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Relative trace formulae toward Bessel and Fourier–Jacobi periods on unitary groups

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Abstract. We propose an approach, via relative trace formulae, toward the global restriction problem involving Bessel or Fourier–Jacobi periods on unitary groups $\mathrm{U}_n \times \mathrm{U}_m$, generalizing the work of Jacquet–Rallis for $m = n-1$ (which is a Bessel period). In particular, when $m = 0$, we recover a relative trace formula proposed by Flicker concerning Kloosterman/Fourier integrals on quasi-split unitary groups. As evidences for our approach, we prove the vanishing part of the fundamental lemmas in all cases, and the full lemma for $\mathrm{U}_n \times \mathrm{U}_n$.

1. Introduction

Recently, Jacquet and Rallis [22] propose a new approach to the global Gan–Gross–Prasad conjecture [8] for unitary groups $\mathrm{U}_n \times \mathrm{U}_{n-1}$, which relates automorphic periods with special L-values. It is based on the comparison of some relative trace formulae. In this article, we extend this approach to all sorts of pairs $\mathrm{U}_n \times \mathrm{U}_m$ where $0 \leq m \leq n$. If $n-m$ is odd (resp. even), the automorphic periods at hand are Bessel (resp. Fourier–Jacobi) periods.

1.1. Periods and special values of L-functions

Let us consider a quadratic extension k/k' of number fields with $\mathbb{A}/\mathbb{A}'$ the corresponding rings of adèles. Denote $\mathscr{M}_k$ (resp. $\mathscr{M}_{k'}$) the set of all places of k (resp. k'). Let τ be the nontrivial element in $\mathrm{Gal}(k/k')$, and $\eta\colon k'\backslash\mathbb{A}'^{\times} \to \{\pm 1\}$ the quadratic character associated to k/k'.

Let $V, (\ ,\ )$ be a (nondegenerate) hermitian space over k (with respect to τ) of dimension n and $W \subset V$ a subspace of dimension m such that the restricted hermitian form $(\ ,\ )|_W$ is nondegenerate. Denote $\mathrm{U}_n = \mathrm{U}(V)$ and $\mathrm{U}_m = \mathrm{U}(W)$ the corresponding unitary groups, respectively. We regard U_m as a subgroup of U_n consisting of elements fixing the orthogonal complement of W in V point-wisely. When $n-m$ is even (resp. odd), we define a unipotent subgroup $U' = U'_{1^r,m} \subset \mathrm{U}_n$ (resp. $U' = U'_{1^r,m+1}$) on which U_m acts through conjugation. We put $H' = U' \rtimes \mathrm{U}_m$, viewed as a subgroup of $\mathrm{U}_n \times \mathrm{U}_m$ via the embedding into the first factor and the

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projection onto the second factor (see Sects. 4.1, 5.1 for the precise definitions). Let π (resp. σ) be an irreducible representation of $\mathrm{U}_n(\mathbb{A}')$ (resp. $\mathrm{U}_m(\mathbb{A}')$) which occurs with multiplicity one in the space of cuspidal automorphic forms $\mathcal{A}_0(\mathrm{U}_n)$ (resp. $\mathcal{A}_0(\mathrm{U}_m)$), and $\mathcal{A}_\pi \subset \mathcal{A}_0(\mathrm{U}_n)$ (resp. $\mathcal{A}_\sigma \subset \mathcal{A}_0(\mathrm{U}_m)$) the unique irreducible π (resp. σ)-isotypic subspace.

First, we consider the case of Bessel periods, that is, $n - m = 2r + 1$ is odd. There is an essentially unique generic character $\nu' \colon U'(k')\backslash U'(\mathbb{A}') \to \mathbb{C}^\times$ that is stabilized by $\mathrm{U}_m(\mathbb{A}')$ and hence can be extended to a character of $H'(k')\backslash H'(\mathbb{A}')$. For $\varphi_\pi \in \mathcal{A}_\pi$ and $\varphi_\sigma \in \mathcal{A}_\sigma$, we define

$$\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma) = \int\limits_{H'(k')\backslash H'(\mathbb{A}')} \varphi_\pi \otimes \varphi_\sigma(h')\nu'(h')^{-1}\mathrm{d}h'$$

to be a Bessel period of $\pi \otimes \sigma$. The global Gan–Gross–Prasad conjecture [8] says that if $\pi \otimes \sigma$ is in a generic Vogan L-packet, then there is a nonzero Bessel period of a representation in the Vogan L-packet of $\pi \otimes \sigma$ *if and only if* the central special L-value $L\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \mathrm{BC}(\sigma)\right) \neq 0$, where BC stands for the standard base change and the L-function is the Rankin–Selberg convolution (see [21]).

Second, we consider the case of Fourier–Jacobi periods, that is, $n - m = 2r$ is even. After choosing a nontrivial character $\psi' \colon k'\backslash\mathbb{A}' \to \mathbb{C}^\times$ and a character $\mu \colon k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ satisfying $\mu|_{\mathbb{A}'^\times} = \eta$, we have an automorphic (essentially Weil) representation $\nu'_{\psi',\mu}$ of $H'(\mathbb{A}')$ realized on some space $\mathcal{S}$ of Schwartz functions. For $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$ and $\phi \in \mathcal{S}$, we define

$$\mathcal{FJ}_r^{\nu'_{\psi',\mu}}(\varphi_\pi, \varphi_\sigma; \phi) = \int\limits_{H'(k')\backslash H'(\mathbb{A}')} \varphi_\pi \otimes \varphi_\sigma(h')\theta(h', \phi)\mathrm{d}h'$$

to be a Fourier–Jacobi period of $\pi \otimes \sigma$ (with respect to μ), where $\theta(h', \phi)$ is a certain theta series on $H'(\mathbb{A}')$ attached to ϕ. The global Gan–Gross–Prasad conjecture [8] says that if $\pi \otimes \sigma$ is in a generic Vogan L-packet, then there is a nonzero Fourier–Jacobi period of a representation in the Vogan L-packet of $\pi \otimes \sigma$ *if and only if* the central special L-value $L\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \mathrm{BC}(\sigma) \otimes \mu^{-1}\right) \neq 0$. In an early work [10], Gelbart and Rogawski studied Fourier–Jacobi periods on $\mathrm{U}_3 \times \mathrm{U}_1$ for endoscopic representations.

There has been significant progress toward these conjectures in a series of works of Ginzburg, Jiang and Rallis including [13] for unitary groups, [11] for symplectic groups and [12] for orthogonal groups, toward the direction that the existence of nontrivial Bessel or Fourier–Jacobi periods implies the non-vanishing of corresponding central L-value. Their approach is to study those periods of certain residue Eisenstein series and some related Fourier coefficients. In a recent preprint [38], Wei Zhang establishes both directions (and even the refined version in [39]; see below) for the pair $\mathrm{U}_n \times \mathrm{U}_{n-1}$ under certain restrictions, by exploring the relative trace formula of Jacquet–Rallis, which is a significant step forward to the conjecture.

One may formulate a refined version of the global Gan–Gross–Prasad conjecture predicting a precise relation between these automorphic periods and central special L-values, known as the Ichino–Ikeda [16] (resp. Harris [15]) conjecture in the context of $\mathrm{SO}_n \times \mathrm{SO}_{n-1}$ (resp. $\mathrm{U}_n \times \mathrm{U}_{n-1}$). In a recent preprint [28] of the author, the refined conjecture has been formulated for all pairs $\mathrm{SO}_n \times \mathrm{SO}_m$ and $\mathrm{U}_n \times \mathrm{U}_m$ with $n-m$ odd (that is, all cases of Bessel periods). The advantage of the relative trace formula approach is the possibility to prove the explicit formula relating $|\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma)|^2$ (or $|\mathcal{FJ}_r{}^{\nu'}_{\psi',\mu}(\varphi_\pi, \varphi_\sigma; \phi)|^2$) and the product of certain local periods of positive type, with L-values as (part of) the scaling factor of these two periods. In particular, one can prove the positivity of the corresponding central special L-value, for example, as in [19].

1.2. The conjecture for $\mathrm{U}_n \times \mathrm{U}_n$

As a special case in our article, we formulate a relative trace formula for the pair $\mathrm{U}_n \times \mathrm{U}_n$, which should be compared with another one on $\mathrm{GL}_n \times \mathrm{GL}_n$. We prove in Theorem 5.15 that the fundamental lemma for $\mathrm{U}_n \times \mathrm{U}_n$ can be reduced to the one for $\mathrm{U}_{n+1} \times \mathrm{U}_n$, which is proved by Yun [37]. Moreover, even the smooth transfer for $\mathrm{U}_n \times \mathrm{U}_n$ can be reduced to the one for $\mathrm{U}_{n+1} \times \mathrm{U}_n$. In other words, the relative trace formula established in the current article provides a way to unify the problem for $\mathrm{U}_{n+1} \times \mathrm{U}_n$ and $\mathrm{U}_n \times \mathrm{U}_n$, which is hard to be seen at the level of global period integrals.

Based on this idea, Hang Xue proves the equivalence of nonvanishing of the periods and the nonvanishing of the central L-values for $\mathrm{U}_n \times \mathrm{U}_n$ [35, Theorem 1.1.1] in a recent work, following the argument of Zhang [38] regarding the pair $\mathrm{U}_{n+1} \times \mathrm{U}_n$, under similar restrictions. Therefore, the Gan–Gross–Prasad conjectures for $\mathrm{U}_n \times \mathrm{U}_n$ (a Fourier–Jacobi case) and $\mathrm{U}_{n+1} \times \mathrm{U}_n$ (a Bessel case) are on the same pace. Moreover, by the machine of theta lifting, one can even prove the conjecture for certain endoscopic cases for $\mathrm{U}_{n+1} \times \mathrm{U}_n$ [35, Theorem 1.1.5], which are not covered by the work of Wei Zhang.

1.3. Relative trace formulae and fundamental lemmas

We briefly describe our relative trace formulae. To be simple for the introduction, we consider only the case of Bessel periods. In other words, we assume that $n-m = 2r+1$ is odd.

Let $f_n \in \mathcal{S}(\mathrm{U}_n(\mathbb{A}'))$ (resp. $f_m \in \mathcal{S}(\mathrm{U}_m(\mathbb{A}'))$) be a Schwartz function on $\mathrm{U}_n(\mathbb{A}')$ (resp. $\mathrm{U}_m(\mathbb{A}')$). We associate to $f_n \otimes f_m$ a kernel function on $(\mathrm{U}_n(k')\backslash \mathrm{U}_n(\mathbb{A}') \times \mathrm{U}_m(k')\backslash \mathrm{U}_m(\mathbb{A}'))^2$ as

$$\mathcal{K}_{f_n \otimes f_m}(g_1', g_2'; g_3', g_4') = \sum_{\zeta' \in \mathrm{U}_n(k')} f_n(g_1'^{-1}\zeta' g_3') \sum_{\xi' \in \mathrm{U}_m(k')} f_m(g_2'^{-1}\xi' g_4'),$$

and consider the following distribution

$$\mathcal{J}(f_n \otimes f_m) = \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \mathcal{K}_{f_n \otimes f_m}\left(h_1', h_1'; h_2', h_2'\right) \nu'(h_1'^{-1} h_2') \mathrm{d}h_1' \mathrm{d}h_2'. \quad (1.1)$$

Such distribution formally computes

$$\sum_{\pi,\sigma}\sum \mathcal{B}_r^{\nu'}(\rho(f_n)\varphi_\pi, \rho(f_m)\varphi_\sigma)\mathcal{B}_r^{\widetilde{\nu'}}(\overline{\varphi_\pi}, \overline{\varphi_\sigma}),$$

where the inner sum is taken over orthonormal bases of $\mathcal{A}_\pi$ and $\mathcal{A}_\sigma$, and ρ denotes the right translation. The integral (1.1) is not absolutely convergent in general and requires regularization. It turns out that the regular part of this distribution has the following decomposition

$$\mathcal{J}_{\mathrm{reg}}(f_n \otimes f_m) = \sum_{\zeta' \in [\mathrm{U}_n(k')_{\mathrm{reg}}]/\mathbf{H}'(k')} \mathcal{J}_{\zeta'}(f),$$

where $[\mathrm{U}_n(k')_{\mathrm{reg}}]/\mathbf{H}'(k')$ is the set of regular orbits in $\mathrm{U}_n(k')$, which will be discussed in Sect. 4.3, and $f \in \mathcal{S}(\mathrm{U}_n(\mathbb{A}'))$ is obtained from $f_n \otimes f_m$. Moreover, each summand $\mathcal{J}_{\zeta'}$ is an adèlic orbital integral

$$\mathcal{J}_{\zeta'}(f) = \int_{\mathrm{U}_m(\mathbb{A}')} \iint_{U'_{1^r,m+1}(\mathbb{A}')^2} f(g'^{-1}u_1'^{-1}\zeta' u_2' g')\nu'(u_1'^{-1}u_2')\mathrm{d}u_1'\mathrm{d}u_2'\mathrm{d}g'.$$

To encode the L-function, one should pass to the general linear groups. Let $\Pi = \mathrm{BC}(\pi)$ (resp. $\Sigma = \mathrm{BC}(\sigma)$) be the base change to $\mathrm{GL}_n(\mathbb{A})$ (resp. $\mathrm{GL}_m(\mathbb{A})$) and assume that it remains cuspidal. We define similarly a unipotent subgroup $U_{1^r,m+1,1^r}$ of GL_n, and put $H = U_{1^r,m+1,1^r} \rtimes \mathrm{GL}_m$, viewed as a subgroup of $\mathrm{GL}_n \times \mathrm{GL}_m$, equipped with a character ν (see Sect. 2.1 for precise definitions). For $\varphi_\Pi \in \mathcal{A}_\Pi$ and $\varphi_\Sigma \in \mathcal{A}_\Sigma$, consider the following version of the Bessel period on general linear groups

$$\mathcal{B}_{r,r}^{\nu}(\varphi_\Pi, \varphi_\Sigma) = \int_{H(k)\backslash H(\mathbb{A})} \varphi_\Pi \otimes \varphi_\Sigma(h)\nu(h)^{-1}\mathrm{d}h.$$

Note that the above integral is the usual Rankin–Selberg convolution for $\mathrm{GL}_n \times \mathrm{GL}_m$ when $r = 0$, but slightly different when $r > 0$. In fact, it is an integral presentation of $L(s, \Pi \times \Sigma)$ as well.

To single out the cuspidal representations that come from unitary groups via base change, we follow [22]. Say that n is odd. Put

$$\mathcal{P}_n(\varphi_\Pi) = \int_{Z_n'(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash\,\mathrm{GL}_n(\mathbb{A}')} \varphi_\Pi(g_1)\mathrm{d}g_1,$$

$$\mathcal{P}_m(\varphi_\Sigma) = \int_{Z_m'(\mathbb{A}')\,\mathrm{GL}_m(k')\backslash\,\mathrm{GL}_m(\mathbb{A}')} \varphi_\Pi(g_2)\eta(\det g_2)\mathrm{d}g_2,$$

where Z_n' (resp. Z_m') denotes the center of $\mathrm{GL}_{n,k'}$ (resp. $\mathrm{GL}_{m,k'}$). As pointed out in [6,7,9], the functional $\mathcal{P}_n$ (resp. $\mathcal{P}_m$) should be nontrivial on Π (resp. Σ) if the representation comes from unitary groups via (standard) base change.

Take $F_n \in \mathcal{S}(\mathrm{GL}_n(\mathbb{A}))$ and $F_m \in \mathcal{S}(\mathrm{GL}_m(\mathbb{A}))$. We introduce another distribution $\mathcal{J}(F_n \otimes F_m)$ which formally computes

$$\sum_{\Pi,\Sigma} \sum \mathcal{B}^{\nu}_{r,r}(\rho(F_n)\varphi_\Pi, \rho(F_m)\varphi_\Sigma)\mathcal{P}_n(\overline{\varphi_\Pi})\mathcal{P}_m(\overline{\varphi_\Sigma}),$$

whose regular part has the following decomposition

$$\mathcal{J}_{\mathrm{reg}}(F_n \otimes F_m) = \sum_{\zeta \in [\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')} \mathcal{J}_\zeta(F),$$

where $[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')$ is the set of regular orbits in the symmetric space $\mathrm{S}_n(k')$, which will be discussed in Sect. 4.3, and $F \in \mathcal{S}(\mathrm{S}_n(\mathbb{A}'))$ is obtained from $F_n \otimes F_m$. Moreover, each summand $\mathcal{J}_\zeta$ is an adèlic orbital integral

$$\mathcal{J}_\zeta(F) = \int_{\mathrm{GL}_m(\mathbb{A})} \int_{U_{1^r,m+1,1^r}(\mathbb{A}')} F(g^{-1}u^{-1}\zeta u^\tau g)\nu(u^{-1})\mathrm{d}u\mathrm{d}g.$$

We prove in Proposition 4.12 that there is a natural bijection

$$\mathbf{N}\colon [\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k') \xrightarrow{\sim} \coprod_\beta [\mathrm{U}_n(k')_{\mathrm{reg}}]/\mathbf{H}'(k'), \tag{1.2}$$

where the disjoint union is taken over all nondegenerate hermitian matrices $\beta \in \mathrm{Her}^\times_m(k')$ of rank m (which determines $W \subset V$) up to similarity. We say the test function F matches the collection $(f^\beta)_\beta$ if $\mathcal{J}_\zeta(F) = \mathcal{J}_{\zeta^\beta}(f^\beta)$ for every ζ, ζ^β such that $\mathbf{N}(\zeta) = \zeta^\beta$ (see Conjecture 4.13).

As the most important and interesting problem in trace formulae, we now discuss the corresponding fundamental lemma for *all* pairs (m, n) including the case of Fourier–Jacobi periods. When $n - m$ is even, we put $\mathrm{S}_{n,m}(k') = \mathrm{S}_n(k') \times \mathrm{Mat}_{1,m}(k') \times \mathrm{Mat}_{m,1}(k')$ and $\mathrm{U}_{n,m}(k') = \mathrm{U}_n(k') \times \mathrm{Mat}_{1,m}(k)$. As in the case of Bessel periods, there are notions of regular elements in both sets and we have a similar bijection as (1.2) (see Sect. 5.3).

Now let k' be a non-archimedean local field and k/k' an unramified quadratic field extension. Let $\mathfrak{o}'$ (resp. $\mathfrak{o}$) be the ring of integers of k' (resp. k). There are only two non-isomorphic hermitian spaces of dimension $m > 0$ over k. Let $\mathrm{U}^+_m \subset \mathrm{U}^+_n$ be the pair associate to $W^+ \subset V^+$ both with trivial discriminant, and $\mathrm{U}^-_m \subset \mathrm{U}^-_n$ be another one. Then W^+ has a selfdual $\mathfrak{o}$-lattice L_W that extends to a selfdual $\mathfrak{o}$-lattice L_V of V^+. The unitary group U^+_m (resp. U^+_n) is unramified and has a model over $\mathfrak{o}'$. The group of $\mathfrak{o}'$-points $\mathrm{U}^+_m(\mathfrak{o}')$ (resp. $\mathrm{U}^+_n(\mathfrak{o}')$) is a hyperspecial maximal subgroup of $\mathrm{U}^+_m(k')$ (resp. $\mathrm{U}^+_n(k')$). Note that $\mathrm{GL}(L_V) \cong \mathrm{GL}_n(\mathfrak{o})$ is a hyperspecial maximal subgroup of $\mathrm{GL}_n(k)$ and we put $\mathrm{S}_n(\mathfrak{o}') = \mathrm{S}_n(k') \cap \mathrm{GL}_n(\mathfrak{o})$.

We propose the following conjecture, where the notion $\mathbb{1}_T$ stands for the characteristic function of a subset T.

Conjecture 1.1. (The fundamental lemma for unit elements). *We have*

(1) *When $n-m$ is odd or $m=0$,*

$$\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')},\zeta)=\begin{cases}\mathbf{t}(\zeta)\mathcal{O}(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')},\zeta^+) & \zeta\leftrightarrow\zeta^+\in\mathrm{U}_n^+(k')\\ 0 & \zeta\leftrightarrow\zeta^-\in\mathrm{U}_n^-(k'),\end{cases}$$

where ζ,ζ^+ are normal, and

$$\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')},\zeta)=\int_{\mathbf{H}(k')}\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}([\zeta]\mathbf{h})\underline{\psi}(\mathbf{h})\eta(\det\mathbf{h})\mathrm{d}\mathbf{h},$$

$$\mathcal{O}(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')},\zeta^+)=\int_{\mathbf{H}^+(k')}\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}([\zeta^+]\mathbf{h}')\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'.$$

Here, $\underline{\psi}$ (resp. $\underline{\psi}'$) is the character induced from a generic character of the unipotent radical of $\mathbf{H}$ (resp. $\mathbf{H}^+$) (see Sect. 4.2), and $\mathbf{t}(\zeta)\in\{\pm1\}$ is a certain transfer factor defined in (4.17). In particular, when $m=0$, the second part of the above identity does not happen.

(2) *When $n-m$ is even and $m\neq0$,*

$$\begin{aligned}&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')};\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\otimes\mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')},[\zeta,x,y])\\&=\begin{cases}\mathbf{t}([\zeta,x,y])\mathcal{O}_\mu(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')};\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})},[\zeta^+,z]) & [\zeta,x,y]\leftrightarrow[\zeta^+,z]\in\mathrm{U}_{n,m}^+(k')\\ 0 & [\zeta,x,y]\leftrightarrow[\zeta^-,z]\in\mathrm{U}_{n,m}^-(k'),\end{cases}\end{aligned}$$

where ζ,ζ^+ are normal, and

$$\begin{aligned}&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')};\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\otimes\mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')},[\zeta,x,y])\\&=\int_{\mathbf{H}(k')}\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}([\zeta]\mathbf{h})\left(\omega^{\dagger}_{\underline{\psi},\overline{\mu}}(\mathbf{h})\left(\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\otimes\mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}\right)\right)(x,y)\underline{\psi}(\mathbf{h})\mathrm{d}\mathbf{h},\\&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')};\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})},[\zeta^+,z])\\&=\int_{\mathbf{H}^+(k')}\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}([\zeta^+]\mathbf{h}')(\omega^{\ddagger}_{\underline{\psi}',\overline{\mu}}(\mathbf{h}')\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})})(z)\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'.\end{aligned}$$

Here, $\underline{\psi}$ (resp. $\underline{\psi}'$) is the character induced from a generic character of the unipotent radical of $\mathbf{H}$ (resp. $\mathbf{H}^+$) (see Sect. 5.2), and $\mathbf{t}([\zeta,x,y])\in\{\pm1\}$ is a certain transfer factor defined in (5.16).

We have the following theorem.

Theorem 1.2. *In the above Conjecture 1.1,*

(1) *The second part of the identities in both cases holds.*
(2) *When $n=m+1$, the fundamental lemma holds if $\mathrm{char}(k)>n$, or $\mathrm{char}(k)=0$ and the residue characteristic is sufficiently large with respect to n.*
(3) *When $n\leq3, m=0$, the fundamental lemma holds if k has odd residue characteristic.*

(4) *When $n = m$, the fundamental lemma holds if* $\mathrm{char}(k) > n$, *or* $\mathrm{char}(k) = 0$ *and the residue characteristic is sufficiently large with respect to n.*

Proof. Part (1) is proved in Propositions 4.16 and 5.14 in this article. Part (2) is proved by Yun [37] where the transfer to characteristic 0 is accomplished by Gordon in the appendix therein. Part (3) is proved by Jacquet [17] when $n = 3$; (essentially) proved by Ye [36] when $n = 2$; and trivial when $n = 1$. Part (4) is proved in Theorem 5.15 by reducing to (2).

Remark 1.3. We have

(1) In general, when $m = 0$, the fundamental lemma can be proved by the argument of Ngô [31], with slight modification, in the case $\mathrm{char}\,(k) > n$, and transferred to the case $\mathrm{char}(k) = 0$ by the work of Cluckers–Loeser [3, Remark 9.2.5].
(2) When $n = m + 1$, the fundamental lemma is the group version of the one proposed by Jacquet–Rallis [22]. When $m = 0$, the fundamental lemma is the one proposed by Flicker [6], which is the unitary group version of the Jacquet–Ye fundamental lemma (see [26]). When $m > 0$ (and $n - m$ is odd), the fundamental lemma is a sort of hybrid of the Jacquet–Rallis fundamental lemma and the Flicker fundamental lemma. We hope that there is a geometric method toward this fundamental lemma as well, which is a sort of hybrid of those in [37] by Yun and [31] by Ngô.
(3) When $n = 3$ and $m = 0$, the fundamental lemma for the whole spherical Hecke algebra is proved by Mao [30].

1.4. Variants of Rankin–Selberg convolutions

As we see in the previous subsection, one needs to consider certain periods of Bessel and Fourier–Jacobi types on general linear groups as well. In Sect. 2 (resp. 3), we generalize the notion of Bessel (resp. Fourier–Jacobi) models and periods for $\mathrm{GL}_n \times \mathrm{GL}_m$ for a pair (r, r^*) of nonnegative integers such that $n = m + 1 + r + r^*$ (resp. $n = m + r + r^*$). When $r = r^*$, they are introduced and considered in [8].

