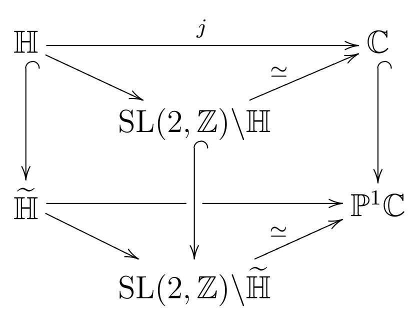
A small survey on the toroidal compactification

Johan Konter

16th June 2011

1 Introduction

We have seen the surjective map j from $\mathbb{H}$, the upper half plane, to $\mathbb{C}$, which sends τ via $\mathbb{C}/(\mathbb{Z}\cdot\tau+\mathbb{Z})$ to the j-invariant of this elliptic curve. The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$. The map j is invariant under this action, meaning that $j(w) = j(w')$ iff $zw = w'$ for some $z \in \mathrm{SL}(2,\mathbb{Z})$. Thus we find that the moduli space of elliptic curves $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is isomorphic to $\mathbb{C}$. To compactify this space we add $\mathbb{P}^1(\mathbb{Q})$ to $\mathbb{H}$ and we call the resulting space $\widetilde{\mathbb{H}}$. We observe that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$. We identify this one extra orbit in $\mathrm{SL}(2,\mathbb{Z})\backslash\widetilde{\mathbb{H}}$ with ∞ in $\mathbb{P}^1\mathbb{C}$.



We already have a topology on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ induced by the one on $\mathbb{H}$. To get a topology on $\widetilde{\mathbb{H}}$ we add to the basis of the topology the following neighborhoods of x: the disks tangent to the real line in x if $x \in \mathbb{Q}$ and the spaces $\mathbb{H}_c = \{w \in \mathbb{C} : \Im(w) > c\}$ if $x = \infty$. This makes $\mathrm{SL}(2,\mathbb{Z})\backslash\widetilde{\mathbb{H}}$ really into a compactification of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.

Satake generalized this procedure for $\mathrm{Sp}(2g, Z)\backslash\mathbb{H}_g$, where $\mathbb{H}_g$ is the Siegel upper half space and $\mathrm{Sp}(2g, Z)$ is the group of symplectic $g \times g$-matrices. Baily and Borel generalized this even further for bounded symmetric domains with the action of an arithmetic subgroup. These Baily-Borel compactifications however are in general very singular and the added spaces have codimension g. A bit better, in this sense, are the Borel-Serre compactifications, which are real analytic manifolds with corners. We'll follow Mumford, [3] [1], in his huge effort to construct smooth compactifications, which are called toroidal compactifications. These in general only have toric singularities, of codimension 1, and often they are even smooth projective varieties. Something noteworthy which we will not show, is that all toroidal compactifications dominate the Baily-Borel compactification, i.e. the identity on the original space extends to a continuous surjective map from the toroidal compactification to the Baily-Borel compactification. Hence a smooth toroidal compactification resolves the singularities of the Baily-Borel compactification.

We continue with these toroidal compactification by first taking a better look at our starting example. After that we talk a bit about toroidal embeddings. Then we go to the example of the universal level k elliptic curve, which serves as an example of the general case.

2 Moduli space of elliptic curves

We have already seen in the introduction that $\mathrm{SL}(2,\mathbb{Z})\backslash\widetilde{\mathbb{H}} \simeq \mathbb{P}^1\mathbb{C}$ by adding one point on both sides of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$. We know that $\mathbb{P}^1\mathbb{C}$ has a richer structure than just the union $\mathbb{C} \cup \{\infty\}$. It can, for example, be described by gluing two copies of $\mathbb{C}$ on their intersection $\mathbb{C}^*$.

The first idea is to factor the map $\mathbb{H} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ over $\Gamma_0\backslash\mathbb{H}$, where $\Gamma_0 = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{Z}\}$ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Because Γ_0 acts on $\mathbb{H}$ by shifting it horizontally over an integral distance, the quotient space $\Gamma_0\backslash\mathbb{H}$ looks like an infinite tube with an open end at the bottom. We identify this tube with $\Delta_1^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. We can do this explicitly with the isomorphism $f : \Gamma_0\backslash\mathbb{H} \to \Delta_1^*$, $f(w) = e^{2\pi i w}$, so complex numbers with a high imaginary part are sent close to zero. We can partially compactify Δ_1^* by adding the origin and we obtain $\Delta_1 = \{z \in \mathbb{C} : |z| < 1\}$. This leads to the attempt of gluing $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ and Δ_1 over $\Delta_1^* \simeq \Gamma_0\backslash\mathbb{H}$.

