 2.3 (Motivic Whitehead). A morphism $f : X \to Y$ of $A^1$-connected spaces is a $A^1$-weak equivalence (isomorphism on the homotopy category, i.e. $\text{Spc}(k)$ by our notation above) if and only if for any choice of base point $x \in X$, the induced morphism on homotopy sheaves is an isomorphism (in the appropriate category of sheaves).

Definition 2.4. A space $X$ is $A^1$-contractible if the structure morphism $X \to \text{Spec } k$ is a $A^1$ weak equivalence.

Let’s think about this a little more. A space is a Nisnevich sheaf $U \mapsto X(U)$. Now every presheaf is a colimit of representables. Hence to define a presheaf map we can just define the map on each representable $[-,U] \to [-,\text{Spec } k]$ so of course such a map exists because our schemes are $k$-schemes. We might have to sheafify things, but we get the desired map in the end.

3 Examples and Computations

So homotopy pushouts can be thought of as a way of computing pushouts using any homotopy (weak) equivalences that you might have proven before. From now on, we assume that we have made $A^1$ contractible. That’s really where things come in handy here:

Proposition 3.1. We have an equivalence $P^1 \simeq S^1 \wedge G_m$. Furthermore, $P^1/A^1 \simeq A^1/(G_m(= A^1 \setminus 0))$

Proof. We may present $P^1$ as a Nisnevich square: $G_m \rightarrow A^1 \rightarrow P^1$

of $P^1$. By inspection we see that the map $A^1 \to P^1$ are open embeddings hence etale. Then we must check excisiviness: $p^{-1}(X-U) = \ast$ and $(X-U) = \ast$ and we can, if we are careful, write down the obvious isomorphism of schemes between them.

Therefore such a diagram is a homotopy pushout in the category $\text{Sch}/k$, hence equivalent to the diagram:

$G_m \rightarrow \ast \rightarrow P^1$

Which means that $P^1 \simeq S^1 \wedge G_m$.

Let’s do stupid examples
**Proposition 3.2.** If \( \zeta : E \to X \) is a geometric vector bundle (the point here is that there is a Zariski cover of \( X \) over which \( E \) looks like \( U \times k^n \), then we have an equivalence. Moreover, if \( X \) is a smooth scheme that is stably isomorphic to affine \( n \)-space then we have that \( X \) is contractible.

**Definition 3.3.** If \( X \) is a smooth scheme over \( k \) and \( E \to X \) is a vector bundle (we should think of this as a coherent sheaf), we denote its Thom Space by \( E/(E \setminus X) \)

One of the most useful things in topology is that we have an identification between the Thom space of the normal bundle (which is a bundle-theoretic construction — one should think “linear” topology) and a tubular neighborhood of the of \( X \). This allows us to do good things like the Thom-Pontrajgin map and transfers! Such a thing also exists in motivic homotopy theory (purity theorem), but let me just mention a result that is easier.

**Proposition 3.4.** Let \( E \to X \) be a vector bundle over \( X \) and \( P(E) \to P(E \oplus O) \) be the closed embedding at infinity. Then the canonical morphism of pointed spaces \( P(E \oplus O)/P(E) \to Th(E) \) is an equivalence.

### 3.1 Some Exotic Elements of the Picard Group

The stable category is obtained from the unstable one by inverting \( P^1 \) under the smash product. Now we note that in the usual situation of the stable homotopy category we have that: \( (S^1 \wedge Map(S^1, S)) \simeq S^1 \wedge \Omega(S) \simeq S \) — the second equivalence being a consequence of the stable situation. We denote the stable category as \( SH(k) \) which is the category we obtain from \( Spc(k) \) by inverting the smash product.

Here’s a result of Po Hu which I like very much. Fix a field \( k \) and a square-free element \( a \in k^\times \). Let \( L_a := k[\sqrt{a}] \). We define two spaces (and think of them as their suspension spectra):

1. Let \( \Sigma L_a \) be the unreduced suspension of \( \text{Spec}(L_a) \), i.e. the cofiber of the map \( \text{Spec}(L_a) \to S^0 \).