Let (r, r^*) be as above, we introduce a unipotent subgroup $U_{1^r,m+1,1^{r^*}}$ of GL_n and put $H = U_{1^r,m+1,1^{r^*}} \rtimes \mathrm{GL}_m$, viewed as a subgroup of $\mathrm{GL}_n \times \mathrm{GL}_m$. We have a character ν of H, which is automorphic if k is a number field. Let π (resp. σ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ (resp. $\mathrm{GL}_m(\mathbb{A})$). We introduce the Bessel integral

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \int_{H(k)\backslash H(\mathbb{A})} \varphi_\Pi \otimes \varphi_\Sigma(h)\nu(h)^{-1}|\det h|_{\mathbb{A}}^{s-\frac{1}{2}}\,\mathrm{d}h,$$

for $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$ and $s \in \mathbb{C}$, and the Bessel period $\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi, \varphi_\sigma) := \mathcal{B}^{\nu}_{r,r^*}\left(\frac{1}{2}, \varphi_\pi, \varphi_\sigma\right)$. Then $\mathcal{B}^{\nu}_{r,r^*}$ is the usual Rankin–Selberg convolution on $\mathrm{GL}_n \times \mathrm{GL}_m$ if and only if $r = 0$. Similarly, we may define the Fourier–Jacobi integral (with respect to μ) $\mathcal{FJ}^{\nu_\mu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi)$ and Fourier–Jacobi period (with respect to μ) $\mathcal{FJ}^{\nu_\mu}_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi) := \mathcal{FJ}^{\nu_\mu}_{r,r^*}\left(\frac{1}{2}; \varphi_\pi, \varphi_\sigma; \Phi\right)$.

The following theorem was known by Jacquet, Piatetskii-Shapiro and Shalika long time ago. Since it is not explicated in any reference, we would like to write it down with proof simply for completeness.

Denote ι to be the outer automorphism of GL_n or GL_m by $\iota(g) = g^\iota := {}^t g^{-1}$. Put $\widetilde{\varphi_\pi}(g) = \varphi_\pi(g^\iota)$ and $\widetilde{\varphi_\sigma}(g) = \varphi_\sigma(g^\iota)$.

Theorem 1.4. (Theorem 2.5, Corollary 2.6, Theorem 3.4, Remark 3.5, Corollary 3.6). *We refer to Sects. 2.2, 3.2 for notation.*

(1) The Bessel integrals are holomorphic in s and satisfy the following functional equation

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \mathcal{B}^{\overline{\nu}}_{r^*,r}(1-s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}).$$

For $\varphi_\pi \in \mathcal{A}_\pi$ and $\varphi_\sigma \in \mathcal{A}_\sigma$ such that $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$ and $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$ are factorizable,

$$\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi, \varphi_\sigma) = L\left(\frac{1}{2}, \pi \times \sigma\right) \prod_{v \in \mathscr{M}_k} \frac{\Psi_{v,r}(s; W_v, W^-_v)}{L_v(s, \pi_v \times \sigma_v)}|_{s=\frac{1}{2}},$$

where in the last product almost all factors are 1. In particular, there is a nontrivial Bessel period of $\pi \otimes \sigma$ if and only if $L\left(\frac{1}{2}, \pi \times \sigma\right) \neq 0$.

(2) The Fourier–Jacobi integrals are meromorphic in s (holomorphic when $n > m$) and satisfy the following functional equation

$$\mathcal{FJ}^{\nu_\mu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \mathcal{FJ}^{\overline{\nu_\mu}}_{r^*,r}(1-s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}; \widehat{\Phi}).$$

For $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$ such that $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$, $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$ and $\Phi = \otimes \Phi_v$ are factorizable,

$$\begin{aligned}&\mathcal{FJ}^{\nu_\mu}_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi)\\ &= L\left(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}\right) \prod_{v \in \mathscr{M}_k} \frac{\Psi_{v,r}(s; W_v, W^-_v \otimes \mu_v^{-1}; \Phi_v)}{L_v(s, \pi_v \times \sigma_v \otimes \mu_v^{-1})}|_{s=\frac{1}{2}},\end{aligned}$$

where in the last product almost all factors are 1. In particular, there is a nontrivial Fourier–Jacobi period of $\pi \otimes \sigma$ for ν_μ if and only if $L\left(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}\right) \neq 0$.

Remark 1.5. The above theorem completely confirms [8, Conjecture 24.1] for split unitary groups, that is, general linear groups.

The following is an outline of the article. In Sect. 2, we focus on the Bessel models and periods on general linear groups, including the proof of Theorem 1.4 for the Bessel case. In Sect. 3, we focus on Fourier–Jacobi models and periods on general linear groups, including a proof of Theorem 1.4 for the Fourier–Jacobi case. After briefly recalling Bessel models and periods for unitary groups, we introduce the relative trace formula in Sect. 4. We prove the matching of orbits and the smooth matching of functions at split places in Sect. 4.3. We formulate the fundamental lemma and prove the vanishing part in Sect. 4.4. In Sect. 5, we repeat the previous section, but for the Fourier–Jacobi models and periods. We also prove the full fundamental lemma for $\mathrm{U}_n \times \mathrm{U}_n$ by reducing to the one for $\mathrm{U}_{n+1} \times \mathrm{U}_n$. Section 5.4 is an appendix on integrals of local Whittaker functions on general linear groups. We collect all the results we need in Sects. 2, 3 from existing literatures. In particular, we have to use various sorts of auxiliary local Whittaker integrals in the theory of Rankin–Selberg convolutions.

1.5. Notation and convention

Here are some general notation and conventions.

- If k is a local field, we denote $|\ |_k$ its normalized absolute value which satisfies $\mathrm{d}ab = |a|_k \mathrm{d}b$ for any Haar measure $\mathrm{d}a$ on the additive group k.
- Let k be a (commutative unital) ring and $\tau: k \to k$ an automorphism. For a k-module M, we put $M_\tau = M \otimes_{k,\tau} k$ and denote $M^\vee = \mathrm{Hom}_k(M, k)$ the dual module.
- All quadratic, symplectic, hermitian, or skew-hermitian spaces are assumed to be nondegenerate.
- For a smooth representation π, we denote $\widetilde{\pi}$ for its (smooth) contragredient representation.
- Denote $\mathcal{A}_0(G)$ the space of cuspidal automorphic forms for a reductive group G defined over a number field k, which is a representation of $G(\mathbb{A})$ by right translation where $\mathbb{A}$ is the ring of adèles of k.
- If G is a linear algebraic group over a number field k, we always use the Tamagawa measure for adèlic integrals over $G(\mathbb{A})$. In particular, if G is unipotent, then the total volume of $G(k)\backslash G(\mathbb{A})$ is 1.

Throughout the article, we will fix a quadratic extension k/k' of number fields with $\mathbb{A}/\mathbb{A}'$ the corresponding rings of adèles. Denote $\mathscr{M}_k$ (resp. $\mathscr{M}_{k'}$) the set of all places of k (resp. k'). Let τ be the nontrivial element in $\mathrm{Gal}(k/k')$ and $\eta: k'\backslash\mathbb{A}'^\times \to \{\pm 1\}$ the quadratic character associated to k/k'. We denote by Tr and Nm the trace and norm of k/k', respectively. Put $k^- = \{x \in k \mid x^\tau = -x\}$. We fix a nonzero element $\jmath \in k^-$ once for all and define $\widetilde{\mathrm{Tr}}(x) = \jmath(x - x^\tau) \in k'$ for $x \in k$. In certain situation, we will also refer k/k' and the related notation to their localizations (or possible general local fields).

- Denote $\mathrm{Mat}_{m,n}$ the affine group scheme over $\mathrm{Spec}\,\mathbb{Z}$ of $m \times n$ matrices; put $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$ with GL_n the open subscheme of invertible matrices. For a scheme S over $\mathrm{Spec}\,\mathbb{Z}$ and a field k, put $S_k = S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,k$.
- For $0 \le m \le n$, denote $\mathbf{1}_n$ the identity matrix of rank n and put

$$\mathsf{w}_0 = \emptyset, \quad \mathsf{w}_n = \begin{bmatrix} & 1 \\ \mathsf{w}_{n-1} & \end{bmatrix}, \quad \mathsf{w}_{n,m} = \begin{bmatrix} \mathbf{1}_m & \\ & \mathsf{w}_{n-m} \end{bmatrix}.$$

- Denote Her_n (resp. $\overline{\mathrm{Her}}_n$) the affine group scheme over $\mathrm{Spec}\,k'$ of $n \times n$-hermitian (resp. skew-hermitian) matrices with respect to the quadratic extension k/k', which is naturally a closed subscheme of $\mathrm{Res}_{k/k'}\mathrm{Mat}_{n,k}$. In particular, $\mathrm{Her}_1(k') = k'$ (resp. $\overline{\mathrm{Her}}_1(k') = k^-$). Put $\mathrm{Her}_n^\times = \mathrm{Her}_n \cap \mathrm{Res}_{k/k'}\mathrm{GL}_{n,k}$.

- Denote $\mathrm{S}_n \subset \mathrm{Res}_{k/k'}\mathrm{GL}_{n,k}$ the symmetric space, which is the closed subscheme defined by the equation $ss^\tau = \mathbf{1}_n$. We have an isomorphism $\mathrm{Res}_{k/k'}\mathrm{GL}_{n,k} / \mathrm{GL}_{n,k'} \cong \mathrm{S}_n$ given by $g \mapsto gg^{\tau,-1}$.
- We have a subgroup $H \subset \mathrm{GL}_{n,k} \times \mathrm{GL}_{m,k}$ (resp. $H' \subset \mathrm{U}_n \times \mathrm{U}_m$) over $\mathrm{Spec}\, k$ (resp. $\mathrm{Spec}\, k'$), whose reductive quotient is $\mathrm{GL}_{m,k}$ (resp. U_m) in Sect. 2.1 (resp. Sect. 3.1). In Sect. 4.2, we will introduce a subgroup $\mathbf{H}$ (resp. $\mathbf{H}'$) of $\mathrm{Res}_{k/k'} H$ (resp. $H' \times_{\mathrm{Spec}\, k'} H'$). Moreover, in Sect. 5.2, we need to consider an auxiliary quotient group $H^\dagger$ (resp. $H^\ddagger$, $\mathbf{H}^\ddagger$) of H (resp. H', $\mathbf{H}'$). Here are two extremal cases. When $n-m \leq 1$, $H = H^\dagger = \mathrm{GL}_{m,k}$ and $\mathbf{H} = \mathrm{GL}_{m,k'}$ (resp. $H' = H^\ddagger = \mathbf{H}' = \mathbf{H}^\ddagger = \mathrm{U}_m$). When $m = 0$, $\mathbf{H} = \mathrm{Res}_{k/k'} H$ (resp. $\mathbf{H}' = H' \times_{\mathrm{Spec}\, k'} H'$); $H^\dagger \simeq \mathrm{Mat}_{1,k}$ (resp. $H^\ddagger \simeq \overline{\mathrm{Her}}_1$ and $\mathbf{H}^\ddagger \simeq \overline{\mathrm{Her}}_1 \times_{\mathrm{Spec}\, k'} \overline{\mathrm{Her}}_1$).

2. Bessel periods on $\mathbf{GL}_n \times \mathbf{GL}_m$

2.1. *Bessel models for general linear groups*

Let k be a local field and V be a k-vector space of dimension n. Suppose that V has a decomposition $V = X \oplus W \oplus E \oplus X^*$, where W, X, X^* and E have dimensions m, r, r^* and 1, respectively. Then $n = m + r + r^* + 1$. Let $P_{r,m+1,r^*}$ be the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing the flag $0 \subset X \subset X \oplus W \oplus E \subset V$, and $U_{r,m+1,r^*}$ its unipotent radical. Then $U_{r,m+1,r^*}$ fits into the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(X^*, X) \longrightarrow U_{r,m+1,r^*} \longrightarrow \mathrm{Hom}(X^*, W \oplus E) + \mathrm{Hom}(W \oplus E, X) \longrightarrow 0,$$

which may be written as

$$0 \longrightarrow (X^*)^\vee \otimes X \longrightarrow U_{r,m+1,r^*} \longrightarrow (X^*)^\vee \otimes (W \oplus E) + (W^\vee \oplus E^\vee) \otimes X \longrightarrow 0.$$

Let $\ell_X\colon X \to k$ (resp. $\ell_{X^*}\colon k \to X^*$) be a nontrivial k-linear homomorphism (if exists), and U_X (resp. U_{X^*}) a maximal unipotent subgroup of $\mathrm{GL}(X)$ (resp. $\mathrm{GL}(X^*)$) stabilizing ℓ_X (resp. ℓ_{X^*}). Moreover, let

$$\ell_W\colon (W \oplus E) + (W^\vee \oplus E^\vee) \to k$$

be a k-linear homomorphism, which is trivial on $W + W^\vee$ and nontrivial on E and $E^\vee$. Let $\ell\colon U_{r,m+1,r^*} \to k$ be the homomorphism as the composition

$$U_{r,m+1,r^*} \to (X^*)^\vee \otimes (W \oplus E) + (W^\vee \oplus E^\vee) \otimes X$$
$$\xrightarrow{\ell_X + \ell_{X^*}^\vee} (W \oplus E) + (W^\vee \oplus E^\vee) \xrightarrow{\ell_W} k,$$

which is fixed by $(U_X \times U_{X^*}) \times \mathrm{GL}(W)$. Therefore, we may extend ℓ trivially to a homomorphism

$$\ell\colon H = U_{r,m+1,r^*} \rtimes ((U_X \times U_{X^*}) \times \mathrm{GL}(W)) \to k.$$

Let $\psi: k \to \mathbb{C}^\times$ be a nontrivial character and $\lambda: U_X \times U_{X^*} \to \mathbb{C}^\times$ a generic character which can be viewed as a character of H. Let δ_W be the character of $\mathrm{GL}(W)$ defined by $\delta_W(g) = |\det g|_k^{r^*-r}$. Then we can form a character $\nu = (\psi \circ \ell) \otimes \lambda \otimes \delta_W^{-\frac{1}{2}}$ of H. There is a natural embedding $\varepsilon: H \to \mathrm{GL}(V)$ and a projection $\kappa: H \to \mathrm{GL}(W)$, which together induce an embedding $(\varepsilon, \kappa): H \to \mathrm{GL}(V) \times \mathrm{GL}(W)$. The pair (H, ν) is uniquely determined up to conjugacy in the group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ by the pair $W \subset V$ and (r, r^*). We have the following theorem.

Theorem 2.1. *Let k be of characteristic* 0. *Let π (resp. σ) be an irreducible admissible representation of* $\mathrm{GL}(V)$ *(resp.* $\mathrm{GL}(W)$*).*

(1) If π and σ are generic, $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \sigma, \nu) \geq 1$.
(2) If $r = r^*$, $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \sigma, \nu) \leq 1$.

It is naturally expected that (2) is true for any r, r^*. For the relative trace formula we are going to consider, r is equal to r^*.

Proof. Part (1) is due to Corollary 6.2(1). Part (2) is proved in [34] (resp. [1,2]) when $r = 1$ (resp. and k non-archimedean, and $k = \mathbb{R}$). The case for general r is reduced to the previous one, as shown in [27].[1] □

Definition 2.2. (*Bessel model*). A nontrivial element in the space $\mathrm{Hom}_H(\pi \otimes \sigma, \nu)$ is called an (r, r^*)-*Bessel model* of $\pi \otimes \sigma$. When $r = r^*$, hence $n - m$ is odd, it is simply the one defined in [8].

Remark 2.3. In Theorem 2.1, if k is archimedean, π and σ will be understood as smooth Fréchet representations of moderate growth (in fact, they are Casselman–Wallach representations) and the tensor product $\pi \otimes \sigma$ will be understood as complete (projective) tensor product. This same convention will be adopted throughout the article, unless otherwise specified.

2.2. *Bessel integrals, functional equations and L-functions*

Now we consider the global situation. Let k be a number field and $|\ |_{\mathbb{A}} = \prod_{v \in \mathscr{M}_k} |\ |_v$ a norm on $\mathbb{A}$. For a place $v \in \mathscr{M}_k$, we denote k_v the completion of k at v. We denote $\mathfrak{o}$ (resp. $\mathfrak{o}_v$) the ring of integers of k (resp. k_v for v finite). For an algebraic group G over k, we denote $G_v = G(k_v)$ the k_v-Lie group for $v \in \mathscr{M}_k$.

We define the pair (H, ν) as in the local case, as well as, ℓ_X, ℓ_{X^*} and ℓ_W. In particular, $\psi: k\backslash\mathbb{A} \to \mathbb{C}^\times$ is a nontrivial character, and $\lambda: (U_X \times U_{X^*})(k)\backslash(U_X \times U_{X^*})(\mathbb{A}) \to \mathbb{C}^\times$ is a generic character. Let π (resp. σ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}(V)(\mathbb{A})$ (resp. $\mathrm{GL}(W)(\mathbb{A})$). Then π (resp. σ) is isomorphic to a unique irreducible sub-representation $\mathcal{A}_\pi$ (resp. $\mathcal{A}_\sigma$) of $\mathcal{A}_0(\mathrm{GL}(V))$ (resp. $\mathcal{A}_0(\mathrm{GL}(W))$).

[1] Although the authors consider the case where k is archimedean, the proof works for non-archimedean case as well without change.

Definition 2.4. (Bessel integral and Bessel period) For $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$, the following integral, with a parameter $s \in \mathbb{C}$:

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \int_{H(k)\backslash H(\mathbb{A})} \varphi_\pi(\varepsilon(h))\varphi_\sigma(\kappa(h))\nu(h)^{-1}|\det h|_{\mathbb{A}}^{s-\frac{1}{2}} \mathrm{d}h,$$

which is absolutely convergent, is an (r, r^*)-*Bessel integral* of $\pi \otimes \sigma$. When $s = \frac{1}{2}$,

$$\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi, \varphi_\sigma) = \mathcal{B}^{\nu}_{r,r^*}\left(\frac{1}{2}; \varphi_\pi, \varphi_\sigma\right)$$

is an (r, r^*)-*Bessel period* of $\pi \otimes \sigma$ (for a pair (H, ν)). If there exist $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$ such that $\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi, \varphi_\sigma) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial (r, r^*)-Bessel period.

It is obvious that $\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi, \varphi_\sigma)$ defines an element in

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \sigma, \nu) = \bigotimes_{v \in \mathscr{M}_k} \mathrm{Hom}_{H_v}(\pi_v \otimes \sigma_v, \nu_v).$$

We now show that the Bessel period is Eulerian. Choose a basis $\{v_1, \dots, v_r\}$ of X under which

- the homomorphism $\ell_X : X \to k$ is given by the coefficient of v_r;
- U_X is the unipotent radical of the parabolic subgroup P_X stabilizing the complete flag $0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_r \rangle = X$;
- the generic character $\lambda|_{U_X}$ is given by

  $$\lambda(u) = \psi\left(u_{1,2} + u_{2,3} + \cdots + u_{r-1,r}\right),$$

  where

$$u = \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,r-1} & u_{1,r} \\ & 1 & u_{2,3} & \cdots & u_{2,r-1} & u_{2,r} \\ & & 1 & \cdots & u_{3,r-1} & u_{3,r} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & u_{r-1,r} \\ & & & & & 1 \end{bmatrix} \in U_X(\mathbb{A}). \tag{2.1}$$

Similarly, we also choose a basis $\left\{v^*_{r^*}, \dots, v^*_1\right\}$ of X^* under which

- the homomorphism $\ell_{X^*} : k \to X^*$ is given by $x \mapsto cxv^*_{r^*}$ for some $c \neq 0$, which will be determined later;
- U_{X^*} is the unipotent radical of the parabolic subgroup P_{X^*} stabilizing the complete flag $0 \subset \langle v^*_{r^*} \rangle \subset \langle v^*_{r^*}, v^*_{r^*-1} \rangle \subset \cdots \subset \langle v^*_{r^*}, \dots, v^*_1 \rangle = X^*$;

- The generic character $\lambda|_{U_{X^*}}$ is given by

$$\lambda(u^*) = \psi\left(u^*_{r^*,r^*-1} + u^*_{r^*-1,r^*-2} + \cdots + u^*_{2,1}\right),$$

where

$$u^* = \begin{bmatrix} 1 & u^*_{r^*,r^*-1} & u^*_{r^*,r^*-2} & \cdots & u^*_{r^*,2} & u^*_{r^*,1} \\ & 1 & u^*_{r^*-1,r^*-2} & \cdots & u^*_{r^*-1,2} & u^*_{r^*-1,1} \\ & & 1 & \cdots & u^*_{r^*-2,2} & u^*_{r^*-2,1} \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & u^*_{2,1} \\ & & & & & 1 \end{bmatrix} \in U_{X^*}(\mathbb{A}). \quad (2.2)$$

Moreover, we choose a basis $\{w_1, \ldots, w_m\}$ of W and $\{w_0\}$ of E under which the homomorphism $\ell_W : (W \oplus E) + (W^\vee \oplus E^\vee) \to k$ is given by $\ell_W(w_i) = \ell_W(w_i^\vee) = 0$ $(1 \le i \le m)$ and $\ell_W(w_0^\vee) = 1$, where $\left\{w_1^\vee, \ldots, w_m^\vee, w_0^\vee\right\}$ is the dual basis. Set $c = \ell_W(w_0)^{-1}$.

We identify $\mathrm{GL}(V)$ (resp. $\mathrm{GL}(W)$) with $\mathrm{GL}_{n,k}$ (resp. $\mathrm{GL}_{m.k}$) under the basis

$$\left\{w_1, \ldots, w_m, v_1, \ldots, v_r, w_0, v^*_{r^*}, \ldots, v^*_1\right\}, \quad (2.3)$$

and view $\mathrm{GL}_{m,k}$ as a subgroup of $\mathrm{GL}_{n,k}$ (through the first m coordinates).

Theorem 2.5. *The Bessel integrals are holomorphic in s and satisfy the following functional equation*

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \mathcal{B}^{\overline{\nu}}_{r^*,r}(1 - s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}).$$

Put $\widetilde{W^{\psi}_{\bullet}}(g) = W^{\psi}_{\bullet}(\mathsf{w}_n g^\iota) \in \mathcal{W}(\widetilde{\pi}, \overline{\psi})$ (resp. $\widetilde{W^{\overline{\psi}}_{\bullet}}(g) = W^{\overline{\psi}}_{\bullet}(\mathsf{w}_m g^\iota) \in \mathcal{W}(\widetilde{\sigma}, \psi)$). If the Whittaker–Fourier coefficient $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$ (resp. $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$) is factorizable, then $\widetilde{W^{\psi}_{\varphi_\pi}} = \otimes_v \widetilde{W_v}$ (resp. $\widetilde{W^{\overline{\psi}}_{\varphi_\sigma}} = \otimes_v \widetilde{W^-_v}$) is also factorizable with $\widetilde{W_v}(g) = W_v(\mathsf{w}_n g^\iota)$ (resp. $\widetilde{W^-_v}(g) = W^-_v(\mathsf{w}_m g^\iota)$). In this case, for $\mathrm{Re}\, s \gg 0$,

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \Psi_r(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma}) = \prod_{v \in \mathscr{M}_k} \Psi_{v,r}(s; W_v, W^-_v),$$

$$\begin{aligned}\mathcal{B}^{\overline{\nu}}_{r^*,r}(s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}) &= \Psi_{r^*}(s; \rho(\mathsf{w}_{n,m})\widetilde{W^{\psi}_{\varphi_\pi}}, \widetilde{W^{\overline{\psi}}_{\varphi_\sigma}}) \\ &= \prod_{v \in \mathscr{M}_k} \Psi_{v,r^*}(s; \rho(\mathsf{w}_{n,m})\widetilde{W_v}, \widetilde{W^-_v}).\end{aligned}$$

We refer readers to Sect. 5.4 for Whittaker functions and $\Psi_{v,r}(s; W_v, W^-_v)$ (6.1).