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & \Gamma_0\backslash\mathbb{H} & \longrightarrow & \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \\ & & \downarrow & & \downarrow \\ & & \Delta_1 & \longrightarrow & \ldots\ldots \end{array}$$

Unfortunately this doesn't work because the map $\Gamma_0\backslash\mathbb{H} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is not injective. To solve this we take at look at $\mathbb{H}_\delta = \{w \in \mathbb{C} : \Im(w) > \delta\}$ for $\delta > 0$. Assume that $w \in \mathbb{H}_\delta$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) w \in \mathbb{H}_\delta$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, Z)$. Writing $w = w_1 + w_2 i$ we see that $w_2 > \delta$ and we can calculate that $\Im(\frac{aw+b}{cw+d}) = \frac{w_2}{(cw_1+d)^2+(cw_2)^2} > \delta$. Hence

$$w_2 > \delta((cw_1 + d)^2 + (cw_2)^2) \geq \delta c^2 w_2^2,$$

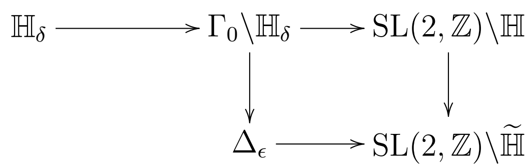
and

$$1 > \delta c^2 w_2 > \delta^2 c^2.$$

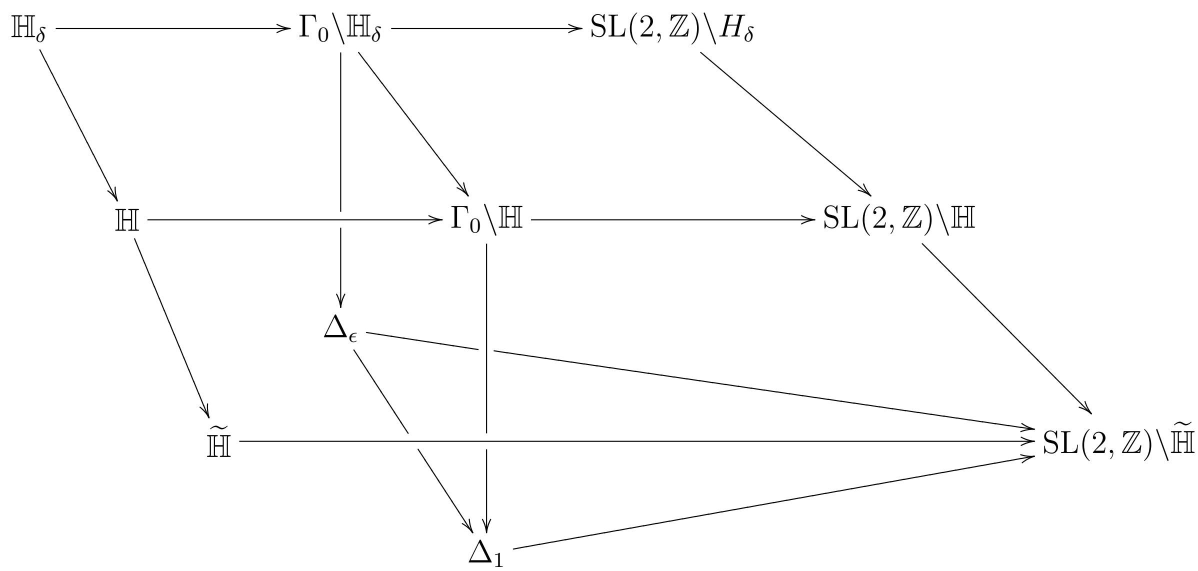
From now on we take $\delta \geq 1$ and we find that $c = 0$, because the entries are integral. Because the determinant is of the matrix is 1 and because we are on the identity component, this implies that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0$. Thus the map $\Gamma_0 \backslash \mathbb{H}_\delta \to \mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}_\delta$ is injective. The map $\mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}_\delta \to \mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}$ is induced by the inclusion $\mathbb{H}_\delta \hookrightarrow \mathbb{H}$ so it is also injective. We conclude that the map