2. \( G_m^a \) be the twisted multiplicative group which is \( \text{Spec}(k[x, y]/(x^2 - ay^2 - 1)) \)

Here the based and unbased context will make a difference. We will denote unreduced suspension by \( \Sigma \) which is computed by adjoining a basepoint and then computing the cofiber and \( S \) by reduced suspension which is computed by the cofiber \( X \to S^0 \) in the category of based maps.

**Proposition 3.5** (Po Hu). We have a canonical equivalence: \( \Sigma L_a \wedge G_m^a \simeq P^1 \) in the stable homotopy category.

**Proof.** The crux of the argument is the following computation in algebraic geometry

**Lemma 3.6.** In the category \( Spc(k) \), we have the homotopy pushout:

\[
\begin{array}{ccc}
\text{Spec}(L_a) \times G_m^a & \xrightarrow{j} & \text{Spec}(L_a) \\
\downarrow{p} & & \downarrow{f} \\
G_m^a & \xrightarrow{i} & P^1
\end{array}
\]

and the following lemma from topology:

**Lemma 3.7.** Let \( X \) be a based space (simplicial set) and \( Y \) an unbased space. We have an equivalence \( S(X \star Y) \simeq SX \wedge \Sigma Y \)
Proof. This is a consequence of the statement that for based spaces \( S(X \ast Y) \cong SX \land SY \).

Therefore, if we have both lemmas, we have that \( SP^1 \cong S(G_m) \ast \Sigma \text{Spec}(L_a) \) which tells us that in the stable homotopy category \( P^1 \cong \Sigma L_a \land G_m^a \) as desired.

Proof of Lemma. We will first describe the map \( i : G_m^a \to P^1 \). Let us describe \( P^1 \) by recalling some basic algebraic geometry.

Lemma 3.8. \( P^1 \) is isomorphic (as schemes) to the projective quadric \( V_+(x^2 - ay^2 = z^2) \subset P^2 \).

Proof. \( V_+(x^2 - ay^2 = z^2) = V_+(x^2 - z^2 = ay^2) = V_+(uw = y^2) \) by making the substitution \( u = x + z, v = x - z \) and then \( w = v/a \). But \( uw = y^2 \) is exactly the image of the 2-uple embedding of \( P^1 \) onto \( P^2 \) described in homogeneous coordinates as \([x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2]\). The square above is obtained by the pullback of \( G_m^a \to P^1 \) embedded as the complement of \((x^2 = a)\). Note here that \((x^2 = a)\) is a point described by the ideal \( z = 0 \) and its residue field is \( \text{Spec} \ k[\sqrt{a}] \) so that it is a \( \text{Spec} \ k[\sqrt{a}] \)-point. This is one of those weird points defined over an algebraic closure.

Now we can define \( A^1_{k[\sqrt{a}]} \to P^1 \) by describing \( A^1_{k[\sqrt{a}]} \) as \( P^1_{k[\sqrt{a}]} \setminus \infty \). We must be a little careful here — the point at 0 in the \( A^1_{k[\sqrt{a}]} \) should be sent to the \( x^2 = a \). The pullback is \( G_m^a \times \text{Spec} \ k[\sqrt{a}] \). And then one checks that the composition of the projection of \( A^1_{k[\sqrt{a}]} \to \text{Spec} \ k[\sqrt{a}] \) is indeed the projection onto the second variable map. This is done via chasing maps on the ring level, and hence we obtain the desired result.

Proposition 3.9. As an object of \( SH(k) \), \( \Sigma(\text{Spec}(L_a)) \) is not generated by \( S^1 \) and \( G_m \) under the smash product.

Proof. Using facts from motivic cohomology, the cohomology of \( \Sigma \text{Spec}(L_a) \) is \( L^\times_a / (k^\times, L^\times_a)^2 \) but by the Milnor conjecture (V’s theorem), the cohomology of \( H^*(Sp \land G_m^a) \) is isomorphic to a direct sum of copies of the mod 2 milnor \( K\)-theory which is a tensor algebra on \( k^\times \) subject to the relations \((a \otimes (1 - a)) \) where \( a \neq 0, 1 \) so it cannot contain the former ring.

4 References

The main references are:

1. Po Hu’s “Picard Group of Stable \( \mathbb{A}^1 \)-homotopy Category”

2. Marcus Severitt’s Thesis