Proof. Under the basis (2.3), the image of $H(\mathbb{A})$ in $\mathrm{GL}(V)(\mathbb{A})$ consists of matrices of the following form

$$
\begin{aligned}
h &= h\left(n, n^*, b; u, u^*; g\right) \\
&= \left[\begin{array}{c|c|c|c}
g & & & \begin{matrix} n^*_{1,r^*} & \cdots & n^*_{1,1} \\ \vdots & & \vdots \\ n^*_{m,r^*} & \cdots & n^*_{m,1} \end{matrix} \\ \hline
\begin{matrix} n_{1,1} & \cdots & n_{1,m} \\ \vdots & & \vdots \\ n_{r,1} & \cdots & n_{r,m} \end{matrix} & u & \begin{matrix} n_{1,0} \\ \vdots \\ n_{r,0} \end{matrix} & b \\ \hline
 & & 1 & \begin{matrix} n^*_{0,r^*} & \cdots & n^*_{0,1} \end{matrix} \\ \hline
 & & & u^*
\end{array}\right],
\end{aligned} \tag{2.4}
$$

where

$$
n = \begin{bmatrix} n_{1,1} & \cdots & n_{1,m} & n_{1,0} \\ \vdots & & \vdots & \vdots \\ n_{r,1} & \cdots & n_{r,m} & n_{r,0} \end{bmatrix} \in \mathrm{Hom}(W \oplus E, X)(\mathbb{A}),
$$

$$
n^* = \begin{bmatrix} n^*_{1,r^*} & \cdots & n^*_{1,1} \\ \vdots & & \vdots \\ n^*_{m,r^*} & \cdots & n^*_{m,1} \\ n^*_{0,r^*} & \cdots & n^*_{0,1} \end{bmatrix} \in \mathrm{Hom}(X^*, W \oplus E)(\mathbb{A}),
$$

$b \in \mathrm{Hom}(X^*, X)(\mathbb{A})$, $u \in U_X(\mathbb{A})$, $u^* \in U_{X^*}(\mathbb{A})$, and $g \in \mathrm{GL}(W)(\mathbb{A})$. Here, u and u^* are upper triangular matrices as in (2.1) and (2.2). Thus the character ν on $H(\mathbb{A})$ is given by

$$
\begin{aligned}
\nu(h) &= \nu\left(h\left(n, n^*, b; u, u^*; g\right)\right) \\
&= |\det g|_{\mathbb{A}}^{\frac{r-r^*}{2}} \psi\left(u_{1,2} + \cdots + u_{r-1,r} + n_{r,0} + n^*_{0,r^*} + u^*_{r^*,r^*-1} + \cdots + u^*_{2,1}\right).
\end{aligned}
$$

Put $U_{1^r,m+1,1^{r^*}} = U_{r,m+1,r^*} \rtimes (U_X \times U_{X^*})$ which is the unipotent radical of H. Precisely,

$$
U_{1^r,m+1,1^{r^*}}(\mathbb{A}) = \left\{\underline{u} = \underline{u}(n, n^*, b; u, u^*) := h(n, n^*, b; u, u^*; \mathbf{1}_m)\right\}.
$$

Then

$$\mathcal{B}^{\nu}_{r,r^*}(s;\varphi_\pi,\varphi_\sigma) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^r,m+1,1^{r^*}}(k)\backslash U_{1^r,m+1,1^{r^*}}(\mathbb{A})} \varphi_\pi\left(\underline{u}g\right)\varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}+\frac{r-r^*}{2}}\overline{\underline{\psi}(\underline{u})}\mathrm{d}\underline{u}\mathrm{d}g, \tag{2.5}$$

where we simply write

$$\underline{\psi}(\underline{u}) = \psi\left(u_{1,2}+\cdots+u_{r-1,r}+n_{r,0}+n^*_{0,r^*}+u^*_{r^*,r^*-1}+\cdots+u^*_{2,1}\right).$$

We are going to use the Fourier transform. Let

$$L_{r+1} = \left\{\underline{u}(n_0,n^*,b;u,u^*)\mid n_0 = \begin{bmatrix} 0 & \cdots & 0 & n_{1,0} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & n_{r,0}\end{bmatrix}\right\}$$

be a subgroup of $U_{1^r,m+1,1^{r^*}}$. For $1\le i\le r$, put

$$L_i = \left\{\underline{l} = \underline{l}(l_i;n_0,n^*,b;u,u^*)\mid l_i = \begin{bmatrix} l_{1,i} & \cdots & l_{1,r-1} & l_{1,r} \\ \vdots & & \vdots & \vdots \\ l_{m,i} & \cdots & l_{m,r-1} & l_{m,r}\end{bmatrix}\right\},$$

where $\underline{l}(l_i;n_0,n^*,b;u,u^*)$ is the one obtained from $\underline{u}(n_0,n^*,b;u,u^*)$ by adding l_i above the entries $[u_{1,i+1},\ldots,u_{1,r},n_{1,0}]$ as in (2.4). In particular, L_1 is the unipotent radical of the (standard) parabolic subgroup stabilizing the flag

$$0\subset W\oplus\langle v_1\rangle\subset W\oplus\langle v_1,v_2\rangle\subset\cdots\subset V.$$

It is clear that for $1\le i\le r$, L_i/L_{i+1} is isomorphic to $\mathrm{Mat}_{m,1,k}$, which may be identified with the group of columns ${}^t[l_{1,i},\cdots,l_{m,i}]$. For $1\le i\le r$, we regard $\underline{\psi}$ as a character of $L_i(\mathbb{A})$ via the quotient group $L_{r+1}(\mathbb{A})$.

By the Fourier inversion formula for $\mathrm{Mat}_{m,1}(k)\backslash\mathrm{Mat}_{m,1}(\mathbb{A})$, we have

$$(2.5) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})}\int_{\mathrm{Mat}_{r,m}(k)\backslash\mathrm{Mat}_{r,m}(\mathbb{A})}\sum_{\epsilon_i\in k} \times\left(\int_{L_r(k)\backslash L_r(\mathbb{A})}\varphi_\pi(\underline{l}\,\underline{n}g)\overline{\psi\left(\sum_{i=1}^m\epsilon_i l_{i,r}\right)\underline{\psi}(\underline{l})}\mathrm{d}\underline{l}\right)\varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}+\frac{r-r^*}{2}}\mathrm{d}n\mathrm{d}g, \tag{2.6}$$

where $\underline{n}$ is the element $\underline{u}\,(n,0,0;\mathbf{1}_r,\mathbf{1}_{r^*})$ with

$$n = \begin{bmatrix} n_{1,1} & \cdots & n_{1,m} & 0 \\ \vdots & & \vdots & \vdots \\ n_{r,1} & \cdots & n_{r,m} & 0\end{bmatrix}.$$

Let $\underline{\epsilon}$ be an element like $\underline{u}(\epsilon, 0, 0; \mathbf{1}_r, \mathbf{1}_{r^*})$, where

$$\epsilon = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \epsilon_1 & \cdots & \epsilon_m & 0 \end{bmatrix}.$$

If we conjugate $\underline{l}$ by $\underline{\epsilon}$ from left to right, we can incorporate $\psi\left(\sum \epsilon_i l_{i,r}\right)$ into $\underline{\psi}(\underline{l})$ and collapse the summation over $\epsilon_i \in k$. Then

$$(2.6) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{Mat}_{r-1,m}(k)\backslash \mathrm{Mat}_{r-1,m}(\mathbb{A})\times \mathrm{Mat}_{1,m}(\mathbb{A})} \times \left(\int_{L_r(k)\backslash L_r(\mathbb{A})} \varphi_\pi(\underline{l}\,\underline{n}g)\overline{\underline{\psi}(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}+\frac{r-r^*}{2}} \mathrm{d}n\mathrm{d}g. \tag{2.7}$$

We repeat the process for $r-1$ more times. Then

$$(2.7) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{Mat}_{r,m}(\mathbb{A})} \times \left(\int_{L_1(k)\backslash L_1(\mathbb{A})} \varphi_\pi(\underline{l}\,\underline{n}g)\overline{\underline{\psi}(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}+\frac{r-r^*}{2}} \mathrm{d}n\mathrm{d}g. \tag{2.8}$$

We will shortly see that when $\operatorname{Re} s \gg 0$, the above integral is absolutely convergent after integrating over $L_1(k)\backslash L_1(\mathbb{A})$. First formally interchange the order of g and n,

$$\begin{aligned} (2.8) &= \int_{\mathrm{Mat}_{r,m}(\mathbb{A})} \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \times \left(\int_{L_1(k)\backslash L_1(\mathbb{A})} \varphi_\pi(\underline{l}g\underline{n})\overline{\underline{\psi}(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}-\frac{r+r^*}{2}} \mathrm{d}g\mathrm{d}n \\ &= \int_{\mathrm{Mat}_{r,m}(\mathbb{A})} \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \times \left(\int_{L_1(k)\backslash L_1(\mathbb{A})} \left(\rho(\underline{n})\varphi_\pi\right)(\underline{l}g)\overline{\underline{\psi}(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}} \mathrm{d}g\mathrm{d}n. \end{aligned} \tag{2.9}$$

Now the inner iterated integral is simply the usual Rankin–Selberg convolution (see [21]). The following calculation is well-known:

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \left(\int_{L_1(k)\backslash L_1(\mathbb{A})} \left(\rho(\underline{n})\varphi_\pi\right)(\underline{l}g)\overline{\underline{\psi}(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g$$
$$= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \sum_{U_{1^m}(k)\backslash \mathrm{GL}_m(k)} W^{\psi}_{\rho(\underline{n})\varphi_\pi}\left(\begin{bmatrix}\gamma & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right)\varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g, \tag{2.10}$$

which is absolutely convergent when $\mathrm{Re}\, s \gg 0$, where U_{1^m} is the standard maximal unipotent subgroup of GL_m and $W^{\psi}_{\bullet}$ stands for the ψ-Whittaker–Fourier coefficient. Thus when $\mathrm{Re}\, s \gg 0$, we may interchange the order of integration and summation in the above expression, and then combine them to obtain

$$(2.10) = \int_{U_{1^m}(k)\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\rho(\underline{n})\varphi_\pi}\left(\begin{bmatrix}\gamma & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right)\varphi_\sigma(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g. \tag{2.11}$$

Factorizing the integral over $U_{1^m}(\mathbb{A})$, we have

$$\begin{aligned}(2.11) &= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^m}(k)\backslash U_{1^m}(\mathbb{A})} W^{\psi}_{\rho(\underline{n})\varphi_\pi}\left(\begin{bmatrix} u & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right)\varphi_\sigma(g)\mathrm{d}u\,|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g\\
&= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\rho(\underline{n})\varphi_\pi}\left(\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right) \int_{U_{1^m}(k)\backslash U_{1^m}(\mathbb{A})} \psi(u)\varphi_\sigma(g)\mathrm{d}u\,|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g\\
&= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\rho(\underline{n})\varphi_\pi}\left(\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right) W^{\overline{\psi}}_{\varphi_\sigma}(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g\\
&= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\varphi_\pi}\left(\begin{bmatrix} g & 0\\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\underline{n}\right) W^{\overline{\psi}}_{\varphi_\sigma}(g)|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\mathrm{d}g.\end{aligned} \tag{2.12}$$

Plugging (2.12) into (2.9), we have

$$(2.9) = \int_{\mathrm{Mat}_{r,m}(\mathbb{A})} \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\varphi_\pi} \times \left(\begin{bmatrix} g & 0 & 0 \\ x & \mathbf{1}_r & 0 \\ 0 & 0 & \mathbf{1}_{n-m-r} \end{bmatrix} \right) W^{\overline{\psi}}_{\varphi_\sigma}(g) |\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}} \mathrm{d}g\mathrm{d}x,$$

which we denote by $\Psi_r(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma})$. By Proposition 6.1 (whose proof uses a gauge estimate in, for example, [20, Sect. 13] and [24, Sect. 3]) and Proposition 6.3, the above integral is absolutely convergent when $\mathrm{Re}\, s \gg 0$ and hence our calculation is valid.

We assume that $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$ and $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W_v^-$ are factorizable with $W_v \in \mathcal{W}(\pi_v, \psi_v)$ and $W_v^- \in \mathcal{W}(\sigma_v, \overline{\psi_v})$ such that for almost all finite places v, W_v, W_v^- are unramified satisfying $W_v(\mathbf{1}_n) = W_v^-(\mathbf{1}_m) = 1$. Denote $\Psi_{v,r}(s; W_v, W_v^-)$ to be

$$\int_{U_{1^m,v}\backslash \mathrm{GL}_{m,v}} \int_{\mathrm{Mat}_{r,m,v}} W_v \left(\begin{bmatrix} g_v & 0 & 0 \\ x_v & \mathbf{1}_r & 0 \\ 0 & 0 & \mathbf{1}_{n-m-r} \end{bmatrix} \right) W_v^-(g_v) |\det g_v|_v^{s-\frac{n-m}{2}} \mathrm{d}x_v \mathrm{d}g_v.$$

Then for $\mathrm{Re}\, s \gg 0$,

$$\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \Psi_r(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma}) = \prod_{v \in \mathscr{M}_k} \Psi_{v,r}(s; W_v, W_v^-).$$

Now we discuss the functional equation for Bessel integrals. We have

$$\begin{aligned} &\mathcal{B}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) \\ &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^r,m+1,1^{r^*}}(k)\backslash U_{1^r,m+1,1^{r^*}}(\mathbb{A})} \\ &\quad \varphi_\pi(\underline{u} g^{\iota}) \varphi_\sigma(g^{\iota}) |\det g^{\iota}|_{\mathbb{A}}^{s-\frac{1}{2}+\frac{r-r^*}{2}} \overline{\underline{\psi}(\underline{u})} \mathrm{d}\underline{u} \mathrm{d}g \\ &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \\ &\quad \int_{U_{1^r,m+1,1^{r^*}}(k)\backslash U_{1^r,m+1,1^{r^*}}(\mathbb{A})} \widetilde{\varphi_\pi}(\underline{u}^{\iota} g) \widetilde{\varphi_\sigma}(g) |\det g|_{\mathbb{A}}^{-s+\frac{1}{2}+\frac{r^*-r}{2}} \overline{\underline{\psi}(\underline{u})} \mathrm{d}\underline{u} \mathrm{d}g, \end{aligned}$$

which equals

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^r,m+1,1^{r^*}}(k)\backslash U_{1^r,m+1,1^{r^*}}(\mathbb{A})} \times \left(\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}\right) (\mathsf{w}_{n,m} \underline{u}^{\iota} \mathsf{w}_{n,m} g) \widetilde{\varphi_\sigma}(g) |\det g|_{\mathbb{A}}^{-s+\frac{1}{2}+\frac{r^*-r}{2}} \overline{\underline{\psi}(\underline{u})} \mathrm{d}\underline{u} \mathrm{d}g. \quad (2.13)$$

Since $\mathsf{w}_{n,m} U_{1^r,m+1,1^{r^*}} \mathsf{w}_{n,m} = U_{1^{r^*},m+1,1^r}$ and $\overline{\underline{\psi}(\underline{u})} = \underline{\psi}\left(\mathsf{w}_{n,m}\underline{u}^t\mathsf{w}_{n,m}\right)$, we have

$$\begin{aligned}(2.13) = &\int\limits_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int\limits_{U_{1^{r^*},m+1,1^r}(k)\backslash U_{1^{r^*},m+1,1^r}(\mathbb{A})} \\ &\times \left(\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}\right)(\underline{u}g)\widetilde{\varphi_\sigma}(g)|\det g|_{\mathbb{A}}^{1-s-\frac{1}{2}+\frac{r^*-r}{2}} \underline{\psi}(\underline{u})\mathrm{d}\underline{u}\mathrm{d}g \\ = &\,\mathcal{B}^{\overline{\nu}}_{r^*,r}(1-s;\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi},\widetilde{\varphi_\sigma}).\end{aligned}$$

The theorem follows. □

By Propositions 6.1 and 6.3, we have the following corollary, which confirms [8, Conjecture 24.1] for Bessel periods on split unitary groups, that is, general linear groups.

Corollary 2.6. *Let the notation be as above.*

(1) Let π (resp. σ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}(V)(\mathbb{A})$ (resp. $\mathrm{GL}(W)(\mathbb{A})$). For a pair (r, r^*) such that $r + r^* = n - m - 1$ and the automorphic representation ν introduced above, we have, for $\varphi_\pi \in \mathcal{A}_\pi$ and $\varphi_\sigma \in \mathcal{A}_\sigma$ such that $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$ and $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$ are factorizable,

$$\mathcal{B}^{\nu}_{r,r^*}(\varphi_\pi,\varphi_\sigma) = L\left(\frac{1}{2},\pi\times\sigma\right)\prod_{v\in\mathscr{M}_k}\frac{\Psi_{v,r}(s;W_v,W^-_v)}{L_v(s,\pi_v\times\sigma_v)}|_{s=\frac{1}{2}},$$

where in the last product almost all factors are 1, and the L-functions are the ones defined by Rankin–Selberg convolutions (see [21]).

(2) There is a nontrivial Bessel period of $\pi\otimes\sigma$ if and only if $L\left(\frac{1}{2},\pi\times\sigma\right)\neq 0$.

3. Fourier–Jacobi periods on $\mathbf{GL}_n \times \mathbf{GL}_m$

3.1. Fourier–Jacobi models for general linear groups

Let k be a local field and V a k-vector space of dimension $n > 0$. Suppose that V has a decomposition $V = X \oplus W \oplus X^*$, where W, X and X^* have dimensions m, r and r^*, respectively. Then $n = m + r + r^*$. Let P_{r,m,r^*} be the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing the flag $0 \subset X \subset X \oplus W \subset V$ and U_{r,m,r^*} its unipotent radical. Then U_{r,m,r^*} fits into the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(X^*, X) \longrightarrow U_{r,m,r^*} \longrightarrow \mathrm{Hom}(X^*, W) + \mathrm{Hom}(W, X) \longrightarrow 0,$$

which may be written as

$$0 \longrightarrow (X^*)^\vee \otimes X \longrightarrow U_{r,m,r^*} \longrightarrow (X^*)^\vee \otimes W + W^\vee \otimes X \longrightarrow 0.$$

Let $\ell_X : X \to k$ (resp. $\ell_{X^*} : k \to X^*$) be a nontrivial k-linear homomorphism (if exists), and U_X (resp. U_{X^*}) a maximal unipotent subgroup of $\mathrm{GL}(X)$ (resp.

$\mathrm{GL}(X^*)$) stabilizing ℓ_X (resp. ℓ_{X^*}). Then the above exact sequence fits into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (X^*)^\vee \otimes X & \longrightarrow & U_{r,m,r^*} & \longrightarrow & W^\vee \otimes X + (X^*)^\vee \otimes W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \ell_{X^*}^\vee \otimes \ell_X} & & \downarrow & & \downarrow{\scriptstyle \ell_X + \ell_{X^*}^\vee} & & \\ 0 & \longrightarrow & k & \longrightarrow & \mathrm{H}(W^\vee + W) & \longrightarrow & W^\vee + W & \longrightarrow & 0, \end{array}$$

which is equivariant under the action of $U_X \times U_{X^*} \times \mathrm{GL}(W)$, where $\mathrm{H}(W^\vee + W) = k + W^\vee + W$ is the Heisenberg group of $W^\vee + W$ whose multiplication is given by

$$(t_1, w_1^\vee, w_1)(t_2, w_2^\vee, w_2) = \left(t_1 + t_2 + \frac{w_1^\vee w_2 - w_2^\vee w_1}{2}, w_1^\vee + w_2^\vee, w_1 + w_2\right).$$

Given a nontrivial character $\psi\colon k \to \mathbb{C}^\times$, there is a unique infinite dimensional irreducible smooth representation ω_ψ of $\mathrm{H}(W^\vee + W)$ with central character ψ, which may be realized on $\mathcal{S}(W^\vee)$, the space of Schwartz functions on $W^\vee$, in the following way. For $\Phi \in \mathcal{S}(W^\vee)$, put

$$\left(\omega_\psi(t, w^\vee, w)\Phi\right)(w^\flat) = \psi\left(t + w^\flat w + \frac{w^\vee w}{2}\right)\Phi(w^\flat + w^\vee)$$

for all $(t, w^\vee, w) \in \mathrm{H}(W^\vee + W)$. Moreover, if we choose a character $\mu\colon k^\times \to \mathbb{C}^\times$, we have a Weil representation ω_μ of $\mathrm{GL}(W)$ on $\mathcal{S}(W^\vee)$ by

$$\left(\omega_\mu(g)\Phi\right)(w^\flat) = \mu(\det g)|\det g|_k^{\frac{1}{2}}\Phi(w^\flat \cdot g),$$

where $g \in \mathrm{GL}(W)$ acts on $W^\vee$ by $(w^\flat \cdot g)w = w^\flat(g \cdot w)$ for all $w \in W$. The two representations ω_ψ and ω_μ together form a representation $\omega_{\psi,\mu}$ of $U_{r,m,r^*} \rtimes \mathrm{GL}(W)$ through the projection $U_{r,m,r^*} \to \mathrm{H}(W^\vee + W)$, and hence a representation of $H := U_{r,m,r^*} \rtimes (U_X \times U_{X^*} \times \mathrm{GL}(W))$ by extending trivially to $U_X \times U_{X^*}$. Choose a generic character $\lambda\colon U_X \times U_{X^*} \to \mathbb{C}^\times$, and define the representation

$$\nu = \nu(\mu, \psi, \lambda) = \omega_{\psi,\mu} \otimes \lambda \otimes \delta_W^{-\frac{1}{2}}$$

of H, which has the Gelfand–Kirillov dimension m. We also define

$$\overline{\nu} = \widetilde{\nu}\delta_W^{-1} = \nu(\mu^{-1}, \psi^{-1}, \lambda^{-1}) = \nu(\mu^{-1}, \overline{\psi}, \overline{\lambda}).$$

As in the Bessel model, we have an embedding $(\varepsilon, \kappa)\colon H \to \mathrm{GL}(V) \times \mathrm{GL}(W)$. Then the pair (H, ν) is uniquely determined up to conjugacy in the group $\mathrm{GL}(V) \times \mathrm{GL}(W)$ by the pair $W \subset V$, (r, r^*) and μ. We have the following theorem.

Theorem 3.1. *Let k be of characteristic* 0. *Let π (resp. σ) be an irreducible admissible representation of* $\mathrm{GL}(V)$ *(resp.* $\mathrm{GL}(W)$*).*

(1) If π and σ are generic, $\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \sigma \otimes \widetilde{\nu}, \mathbb{C}) \geq 1$.
(2) If $r = r^*$, $\mathrm{Hom}_H(\pi \otimes \sigma \otimes \widetilde{\nu}, \mathbb{C}) \leq 1$.

It is naturally expected that (2) is true for any r, r^*. For the relative trace formula we are going to consider, r is equal to r^*.

Proof. Part (1) is due to Corollary 6.2(2). Part (2) is proved in [33] (for k non-archimedean) and [34] (for k archimedean) when $r = 0$. The case for general r is reduced to the previous one, as shown in [29]. □

Definition 3.2. (Fourier–Jacobi model). A nontrivial element in the space $\mathrm{Hom}(\pi \otimes \sigma \otimes \widetilde{\nu}, \mathbb{C})$ is called an (r, r^*)-*Fourier–Jacobi model* of $\pi \otimes \sigma$. When $r = r^*$, hence $n - m$ is even, it is simply the one defined in [8].

3.2. Fourier–Jacobi integrals, functional equations and L-functions

Let k be a number field, (H, ν) the pair associated with a nontrivial character $\psi\colon k\backslash\mathbb{A} \to \mathbb{C}^\times$ and a character $\mu\colon k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. Let λ be a generic character $(U_X \times U_{X^*})(k)\backslash(U_X \times U_{X^*})(\mathbb{A}) \to \mathbb{C}^\times$. We have the representation ν that realizes on the space $\mathcal{S}(W^\vee(\mathbb{A}))$.

For $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$, define the theta series as

$$\theta_\nu(h, \Phi) = \theta_{\nu(\mu,\psi,\lambda)}(h, \Phi) = \sum_{w^\flat \in W^\vee(k)} \lambda(h)\,(\nu(h)\Phi)\,(w^\flat),$$

which is an automorphic form of H. Let π (resp. σ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}(V)(\mathbb{A})$ (resp. $\mathrm{GL}(W)(\mathbb{A})$).

Definition 3.3. (Fourier–Jacobi integral and Fourier–Jacobi period). Assume $n > m$. For $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$, the following integral, with a parameter $s \in \mathbb{C}$:

$$\mathcal{FJ}^\nu_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi) = \int_{H(k)\backslash H(\mathbb{A})} \varphi_\pi(\varepsilon(h))\varphi_\sigma(\kappa(h))\theta_{\widetilde{\nu}}(h, \Phi)|\det h|_{\mathbb{A}}^{s-\frac{1}{2}+r^*-r}\,\mathrm{d}h$$

which is absolutely convergent, is an (r, r^*)-*Fourier–Jacobi integral* of $\pi \otimes \sigma$. When $s = \frac{1}{2}$,

$$\mathcal{FJ}^\nu_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi) = \mathcal{FJ}^\nu_{r,r^*}\left(\frac{1}{2}; \varphi_\pi, \varphi_\sigma; \Phi\right)$$

is an (r, r^*)-*Fourier–Jacobi period* of $\pi \otimes \sigma$ (for a pair (H, ν)). If there exist $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$ such that $\mathcal{FJ}^\nu_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial (r, r^*)-Fourier–Jacobi period. The case $m = n$ will be discussed in Remark 3.5.