$$\Gamma_0 \backslash \mathbb{H}_\delta \to \mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}_\delta \to \mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}$$

is injective. The map $f : \Gamma_0 \backslash \mathbb{H} \to \Delta_1^*$ restricts to $f : \Gamma_0 \backslash \mathbb{H}_\delta \to \Delta_\epsilon^*$ where $\epsilon = e^{-2\pi\delta}$. This we use and we construct $\mathrm{SL}(2,\mathbb{Z})\backslash \widetilde{\mathbb{H}}$ by gluing Δ_ϵ and $\mathrm{SL}(2,\mathbb{Z})\backslash \mathbb{H}$ over Δ_ϵ^*, as follows.



We summarize this discussion in the following diagram.



3 Toroidal embeddings

In general an algebraic torus over a base scheme S is defined as a group scheme over S such that it is fpqc locally isomorphic to a finite product of multiplicative groups. Let

T be an algebraic torus over $\mathbb{C}$. Then this definition boils down to $T \simeq (\mathbb{C}^*)^n$ where n is the dimension of the torus. The fundamental group $N = \pi_1(T)$ is a free abelian group and can be viewed as lattice in $N_\mathbb{R} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$. This lattice defines a rational structure on $N_\mathbb{R}$. On the other hand we can recover T from this lattice by $T \simeq N_\mathbb{C}/N$, with $N_\mathbb{C} = N \otimes \mathbb{C} \simeq \mathbb{C}^n$.

A *closed convex rational polyhedral cone* is a subset σ of $N_\mathbb{R}$ such that there exists a finite number of elements $n_1, \ldots, n_s$ in the lattice N with

$$\sigma = \{a_1 n_1 + \cdots + a_s n_s \ : \ a_i \geq 0, i = 1, \ldots, s\}.$$

A *rational partial cone decomposition* of $N_\mathbb{R}$ is a collection $\Sigma = \{\sigma_i\}$ of closed convex rational polyhedral cones in $N_\mathbb{R}$ satisfying

1. Every face of any $\sigma_i \in \Sigma$ is again a cone $\sigma_j \in \Sigma$.
2. For every pair $\sigma_i, \sigma_j \in \Sigma$, the intersection $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j (and hence is also in Σ).

Now $\operatorname{span}(\sigma) = \sigma + (-\sigma)$ is a linear subspace of $N_\mathbb{R}$ for any $\sigma \in \Sigma$. The complex subspace $\operatorname{span}_\mathbb{C}(\sigma) = \operatorname{span}(\sigma) \otimes \mathbb{C}$ acts on the torus T by translation. We define the boundary component $\bar{T}_\sigma$ as the quotient $T/\operatorname{span}_\mathbb{C}(\sigma)$, which is a complex torus of smaller dimension. We define a partial compactification of T, called the *toroidal embedding of T associated to* Σ, as

$$\bar{T}_\Sigma = T \cup \coprod_{\sigma \in \Sigma} \bar{T}_\sigma.$$

To define the topology of $\bar{T}_\Sigma$, we observe that for every closed convex rational polyhedral cone $\sigma \in \Sigma$ there exist finitely many linear $l_1, \ldots, l_k$ on $N_\mathbb{R}$ and a $p \leq k$ such that

$$\sigma = \{n : l_1(n) \geq 0, \ldots, l_p(n) \geq 0, l_{p+1}(n) = 0, \ldots, l_k(n) = 0\},$$
$$\sigma^\circ = \{n : l_1(n) > 0, \ldots, l_p(n) > 0, l_{p+1}(n) = 0, \ldots, l_k(n) = 0\}.$$

Then a sequence $z_i = x_i + iy_i$ in $T \simeq N_\mathbb{C}/N$ converges to a point $z \in \bar{T}_\sigma$, per definition if and only if

1. the sequences $l_1(x_i), \ldots l_p(x_i)$ all diverge, while the sequences $l_{p+1}(x_i), \ldots, l_k(x_i)$ are bounded,
2. the projection of z_i onto $\bar{T}_\sigma$ converges in $\bar{T}_\sigma$ to z.