It is obvious that $\mathcal{FJ}^\nu_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi)$ defines an element in

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \sigma \otimes \widetilde{\nu}, \mathbb{C}) = \bigotimes_{v \in \mathcal{M}_k} \mathrm{Hom}_{H_v}(\pi_v \otimes \sigma_v \otimes \widetilde{\nu_v}, \mathbb{C}).$$

We now show that the Fourier–Jacobi period is Eulerian. Choose a basis $\{v_1, \ldots, v_r\}$ (resp. $\{v_1^*, \ldots, v_{r^*}^*\}$) for X (resp. X^*) in the same way as in Sect. 2.2 (with $c = 1$). We also choose a basis $\{w_1, \ldots, w_m\}$ of W with dual basis $\{w_1^\vee, \ldots, w_m^\vee\}$ of $W^\vee$. We identify $\mathrm{GL}(V)$ (resp. $\mathrm{GL}(W)$) with $\mathrm{GL}_{n,k}$ (resp. $\mathrm{GL}_{m.k}$) under the basis

$$\left\{w_1, \ldots, w_m, v_1, \ldots, v_r, v_{r^*}^*, \ldots, v_1^*\right\}. \tag{3.1}$$

and view $\mathrm{GL}_{m,k}$ as a subgroup of $\mathrm{GL}_{n,k}$ (through the first m coordinates).

Theorem 3.4. *Assume $n > m$. the Fourier–Jacobi integrals are holomorphic in s and satisfy the following functional equation*

$$\mathcal{FJ}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \mathcal{FJ}^{\overline{\nu}}_{r^*,r}(1-s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}; \widehat{\Phi}),$$

where the Fourier transform $\widehat{\Phi}$ is explicated as (3.9) in the proof below. Put $\widetilde{W^{\psi}_{\bullet}}(g) = W^{\psi}_{\bullet}(\mathsf{w}_n g^{\iota}) \in \mathcal{W}(\widetilde{\pi}, \overline{\psi})$ (resp. $\widetilde{W^{\overline{\psi}}_{\bullet}}(g) = W^{\overline{\psi}}_{\bullet}(\mathsf{w}_m g^{\iota}) \in \mathcal{W}(\widetilde{\sigma}, \psi)$). If $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$ (resp. $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W_v^-$, $\Phi = \otimes_v \Phi_v$) is factorizable, then $\widetilde{W^{\psi}_{\varphi_\pi}} = \otimes_v \widetilde{W_v}$ (resp. $\widetilde{W^{\overline{\psi}}_{\varphi_\sigma}} = \otimes_v \widetilde{W_v^-}$, $\widehat{\Phi} = \otimes_v \widehat{\Phi}_v$) is also factorizable with $\widetilde{W_v}(g) = W_v(\mathsf{w}_n g^{\iota})$ (resp. $\widetilde{W_v^-}(g) = W_v^-(\mathsf{w}_m g^{\iota})$, $\widehat{\Phi}_v = \widehat{\Phi_v}$). In this case, for $\mathrm{Re}\, s \gg 0$,

$$\begin{aligned}\mathcal{FJ}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi) &= \Psi_r(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma} \otimes \mu^{-1}; \Phi)\\ &= \prod_{v \in \mathcal{M}_k} \Psi_{v,r}(s; W_v, W_v^- \otimes \mu_v^{-1}; \Phi_v),\end{aligned}$$

$$\begin{aligned}\mathcal{FJ}^{\overline{\nu}}_{r^*,r}(s; \rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}, \widetilde{\varphi_\sigma}; \widehat{\Phi}) &= \Psi_{r^*}(s; \rho(\mathsf{w}_{n,m})\widetilde{W^{\psi}_{\varphi_\pi}}, \widetilde{W^{\overline{\psi}}_{\varphi_\sigma}} \otimes \mu; \widehat{\Phi})\\ &= \prod_{v \in \mathcal{M}_k} \Psi_{v,r^*}(s; \rho(\mathsf{w}_{n,m})\widetilde{W_v}, \widetilde{W_v^-} \otimes \mu_v; \widehat{\Phi_v}).\end{aligned}$$

We refer readers to Sect. 5.4 for Whittaker functions and $\Psi_{v,r}(s; W_v, W_v^-; \Phi_v)$ (6.2), (6.3).

Proof. Under the basis (3.1), the image of $H(\mathbb{A})$ in $\mathrm{GL}_n(\mathbb{A})$ consists of matrices of the following form

$$h = h(n, n^*, b; u, u^*; g) = \left[\begin{array}{ccc|c|ccc} & g & & & n^*_{1,r^*} & \cdots & n^*_{1,1}\\ & & & & \vdots & & \vdots\\ & & & & n^*_{m,r^*} & \cdots & n^*_{m,1}\\ \hline n_{1,1} & \cdots & n_{1,m} & & & &\\ \vdots & & \vdots & u & & b &\\ n_{r,1} & \cdots & n_{r,m} & & & &\\ \hline & & & & & u^* & \end{array}\right], \tag{3.2}$$

where n, n^*, b, u, u^* and g are similar to those in Sect. 2.2, but without entries related to w_0. Put $U_{1^r,m,1^{r^*}} = U_{r,m,r^*} \rtimes (U_X \times U_{X^*})$ which is the unipotent radical of H. Precisely,

$$U_{1^r,m,1^{r^*}}(\mathbb{A}) = \left\{\underline{u} = \underline{u}(n, n^*, b; u, u^*) := h(n, n^*, b; u, u^*; \mathbf{1}_m)\right\}.$$

For $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$, we have

$$\left(\nu(\underline{u})\Phi\right)(w^\flat) = \underline{\psi}(\underline{u})\psi\left(w^{\flat t}[n^*_{1,r^*}, \ldots, n^*_{m,r^*}]\right)\Phi(w^\flat + [n_{r,1}, \ldots, n_{r,m}]),$$

where $\underline{\psi}(\underline{u}) = \psi(u_{1,2} + \cdots + u_{r-1,r} + b_{r,r^*} + u^*_{r^*,r^*-1} + \cdots + u^*_{2,1})$. Then

$$\begin{aligned}
&\mathcal{F}\mathcal{J}^\nu_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi) \\
&= \int\limits_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int\limits_{U_{1^r,m,1^{r^*}}(k)\backslash U_{1^r,m,1^{r^*}}(\mathbb{A})} \varphi_\pi(\underline{u}g)\varphi_\sigma(g)\theta_{\tilde{\nu}}(\underline{u}g, \Phi)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}} \mathrm{d}\underline{u}\mathrm{d}g.
\end{aligned} \tag{3.3}$$

There are two cases.

Case 1: $r > 0$. We have

$$\begin{aligned}
\theta_\nu(\underline{u}g, \Phi) &= \sum_{w^\flat \in W^\vee(k)} \lambda(\underline{u})\left(\omega_{\psi,\mu}(\underline{u}g)\Phi\right)(w^\flat)|\det g|_{\mathbb{A}}^{\frac{r-r^*}{2}} \\
&= \sum_{n_r \in \mathrm{Mat}_{1,m}(k)} \lambda(\underline{u})\left(\omega_{\psi,\mu}(\underline{n}_r\underline{u}g)\Phi\right)(0)|\det g|_{\mathbb{A}}^{\frac{r-r^*}{2}},
\end{aligned}$$

where $\underline{n}_r = \underline{u}(n_r, 0, 0; \mathbf{1}_r, \mathbf{1}_{r^*})$, and

$$n_r = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ n_{r,1} & \cdots & n_{r,m} \end{bmatrix}.$$

Let $L_{r+1} = \left\{\underline{u}(0, n^*, b; u, u^*)\right\}$ be a subgroup of $U_{1^r,m,1^{r^*}}$. For $1 \le i \le r$, put

$$L_i = \left\{\underline{l} = \underline{l}(l_i; 0, n^*, b; u, u^*) \mid l_i = \begin{bmatrix} l_{1,i} & \cdots & l_{1,r} \\ \vdots & & \vdots \\ l_{m,i} & \cdots & l_{m,r} \end{bmatrix}\right\},$$

where $\underline{l}(l_i; 0, n^*, b; u, u^*)$ is the one obtained from $\underline{u}(0, n^*, b; u, u^*)$ by adding l_i above the entries $[u_{1,i}, \ldots, u_{1,r}]$ as in (3.2). It is clear that for $1 \le i \le r$, L_i/L_{i+1} is isomorphic to $\mathrm{Mat}_{m,1,k}$, which may be identified with the group of columns ${}^t[l_{i,i}, \cdots, l_{m,i}]$. For $1 \le i \le r$, we regard $\underline{\psi}$ as a character of $L_i(\mathbb{A})$ via the quotient group $L_{r+1}(\mathbb{A})$.

Expand the theta series and summate the last row in n over $\mathrm{Mat}_{1,m}(k)$. We have

$$(3.3) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{Mat}_{r-1,m}(k)\backslash \mathrm{Mat}_{r-1,m}(\mathbb{A})\times \mathrm{Mat}_{1,m}(\mathbb{A})} \int_{L_{r+1}(k)\backslash L_{r+1}(\mathbb{A})} \varphi_\pi(\underline{l}\,\underline{n}g)\varphi_\sigma(g)\Phi(\underline{n}_r g)\overline{\psi(\underline{l})}\mu(\det g)^{-1}|\det g|_{\mathbb{A}}^{s+\frac{r-r^*}{2}}\,\mathrm{d}\underline{l}\,\mathrm{d}n\,\mathrm{d}g, \tag{3.4}$$

where n_r only remembers the last row of n. If we repeat the process (2.6), (2.7), (2.8) and (2.9), then

$$(3.4) = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{Mat}_{r,m}(\mathbb{A})} \times \left(\int_{L_1(k)\backslash L_1(\mathbb{A})} \left(\rho(\underline{n})\varphi_\pi\right)(\underline{l}g)\overline{\psi(\underline{l})}\mathrm{d}\underline{l} \right) \varphi_\sigma(g)\Phi(\underline{n}_r)\mu(\det g)^{-1}|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\,\mathrm{d}n\,\mathrm{d}g. \tag{3.5}$$

We will shortly see that when $\mathrm{Re}\, s \gg 0$, the above integral is absolutely convergent after integrating over $L_1(k)\backslash L_1(\mathbb{A})$. First formally interchange the order of g and n. By the classical argument for the Rankin–Selberg convolution, we have that for $\mathrm{Re}\, s \gg 0$

$$(3.5) = \Psi_r(s; W_{\varphi_\pi}^{\psi}, W_{\varphi_\sigma}^{\overline{\psi}} \otimes \mu^{-1}; \Phi) := \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{Mat}_{r-1,m}(\mathbb{A})} \int_{\mathrm{Mat}_{1,m}(\mathbb{A})} W_{\varphi_\pi}^{\psi}\left(\begin{bmatrix} g & 0 & 0 & 0 \\ x & \mathbf{1}_{r-1} & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{r^*} \end{bmatrix}\right) W_{\varphi_\sigma}^{\overline{\psi}}(g)\Phi(y)\mu(\det g)^{-1}|\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}g,$$

which is absolutely convergent and hence our calculation is valid. If $W_{\varphi_\pi}^{\psi} = \otimes_v W_v$, $W_{\varphi_\sigma}^{\overline{\psi}} = \otimes_v W_v^-$ and $\Phi = \otimes_v \Phi_v$ are factorizable, we have

$$\begin{aligned} \mathcal{FJ}_{r,r^*}^{\nu}(s; \varphi_\pi, \varphi_\sigma; \Phi) &= \Psi_r(s; W_{\varphi_\pi}^{\psi}, W_{\varphi_\sigma}^{\overline{\psi}} \otimes \mu^{-1}; \Phi) \\ &= \prod_{v \in \mathscr{M}_k} \Psi_{v,r}(s; W_v, W_v^- \otimes \mu_v^{-1}; \Phi_v). \end{aligned}$$

where

$$\Psi_{v,r}(s; W_v, W_v^- \otimes \mu_v^{-1}; \Phi_v) = \int_{U_{1^m,v}\backslash \mathrm{GL}_{m,v}} \int_{\mathrm{Mat}_{r-1,m,v}} \int_{\mathrm{Mat}_{1,m,v}} W_v\left(\begin{bmatrix} g_v & 0 & 0 & 0 \\ x_v & \mathbf{1}_{r-1} & 0 & 0 \\ y_v & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{r^*} \end{bmatrix}\right) W_v^-(g_v)\Phi_v(y_v)\mu_v(\det g_v)^{-1}|\det g_v|_v^{s-\frac{n-m}{2}}\,\mathrm{d}y_v\,\mathrm{d}x_v\,\mathrm{d}g_v.$$

Case 2: $r = 0$ **but** $r^* > 0$**.** Let P_m be the standard mirabolic subgroup of GL_m consisting of (invertible) matrices whose last row is $e_m = [0, \ldots, 0, 1] \in \mathrm{Mat}_{1,m}(k)$. Then

$$\begin{aligned}\theta_\nu(g\underline{u}, \Phi) &= \sum_{w^\flat \in W^\vee(k)} \lambda(\underline{u}) \left(\omega_{\psi,\mu}(g\underline{u})\Phi\right)(w^\flat)|\det g|_{\mathbb{A}}^{\frac{m-n}{2}} \\ &= \lambda(\underline{u})\mu(\det g)|\det g|_{\mathbb{A}}^{\frac{1}{2}}\Phi(0)|\det g|_{\mathbb{A}}^{\frac{m-n}{2}} \\ &\quad + \mu(\det g)|\det g|_{\mathbb{A}}^{\frac{1}{2}} \sum_{\gamma \in P_m(k)\backslash \mathrm{GL}_m(k)} \lambda(\underline{u}) \left(\omega_{\mu,\psi}(\underline{u})\Phi\right)(e_m\gamma g)|\det g|_{\mathbb{A}}^{\frac{m-n}{2}},\end{aligned}$$

where the first term (for $\theta_{\overline{\nu}}$) contributes 0 to $\mathcal{F}\mathcal{J}^{\nu}_{0,n-m}(s; \varphi_\pi, \varphi_\sigma; \Phi)$ since φ_π is a cusp form. Therefore,

$$\mathcal{F}\mathcal{J}^{\nu}_{0,n-m}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \int_{U_{m,1^{n-m}}(k)\backslash U_{m,1^{n-m}}(\mathbb{A})} \int_{P_m(k)\backslash \mathrm{GL}_m(\mathbb{A})}$$

$$\varphi_\pi(g\underline{u})\varphi_\sigma(g)\Phi(e_m g)\overline{\psi(e_m g n^* e_m^*)}\ \overline{\underline{\psi}(\underline{u})}\mu(\det g)^{-1}|\det g|_{\mathbb{A}}^{s+\frac{n-m}{2}}\,\mathrm{d}g\mathrm{d}\underline{u}, \quad (3.6)$$

where $\underline{u} = \underline{u}(\emptyset, n^*, \emptyset; \emptyset, u^*)$, since $r = 0$ and $e_m^* = {}^t[1, 0, \ldots, 0]$. Applying the Fourier inversion formula to φ_σ, we have for $\mathrm{Re}\, s \gg 0$,

$$(3.6) = \int_{U_{m,1^{n-m}}(k)\backslash U_{m,1^{n-m}}(\mathbb{A})} \int_{U_{1^m}(k)\backslash \mathrm{GL}_m(\mathbb{A})}$$

$$\varphi_\pi(g\underline{u})W^{\overline{\psi}}_{\varphi_\sigma}(g)\Phi(e_m g)\overline{\psi(e_m g n^* e_m^*)}\ \overline{\underline{\psi}(\underline{u})}\mu(\det g)^{-1}|\det g|_{\mathbb{A}}^{s+\frac{n-m}{2}}\,\mathrm{d}g\mathrm{d}\underline{u}. \quad (3.7)$$

Factorizing the inner integral through $U_{1^m}(k)\backslash U_{1^m}(\mathbb{A})$ and incorporating this unipotent part into $\underline{u}$, we get the integral over $U_{1^n}(k)\backslash U_{1^n}(\mathbb{A})$, where U_{1^n} is the standard maximal unipotent subgroup of GL_n. Moreover, if we interchange the order of g and $u \in U_{1^n}(k)\backslash U_{1^n}(\mathbb{A})$ when $\mathrm{Re}\, s \gg 0$, all terms involving ψ will form a generic character $\underline{\psi}$ of $U_{1^n}(k)\backslash U_{1^n}(\mathbb{A})$ as

$$\underline{\psi}(u) = \psi\left(u_{1,2} + \cdots + u_{n-1,n}\right).$$

In all, for $\mathrm{Re}\, s \gg 0$,

$$\begin{aligned}(3.7) &= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^n}(k)\backslash U_{1^n}(\mathbb{A})} \varphi_\pi(ug)\overline{\underline{\psi}(u)}W^{\overline{\psi}}_{\varphi_\sigma}(g)\Phi(e_m g)\mu(\det g)^{-1} \\ &\qquad |\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\,\mathrm{d}u\mathrm{d}g \\ &= \int_{U_{1^m}(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W^{\psi}_{\varphi_\pi}\left(\begin{bmatrix} g & 0 \\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right) W^{\overline{\psi}}_{\varphi_\sigma}(g)\Phi(e_m g)\mu(\det g)^{-1} \\ &\qquad |\det g|_{\mathbb{A}}^{s-\frac{n-m}{2}}\,\mathrm{d}g. \qquad (3.8)\end{aligned}$$

We denote (3.8) by $\Psi_0(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma} \otimes \mu^{-1}; \Phi)$, which is absolutely convergent when $\operatorname{Re} s \gg 0$. Moreover, if $W^{\psi}_{\varphi_\pi} = \otimes_v W_v$, $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$ and $\Phi = \otimes_v \Phi_v$ are factorizable, we have

$$\begin{aligned}\mathcal{FJ}^{\nu}_{0,n-m}(s; \varphi_\pi, \varphi_\sigma; \Phi) &= \Psi_0(s; W^{\psi}_{\varphi_\pi}, W^{\overline{\psi}}_{\varphi_\sigma} \otimes \mu^{-1}; \Phi) \\ &= \prod_{v \in \mathscr{M}_k} \Psi_{v,0}(s; W_v, W^-_v \otimes \mu_v^{-1}; \Phi_v),\end{aligned}$$

where

$$\begin{aligned}&\Psi_{v,0}(s; W_v, W^-_v \otimes \mu_v^{-1}; \Phi_v) \\ &= \int_{U_{1^m,v} \backslash \mathrm{GL}_{m,v}} W_v\left(\begin{bmatrix} g_v & 0 \\ 0 & \mathbf{1}_{n-m}\end{bmatrix}\right) W^-_v(g_v)\Phi_v(e_m g_v)\mu_v(\det g_v)^{-1}|\det g_v|_v^{s-\frac{n-m}{2}}\,\mathrm{d}g_v.\end{aligned}$$

Now we discuss the functional equations of the (r, r^*)-*Fourier–Jacobi integrals* $\mathcal{FJ}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi)$. There is a linear map $\widehat{\ }: \mathcal{S}(W^\vee(\mathbb{A})) \to \mathcal{S}(W(\mathbb{A}))$ given by

$$\widehat{\Phi}(w^\sharp) = \int_{W^\vee(\mathbb{A})} \Phi(w^\flat)\psi\left(w^\flat(w^\sharp)\right)\mathrm{d}w^\flat. \tag{3.9}$$

If we identify W with $W^\vee$ through the basis $\{w_1, \ldots, w_m\}$, then $\widehat{\ }$ is an endomorphism of $\mathcal{S}(W^\vee(\mathbb{A}))$. Consider the group isomorphism $\iota_{r^*,r}: H \to H$ given by $\iota_{r^*,r}(\underline{u}g) = \mathsf{w}_{n,m}\underline{u}^\iota \mathsf{w}_{n,m} g^\iota$. For $h \in H(\mathbb{A})$, we have the following commutative diagram, which can be checked directly

$$\begin{array}{ccc} \mathcal{S}(W^\vee(\mathbb{A})) & \xrightarrow{\ \widehat{\ }\ } & \mathcal{S}(W^\vee(\mathbb{A})) \\ \Big\downarrow {\scriptstyle \overline{\lambda}\cdot\omega_{\overline{\psi},\mu^{-1}}(\iota_{r^*,r}(h))} & & \Big\downarrow {\scriptstyle \lambda\cdot\omega_{\psi,\mu}(h)} \\ \mathcal{S}(W^\vee(\mathbb{A})) & \xrightarrow{\ \widehat{\ }\ } & \mathcal{S}(W^\vee(\mathbb{A})). \end{array} \tag{3.10}$$

Then

$$\begin{aligned}&\mathcal{FJ}^{\nu}_{r,r^*}(s; \varphi_\pi, \varphi_\sigma; \Phi) \\ &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^{r^*},m,1^r}(k)\backslash U_{1^{r^*},m,1^r}(\mathbb{A})} \varphi_\pi(\underline{u}g^\iota)\varphi_\sigma(g^\iota)\theta_{\overline{\nu}}(\underline{u}g^\iota, \Phi)|\det g|_{\mathbb{A}}^{-s+\frac{1}{2}}\,\mathrm{d}\underline{u}\mathrm{d}g \\ &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^{r^*},m,1^r}(k)\backslash U_{1^{r^*},m,1^r}(\mathbb{A})} \\ &\left(\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}\right)(\mathsf{w}_{n,m}\underline{u}^\iota \mathsf{w}_{n,m} g)\widetilde{\varphi_\sigma}(g)\theta_{\overline{\nu}}(\underline{u}g^\iota, \Phi)|\det g|_{\mathbb{A}}^{-s+\frac{1}{2}}\,\mathrm{d}\underline{u}\mathrm{d}g,\end{aligned}$$

which equals

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^{r*},m,1^r}(k)\backslash U_{1^{r*},m,1^r}(\mathbb{A})} \left(\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}\right)(\underline{u}g)\widetilde{\varphi_\sigma}(g)\theta_{\widetilde{\nu}}\left(\iota_{r^*,r}(\underline{u}g),\Phi\right) |\det g|_{\mathbb{A}}^{-s+\frac{1}{2}}\mathrm{d}\underline{u}\mathrm{d}g. \tag{3.11}$$

But by the Poisson summation formula and (3.10),

$$\begin{aligned}\theta_{\widetilde{\nu}}\left(\iota_{r^*,r}(\underline{u}g),\Phi\right) &= \sum_{w^\flat\in W^\vee(k)} \overline{\lambda(\underline{u})}\left(\omega_{\overline{\psi},\mu^{-1}}\left(\iota_{r^*,r}(\underline{u}g)\right)\Phi\right)(w^\flat)|\det g^\iota|_{\mathbb{A}}^{\frac{r^*-r}{2}}\\ &= \sum_{w^\sharp\in W^\vee(k)} \overline{\lambda(\underline{u})}\left(\omega_{\overline{\psi},\mu^{-1}}\left(\iota_{r^*,r}(\underline{u}g)\right)\Phi\right)^\wedge(w^\sharp)|\det g^\iota|_{\mathbb{A}}^{\frac{r^*-r}{2}}\\ &= \sum_{w^\sharp\in W^\vee(k)} \lambda(\underline{u})\left(\omega_{\psi,\mu}(\underline{u}g)\widehat{\Phi}\right)(w^\sharp)|\det g|_{\mathbb{A}}^{\frac{r-r^*}{2}}\\ &= \theta_{\widetilde{\overline{\nu}}}(\underline{u}g,\widehat{\Phi}).\end{aligned}$$

Therefore,

$$\begin{aligned}(3.11) &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \int_{U_{1^{r*},m,1^r}(k)\backslash U_{1^{r*},m,1^r}(\mathbb{A})} \left(\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi}\right)(\underline{u}g)\widetilde{\varphi_\sigma}(g)\theta_{\widetilde{\overline{\nu}}}(\underline{u}g,\widehat{\Phi}) |\det g|_{\mathbb{A}}^{1-s-\frac{1}{2}}\mathrm{d}\underline{u}\mathrm{d}g\\ &= \mathcal{F}\mathcal{J}^{\overline{\nu}}_{r^*,r}(1-s;\rho(\mathsf{w}_{n,m})\widetilde{\varphi_\pi},\widetilde{\varphi_\sigma};\widehat{\Phi}).\end{aligned}$$

The theorem follows. □

Remark 3.5. (The case $n=m$). We have $V=W$ and $H=\mathrm{GL}_n$. We will see that this is exactly the case of Rankin–Selberg convolution for $\mathrm{GL}_n\times\mathrm{GL}_n$. For simplicity, we assume that $\pi\boxtimes\sigma\otimes\mu^{-1}$ is unitary. Fix a basis $\{v_1,\dots,v_n\}$ for V and identify $V^\vee$ with $\mathrm{Mat}_{1,n,k}$. In this case, there is no need to choose of λ.

For $\Phi\in\mathcal{S}(V^\vee(\mathbb{A}))$, and a character $\chi: k^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$ such that $\mu\cdot\chi$ is unitary, we define

$$\theta^*_\nu(s;g,\Phi,\chi) = |\det g|_{\mathbb{A}}^{s-\frac{1}{2}} \int_{k^\times\backslash\mathbb{A}^\times} \sum_{v^\flat\in\mathrm{Mat}_{1,n}(k)-\{0\}} \left(\omega_{\psi,\mu}(ag)\Phi\right)(v^\flat)|a|_{\mathbb{A}}^{n\left(s-\frac{1}{2}\right)}\chi(a)\mathrm{d}a,$$

which is absolutely convergent when $\mathrm{Re}\, s>1$ and has a meromorphic continuation to the entire complex plane that is holomorphic at $s=\frac{1}{2}$ (see [23]). For a holomorphic point s, $\theta^*_\nu(s;g,\Phi,\chi)$ is in $\mathcal{A}\left(\mathrm{GL}_{n,k}\right)$ with the central character χ^{-1}. Moreover,

$$\theta^*_\nu\left(\frac{1}{2};g,-,\chi\right):\ \left(\nu,\mathcal{S}(V^\vee(\mathbb{A}))\right)\to\mathcal{A}\left(\mathrm{GL}_{n,k}\right)$$

is $\mathrm{GL}_n(\mathbb{A})$-equivariant. We denote Z_n the center of GL_n, χ_π (resp. χ_σ) the central character of π (resp. σ), and put

$$\mathfrak{FJ}^{\nu}_{0,0}(s;\varphi_\pi,\varphi_\sigma;\Phi)=\int\limits_{Z_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\varphi_\pi(g)\varphi_\sigma(g)\theta^*_\nu(s;g,\Phi,\chi_\pi\cdot\chi_\sigma)\mathrm{d}g,\tag{3.12}$$

which is the usual Rankin–Selberg integral (see [21,23]). It is absolutely convergent when $\mathrm{Re}\, s>1$. The *Fourier–Jacobi period*

$$\mathfrak{FJ}^{\nu}_{0,0}(\varphi_\pi,\varphi_\sigma;\Phi):=\mathfrak{FJ}^{\nu}_{0,0}\left(\frac{1}{2};\varphi_\pi,\varphi_\sigma;\Phi\right)$$

defines an element in

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})}(\pi\otimes\sigma\otimes\widetilde{\nu_\mu},\mathbb{C})=\bigotimes_{v\in\mathscr{M}_k}\mathrm{Hom}_{H_v}(\pi_v\otimes\sigma_v\otimes\widetilde{\nu_v},\mathbb{C}).$$

Unfolding $\theta^*_{\overline{\nu}}$, we see that

$$\begin{aligned}(3.12)&=\int\limits_{P_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\varphi_\pi(g)\varphi_\sigma(g)\Phi(e_ng)\mu(\det g)^{-1}|\det g|^s_{\mathbb{A}}\mathrm{d}g\\&=\int\limits_{U_{1^n}(\mathbb{A})\backslash\mathrm{GL}_n(\mathbb{A})}W^{\psi}_{\varphi_\pi}(g)W^{\overline{\psi}}_{\varphi_\sigma}(g)\Phi(e_ng)\mu(\det g)^{-1}|\det g|^s_{\mathbb{A}}\mathrm{d}g.\end{aligned}\tag{3.13}$$

Denote (3.13) by $\Psi_0(s;W^{\psi}_{\varphi_\pi},W^{\overline{\psi}}_{\varphi_\sigma}\otimes\mu^{-1};\Phi)$, which is absolutely convergent when $\mathrm{Re}\, s\gg 0$. Moreover, if $W^{\psi}_{\varphi_\pi}=\otimes_v W_v$, $W^{\overline{\psi}}_{\varphi_\sigma}=\otimes_v W^-_v$ and $\Phi=\otimes_v\Phi_v$ are factorizable, we have

$$\begin{aligned}&\mathfrak{FJ}^{\nu}_{0,0}(s;\varphi_\pi,\varphi_\sigma;\Phi)\\&\quad=\Psi_0(s;W^{\psi}_{\varphi_\pi},W^{\overline{\psi}}_{\varphi_\sigma}\otimes\mu^{-1};\Phi)=\prod_{v\in\mathscr{M}_k}\Psi_{v,0}(s;W_v,W^-_v\otimes\mu_v^{-1};\Phi_v),\end{aligned}$$

where

$$\begin{aligned}&\Psi_{v,0}(s;W_v,W^-_v\otimes\mu_v^{-1};\Phi_v)\\&\quad=\int\limits_{U_{1^n,v}\backslash\mathrm{GL}_{n,v}}W_v(g_v)W^-_v(g_v)\Phi_v(e_ng_v)\mu_v(\det g_v)^{-1}|\det g_v|^s_v\mathrm{d}g_v.\end{aligned}$$

Moreover, we have the following well-known functional equation

$$\mathfrak{FJ}^{\nu}_{0,0}(s;\varphi_\pi,\varphi_\sigma;\Phi)=\mathfrak{FJ}^{\overline{\nu}}_{0,0}(1-s;\widetilde{\varphi_\pi},\widetilde{\varphi_\sigma};\widehat{\Phi}).$$

The function $\mathfrak{FJ}^{\nu}_{0,0}(s;\varphi_\pi,\varphi_\sigma;\Phi)$ will have possible simple poles at $s=-\mathrm{i}\sigma$ and $s=1-\mathrm{i}\sigma$ with σ real only if $\pi\cong\widetilde{\sigma}\otimes\mu|\det|^{\mathrm{i}\sigma}_{\mathbb{A}}$.

By Proposition 6.1 and (6.3), we have the following corollary, which confirms [8, Conjecture 24.1] for Fourier–Jacobi periods on split unitary groups, that is, general linear groups.

Corollary 3.6. *Let the notation be as above.*

(1) Let π (resp. σ) be an irreducible cuspidal automorphic representation[2] *of* $\mathrm{GL}(V)(\mathbb{A})$ *(resp.* $\mathrm{GL}(W)(\mathbb{A})$*). For a pair (r, r^*) such that $r+r^* = n-m$ and the representation $\nu = \nu(\mu, \psi, \lambda)$ introduced above, we have, for $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$ such that $W^\psi_{\varphi_\pi} = \otimes_v W_v$, $W^{\overline{\psi}}_{\varphi_\sigma} = \otimes_v W^-_v$ and $\Phi = \otimes \Phi_v$ are factorizable,*

$$\mathcal{FJ}^\nu_{r,r^*}(\varphi_\pi, \varphi_\sigma; \Phi) = L\left(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}\right) \times \prod_{v \in \mathcal{M}_k} \frac{\Psi_{v,r}(s; W_v, W^-_v \otimes \mu^{-1}_v; \Phi_v)}{L_v(s, \pi_v \times \sigma_v \otimes \mu^{-1}_v)}\Big|_{s=\frac{1}{2}},$$

where in the last product almost all factors are 1*, and the L-functions are the ones defined by Rankin–Selberg convolutions (see* [21]*).*

(2) There is a nontrivial Fourier–Jacobi period of $\pi \otimes \sigma$ for ν if and only if $L\left(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}\right) \neq 0$.

4. A relative trace formula for $\mathrm{U}_n \times \mathrm{U}_m$: Bessel periods

4.1. Bessel models and periods

Let us briefly recall the definition of Bessel models and periods for unitary groups in [8]. We will fix a (nondegenerate) hermitian line E over k. First, let us consider the local situation. Let k' be a local field and k/k' an étale algebra of degree 2 with the unique nontrivial involution τ. Let V, $(\ ,\ )$ be a hermitian space over k of dimension n and $W \subset V$ a subspace of dimension m such that the restricted hermitian form $(\ ,\ )|_W$ is nondegenerate. We assume that the orthogonal complement of W in V has the decomposition $W^\perp = X \oplus X^* \oplus E$ such that X, X^* are both r-dimensional isotropic subspaces orthogonal to E. Then $n = m + 2r + 1$. The hermitian form restricted on W (resp. $X \oplus X^*$) identifies W (resp. X^*) with $W^\vee_\tau$ (resp. $X^\vee_\tau$). We denote $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) the unitary group of V (resp. W) which is a reductive group over k'. Let $P'_{r,m+1}$ be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing X and $U'_{r,m+1}$ its unipotent radical. Then $U'_{r,m+1}$ fits into the following exact sequence

$$0 \longrightarrow \wedge^2_\tau X \longrightarrow U'_{r,m+1} \longrightarrow \mathrm{Hom}_k(W \oplus E, X) \longrightarrow 0,$$

where $\wedge^2_\tau X \subset X_\tau \otimes X = \mathrm{Hom}_k(X^\vee_\tau, X)$ consists of homomorphisms b such that $b^\vee_\tau = -b$. Here, $b^\vee_\tau$ is simply $b^\vee\colon X^\vee \to X_\tau$ but viewed as an element in $\mathrm{Hom}_k(X^\vee_\tau, X)$.

[2] When $n = m$, to prevent the occurrence of a pole at $s = \frac{1}{2}$, we assume that the character $\chi_\pi \otimes \chi_\sigma \otimes \mu^{-1}$ is unitary for simplicity.

Let $\ell'_X: X \to k$ be a nontrivial k-linear homomorphism (if exists), and U'_X a maximal unipotent subgroup of $\mathrm{GL}(X)$ stabilizing ℓ'_X. Let $\ell'_W: k \to W \oplus E$ be a nontrivial k-linear homomorphism whose image is contained in E. We have a homomorphism

$$\ell': U'_{r,m+1} \to \mathrm{Hom}_k(W \oplus E, X) \xrightarrow{(\ell'_W)^\vee \otimes \ell'_X} k,$$

which is fixed by $U'_X \times \mathrm{U}(W)$. Thus we may extend ℓ' trivially to $H = U'_{r,m+1} \rtimes \big(U'_X \times \mathrm{U}(W)\big)$, which is a homomorphism to k. Let $\psi': k' \to \mathbb{C}^\times$ be a nontrivial character and $\lambda': U'_X \to \mathbb{C}^\times$ a generic character. Put $\nu' = (\psi' \circ \widetilde{\mathrm{Tr}} \circ \ell') \otimes \lambda'$ which is a character of H'. We have an embedding $(\varepsilon, \kappa): H' \to \mathrm{U}(V) \times \mathrm{U}(W)$. Up to $\mathrm{U}(V) \times \mathrm{U}(W)$-conjugacy, the pair (H', ν') is uniquely determined by $W \subset V$.

Let π (resp. σ) be an irreducible admissible representation of $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$). A nontrivial element in $\mathrm{Hom}_{H'}(\pi \otimes \sigma, \nu')$ is called a *Bessel model* of $\pi \otimes \sigma$. In particular, when k/k' is split, the Bessel model is simply the (r, r)-Bessel model for general linear groups introduced in Sect. 2.1. We have the following multiplicity one result.

Theorem 4.1. *Let k be of characteristic* 0 *and π, σ as above. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H'}(\pi \otimes \sigma, \nu') \le 1.$$

Proof. When $m = n - 1$, this is due to [34], or [2] when k is non-archimedean. For general (n, m), this is due to [27], or [8, Sect. 14] when k is non-archimedean. □

Now we discuss the global case. Let k/k' be a quadratic extension of number fields. We have the notions $\mathfrak{o}'$, $k'_{v'}$, $\mathfrak{o}'_{v'}$ for $v' \in \mathscr{M}_{k'}$. Let $\psi': k'\backslash\mathbb{A}' \to \mathbb{C}^\times$ be a nontrivial character and $\lambda': U'_X(k)\backslash U'_X(\mathbb{A}) \to \mathbb{C}^\times$ a generic character, which give rise to the pair (H', ν') similarly in the global situation.

Let π (resp. σ) be an irreducible representation of $\mathrm{U}(V)(\mathbb{A}')$ (resp. $\mathrm{U}(W)(\mathbb{A}')$) which occurs with multiplicity one in the space $\mathcal{A}_0(\mathrm{U}(V))$ (resp. $\mathcal{A}_0(\mathrm{U}(W))$). We denote by $\mathcal{A}_\pi \subset \mathcal{A}_0(\mathrm{U}(V))$ (resp. $\mathcal{A}_\sigma \subset \mathcal{A}_0(\mathrm{U}(W))$) the unique irreducible π (resp. σ)-isotypic subspace.

Definition 4.2. (*Bessel period*). For $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$, we define the following integral

$$\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma) = \int_{H'(k')\backslash H'(\mathbb{A}')} \varphi_\pi(\varepsilon(h'))\varphi_\sigma(\kappa(h'))\nu'(h')^{-1}\mathrm{d}h',$$

which is absolutely convergent, to be a *Bessel period* of $\pi \otimes \sigma$ (for a pair (H', ν')). If there exist $\varphi_\pi \in \mathcal{A}_\pi$, $\varphi_\sigma \in \mathcal{A}_\sigma$ such that $\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma) \ne 0$, then we say $\pi \otimes \sigma$ has a nontrivial Bessel period. It is obvious that $\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma)$ defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A})}(\pi \otimes \sigma, \nu') = \bigotimes_{v' \in \mathscr{M}_{k'}} \mathrm{Hom}_{H'_{v'}}(\pi_{v'} \otimes \sigma_{v'}, \nu'_{v'}).$$

We choose a basis $\{v_1, \dots, v_r\}$ of X under which

- the homomorphism $\ell'_X : X \to k$ is given by the coefficient of v_r;
- U'_X is the unipotent radical of the parabolic subgroup P'_X stabilizing the complete flag $0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_r \rangle = X$;
- the generic character λ' is given by

$$\lambda'(u') = \psi'\left(\widetilde{\mathrm{Tr}}\left(u'_{1,2} + u'_{2,3} + \cdots + u'_{r-1,r}\right)\right).$$

We denote by $\{\check{v}_1, \ldots, \check{v}_r\}$ the dual basis of $X_\tau^\vee$. We also choose a basis $\{w_1, \ldots, w_m\}$ of W and $\{w_0\}$ of E, under which the homomorphism $\ell'_W : k \to W \oplus E$ is given by $a \mapsto aw_0$. Put $\beta = [(w_i, w_j)]_{i,j=1}^m \in \mathrm{Her}_m^\times(k')$, $\beta_0 = (w_0, w_0) \in k'^\times$, and

$$\beta' = \begin{bmatrix} \beta & \\ & \beta_0 \end{bmatrix} \in \mathrm{Her}_{m+1}^\times(k'). \tag{4.1}$$

We identify $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) with a unitary group U_n (resp. U_m) of n (resp. m) variables under the basis $\{v_1, \ldots, v_r, w_1, \ldots, w_m, w_0, \check{v}_r, \ldots, \check{v}_1\}$, and view U_m as a subgroup of U_n. Let $U'_{1^r,m+1} = U'_{r,m+1} \rtimes U'_X$ be the unipotent radical of H'. The image of $H'(\mathbb{A}')$ in $\mathrm{U}_n(\mathbb{A}')$ consists of the matrices $h' = h'(n', b'; u'; g') = \underline{u}'(n', b'; u') \cdot g'$, where $g' \in \mathrm{U}_m(\mathbb{A}')$ and

$$\underline{u}' = \underline{u}'(n', b'; u') = \begin{bmatrix} \mathbf{1}_r & n' & \mathsf{w}_r\left(b' + \frac{n' n'_{\beta'}}{2}\right) \\ & \mathbf{1}_{m+1} & n'_{\beta'} \\ & & \mathbf{1}_r \end{bmatrix}$$
$$\times \begin{bmatrix} u' & & \\ & \mathbf{1}_{m+1} & \\ & & \check{u}' \end{bmatrix} \in U'_{1^r,m+1}(\mathbb{A}')$$

for $n' \in \mathrm{Mat}_{r,m+1}(\mathbb{A})$, $b' \in \overline{\mathrm{Her}}_r(\mathbb{A}')$, $u' \in U'_X(\mathbb{A})$; $n'_{\beta'} = -\beta'^{-1}\,{}^t n'^\tau \mathsf{w}_r$, and $\check{u}' = \mathsf{w}_r\,{}^t u'^{\tau,-1} \mathsf{w}_r$. The character ν' on $H'(\mathbb{A}')$ is given by

$$\nu'(h') = \nu'(h'(n', b'; u'; g')) = \underline{\psi}'(\underline{u}') := \psi'\left(\widetilde{\mathrm{Tr}}\left(u'_{1,2} + \cdots + u'_{r-1,r} + n'_{r,m+1}\right)\right),$$

and the Bessel period

$$\mathcal{B}_r^{\nu'}(\varphi_\pi, \varphi_\sigma) = \int_{\mathrm{U}_m(k')\backslash \mathrm{U}_m(\mathbb{A}')} \int_{U'_{1^r,m+1}(k')\backslash U'_{1^r,m+1}(\mathbb{A}')} \varphi_\pi(\underline{u}' g') \varphi_\sigma(g') \overline{\underline{\psi}'(\underline{u}')} \mathrm{d}\underline{u}' \mathrm{d}g'.$$

4.2. Decomposition of distributions

We describe the relative trace formula on unitary groups concerning Bessel periods. Let $f_n \in \mathcal{S}(\mathrm{U}_n(\mathbb{A}'))$ (resp. $f_m \in \mathcal{S}(\mathrm{U}_m(\mathbb{A}'))$) be a Schwartz function on $\mathrm{U}_n(\mathbb{A}')$ (resp. $\mathrm{U}_m(\mathbb{A}')$). We associate to $f_n \otimes f_m$ a kernel function on $(\mathrm{U}_n(k')\backslash \mathrm{U}_n(\mathbb{A}') \times \mathrm{U}_m(k')\backslash \mathrm{U}_m(\mathbb{A}'))^2$ as

$$\mathcal{K}_{f_n \otimes f_m}(g'_1, g'_2; g'_3, g'_4) = \sum_{\zeta' \in \mathrm{U}_n(k')} f_n(g'^{-1}_1 \zeta' g'_3) \sum_{\xi' \in \mathrm{U}_m(k')} f_m(g'^{-1}_2 \xi' g'_4), \tag{4.2}$$

and consider the following distribution

$$\mathcal{J}(f_n \otimes f_m) = \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \mathcal{K}_{f_n \otimes f_m}\left(\varepsilon(h_1'), \kappa(h_1'); \varepsilon(h_2'), \kappa(h_2')\right) \nu'(h_1'^{-1} h_2') \mathrm{d}h_1' \mathrm{d}h_2'. \quad (4.3)$$

Such distribution formally computes

$$\sum_{\pi,\sigma} \sum \mathcal{B}_r^{\nu'}(\rho(f_n)\varphi_\pi, \rho(f_m)\varphi_\sigma) \mathcal{B}_r^{\widetilde{\nu'}}(\overline{\varphi_\pi}, \overline{\varphi_\sigma}),$$

where the inner sum is taken over orthonormal bases of $\mathcal{A}_\pi$ and $\mathcal{A}_\sigma$.

Remark 4.3. The integral (4.3) is *not* absolutely convergent in general and needs regularization. In order to see what we should expect for these distributions, we will not treat the convergence problem in this article. In particular, the following calculation of decomposition into orbital integrals will be formal, unless we put certain restrictions on $f_n \otimes f_m$ to make it convergent—for example, we may assume that the function f (4.6) supports only on regular elements. The remark applies similarly to later distributions (4.9), (5.5), and (5.10).

Plugging in (4.2), we have

$$\begin{aligned}(4.3) &= \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \sum_{\zeta' \in \mathrm{U}_n(k')} f_n(\varepsilon(h_1')^{-1}\zeta'\varepsilon(h_2')) \\ &\quad \sum_{\xi' \in \mathrm{U}_m(k')} f_m(\kappa(h_1')^{-1}\xi'\kappa(h_2'))\nu'(h_1'^{-1}h_2')\mathrm{d}h_1'\mathrm{d}h_2' \\ &= \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \sum_{\xi' \in \mathrm{U}_m(k')} \\ &\quad \sum_{\zeta' \in \mathrm{U}_n(k')} f_n(\varepsilon(h_1')^{-1}\xi'\zeta'\varepsilon(h_2')) f_m(\kappa(h_1')^{-1}\xi'\kappa(h_2'))\nu'(h_1'^{-1}h_2')\mathrm{d}h_1'\mathrm{d}h_2' \\ &= \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \sum_{\xi' \in H'(k')} \sum_{\zeta' \in U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')} \\ &\quad f_n(\varepsilon(\xi'h_1')^{-1}\zeta'\varepsilon(h_2')) f_m(\kappa(\xi'h_1')^{-1}\kappa(h_2'))\nu'(h_1'^{-1}h_2')\mathrm{d}h_1'\mathrm{d}h_2',\end{aligned}$$

which equals

$$\int_{H'(k')\backslash H'(\mathbb{A}')} \int_{H'(\mathbb{A}')} \sum_{\zeta' \in U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')} f_n(\varepsilon(h_1')^{-1}\zeta'\varepsilon(h_2')) f_m(\kappa(h_1')^{-1}\kappa(h_2'))\nu'(h_1'^{-1}h_2')\mathrm{d}h_1'\mathrm{d}h_2'. \quad (4.4)$$

If we write $h'_i = \underline{u}'_i g'_i$, then

$$(4.4) = \int_{H'(k')\backslash H'(\mathbb{A}')} \int_{U'_{1^r,m+1}(\mathbb{A}')} \int_{\mathrm{U}_m(\mathbb{A}')} \sum_{\zeta' \in U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')} f_n(g_1'^{-1}\underline{u}_1'^{-1}\zeta'\varepsilon(h'_2)) f_m(g_1'^{-1}\kappa(h'_2)) \underline{\psi}'(\underline{u}_1'^{-1})\nu'(h'_2)\mathrm{d}g'_1\mathrm{d}\underline{u}'_1\mathrm{d}h'_2,$$

which equals

$$\int_{H'(k')\backslash H'(\mathbb{A}')} \int_{U'_{1^r,m+1}(\mathbb{A}')} \int_{\mathrm{U}_m(\mathbb{A}')} \sum_{\zeta' \in U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')} f_n(g_1'^{-1}\kappa(h'_2)^{-1}\underline{u}_1'^{-1}\zeta'\varepsilon(h'_2)) f_m(g_1'^{-1}) \underline{\psi}'(\underline{u}_1'^{-1})\nu'(h'_2)\mathrm{d}g'_1\mathrm{d}\underline{u}'_1\mathrm{d}h'_2. \quad (4.5)$$

Define a function $f \in \mathcal{S}(\mathrm{U}_n(\mathbb{A}'))$ by

$$f(g') = \int_{\mathrm{U}_m(\mathbb{A}')} f_n(g'_1 g') f_m(g'_1)\mathrm{d}g'_1. \quad (4.6)$$

Then

$$(4.5) = \sum_{\zeta' \in U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')} \int_{H'(k')\backslash H'(\mathbb{A}')} \int_{U'_{1^r,m+1}(\mathbb{A}')} f(\kappa(h'_2)^{-1}\underline{u}_1'^{-1}\zeta'\varepsilon(h'_2)) \underline{\psi}'(\underline{u}_1'^{-1})\nu'(h'_2)\mathrm{d}\underline{u}'_1\mathrm{d}h'_2.$$

The group H' acts on $U'_{1^r,m+1}\backslash \mathrm{U}_n$ by conjugation. Denote $(U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k')) /\!\!/ H'(k')$ the set of conjugacy classes of k'-points and then the above expression equals

$$\sum_{\zeta' \in (U'_{1^r,m+1}(k')\backslash \mathrm{U}_n(k'))/\!\!/ H'(k')} \int_{\mathrm{Stab}^{H'}_{\zeta'}(k')\backslash H'(\mathbb{A}')} \int_{U'_{1^r,m+1}(\mathbb{A}')} f(\kappa(h'_2)^{-1}\underline{u}_1'^{-1}\zeta'\varepsilon(h'_2)) \underline{\psi}'(\underline{u}_1'^{-1})\nu'(h'_2)\mathrm{d}\underline{u}'_1\mathrm{d}h'_2. \quad (4.7)$$

We introduce a k'-algebraic group

$$\mathbf{H}' = H' \underset{\mathrm{U}_m}{\times} H' \subset H' \times_{\mathrm{Spec}\, k'} H',$$

which acts on U_n from right in the following way. For a k'-algebra R, $\mathbf{h}' = \mathbf{h}'(\underline{u}'_1, \underline{u}'_2; g') \in \mathbf{H}'(R)$ with $\underline{u}'_i \in U'_{1^r,m+1}(R)$, $g' \in \mathrm{U}_m(R)$ and $g \in \mathrm{U}_n(R)$, the right action is given by $[g]\mathbf{h}' = g'^{-1}\underline{u}_1'^{-1} g \underline{u}'_2 g'$. We denote by $[\mathrm{U}_n(k')]/\mathbf{H}'(k')$ the set of k'-orbits under this action. Define a character (also denoted by) $\underline{\psi}'$ of $\mathbf{H}'(\mathbb{A}')$ by

$$\underline{\psi}'(\mathbf{h}') = \underline{\psi}'\left(\mathbf{h}'(\underline{u}'_1, \underline{u}'_2; g')\right) := \underline{\psi}'(\underline{u}_1'^{-1}\underline{u}'_2).$$

Then

$$(4.7) = \mathcal{J}(f) := \sum_{\zeta' \in [\mathrm{U}_n(k')]/\mathbf{H}'(k')} \mathcal{J}_{\zeta'}(f),$$

where

$$\mathcal{J}_{\zeta'}(f) = \int_{\mathrm{Stab}^{\mathbf{H}'}_{\zeta'}(k') \backslash \mathbf{H}'(\mathbb{A}')} f([\zeta']\mathbf{h}')\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'.$$

We denote $\mathrm{U}_n(k')_{\mathrm{reg}}$ the set of regular k'-elements, a notion that will be defined in Definition 4.9. In particular, the $\mathbf{H}'$-stabilizer $\mathrm{Stab}^{\mathbf{H}'}_{\zeta'}$ is trivial for $\zeta' \in \mathrm{U}_n(k')_{\mathrm{reg}}$ by Proposition 4.12, and the corresponding term $\mathcal{O}(f, \zeta') := \mathcal{J}_{\zeta'}(f)$ is an orbital integral. If $f = \otimes_{v'} f_{v'}$ is factorizable, then

$$\mathcal{O}(f, \zeta') = \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}(f_{v'}, \zeta'),$$

where

$$\mathcal{O}(f_{v'}, \zeta') = \int_{\mathbf{H}'_{v'}} f_{v'}([\zeta']\mathbf{h}'_{v'})\underline{\psi}'_{v'}(\mathbf{h}'_{v'})\mathrm{d}\mathbf{h}'_{v'}.$$

In particular, if f supports only on regular elements, then

$$\mathcal{J}(f) = \mathcal{J}_{\mathrm{reg}}(f) = \sum_{\zeta' \in [\mathrm{U}_n(k')_{\mathrm{reg}}]/\mathbf{H}'(k')} \mathcal{J}_{\zeta'}(f) = \sum_{\zeta' \in [\mathrm{U}_n(k')_{\mathrm{reg}}]/\mathbf{H}'(k')} \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}(f_{v'}, \zeta').$$

Now we discuss the relative trace formula on general linear groups concerning Bessel integrals. We identify $\mathrm{GL}_{m,k} \subset \mathrm{GL}_{n,k}$ with $\mathrm{GL}(W) \subset \mathrm{GL}(V)$, and view $\mathrm{GL}_{n,k'} \subset \mathrm{Res}_{k/k'} \mathrm{GL}_{n,k}$ (resp. $\mathrm{GL}_{m,k'} \subset \mathrm{Res}_{k/k'} \mathrm{GL}_{m,k}$) through the basis $\{v_1, \dots, v_r, w_1, \dots, w_m, w_0, \check{v}_r, \dots, \check{v}_1\}$.[3] Let Z'_n (resp. Z'_m) be the center of $\mathrm{GL}_{n,k'}$ (resp. $\mathrm{GL}_{m,k'}$). Put $\psi = \psi' \circ \widetilde{\mathrm{Tr}}$ and

$$\nu(h) = \nu\left(\underline{u}g\right) = \nu\left(h\left(n, n^*, b; u, u^*; g\right)\right) = \underline{\psi}(\underline{u}),$$

where $\underline{\psi}(\underline{u}) = \psi(u_{1,2} + \cdots + u_{r-1,r} + n_{r,0} + \beta_0 n^*_{0,r} + u^*_{r,r-1} + \cdots + u^*_{2,1})$.