As the name suggests, with this topology, $\bar{T}_\Sigma$ is an toroidal embedding. A *torus embedding* is a normal variety containing a algebraic torus as a dense subset such that the action of the torus extends to whole space. A *toroidal embedding* is a pair $U \subset X$, with X a normal analytic space and U an open in the complex topology, such that it is locally isomorphic to a torus embedding for every $x \in X$.

4 Universal level k elliptic curve

We start by looking at the *moduli space of level k elliptic curves*. So we define

$$\Gamma_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod k \right\} \subset \mathrm{SL}(2,\mathbb{Z}),$$

and we are interested in $\Gamma_k \backslash \mathbb{H}$. We basically do the same thing as before, but this time we have to realize that $\mathbb{P}^1\mathbb{Q}$ is not a single orbit for the action of Γ_k as it is for the action of $\mathrm{SL}(2,\mathbb{Z})$. Therefore $\Gamma_k\backslash\mathbb{H}$ has multiple cusps and $\Gamma_k\backslash\mathbb{H} \not\simeq \mathbb{C}$.

We do exactly the same thing as above by introducing $\Gamma_{0,k} = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \equiv 0 \mod k\} \subset \Gamma_k$. Then we factor $\mathbb{H} \to \Gamma_k\backslash\mathbb{H}$ through $\Gamma_{0,k}\backslash\mathbb{H}$. Knowing what will come of this, we restrict ourselves to $\mathbb{H}_\delta$ with $\delta \geq 1$ and we see that

$$\Gamma_{0,k}\backslash\mathbb{H}_\delta \to \Gamma_k\backslash\mathbb{H}_\delta \to \Gamma_k\backslash\mathbb{H}$$

is again injective. We adjust our f from above slightly to be defined as $f(w) = e^{2\pi i w/k}$. So it is an isomorphism between $\Gamma_{0,k}\backslash\mathbb{H}_\delta$ and $\Delta^*_\epsilon = \{z \in \mathbb{C} : 0 < |z| < \epsilon\}$ with $\epsilon = e^{-2\pi\delta}$. Hence we can glue $\Gamma_k\backslash\mathbb{H}$ and Δ_ϵ together over $\Gamma_{0,k}\backslash\mathbb{H}_\delta$. This doesn't give a compact space immediately, because there are more cusps to be taken care of.

We have seen that $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\mathbb{P}^1\mathbb{Q}$, so the group $\mathrm{SL}(2,\mathbb{Z})/\Gamma_k$ acts transitively on the cusps of $\Gamma_k\backslash\mathbb{H}$. So with the action of $\mathrm{SL}(2,\mathbb{Z})/\Gamma_k$ we can apply the above procedure for the other cusps and we find a smooth compactification of $\Gamma_k\backslash\mathbb{H}$ which is equivalent to $\Gamma_k\backslash\widetilde{\mathbb{H}}$.

We define the semi-direct product $\Gamma_k^A = \Gamma_k \ltimes \mathbb{Z}^2$, where the action of Γ_k on $\mathbb{Z}^2$ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(m,n) = (am + cn, bm + dn)$. This product on its turn acts in its turn on $\mathbb{H} \times \mathbb{C}$ by

$$(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), m, n)(w,z) = (\frac{aw+b}{cw+d}, \frac{z+mw+n}{cw+d}).$$

The A in Γ_k^A stands for *affine*, and $\Gamma_k^A\backslash(\mathbb{H}\times\mathbb{C})$, together with the projection on $\Gamma_k\backslash\mathbb{H}$, is called the *universal level k elliptic curve over* $\Gamma_k\backslash\mathbb{H}$. The problem is now to generalize the above approach to find a compactification of $\Gamma_k^A\backslash(\mathbb{H}\times\mathbb{C})$.

First we define $\Gamma_{1,k}^A = \Gamma_{0,k} \ltimes (0 \times \mathbb{Z}) \subset \Gamma_k^A$. We see that $0 \times \mathbb{Z}$ is closed under the action from $\Gamma_{0,k}$ so this is well defined. Furthermore the $\Gamma_{1,k}^A$ acts on $\mathbb{H}\times\mathbb{C}$ by sending (w,z) to an element $(w+b, z+n)$ for $b,n \in \mathbb{Z}$ with $b \equiv 0 \mod k$. In this light it is immediate that $\Gamma_{1,k}^A\backslash(\mathbb{H}\times\mathbb{C})$ is isomorphic to $\Delta_1^* \times \mathbb{C}^*$ via the map $g(w,z) = (e^{2\pi i w/k}, e^{2\pi i z})$. Of course the map $\Gamma_{1,k}^A\backslash(\mathbb{H}\times\mathbb{C}) \to \Gamma_k^A\backslash(\mathbb{H}\times\mathbb{C})$ is not injective, but our usual approach of restricting to $\Gamma_{1,k}^A\backslash(\mathbb{H}_\delta\times\mathbb{C}) \to \Gamma_k^A\backslash(\mathbb{H}_\delta\times\mathbb{C})$ is not enough to fix this. We need an intermediate group

$\Gamma^A_{1,k} \subset \Gamma^A_{2,k} \subset \Gamma^A_k$ which we will define as $\Gamma^A_{2,k} = \Gamma_{0,k} \ltimes \mathbb{Z}^2$. So now we have all the following spaces and maps.

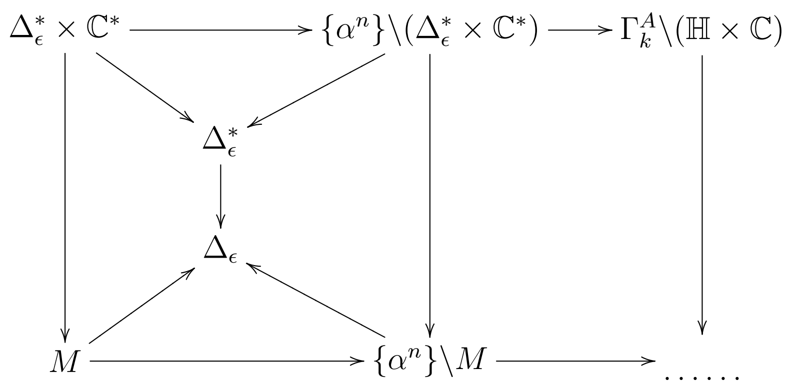
$$\begin{array}{ccccccc}
\mathbb{H}_\delta \times \mathbb{C} & \longrightarrow & \Gamma^A_{1,k}\backslash(\mathbb{H}_\delta \times \mathbb{C}) & \longrightarrow & \Gamma^A_{2,k}\backslash(\mathbb{H}_\delta \times \mathbb{C}) & \longrightarrow & \Gamma^A_k\backslash(\mathbb{H}_\delta \times \mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{H} \times \mathbb{C} & \longrightarrow & \Gamma^A_{1,k}\backslash(\mathbb{H} \times \mathbb{C}) & \longrightarrow & \Gamma^A_{2,k}\backslash(\mathbb{H} \times \mathbb{C}) & \longrightarrow & \Gamma^A_k\backslash(\mathbb{H} \times \mathbb{C})
\end{array}$$

When $(w,z) \in \mathbb{H}_\delta \times \mathbb{C}$ and $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), m, n)(w,z) \in \mathbb{H}_\delta \times \mathbb{C}$ with $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), m, n) \in \Gamma^A_k$, we get that $\frac{aw+b}{cw+d} \in \mathbb{H}_\delta$. We have seen before that this implies that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{0,k}$, so we find that $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), m, n) \in \Gamma^A_{2,k}$. This implies that the map

$$\Gamma^A_{2,k}\backslash(\mathbb{H}_\delta \times \mathbb{C}) \to \Gamma^A_k\backslash(\mathbb{H}_\delta \times \mathbb{C}) \to \Gamma^A_k\backslash(\mathbb{H} \times \mathbb{C})$$

ís injective. The group $\Gamma^A_{1,k}$ is a normal subgroup of $\Gamma^A_{2,k}$ and clearly $\Gamma^A_{2,k}/\Gamma^A_{1,k} \simeq \mathbb{Z}$. Let α denote the generator of this group, so $\alpha(w,z) = (w, z+kw)$, and we write $\Gamma^A_{2,k}/\Gamma^A_{1,k} = \{\alpha^n\}$.