Take $F_n \in \mathcal{S}(\mathrm{GL}_n(\mathbb{A}))$ and $F_m \in \mathcal{S}(\mathrm{GL}_m(\mathbb{A}))$. Associate to $F_n \otimes F_m$ a kernel function $\mathcal{K}_{F_n \otimes F_m}(g_1, g_2; g_3, g_4)$ on $(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A}))^2$ (averaged by $Z'_n \times Z'_m$) by the formula

$$\int_{Z'_n(k')\backslash Z'_n(\mathbb{A}')} \sum_{\zeta \in \mathrm{GL}_n(k)} F_n(g_1^{-1} z_1 \zeta g_3)\mathrm{d}z_1 \int_{Z'_m(k')\backslash Z'_m(\mathbb{A}')} \sum_{\xi \in \mathrm{GL}_m(k)} F_m(g_2^{-1} z_2 \xi g_4)\mathrm{d}z_2. \tag{4.8}$$

[3] Note that this basis is different from (2.3).

Consider the following distribution

$$\mathcal{J}(s; F_n \otimes F_m) = \int\limits_{Z'_m(\mathbb{A}')\,\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int\limits_{Z'_n(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int\limits_{H(k)\backslash H(\mathbb{A})}$$
$$\mathcal{K}_{F_n\otimes F_m}(\varepsilon(h), \kappa(h); g_1, g_2)\nu(h^{-1})|\det h|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det g_2)\mathrm{d}h\mathrm{d}g_1\mathrm{d}g_2,$$
$$= \int\limits_{Z'_m(\mathbb{A}')\,\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int\limits_{Z'_n(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int\limits_{Z'_m(k')\backslash Z'_m(\mathbb{A}')} \int\limits_{Z'_n(k')\backslash Z'_n(\mathbb{A}')} \int\limits_{H(k)\backslash H(\mathbb{A})}$$
$$\sum_{\zeta\in \mathrm{GL}_n(k)} F_n(\varepsilon(h)^{-1}z_1\zeta g_1) \sum_{\xi\in \mathrm{GL}_m(k)} F_m(\kappa(h)^{-1}z_2\xi g_2)$$
$$\nu(h^{-1})|\det h|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det g_2)\mathrm{d}h\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}g_1\mathrm{d}g_2,$$

which equals

$$\int\limits_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int\limits_{\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int\limits_{U_{1^r,m+1,1^r}(k)\backslash H(\mathbb{A})}$$
$$\sum_{\zeta\in \mathrm{GL}_n(k)} F_n(\varepsilon(h)^{-1}\zeta g_1)F_m(\kappa(h)^{-1}g_2)\nu(h^{-1})|\det h|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det g_2)\mathrm{d}h\mathrm{d}g_1\mathrm{d}g_2. \tag{4.9}$$

Such distribution formally computes

$$\sum_{\Pi,\Sigma}\sum \mathcal{B}^{\nu}_{r,r}(s; \rho(F_n)\varphi_\Pi, \rho(F_m)\varphi_\Sigma)\mathcal{P}_n(\overline{\varphi_\Pi})\mathcal{P}_m(\overline{\varphi_\Sigma}),$$

where the inner sum is taken over orthonormal bases of $\mathcal{A}_\Pi$ and $\mathcal{A}_\Sigma$. Decompose $h = \underline{u}g$ and note that the group $H(\mathbb{A})$ is unimodular. We make a change of variable as $g \mapsto g_2^{-1}g$. Then

$$(4.9) = \int\limits_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int\limits_{\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int\limits_{U_{1^r,m+1,1^r}(k)\backslash U_{1^r,m+1,1^r}(\mathbb{A})} \int\limits_{\mathrm{GL}_m(\mathbb{A})}$$
$$\sum_{\zeta\in \mathrm{GL}_n(k)} F_n(g^{-1}g_2^{-1}\underline{u}^{-1}\zeta g_1)F_m(g^{-1})\underline{\psi}(\underline{u}^{-1})$$
$$|\det g|_{\mathbb{A}}^{s-\frac{1}{2}}|\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det g_2)\mathrm{d}g\mathrm{d}\underline{u}\mathrm{d}g_1\mathrm{d}g_2. \tag{4.10}$$

Define a function $\tilde{F}_s$ on $\mathrm{GL}_n(\mathbb{A})$, which is holomorphic in s, by

$$\tilde{F}_s(\tilde{g}) = \int\limits_{\mathrm{GL}_m(\mathbb{A})} F_n(g^{-1}\tilde{g})F_m(g^{-1})|\det g|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g.$$

We have a surjective linear map $\sigma : \mathcal{S}(\mathrm{GL}_n(\mathbb{A})) \to \mathcal{S}(\mathrm{S}_n(\mathbb{A}'))$ given by

$$\sigma(F)(gg^{\tau,-1}) = \int\limits_{\mathrm{GL}_n(\mathbb{A}')} F(g\tilde{g})\mathrm{d}\tilde{g}. \tag{4.11}$$

Put $F_s = \sigma(\tilde{F}_s)$. Using the isomorphism $\mathrm{GL}_n(k)/\mathrm{GL}_n(k') \simeq \mathrm{S}_n(k')$ and combining the previous two operations together, we obtain

$$(4.10) = \sum_{\zeta\in \mathrm{S}_n(k')} \int_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{U_{1^r,m+1,1^r}(k)\backslash U_{1^r,m+1,1^r}(\mathbb{A})} F_s(g_2^{-1}\underline{u}^{-1}\zeta\underline{u}^{\tau}g_2)\underline{\psi}(\underline{u}^{-1})|\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det g_2)\mathrm{d}\underline{u}\mathrm{d}g_2. \quad (4.12)$$

Similar to the case of unitary groups, we introduce the following k'-algebraic group

$$\mathbf{H} = \mathrm{Res}_{k/k'}(U_{1^r,m+1,1^r}) \rtimes \mathrm{GL}_{m,k'} \subset \mathrm{Res}_{k/k'} H,$$

which acts on S_n from right in the following way. For a k'-algebra R, $\mathbf{h} = \mathbf{h}(\underline{u}; g)$ with $\underline{u} \in U_{1^r,m+1,1^r}(R \otimes k)$, $g \in \mathrm{GL}_m(R)$ and $s \in \mathrm{S}_n(R)$, the right action is given by $[s]\mathbf{h} = g^{-1}\underline{u}^{-1}s\underline{u}^{\tau}g$. We denote by $[\mathrm{S}_n(k')]/\mathbf{H}(k')$ the set of k'-orbits under this action. Define a character (also denoted by $\underline{\psi}$) of $\mathbf{H}(\mathbb{A}')$ by

$$\underline{\psi}(\mathbf{h}) = \underline{\psi}(\mathbf{h}(\underline{u}g)) = \underline{\psi}(\underline{u}^{-1}),$$

and put $\det \mathbf{h} = \det g$. We have

$$(4.12) = \mathcal{J}(s; F_s) := \sum_{\zeta\in[\mathrm{S}_n(k')]/\mathbf{H}(k')} \mathcal{J}_\zeta(s; F_s),$$

where

$$\mathcal{J}_\zeta(s; F_s) = \int_{\mathbf{H}(\mathbb{A}')/\mathrm{Stab}_\zeta^{\mathbf{H}}(k')} F_s([\zeta]\mathbf{h})\underline{\psi}(\mathbf{h})|\det \mathbf{h}|_{\mathbb{A}}^{s-\frac{1}{2}}\eta(\det \mathbf{h})\mathrm{d}\mathbf{h}.$$

We denote $\mathrm{S}_n(k')_{\mathrm{reg}}$ the set of regular k'-elements, a notion that will be defined in Definition 4.9. In particular, the $\mathbf{H}$-stabilizer $\mathrm{Stab}_\zeta^{\mathbf{H}}$ is trivial for $\zeta \in \mathrm{S}_n(k')_{\mathrm{reg}}$ by Proposition 4.12, and the corresponding term $\mathcal{O}(s; F_s, \zeta) := \mathcal{J}_\zeta(s; F_s)$ is an orbital integral. If $F_s = \otimes_{v'} F_{s,v'}$ is factorizable, then

$$\mathcal{O}(s; F_s, \zeta) = \prod_{v'\in\mathscr{M}_{k'}} \mathcal{O}(s; F_{s,v'}, \zeta),$$

where

$$\mathcal{O}(s; F_{s,v'}, \zeta) = \int_{\mathbf{H}_{v'}} F_{s,v'}([\zeta]\mathbf{h}_{v'})\underline{\psi}_{v'}(\mathbf{h}_{v'})\eta_{v'}(\det \mathbf{h}_{v'})|\det \mathbf{h}_{v'}|_{v'}^{s-\frac{1}{2}}\mathrm{d}\mathbf{h}_{v'}.$$

In particular, if F_s supports only on regular elements, then

$$\mathcal{J}(s; F_s) = \mathcal{J}_{\mathrm{reg}}(s; F_s) = \sum_{\zeta\in[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')} \mathcal{J}_\zeta(s; F_s)$$
$$= \sum_{\zeta\in[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')} \prod_{v'\in\mathscr{M}_{k'}} \mathcal{O}(s; F_{s,v'}, \zeta).$$

When $s = \frac{1}{2}$, the terms involving $|\det|$ disappear and we suppress s in notation. In particular,

$$\mathcal{J}_\zeta(F) = \mathcal{O}(F, \zeta) = \int\limits_{\mathbf{H}(\mathbb{A}')} F([\zeta]\mathbf{h})\underline{\psi}(\mathbf{h})\eta(\det \mathbf{h})\mathrm{d}\mathbf{h}.$$

We expect that the above orbital integral has a close relation with the following one

$$\mathcal{J}_{\zeta'}(f) = \mathcal{O}(f, \zeta') = \int\limits_{\mathbf{H}'(\mathbb{A}')} f([\zeta']\mathbf{h}')\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'$$

introduced previously, assuming that ζ and ζ' are both regular, and match in a natural sense defined in the next subsection.

Remark 4.4. If the original functions $F_n = \otimes_v F_{n,v}$ and $F_m = \otimes_v F_{m,v}$ are factorizable, then $\tilde{F}_s = \otimes_v \tilde{F}_{s,v}$ and $F_s = \otimes_{v'} F_{s,v'}$ are also factorizable. If for some (finite) place v', $F_{m,v'}$ has the property that $\left\{|\det g|_{v'} \mid F_{m,v'}(g) \neq 0\right\} = \{N\}$ is a singleton, then $F_{s,v'} = F_{v'} \cdot N^{s-\frac{1}{2}}$ for a function $F_{v'}$ that is independent of s. In particular, this is the case for almost all v'.

4.3. Matching of orbits and functions

Suppose we are in either local or global situations. We say two elements $\beta_1, \beta_2 \in \mathrm{Her}_m^\times(k')$ are *similar*, denoted by $\beta_1 \sim \beta_2$, if there exists $g \in \mathrm{GL}_m(k)$ such that $\beta_2 = {}^t g^\tau \beta_1 g$. We denote $[\mathrm{Her}_m^\times(k')]$ the set of similarity classes. We write $W^\beta = W$ and $V^\beta = V = W^\beta \oplus X \oplus X^* \oplus E$, if the matrix representing the hermitian form on W is in the class $\beta \in [\mathrm{Her}_m^\times(k')]$, also U_m^β (resp. U_n^β, $\mathbf{H}^\beta$) for $\mathrm{U}(W^\beta)$ (resp. $\mathrm{U}(V^\beta)$, $\mathbf{H}'$). Define

$$\epsilon(\beta) = \eta\left((-1)^{\frac{m(m-1)}{2}} \det \beta\right) \in \{\pm 1\}$$

to be the ϵ-factor of β, which depends only on its similarity class.

We first define the notion of pre-regular orbits. Recall that we have the action of $\mathbf{H}$ (resp. $\mathbf{H}'$), and hence its unipotent radical $\mathrm{Res}_{k/k'} U_{1^r,m+1,1^r}$ (resp. $(U'_{1^r,m+1})^2$), on S_n (resp. U_n^β).

Definition 4.5. (*Pre-regular element*). An element $\zeta \in \mathrm{S}_n(k')$ (resp. $\zeta^\beta \in \mathrm{U}_n^\beta(k')$) is called *pre-regular* if its stabilizer under the action of $\mathrm{Res}_{k/k'} U_{1^r,m+1,1^r}$ (resp. $(U'_{1^r,m+1})^2$) is trivial.

Let $\mathbf{B}$ be the Borel subgroup of GL_n consisting of upper-triangular matrices and $\mathbf{A} \cong (\mathrm{GL}_1)^n$ be the maximal torus consisting of diagonal matrices. Let $\mathbf{W}_n$ be the Weyl group of GL_n, which we identify with the subgroup of permutation matrices in GL_n. Moreover, let $\mathbf{W}_n^{\mathrm{S}} \subset \mathbf{W}_n$ be the subgroup consisting of elements whose square is $\mathbf{1}_n$. Let $\mathbf{P}$ be a standard parabolic subgroup of $\mathrm{GL}_{n,k}$ whose unipotent radical is $\mathbf{U}$. Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{P}$ consisting of diagonal blocks. The group $\mathrm{Res}_{k/k'} \mathbf{P}$ acts on S_n from right by $[s]p = p^{-1} s p^\tau$. First, we have the following lemma.

Lemma 4.6. *An element* $\zeta \in \mathrm{S}_n(k')$ *has trivial stabilizer under the action of* $\mathbf{U}(k) \subset \mathbf{P}(k)$ *if and only if its orbit intersects with* $[\mathbf{w}]\mathbf{M}(k)$*, where* $\mathbf{w} = \mathsf{w}_n$ *is the longest element in* $\mathbf{W}_n$*. Moreover, the intersection contains at most one element.*

Proof. By [7, Proposition 3], we have the following Bruhat decomposition for $\mathrm{S}_n(k')$

$$\mathrm{S}_n(k') = \coprod_{w \in \mathbf{W}_n^{\mathrm{S}}} [w]\mathbf{B}(k).$$

It implies that for general $\mathbf{P}$, we have

$$\mathrm{S}_n(k') = \bigcup_{w \in \mathbf{W}_n^{\mathrm{S}}} [w]\mathbf{P}(k) = \bigcup_{w \in \mathbf{W}_n^{\mathrm{S}}} [w]\mathbf{M}(k)\mathbf{U}(k).$$

Therefore, in a $\mathbf{U}(k)$-orbit, there is a representative of the form $[w]m$. We assume that $\zeta = [w]m = m^{-1}wm^{\tau}$. Then its stabilizer is trivial if and only if

$$\left\{u^{-1}wu^{\tau} = w \mid u \in \mathbf{U}(k)\right\} = \{\mathbf{1}_n\}. \tag{4.13}$$

Note that $u^{-1}wu^{\tau} = w$ is equivalent to $wuw = u^{\tau}$. Therefore, if $w = \mathbf{w}$ is the longest Weyl element, $u = \mathbf{1}_n$ and $[w]m$ is the only point where its orbit and $[w]\mathbf{M}(k)$ intersect.

Conversely, we need to show that if (4.13) holds, then $w \in [\mathbf{w}]\mathbf{W_M}$, where $\mathbf{W_M} \subset \mathbf{W}_n \cap \mathbf{M}(k)$ is isomorphic to the Weyl group of $\mathbf{M}$. We observe that $w \in \mathbf{W}_n^{\mathrm{S}}$ is a disjoint union of transpositions. We use induction on n. The case $n = 1$ is trivial. Suppose that the above assertion holds for numbers less than n. If the transposition $(1, n)$ appears in w, then we reduce to the case of $n-2$ and we are done. Otherwise, $(1, a)$ will appear in w with $1 \le a < n$. Suppose that $\mathbf{M} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t}$ (arranged from upper-left to lower-right) with $n = n_1 + \cdots + n_t$, $n_i > 0$, and $t > 1$ (otherwise, the proof is trivial). If $n - a < n_t$, then $w' = (a, n)$ is an element in $\mathbf{W_M} \subset \mathbf{M}(k)$. The conjugation $w'^{-1}ww' \in \mathbf{W}_n^{\mathrm{S}}$ will contain the transportation $(1, n)$ and we are done. Otherwise, $n - a \ge n_t$, and we consider the transportation (b, n) in w' with $1 < b \le n$. If $b-1 < n_1$, then we can conjugate w by $(1, b) \in \mathbf{W_M}$ and we are again done. The remaining case is that $b-1 \ge n_1$. We define an element $u \in \mathbf{U}(k)$ whose entries are 1 at diagonals and positions $(1, b)$, (a, n); 0 elsewhere. Then $wuw = u = u^{\tau}$ which contradicts (4.13). □

Applying the above lemma to the parabolic subgroup $\mathbf{P} = P_{1^r,m+1,1^r}$ stabilizing the flag

$$0 \subset \{v_1\} \subset \cdots \subset X \subset X \oplus W \oplus E \subset X \oplus W \oplus E \oplus \left\{\check{v}_r\right\} \subset \cdots \subset V,$$

which is standard under the basis $\{v_1, \ldots, v_r, w_1, \ldots, w_m, w_0, \check{v}_r, \ldots, \check{v}_1\}$ adopted in this section. Since $\mathbf{w}$ normalizes $P_{1^r,m+1,1^r}$, the $U_{1^r,m+1,1^r}(k)$-orbit of a pre-regular element ζ must contain a unique element of the form

$$\begin{bmatrix} & & & & & & t_1(\zeta) \\ & & & & & \cdots & \\ & & & & t_r(\zeta) & & \\ & & & \Pr(\zeta) & & & \\ & & t_r(\zeta)^{\tau,-1} & & & & \\ & \cdots & & & & & \\ t_1(\zeta)^{\tau,-1} & & & & & & \end{bmatrix}, \tag{4.14}$$

with $t_i(\zeta) \in k^\times$ and $\Pr(\zeta) \in \mathrm{S}_{m+1}(k')$. We call it the *normal form* of ζ. We say ζ is *normal* if it is in the above form.

Now we consider the unitary group U_n^β. We fix a minimal parabolic subgroup $\mathbf{P}_0^\beta$ such that its unipotent radical $\mathbf{U}_0^\beta$ contains $U'_{1^r,m+1}$. Let $\mathbf{A}_0^\beta$ be a maximal torus inside $\mathbf{P}_0^\beta$, and $\mathbf{W}_n^\beta$ the Weyl group. Let $\mathbf{P}'$ be a standard parabolic subgroup of U_n^β with $\mathbf{U}'$ its unipotent radical, and $\mathbf{M}' \supset \mathbf{A}_0^\beta$ a Levi subgroup. The group $(\mathbf{P}')^2$ acts on U_n^β from right by $[g](p_1, p_2) = p_1^{-1} g p_2$. We have the following lemma similar to the one for symmetric spaces.

Lemma 4.7. *An element $\zeta' \in \mathrm{U}_n^\beta(k')$ has trivial stabilizer under the action of $\mathbf{U}'(k')^2 \subset \mathbf{P}'(k')^2$ if and only if its orbit intersects $[\mathbf{w}^\beta]\mathbf{M}'(k')^2 = \mathbf{M}'(k')\mathbf{w}^\beta\mathbf{M}'(k')$, where $\mathbf{w}^\beta$ is the longest element in $\mathbf{W}_n^\beta$. Moreover, the intersection contains at most one element.*

Proof. We have the usual Bruhat decomposition

$$\begin{aligned} \mathrm{U}_n^\beta(k') &= \coprod_{w \in \mathbf{W}_{\mathbf{M}'} \backslash \mathbf{W}_n^\beta / \mathbf{W}_{\mathbf{M}'}} \mathbf{P}'(k') w \mathbf{P}'(k') \\ &= \coprod_{w \in \mathbf{W}_{\mathbf{M}'} \backslash \mathbf{W}_n^\beta / \mathbf{W}_{\mathbf{M}'}} \mathbf{U}'(k')\mathbf{M}'(k') w \mathbf{M}'(k')\mathbf{U}'(k'). \end{aligned}$$

Therefore, in a $\mathbf{U}'(k')^2$-orbit, there is a representative of the form $m_1 w m_2$. We assume that $\zeta' = m_1 w m_2$. Then its stabilizer is trivial if and only if

$$w\mathbf{U}'(k')w^{-1} \cap \mathbf{U}'(k') = \{\mathbf{1}_n\}. \tag{4.15}$$

Let $R^+(\mathbf{A}_0^\beta, \mathrm{U}_n^\beta)$ (resp. $R^+(\mathbf{A}_0^\beta, \mathbf{M}')$) be the set of positive roots of $\mathbf{A}_0^\beta$ (resp. in $\mathbf{M}'$). Then a double coset of $\mathbf{W}_{\mathbf{M}'} \backslash \mathbf{W}_n^\beta / \mathbf{W}_{\mathbf{M}'}$ has a unique representative w satisfying $w(\alpha) < 0$ and $w^{-1}(\alpha) < 0$ for all $\alpha \in R^+(\mathbf{A}_0^\beta, \mathbf{M}')$. Assume that w satisfies (4.15) and the above condition. Then $w(\alpha) < 0$ for $\alpha \in R^+(\mathbf{A}_0^\beta, \mathrm{U}_n^\beta)$. Therefore, $w = \mathbf{w}^\beta$.

Conversely, if $w = \mathbf{w}^\beta$, then (4.15) holds, and the intersection is a singleton. □

Applying the above lemma to $\mathbf{P}' = P_{1^r,m+1}^\beta$, the standard parabolic subgroup stabilizing the flag $0 \subset \{v_1\} \subset \cdots \subset X \subset V$. Since $\mathbf{w}^\beta$ normalizes $P_{1^r,m+1}^\beta$, the

$U'_{1^r,m+1}(k')^2$-orbit of a pre-regular element ζ^β must contain a unique element of the form

$$\begin{bmatrix} & & & & & & t_1(\zeta^\beta) \\ & & & & & \cdots & \\ & & & & t_r(\zeta^\beta) & & \\ & & & \Pr(\zeta^\beta) & & & \\ & & t_r(\zeta^\beta)^{\tau,-1} & & & & \\ & \cdots & & & & & \\ t_1(\zeta^\beta)^{\tau,-1} & & & & & & \end{bmatrix}, \tag{4.16}$$

with $t_i(\zeta^\beta) \in k^\times$ and $\Pr(\zeta^\beta) \in \mathrm{U}^\beta_{m+1}(k')$, where $\mathrm{U}^\beta_{m+1} = \mathrm{U}(W^\beta \oplus E)$. We call it the *normal form* of ζ^β. We say ζ^β is *normal* if it is in the above form.