We have just seen that $\Gamma^A_{1,k}\backslash(\mathbb{H}\times\mathbb{C}) \simeq \Delta^*_1 \times \mathbb{C}^*$ and by restricting g we also have $\Gamma^A_{1,k}\backslash(\mathbb{H}_\delta \times \mathbb{C}) \simeq \Delta^*_\epsilon \times \mathbb{C}^*$ with $\epsilon = e^{-2\pi\delta}$. This is nice, because it gives a description of $\Gamma^A_{2,k}\backslash(\mathbb{H}_\delta \times \mathbb{C})$ as $\{\alpha^n\}\backslash(\Delta^*_\epsilon \times \mathbb{C}^*)$. And α acts only on $\mathbb{C}^*$, not on Δ^*_ϵ, so to construct a compactification of $\Gamma^A_k\backslash(\mathbb{H} \times \mathbb{C})$ we are looking for an analytic manifold M fitting in the following diagram.



Here $\Delta^*_\epsilon \times \mathbb{C}^* \to M$ should be injective, and this makes sure that $\{\alpha^n\}\backslash\Delta^*_\epsilon \times \mathbb{C}^* \to \{\alpha^n\}\backslash M$ is also injective.

For this we introduce the toroidal embeddings. We view $\Delta^*_\epsilon \times \mathbb{C}^*$ as a subset of the complex torus $T = \mathbb{C}^* \times \mathbb{C}^*$, so we are in the case $n = 2$. We identify the lattice N in $\mathbb{R} \times \mathbb{R}$ with $\mathbb{Z} \times \mathbb{Z}$. Let n_i denote the point $(1, i)$ in N and define Σ as the collection of all closed convex rational polyhedral cones of the form $\sigma_i = \{a_0 n_i + a_1 n_{i+1} : a_0, a_1 \geq 0\}$ and $\tau_i = \{a_0 n_i : a_0 \geq 0\}$. We view T as a complex line bundle over the base space C^* by projection on the first component. Then $\bar{T}_\Sigma$ has one extra fiber, over 0, which looks like a collection of rational curves, labelled by $\mathbb{Z}$, each intersecting once with its predecessor and its successor.

We define the complex manifold $M = \overline{\Delta_\epsilon^* \times \mathbb{C}^*}$ as the closure of $\Delta_\epsilon^* \times \mathbb{C}^*$ in $\bar{T}_\Sigma$. And even though α doesn't act discontinuously on $\mathbb{C}^* \times \mathbb{C}^*$ it does act properly discontinuously on M. We have seen in class that this implies that $\{\alpha^n\}\backslash M$ is a normal analytic space, so $\{\alpha^n\}\backslash(\Delta_\epsilon^* \times \mathbb{C}^*) \subset \{\alpha^n\}\backslash M$ is a toroidal embedding. Over every element of Δ_ϵ^* the fibers look like $\{\alpha^n\}\backslash\mathbb{C}^*$ and over 0 the fiber is an k-gon of rational curves. This M satisfies all imposed conditions and we use M to define the boundary component over ∞. Again we can use the action of $(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)/\Gamma_k^A \simeq \mathrm{SL}(2,\mathbb{Z})/\Gamma_k$ to apply this construction on the other cusps. The resulting space is called the *toroidal compactification* of $(\Gamma_k \ltimes \mathbb{Z}^2)\backslash(\mathbb{H}\times\mathbb{C})$.

Notice that this construction depends on the rational partial cone decomposition Σ. It is even possible to take a different cone decomposition for every cusp. Then in general one must check whether these partial compactifications are compatible in the sense that the are part of one big compactification. Mumford expresses this compatibility algebraically and this leads to considering Γ-admissible families of rational partial cone decompositions, where every element of the family takes care of one of the boundary components. In the above case we have of course $\Gamma = \Gamma_k \ltimes \mathbb{Z}^2$ and our family consists only of the described Σ, but we won't go into checking whether this setup satisfies the necessary conditions.

References

[1] Ash, A., Mumford, D., Rapoport, M., and Tai, Y.-S. *Smooth Compactification of Locally Symmetric Varieties*, vol. IV of *Lie groups: history, frontiers and applications*. Math Sci Press, 1975.

[2] Ji, L. The greatest common quotient of Borel-Serre and the the toroidal compactifications of locally symmetric spaces. *GAFA, Geometry and functional analysis 8* (1998), 978–1015.

[3] Mumford, D. A new approach to compactifying locally symmetric varieties. In *Discrete Subgroups of Lie Groups and Applications to Moduli (International Colloquium, Bombay, 1973)* (1975), pp. 211–224.

[4] Namikawa, Y. *Toroidal Compactification of Siegel Spaces*, vol. 812 of *Lecture Notes in Mathematics*. Springer-Verlag, 1980.