Remark 4.8. There is a more natural way to define the invariants t_i. For $\zeta \in \mathrm{Mat}_n(k)$ and $i = 1, \dots, r$, let $\zeta[i]$ be the left-lower $i \times i$ block of ζ, and $s_i(\zeta) = \det \zeta[i]$ which is invariant under the action $\zeta \mapsto u\zeta u'$ for $u, u' \in U_{1^r,m+1,1^r}(k)$. Then $\zeta \in \mathrm{S}_n(k')$ (resp. $\zeta^\beta \in \mathrm{U}^\beta_n(k')$) is pre-regular if and only if $s_i(\zeta) \in k^\times$ (resp. $s_i(\zeta^\beta) \in k^\times$) for all i, where we view ζ (resp. ζ^β) as elements in $\mathrm{Mat}_n(k)$ through the natural inclusion $\mathrm{S}_n \subset \mathrm{Res}_{k/k'} \mathrm{Mat}_{n,k} = \mathrm{End}(V)$ (resp. $\mathrm{U}^\beta_n \subset \mathrm{Res}_{k/k'} \mathrm{Mat}_{n,k}$). Moreover, the invariants t_i and s_i are related by $t_i(\zeta) = s_{i-1}(\zeta)^\tau s_i(\zeta)^{\tau,-1}$ ($s_0 = 1$), and similarly for ζ^β.

Recall that we have natural inclusions $\mathrm{S}_{m+1} \subset \mathrm{Res}_{k/k'} \mathrm{Mat}_{m+1,k} = \mathrm{End}(W \oplus E)$ and $\mathrm{U}^\beta_{m+1} \subset \mathrm{Res}_{k/k'} \mathrm{Mat}_{m+1,k}$.

Definition 4.9. (*Regular element*). An element $\xi \in \mathrm{Mat}_{m+1}(k)$ is called *regular* if it satisfies

- ξ is regular semisimple as an element of $\mathrm{Mat}_{m+1}(k)$;
- the vectors $\{w_0, \xi w_0, \dots, \xi^m w_0\}$ span $W \oplus E$;
- the vectors $\left\{w_0^\vee, w_0^\vee \xi, \dots, w_0^\vee \xi^m\right\}$ span $W^\vee \oplus E^\vee$.

An element $\zeta \in \mathrm{S}_n(k')$ (resp. $\zeta^\beta \in \mathrm{U}^\beta_n(k')$) is called *regular*, if it is pre-regular and $\Pr(\zeta) \in \mathrm{Mat}_{m+1}(k)$ (resp. $\Pr(\zeta^\beta) \in \mathrm{U}^\beta_{m+1}(k)$) is regular. An $\mathbf{H}$-orbit $\zeta \in [\mathrm{S}_n(k')]/\mathbf{H}(k')$ (resp. $\mathbf{H}^\beta$-orbit $\zeta^\beta \in [\mathrm{U}^\beta_n(k')]/\mathbf{H}^\beta(k')$) is called *regular* if some and hence all elements it contains are regular. We denote by $\mathrm{Mat}_{m+1}(k)_{\mathrm{reg}}$, $\mathrm{GL}_{m+1}(k)_{\mathrm{reg}} := \mathrm{Mat}_{m+1}(k)_{\mathrm{reg}} \cap \mathrm{GL}_{m+1}(k)$, $\mathrm{S}_n(k')_{\mathrm{reg}}$ (resp. $[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')$) and $\mathrm{U}^\beta_n(k')_{\mathrm{reg}}$ (resp. $[\mathrm{U}^\beta_n(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k')$) the various sets of regular elements (resp. orbits).

To proceed, we recall some results from [32, Sect. 6]; see also [22,37,38]. To include the whole action of $\mathbf{H}$ (resp. $\mathbf{H}^\beta$), we need to consider the conjugation action (from right) of $\mathrm{GL}_{m,k'}$ (resp. U^β_m). We consider more generally the conjugation action of $\mathrm{GL}_{m,k}$. Recall that, by our choice of coordinates, the group $\mathrm{GL}_{m,k}$ embeds into $\mathrm{GL}_{m+1,k}$ via

$$g \mapsto \begin{bmatrix} g & \\ & 1 \end{bmatrix}.$$

For $\xi \in \mathrm{Mat}_{m+1}(k)$, put

- $a_i(\xi) = \mathrm{Tr} \bigwedge^i \xi$ for $1 \le i \le m+1$;
- $b_i(\xi) = w_0^\vee \xi^i w_0$ for $1 \le i \le m$;
- $\mathbf{D}_\xi$ to be the matrix $[(w_0^\vee \xi^{i-1})(\xi^{j-1} w_0)]_{i,j=1}^{m+1}$; and
- $\mathbf{T}_\xi = \det[(w_0^\vee \xi^{i-1})(w_{j-1})]_{i,j=1}^{m+1}$.

It is clear that if ξ is regular, $\Delta_\xi \neq 0$. Moreover, we have the following lemma.

Lemma 4.10. *Two regular elements ξ, ξ' in $\mathrm{GL}_{m+1}(k)$ are conjugate under $\mathrm{GL}_m(k)$ if and only if $a_i(\xi) = a_i(\xi')$ $(1 \le i \le m+1)$ and $b_i(\xi) = b_i(\xi')$ $(1 \le i \le m)$. The GL_m-stabilizer of a regular element is trivial.*

Proof. See [32, Proposition 6.2 and Theorem 6.1] for the (equivalent version of the) first and second statements, respectively. □

To include all unitary groups at the same time, we consider the set $\mathfrak{U}_{m+1}$ of pairs (β, ξ^β) where $\beta \in \mathrm{Her}_m^\times(k')$, and

$$\xi^\beta \in \mathrm{U}_{m+1}^\beta(k')_{\mathrm{reg}} = \left\{ \xi^\beta \in \mathrm{Mat}_{m+1}(k)_{\mathrm{reg}} \mid {}^t(\xi^\beta)^\tau \beta' \xi^\beta = \beta' \right\},$$

where β' is defined in (4.1). The group $\mathrm{GL}_m(k)$ acts on $\mathfrak{U}_{m+1}$ by $(\beta, \xi^\beta)g = ({}^t g^\tau \beta g, g^{-1}\xi^\beta g)$. For $\xi \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}} := \mathrm{S}_{m+1}(k') \cap \mathrm{Mat}_{m+1}(k)_{\mathrm{reg}}$, we denote by $\xi \Leftrightarrow (\beta, \xi^\beta)$ if there exists $g \in \mathrm{GL}_m(k)$ such that $\xi = g^{-1}\xi^\beta g$. The following lemma is also considered in [22,38].

Lemma 4.11. *For $\xi \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$, there exists a pair (β, ξ^β), unique up to above action of $\mathrm{GL}_m(k)$, such that $\xi \Leftrightarrow (\beta, \xi^\beta)$. Conversely, for every pair $(\beta, \xi^\beta) \in \mathfrak{U}_{m+1}$, there exists an element $\xi \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$, unique up to the conjugation action of $\mathrm{GL}_m(k')$, such that $\xi \Leftrightarrow (\beta, \xi^\beta)$.*

Proof. We first point out that two elements $\xi, \xi' \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$ are conjugate under $\mathrm{GL}_m(k)$ if and only if they are conjugate under $\mathrm{GL}_m(k')$. In fact, assume $g^{-1}\xi g = \xi'$. Then $g^{-1}\xi g = g^{\tau,-1}\xi^{\tau,-1}g^\tau = g^{\tau,-1}\xi g^\tau$, which implies $g = g^\tau$.

It is easy too see that for $\xi \in \mathrm{Mat}_{m+1}(k)_{\mathrm{reg}}$, ξ and ${}^t\xi$ have the same invariants a_i, b_i. By the above lemma, there is a unique element $g \in \mathrm{GL}_m(k)$ such that $g^{-1}\xi g = {}^t\xi$. If $\xi \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$, then ${}^t\xi \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$. Therefore, we have $g^\tau = g$. Also, ${}^t g\, {}^t\xi\, {}^t g^{-1} = \xi$; $g\, {}^t\xi g^{-1} = \xi$, which imply that $g = {}^t g$. Together, we have $g \in \mathrm{Her}_m^\times(k')$. Moreover, since ${}^t\xi^\tau(g^{-1})\xi = g^{-1}$, we have $\xi \in \mathrm{U}_{m+1}^{g^{-1}}(k')_{\mathrm{reg}}$ and $\xi \Leftrightarrow (g^{-1}, \xi)$.

Conversely, given $(\beta, \xi) \in \mathfrak{U}_{m+1}$, we have ${}^t\xi\beta^\tau\xi^\tau = \beta^\tau$, and hence $\beta^{\tau,-1}\,{}^t\xi\beta^\tau = \xi^{\tau,-1}$. Moreover, there is an element $\gamma \in \mathrm{GL}_m(k)$ such that $\gamma^{-1}\xi\gamma = {}^t\xi$. We have $\beta^{\tau,-1}\gamma^{-1}\xi\gamma\beta^\tau = \xi^{\tau,-1}$, that is, $(\gamma\beta^\tau)^{-1}\xi(\gamma\beta^\tau) = \xi^{\tau,-1}$. By regularity, $\gamma\beta^\tau \in \mathrm{S}_m(k')$. Therefore, there exists $g \in \mathrm{GL}_m(k)$ such that $\gamma\beta^\tau = gg^{\tau,-1}$. Then

$$\begin{aligned} g^\tau g^{-1}\xi g g^{\tau,-1}\xi^\tau = \mathbf{1}_{m+1} \Longrightarrow g^{-1}\xi g g^{\tau,-1}\xi^\tau g^\tau \\ = \mathbf{1}_{m+1} \Longrightarrow (g^{-1}\xi g)(g^{-1}\xi g)^\tau = \mathbf{1}_{m+1}, \end{aligned}$$

that is, $g^{-1}\xi g \in \mathrm{S}_{m+1}(k')_{\mathrm{reg}}$. The uniqueness part is obvious. □

We have the following proposition.

Proposition 4.12. *Let the notation be as above.*

(1) There is a natural bijection

$$[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k') \overset{\mathbf{N}}{\longleftrightarrow} \coprod_{\beta\in[\mathrm{Her}_m^\times(k')]} [\mathrm{U}_n^\beta(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k').$$

If $\mathbf{N}\zeta^\beta=\zeta$, *we say that they* match *and denote by* $\zeta\leftrightarrow\zeta^\beta$.

(2) The set $\mathrm{S}_n(k')_{\mathrm{reg}}$ *(resp.* $\mathrm{U}_n^\beta(k')_{\mathrm{reg}}$*) is non-empty and Zariski open in* S_n *(resp.* U_n^β*). Moreover, the* $\mathbf{H}$*-stabilizer (resp.* $\mathbf{H}^\beta$*-stabilizer) of regular* ζ *(resp.* ζ^β*) is trivial.*

Proof. For (1), we start with an element $\zeta\in\mathrm{S}_n(k')_{\mathrm{reg}}$ and consider its normal form. We obtain invariants $t_1(\zeta),\dots,t_r(\zeta)$ and an element $\mathrm{Pr}(\zeta)\in\mathrm{S}_{m+1}(k')_{\mathrm{reg}}$. By Lemma 4.11, there is a pair (β,ξ^β) such that $\xi=g^{-1}\xi^\beta g$. We fix β, and then ξ^β is uniquely determined up to the $\mathrm{U}_m^\beta(k')$-conjugation. Define an element $\zeta^\beta\in\mathrm{U}_n^\beta(k')_{\mathrm{reg}}$ by

$$\zeta^\beta=\begin{bmatrix} &&&&&&t_1(\zeta)\\ &&&&&\cdots&\\ &&&&t_r(\zeta)&&\\ &&&\xi^\beta&&&\\ &&t_r(\zeta)^{\tau,-1}&&&&\\ &\cdots&&&&&\\ t_1(\zeta)^{\tau,-1}&&&&&& \end{bmatrix}.$$

By construction, $\zeta\mapsto\zeta^\beta$ defines a map

$$\mathbf{N}\colon [\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k') \to \coprod_{\beta\in[\mathrm{Her}_m^\times(k')]} [\mathrm{U}_n^\beta(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k'),$$

which is injective. The other direction is similar.

For (2), the openness is due to the following two facts: the set of pre-regular elements in both cases corresponds to the unique open cell in the Bruhat decomposition; and the three conditions in Definition 4.9 are open. The non-emptiness is essentially exhibited in [22, Sect. 3] combining with the exponential map. For the last part, we prove for $\zeta\in\mathrm{S}_n(k')_{\mathrm{reg}}$, and the case of unitary groups is similar. Write ζ in its normal form. If $g^{-1}u^{-1}\zeta u^\tau g=\zeta$, then $u_1^{-1}g^{-1}\zeta g u_1^\tau=\zeta$ with $u_1=g^{-1}ug\in U_{1^r,m+1,1^r}(k)$. We have $g^{-1}\zeta g=\zeta$, and then $g^{-1}\mathrm{Pr}(\zeta)g=\mathrm{Pr}(\zeta)$ which implies that $g=\mathbf{1}_n$, and hence $u=\mathbf{1}_n$. □

It is clear from the above discussion that the regular orbit $\zeta\in[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')$ or $[\mathrm{U}_n^\beta(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k')$ is determined by its invariants $t_i(\zeta)$ $(i=1,\dots,r)$, $a_i(\zeta):=a_i(\mathrm{Pr}(\zeta))$ $(i=1,\dots,m+1)$ and $b_i(\zeta):=b_i(\mathrm{Pr}(\zeta))$ $(i=1,\dots,m)$. We have $\zeta\leftrightarrow\zeta^\beta$ if and only if they have the same invariants. For simplicity, put $\mathbf{T}_\zeta=\mathbf{T}_{\mathrm{Pr}(\zeta)}$, $\mathbf{D}_\zeta=\mathbf{D}_{\mathrm{Pr}(\zeta)}$ and $\Delta_\zeta=\Delta_{\mathrm{Pr}(\zeta)}$.

Conjecture 4.13. (Smooth matching). *Let $k' = k'_{v'}$ be a localization. Given a Schwartz function $F \in \mathcal{S}(\mathrm{S}_n(k'))$, there exist Schwartz functions $(f^\beta \in \mathcal{S}(\mathrm{U}_n^\beta(k')))_\beta$ for each $\beta \in [\mathrm{Her}_m^\times(k')]$ such that*

$$\mathcal{O}(F, \zeta) = \mathbf{t}(\zeta)\mathcal{O}(f^\beta, \zeta^\beta)$$

for all $\zeta \in [\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')$ and $\zeta \leftrightarrow \zeta^\beta$ which are both normal. Conversely, given functions $f^\beta \in \mathcal{S}(\mathrm{U}_n^\beta(k'))$ for each $\beta \in [\mathrm{Her}_m^\times(k')]$, there exists a function $F \in \mathcal{S}(\mathrm{S}_n(k'))$ such that the above identity holds. Here, $\mathbf{t}$ is a certain "transfer factor" on $[\mathrm{S}_n(k')_{\mathrm{reg}}]/\mathbf{H}(k')$; for example, when n is odd (and hence m is even), $\mathbf{t}(\zeta) = \eta(\mathbf{T}_\zeta \cdot (\det \mathrm{Pr}(\zeta))^{-\frac{m}{2}})$.

If F and $(f^\beta)_\beta$ satisfy the property in the above conjecture, we say they *match* and denote by $F \leftrightarrow (f^\beta)_\beta$.

Proposition 4.14. *If v' is split in k, then the above conjecture of smoothing matching holds.*

Proof. Suppose that v' splits into two places $v_\bullet, v_\circ \in \mathscr{M}_k$. In this case, $\mathbf{t}$ is trivial. We may identify $\mathrm{S}_{n,v'}$ with the set of pairs $(g_\bullet, g_\circ) \in \mathrm{GL}_{n,v_\bullet} \times \mathrm{GL}_{n,v_\circ}$ with $g_\bullet g_\circ = \mathbf{1}_n$, hence with $\mathrm{GL}_{n,v'}$ by $(g_\bullet, g_\circ) \mapsto g_\bullet$. Then $F_{v'}$ becomes a function on $\mathrm{GL}_{n,v'}$, and

$$\mathcal{O}(F_{v'}, \zeta) = \int_{\mathrm{GL}_{m,v'}} \iint_{(U_{1^r,m+1,1^r,v'})^2} F_{v'}(g^{-1}\underline{u}_\bullet^{-1}\zeta\underline{u}_\circ g)\underline{\underline{\psi}}'(\underline{u}_\bullet^{-1}\underline{u}_\circ)\mathrm{d}\underline{u}_\bullet \mathrm{d}\underline{u}_\circ \mathrm{d}g,$$

for the generic character

$$\underline{\underline{\psi}}'(\underline{u}) = \psi'\left(j\left(u_{1,2} + \cdots + u_{r-1,r} + n_{r,0} + \beta_0 n_{0,r}^* + u_{r,r-1}^* + \cdots + u_{2,1}^*\right)\right),$$

where $\jmath = (j, -j)$.

On the other hand, we may identify $\mathrm{U}_{n,v'}$ with the pairs $(g_\bullet, g_\circ)$ such that $g_\circ = \mathsf{w}_{\beta'_\circ}^{-1}\,{}^t g_\bullet^{-1}\mathsf{w}_{\beta'_\circ}$, hence with $\mathrm{GL}_{n,v'}$ by $(g_\bullet, g_\circ) \mapsto g_\bullet$. Here,

$$\mathsf{w}_{\beta'_\circ} = \begin{bmatrix} & & \mathsf{w}_r \\ & \beta'_\circ & \\ \mathsf{w}_r & & \end{bmatrix},$$

where $\beta' = (\beta'_\bullet, \beta'_\circ)$. Then $f_{v'}$ becomes a function on $\mathrm{GL}_{n,v'}$, and

$$\mathcal{O}(f_{v'}, \zeta') = \int_{\mathrm{GL}_{m,v'}} \iint_{(U_{1^r,m+1,1^r,v'})^2} f_{v'}(g'^{-1}\underline{u}_\bullet'^{-1}\zeta'\underline{u}'_\circ g')\underline{\underline{\psi}}'(\underline{u}_\bullet'^{-1}\underline{u}'_\circ)\mathrm{d}\underline{u}'_\bullet \mathrm{d}\underline{u}'_\circ \mathrm{d}g'.$$

Moreover, in this case, that ζ and ζ' match exactly means $\zeta = \zeta' \in \mathrm{GL}_{n,v'}$. Therefore, if $f_{v'} = F_{v'}$, we have $\mathcal{O}(F_{v'}, \zeta) = \mathcal{O}(f_{v'}, \zeta)$ for all regular ζ. □

4.4. The fundamental lemma

To establish the equality between two relative trace formulae, one needs to prove the corresponding fundamental lemma. Let m be a nonnegative integer. Let k' be a non-archimedean local field and k/k' a separable quadratic extension of fields. There are only two non-isomorphic hermitian spaces of dimension m, if > 0, which are distinguished by the factor $\epsilon(\beta)$. In the following discussion, we use the superscript $\pm$ instead of β in the way that $\epsilon(\beta) = \pm 1$.

Assume that the extension k/k' and the character $\psi' \colon k' \to \mathbb{C}^\times$ are both unramified. As before, we write $\mathfrak{o}'$ (resp. $\mathfrak{o}$) for the ring of integers of k' (resp. k). Denote $\mathrm{val}\colon k^\times \to \mathbb{Z}$ the valuation map. We also assume that $\beta_0 \in \mathfrak{o}'$. Thus, W^+ contains a selfdual $\mathfrak{o}$-lattice L_W that extends to a selfdual $\mathfrak{o}$-lattice L_V in V^+. The unitary group U_m^+ (resp. U_n^+) is unramified and has a smooth model over $\mathrm{Spec}\,\mathfrak{o}'$ defined by L_W (resp. L_V). The group of $\mathfrak{o}'$-points $\mathrm{U}_m^+(\mathfrak{o}')$ (resp. $\mathrm{U}_n^+(\mathfrak{o}')$) is a hyperspecial maximal subgroup of $\mathrm{U}_m^+(k')$ (resp. $\mathrm{U}_n^+(k')$). We identify $\mathrm{GL}_n(\mathfrak{o})$ with $\mathrm{GL}_n(L_V)$, a hyperspecial maximal subgroup of $\mathrm{GL}_n(k)$, and put $\mathrm{S}_n(\mathfrak{o}') = \mathrm{S}_n(k') \cap \mathrm{GL}_n(\mathfrak{o})$.

We denote by $\mathcal{S}(\mathrm{U}_n^+(k) /\!\!/ \mathrm{U}_n^+(\mathfrak{o}'))$ (resp. $\mathcal{S}(\mathrm{GL}_n(k) /\!\!/ \mathrm{GL}_n(\mathfrak{o}))$) the spherical Hecke algebra of U_n^+ (resp. $\mathrm{GL}_{n,k}$). There is a base change map $b\colon \mathcal{S}(\mathrm{GL}_n(k) /\!\!/ \mathrm{GL}_n(\mathfrak{o})) \to \mathcal{S}(\mathrm{U}_n^+(k') /\!\!/ \mathrm{U}_n^+(\mathfrak{o}'))$, and also a linear map $\sigma : \mathcal{S}(\mathrm{GL}_n(k) /\!\!/ \mathrm{GL}_n(\mathfrak{o})) \to \mathcal{S}(\mathrm{S}_n(k'))$ defined by (4.11), for the local situation. For the transfer factor, we have

$$\mathbf{t}(\zeta) = \begin{cases} (-1)^{\mathrm{val}(\mathbf{T}_\zeta \cdot \prod_{i=1}^r t_i(\zeta))} & m \text{ is odd} \\ (-1)^{\mathrm{val}(\mathbf{T}_\zeta)} & m \text{ is even.} \end{cases} \tag{4.17}$$

Conjecture 4.15. (The fundamental lemma) *For an element $\tilde{F} \in \mathcal{S}(\mathrm{GL}_n(k) /\!\!/ \mathrm{GL}_n(\mathfrak{o}))$, the functions $F = \sigma(\tilde{F})$ and (f^+, f^-) match, where $f^+ = b(\tilde{F})$ and $f^- = 0$. In particular, we have*

$$\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}, \zeta) = \begin{cases} \mathbf{t}(\zeta)\mathcal{O}(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}, \zeta^+) & \zeta \leftrightarrow \zeta^+ \in \mathrm{U}_n^+(k') \\ 0 & \zeta \leftrightarrow \zeta^- \in \mathrm{U}_n^-(k'), \end{cases}$$

where ζ, ζ^+ are normal, and

$$\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}, \zeta) = \int_{\mathbf{H}(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}([\zeta]\mathbf{h})\underline{\psi}(\mathbf{h})\eta(\det \mathbf{h})\mathrm{d}\mathbf{h};$$

$$\mathcal{O}(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}, \zeta^+) = \int_{\mathbf{H}^+(k')} \mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}([\zeta^+]\mathbf{h}')\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'.$$

It is easy to see that ζ matches some element $\zeta^+ \in \mathrm{U}_n^+(k')$ (resp. $\zeta^- \in \mathrm{U}_n^-(k')$) if and only if $\mathrm{val}(\Delta_\zeta)$ is even (resp. odd).

Proposition 4.16. *If* $\mathrm{val}(\Delta_\zeta)$ *is odd, then* $\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}, \zeta) = 0$.

Proof. The following argument is modified from the one in [38]. Put

$$\mathsf{w} = \begin{bmatrix} & & \mathsf{w}_r \\ & \mathbf{1}_{m+1} & \\ \mathsf{w}_r & & \end{bmatrix}.$$

It is easy to see that $\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(s) = \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(\mathsf{w}\,{}^t s \mathsf{w})$. Since ζ is normal,

$$\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}, \zeta) = \int\limits_{U_{1^r,m+1,1^r}(k)} \int\limits_{\mathrm{GL}_m(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')} \left(\mathsf{w}\,{}^t \underline{u}^{\tau}\,{}^t g\,{}^t \zeta\,{}^t g^{-1}\,{}^t \underline{u}^{-1} \mathsf{w}\right) \underline{\psi}(\underline{u}^{-1}) \eta(\det g) \mathrm{d}g \mathrm{d}\underline{u}$$

which equals

$$\int\limits_{U_{1^r,m+1,1^r}(k)} \int\limits_{\mathrm{GL}_m(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')} \left((\mathsf{w}\,{}^t \underline{u}^{\tau,-1} \mathsf{w})^{-1}\,{}^t g (\mathsf{w}\,{}^t \zeta \mathsf{w})\,{}^t g^{-1} (\mathsf{w}\,{}^t \underline{u}^{\tau,-1} \mathsf{w})^{\tau}\right)$$
$$\underline{\psi}(\underline{u}^{-1}) \eta(\det g) \mathrm{d}g \mathrm{d}\underline{u}. \tag{4.18}$$

Since $\mathsf{w}\,{}^t \zeta \mathsf{w}$ and ζ are both normal and have the same invariants, in the proof of Lemma 4.11, we see that there exists $h \in \mathrm{GL}_m(k')$ such that $\mathsf{w}\,{}^t \zeta \mathsf{w} = h^{-1} \zeta h$ and $\eta(\det h) = -1$. Moreover, $\underline{\psi}(\underline{u}) = \underline{\psi}(\mathsf{w}\,{}^t \underline{u}^{\tau,-1} \mathsf{w})$. After making change of variables as $\mathsf{w}\,{}^t \underline{u}^{\tau,-1} \mathsf{w} \mapsto \underline{u}$ and $h\,{}^t g^{-1} \mapsto g$, we have

$$(4.18) = -\int\limits_{U_{1^r,m+1,1^r}(k)} \int\limits_{\mathrm{GL}_m(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')} \left(\underline{u}^{-1} g^{-1} \zeta g \underline{u}^{\tau}\right) \underline{\psi}(\underline{u}^{-1}) \eta(\det g) \mathrm{d}g \mathrm{d}\underline{u},$$

which implies $\mathcal{O}(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}, \zeta) = 0$.

5. A relative trace formula for $\mathrm{U}_n \times \mathrm{U}_m$: Fourier–Jacobi periods

5.1. Fourier–Jacobi models and periods

Let us briefly recall the definition of Fourier–Jacobi models and periods for unitary groups in [8]. We keep the setup in Sect. 4.1. Let $V, (\ ,\ )$ be a hermitian space over k of dimension n and $W \subset V$ a subspace of dimension m, such that the restricted hermitian form $(\ ,\ )|_W$ is nondegenerate. We assume that the orthogonal complement of W in V has the decomposition $W^{\perp} = X \oplus X^*$ where X, X^* are both r-dimensional isotropic subspaces. Then $n = m + 2r$. The hermitian form restricted on W (resp. $X \oplus X^*$) identifies W (resp. X^*) with $W_{\tau}^{\vee}$ (resp. $X_{\tau}^{\vee}$). We denote $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) the unitary group of V (resp. W) which is a reductive group over k'. Let $P'_{r,m}$ be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing X and $U'_{r,m}$ its unipotent radical. Then $U'_{r,m}$ fits into the following exact sequence

$$0 \longrightarrow \wedge_{\tau}^2 X \longrightarrow U'_{r,m} \longrightarrow \mathrm{Hom}_k(W, X) \longrightarrow 0.$$

Let $\ell'_X : X \to k$ be a nontrivial k-linear homomorphism (if exists), and U'_X a maximal unipotent subgroup of $\mathrm{GL}(X)$ stabilizing ℓ'_X. The homomorphism ℓ'_X induces two homomorphisms

$$\wedge_{\tau}^2 \ell'_X :\ \wedge_{\tau}^2 X \to \wedge_{\tau}^2 k = k^- \xrightarrow{\jmath\cdot} k',$$
$$\mathrm{Res}_{k/k'} \ell'_X :\ \mathrm{Hom}_k(W, X) \to \mathrm{Hom}_k(W, k) = \mathrm{Hom}_{k'}(\mathrm{Res}_{k/k'} W, k') = (\mathrm{Res}_{k/k'} W)^{\vee}.$$

The hermitian pair $(\ ,\ )$ of W induces a symplectic pair $\widetilde{\mathrm{Tr}}(\ ,\ )$ on the $2m$-dimensional k'-vector space $\mathrm{Res}_{k/k'} W$, through which $(\mathrm{Res}_{k/k'} W)^\vee$ is identified with $\mathrm{Res}_{k/k'} W$. Let $\mathrm{H}(\mathrm{Res}_{k/k'} W)$ be the Heisenberg group. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^2_\tau X & \longrightarrow & U'_{r,m} & \longrightarrow & \mathrm{Hom}_k(W, X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \wedge^2_\tau \ell'_X} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{k/k'} \ell'_X} & & \\ 0 & \longrightarrow & k' & \longrightarrow & \mathrm{H}(\mathrm{Res}_{k/k'} W) & \longrightarrow & \mathrm{Res}_{k/k'} W & \longrightarrow & 0. \end{array} \tag{5.1}$$

For a nontrivial character $\psi' \colon k' \to \mathbb{C}^\times$, we have the Weil representation $\omega'_{\psi'}$ of $\mathrm{H}(\mathrm{Res}_{k/k'} W) \rtimes \mathrm{Mp}(\mathrm{Res}_{k/k'} W)$. If we choose a character $\mu \colon k^\times \to \mathbb{C}^\times$ such that $\mu|_{k'^\times} = \eta$, we will have a splitting map

$$\begin{array}{ccc} & & \mathrm{Mp}(\mathrm{Res}_{k/k'} W) \\ & \nearrow^{\iota_\mu} & \downarrow \\ \mathrm{U}(W) & \xhookrightarrow{\ \iota\ } & \mathrm{Sp}(\mathrm{Res}_{k/k'} W) \end{array} \tag{5.2}$$

(see [14, §§1, 2]). By restriction, we obtain the representation $\omega'_{\psi',\mu}$ of $\mathrm{H}(\mathrm{Res}_{k/k'} W) \rtimes \mathrm{U}(W)$, and hence a representation of $U'_{r,m} \rtimes \mathrm{U}(W)$ through the middle vertical map in (5.1). Let $\lambda' \colon U'_X \to \mathbb{C}^\times$ be a generic character. Put $\nu' = \nu'(\mu, \psi', \lambda') = \omega'_{\psi',\mu} \otimes \lambda'$ which is a smooth representation of $H' := U'_{r,m} \rtimes \left(U'_X \times \mathrm{U}(W)\right)$. Then $\widetilde{\nu'} \cong \nu'(\mu^{-1}, \overline{\psi'}, \overline{\lambda'})$. As before, we have an embedding $H' \to \mathrm{U}(V) \times \mathrm{U}(W)$. Up to conjugation by the normalizer of H' in $\mathrm{U}(V) \times \mathrm{U}(W)$, ν' is determined by ψ' modulo $\mathrm{Nm}\, k^\times$ and μ.

Let π (resp. σ) be an irreducible admissible representation of $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$). A nontrivial element in $\mathrm{Hom}_{H'}(\pi \otimes \sigma \otimes \widetilde{\nu'}, \mathbb{C})$ is called a *Fourier–Jacobi model* of $\pi \otimes \sigma$. In particular, when k/k' is split, the Fourier–Jacobi model is simply the (r, r)-Fourier–Jacobi model for general linear groups introduced in Sect. 3.1. We have the following multiplicity one result.

Theorem 5.1. *Let k be of characteristic* 0 *and π, σ as above. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H'}(\pi \otimes \sigma \otimes \widetilde{\nu'}, \mathbb{C}) \le 1.$$

Proof. It is proved in [33] (for k non-archimedean) and [34] (for k archimedean) when $r = 0$. The case for general r is reduced to the previous one, as shown in [29], or [8, Sect. 15] when k is non-archimedean. □

Now we discuss the global case. Let k/k' be a quadratic extension of number fields, $\psi' \colon k'\backslash\mathbb{A}' \to \mathbb{C}^\times$ nontrivial, $\mu \colon k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ such that $\mu|_{\mathbb{A}'^\times} = \eta$, and $\lambda' \colon U'_X(k)\backslash U'_X(\mathbb{A}) \to \mathbb{C}^\times$ a generic character, which give rise to the pair (H', ν') similarly in the global situation. To define a global period, we need to fix a model for the Weil representation. Let $\mathbf{L} \subset (\mathrm{Res}_{k/k'} W)^\vee$ be a Lagrangian (k'-)subspace, and $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$ the space of Schwartz functions on $\mathbf{L}(\mathbb{A}')$. For $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, define the theta series to be

$$\theta_{\nu'}(h', \phi) = \sum_{w \in \mathbf{L}(k')} \lambda'(h') \left(\omega'_{\psi',\mu}(h')\phi \right)(w),$$

which is an automorphic form on H'.

Let π (resp. σ) be an irreducible representation of $\mathrm{U}(V)(\mathbb{A}')$ (resp. $\mathrm{U}(W)(\mathbb{A}')$) which occurs with multiplicity one in the space $\mathcal{A}_0(\mathrm{U}(V))$ (resp. $\mathcal{A}_0(\mathrm{U}(W))$). We denote by $\mathcal{A}_\pi$ (resp. $\mathcal{A}_\sigma$) the unique irreducible $\pi \subset \mathcal{A}_0(\mathrm{U}(V)$ (resp. $\sigma \subset \mathcal{A}_0(\mathrm{U}(W))$)-isotypic subspace.

Definition 5.2. (*Fourier–Jacobi period*). For $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$, and $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, we define the following integral

$$\mathcal{FJ}_r^{\nu'}(\varphi_\pi, \varphi_\sigma; \phi) = \int\limits_{H'(k')\backslash H'(\mathbb{A}')} \varphi_\pi(\varepsilon(h'))\varphi_\sigma(\kappa(h'))\theta_{\widetilde{\nu'}}(h'; \phi)\mathrm{d}h',$$

which is absolutely convergent, to be a *Fourier–Jacobi period* of $\pi \otimes \sigma$ (for a pair (H', ν')). If there exist $\varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma$, and $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$ such that $\mathcal{FJ}_r^{\nu'}(\varphi_\pi, \varphi_\sigma; \phi) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial Fourier–Jacobi period. It is obvious that $\mathcal{FJ}_r^{\nu'}$ defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}')}(\pi \otimes \sigma \otimes \widetilde{\nu'}, \mathbb{C}) = \bigotimes_{v' \in \mathcal{M}_{k'}} \mathrm{Hom}_{H'_{v'}}(\pi_{v'} \otimes \sigma_{v'} \otimes \widetilde{\nu'_{v'}}, \mathbb{C}).$$

We choose a basis $\{v_1, \dots, v_r\}$ of X as in Sect. 4.1, and denote $\{\check{v}_1, \dots, \check{v}_r\}$ the dual basis of $X_\tau^\vee$. We also choose a basis $\{w_1, \dots, w_m\}$ of W. Put $\beta = [(w_i, w_j)]_{i,j=1}^m \in \mathrm{Her}_m^\times(k')$. We identify $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) with a unitary group U_n (resp. U_m) of n (resp. m) variables under the basis $\{v_1, \dots, v_r, w_1, \dots, w_m, \check{v}_r, \dots, \check{v}_1\}$, and view U_m as a subgroup of U_n. Let $U'_{1^r,m} = U'_{r,m} \rtimes U'_X$ be the unipotent radical of H'. The image of $H'(\mathbb{A}')$ in $\mathrm{U}_n(\mathbb{A}')$ consists of the matrices $h' = h'(n', b'; u'; g') = \underline{u}'(n', b'; u') \cdot g'$, where $g' \in \mathrm{U}_m(\mathbb{A}')$ and

$$\underline{u}' = \underline{u}'(n', b'; u') = \begin{bmatrix} \mathbf{1}_r & n' & \mathsf{w}_r\left(b' + \frac{n'n'_\beta}{2}\right) \\ & \mathbf{1}_m & n'_\beta \\ & & \mathbf{1}_r \end{bmatrix} \begin{bmatrix} u' & & \\ & \mathbf{1}_m & \\ & & \check{u}' \end{bmatrix} \in U'_{1^r,m}(\mathbb{A}')$$

for $n' \in \mathrm{Mat}_{r,m}(\mathbb{A})$, $b' \in \overline{\mathrm{Her}}_r(\mathbb{A}')$, $u' \in U'_X(\mathbb{A})$, $n'_\beta = -\beta^{-1}\,{}^t n'^\tau \mathsf{w}_r$ and $\check{u}' = \mathsf{w}_r\,{}^t u'^{\tau,-1}\mathsf{w}_r$. If $r > 0$, let $U^\ddagger$ be the unipotent radical of the parabolic subgroup of $\mathrm{U}(\{v_r, \check{v}_r\} \oplus W)$ stabilizing the flag $0 \subset \{v_r\}$. Put $H^\ddagger = U^\ddagger \rtimes \mathrm{U}(W)$ (resp. $H^\ddagger = \mathrm{U}(W)$) if $r > 0$ (resp. $r = 0$). There is a natural map $H' \to H^\ddagger$. We write $h^\ddagger = \underline{u}^\ddagger g'$ to be the image of h' under this map. Then we have

$$\nu'(h') = \underline{\psi}'(\underline{u}')\omega'_{\psi',\mu}(h^\ddagger) = \psi'\left(\widetilde{\mathrm{Tr}}\left(u'_{1,2} + \cdots + u'_{r-1,r}\right)\right)\omega'_{\psi',\mu}(h^\ddagger).$$

5.2. Decomposition of distributions

This time, we start from the relative trace formula on general linear groups. We identify $\mathrm{GL}_{m,k} \subset \mathrm{GL}_{n,k}$ with $\mathrm{GL}(W) \subset \mathrm{GL}(V)$, and view $\mathrm{GL}_{n,k'}$ (resp. $\mathrm{GL}_{m,k'}$) as a subgroup of $\mathrm{Res}_{k/k'}\mathrm{GL}_{n,k}$ (resp. $\mathrm{Res}_{k/k'}\mathrm{GL}_{m,k}$) via the basis $\{v_1, \dots, v_r, w_1, \dots, w_m, \check{v}_r, \dots, \check{v}_1\}$.

Recall that we realize the Weil representation $\omega_{\psi,\mu}$ on the space $\mathcal{S}(W^\vee(\mathbb{A}))$ where $\psi = \psi' \circ \widetilde{\mathrm{Tr}}$. Put

$$W^+ = \bigoplus_{i=1}^r k' w_i^\vee, \quad W^- = \bigoplus_{i=1}^r k^- w_i^\vee, \quad W_+ = \bigoplus_{i=1}^r k' w_i, \quad W_- = \bigoplus_{i=1}^r k^- w_i,$$

which are vector spaces over k'. Then $\mathrm{Res}_{k/k'} W = W_+ \oplus W_-$ and $\mathrm{Res}_{k/k'} W^\vee = W^+ \oplus W^-$. We also put $W^\dagger = W^+ \oplus W_+$, which is a vector space over k'. Define a linear isomorphism from $\mathcal{S}(W^\vee(\mathbb{A}))$ to $\mathcal{S}(W^\dagger(\mathbb{A}'))$ by $\Phi \mapsto \Phi^\dagger$, where for $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$,

$$\Phi^\dagger(w^+, w_+) = \int_{W^-(\mathbb{A}')} \Phi(w^+, w^-)\psi(w^- w_+)\mathrm{d}w^-. \tag{5.3}$$

If $r > 0$, let $U^\dagger$ be the unipotent radical of the parabolic subgroup of $\mathrm{GL}(\{v_r, \check{v}_r\} \oplus W)$ stabilizing the flag $0 \subset \{v_r\} \subset \{v_r\} \oplus W$. Put $H^\dagger = U^\dagger \rtimes \mathrm{GL}_m$ (resp. $H^\dagger = \mathrm{GL}_m$) if $r > 0$ (resp. $r = 0$). It consists of elements $h^\dagger = \underline{u}^\dagger g = \underline{u}^\dagger(n^+, n^-, n_+, n_-, b^\dagger)g$ where

$$\underline{u}^\dagger(n^+, n^-, n_+, n_-, b^\dagger) = \begin{bmatrix} 1 & n^+ + n^- & b^\dagger + \frac{(n^+ + n^-)(n_+ + n_-)}{2} \\ & \mathbf{1}_m & n_+ + n_- \\ & & 1 \end{bmatrix}$$

with $n^+ \in \mathrm{Mat}_{1,m}(k')$, $n^- \in \mathrm{Mat}_{1,m}(k^-)$, $n_+ \in \mathrm{Mat}_{m,1}(k')$, and $n_- \in \mathrm{Mat}_{m,1}(k^-)$. We have a natural quotient homomorphism $H \to H^\dagger$.

Define a representation $\omega^\dagger_{\overline{\psi},\overline{\mu}}$ of $H(\mathbb{A})$ on $\mathcal{S}(W^\dagger(\mathbb{A}'))$ by

$$\omega^\dagger_{\overline{\psi},\overline{\mu}}(h)\Phi^\dagger = \left(\omega_{\overline{\psi},\overline{\mu}}(h)\Phi\right)^\dagger.$$

It is easy to see that $\omega^\dagger_{\overline{\psi},\overline{\mu}}$ factors through $H^\dagger$ and

$$\begin{aligned}&\left(\omega^\dagger_{\overline{\psi},\overline{\mu}}\left(\underline{u}^\dagger\left(n^+, n^-, n_+, n_-, b^\dagger\right) g\right)\Phi^\dagger\right)(w^+, w_+) \\ &= \eta(\det g)\overline{\psi}\left(b^\dagger + w^+ n_- + n^- w_+ + \frac{n^+ n_- - n^- n_+}{2}\right)\Phi^\dagger \\ &\times \left((w^+ + n^+)g, g^{-1}(w_+ - n_+)\right)\end{aligned} \tag{5.4}$$

if $g \in \mathrm{GL}_m(k')$. Moreover, we have the Poisson summation formula

$$\sum_{w^\dagger \in W^\dagger(k')} \Phi^\dagger(w^\dagger) = \sum_{w^\flat \in W^\vee(k)} \Phi(w^\flat).$$

Until the end of this subsection, we assume that n is odd, and hence m is odd as well. Since the other case is similar and will lead to the same fundamental lemma, we omit it in the following discussion. We proceed exactly as in Sect. 4.2 and take μ to be the character used in (5.2) which is assumed to be unitary. For $F_n \in \mathcal{S}(\mathrm{GL}_n(\mathbb{A}))$, $F_m \in \mathcal{S}(\mathrm{GL}_m(\mathbb{A}))$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$, we associate to $F_n \otimes F_m$ a kernel function $\mathcal{K}_{F_n\otimes F_m}(g_1, g_2; g_3, g_4)$ as (4.8) and consider the distribution

$$\mathcal{J}^\mu(s; F_n \otimes F_m \otimes \Phi) = \int_{Z'_m(\mathbb{A}')\,\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{Z'_n(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int_{H(k)\backslash H(\mathbb{A})} \mathcal{K}_{F_n\otimes F_m}(\varepsilon(h), \kappa(h); g_1, g_2)\theta_{\widetilde{\nu}}(h, \Phi)|\det h|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}h\mathrm{d}g_1\mathrm{d}g_2. \tag{5.5}$$

Such distribution formally computes

$$\sum_{\Pi,\Sigma}\sum \mathcal{F}\mathcal{J}^\nu_{r,r}(s; \rho(F_n)\varphi_\Pi, \rho(F_m)\varphi_\Sigma; \Phi)\mathcal{P}(\overline{\varphi_\Pi})\mathcal{P}(\overline{\varphi_\Sigma}),$$

where the inner sum is taken over orthonormal bases of $\mathcal{A}_\Pi$ and $\mathcal{A}_\Sigma$. Proceeding similarly as in (4.9), we have

$$(5.5) = \int_{Z'_m(\mathbb{A}')\,\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{Z'_n(\mathbb{A}')\,\mathrm{GL}_n(k')\backslash \mathrm{GL}_n(\mathbb{A}')} \int_{U_{1^r,m,1^r}(k)\backslash U_{1^r,m,1^r}(\mathbb{A})} \int_{\mathrm{GL}_m(\mathbb{A})} \sum_{\zeta\in\mathrm{GL}_n(k)} F_n(g^{-1}g_2^{-1}\underline{u}^{-1}\zeta g_1)F_m(g^{-1})\theta_{\widetilde{\nu}}(\underline{u}g_2g, \Phi)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}} |\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}\underline{u}\mathrm{d}g_1\mathrm{d}g_2,$$

which equals

$$\int_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{U_{1^r,m,1^r}(k)\backslash U_{1^r,m,1^r}(\mathbb{A})} \int_{\mathrm{GL}_m(\mathbb{A})} \sum_{\zeta\in\mathrm{S}_n(k')} \sigma(F_n)(g^{-1}g_2^{-1}\underline{u}^{-1}\zeta \underline{u}^\tau g_2 g^\tau)F_m(g^{-1})\theta_{\widetilde{\nu}}(\underline{u}g_2g, \Phi) |\det g|_{\mathbb{A}}^{s-\frac{1}{2}}|\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}\underline{u}\mathrm{d}g_2. \tag{5.6}$$

Unfolding $U_{1^r,m,1^r}(k)$, we have

$$(5.6) = \sum_{\zeta\in[\mathrm{S}_n(k')]/U_{1^r,m,1^r}(k)} \int_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{\mathrm{Stab}_\zeta^{U_{1^r,m,1^r}}(k')\backslash U_{1^r,m,1^r}(\mathbb{A})} \int_{\mathrm{GL}_m(\mathbb{A})} \sigma(F_n)(g^{-1}g_2^{-1}([\zeta]\underline{u})g_2 g^\tau)F_m(g^{-1})\theta_{\widetilde{\nu}}(\underline{u}g_2g, \Phi)|\det g|_{\mathbb{A}}^{s-\frac{1}{2}} |\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}\underline{u}\mathrm{d}g_2,$$

which equals

$$\sum_{\zeta\in[\mathrm{S}_n(k')]/U_{1^r,m,1^r}(k)} \int_{\mathrm{Stab}_\zeta^{U_{1^r,m,1^r}}(k')\backslash U_{1^r,m,1^r}(\mathbb{A})} \int_{\mathrm{GL}_m(k')\backslash \mathrm{GL}_m(\mathbb{A}')} \int_{\mathrm{GL}_m(\mathbb{A})}$$

$$\sigma(F_n)(g^{-1}([g_2^{-1}\zeta g_2]\underline{u})g^\tau)F_m(g^{-1})\theta_{\widetilde{\nu}}\left(g_2, \omega_{\overline{\psi},\overline{\mu}}(\underline{u}g)\Phi\right)\underline{\psi}(\underline{u}^{-1})|\det g|_{\mathbb{A}}^{s-\frac{1}{2}}$$

$$|\det g_2|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}g_2\mathrm{d}\underline{u}. \tag{5.7}$$

On the other hand, we have

$$\begin{aligned}\theta_{\widetilde{\nu}}\left(g_2, \omega_{\overline{\psi},\overline{\mu}}(\underline{u}g)\Phi\right) &= \sum_{w^\flat\in W^\vee(k)} \left(\omega_{\overline{\psi},\overline{\mu}}(g_2)\omega_{\overline{\psi},\overline{\mu}}(\underline{u}g)\Phi\right)(w^\flat)\\ &= \sum_{w^\dagger\in W^\dagger(k')} \left(\omega^\dagger_{\overline{\psi},\overline{\mu}}(g_2)\left(\omega_{\overline{\psi},\overline{\mu}}(\underline{u}g)\Phi\right)^\dagger\right)(w^\dagger)\\ &= \sum_{x\in\mathrm{Mat}_{1,m}(k')}\sum_{y\in\mathrm{Mat}_{m,1}(k')} \eta(\det g_2)\left(\omega_{\overline{\psi},\overline{\mu}}(\underline{u}g)\Phi\right)^\dagger(xg_2, g_2^{-1}y).\end{aligned}$$

To proceed, we introduce a k'-variety

$$\mathrm{S}_{n,m} = \mathrm{S}_n \times \mathrm{Mat}_{1,m,k'} \times \mathrm{Mat}_{m,1,k'}.$$

As before, we put $\mathbf{H} = \mathrm{Res}_{k/k'}(U_{1^r,m,1^r}) \rtimes \mathrm{GL}_{m,k'}$ which acts on $\mathrm{S}_{n,m}$ from right in the following way. For a k'-algebra R, $\mathbf{h} = \mathbf{h}(\underline{u}, g)$ with $\underline{u} \in U_{1^r,m,1^r}(R\otimes k)$, $g \in \mathrm{GL}_m(R)$ and $[s, x, y] \in \mathrm{S}_{n,m}(R)$, the right action is given by $[s, x, y]\mathbf{h} = [g^{-1}\underline{u}^{-1}su^\tau g, xg, g^{-1}y]$. Denote $[\mathrm{S}_{n,m}(k')]/\mathbf{H}(k')$ the set of k'-orbits under this action. We also put $\underline{\psi}(\mathbf{h}) = \underline{\psi}(\underline{u}^{-1})$, and $\det\mathbf{h} = \det g$. Then

$$(5.7) = \mathfrak{J}^\mu(s; F_n\otimes F_m\otimes\Phi) := \sum_{[\zeta,x,y]\in[\mathrm{S}_{n,m}(k')]/\mathbf{H}(k')} \mathfrak{J}^\mu_{[\zeta,x,y]}(s; F_n\otimes F_m\otimes\Phi),$$

where

$$\mathfrak{J}^\mu_{[\zeta,x,y]}(s; F_n\otimes F_m\otimes\Phi) = \int_{\mathrm{Stab}^{\mathbf{H}}_{[\zeta,x,y]}(k')\backslash\mathbf{H}(\mathbb{A}')} \int_{\mathrm{GL}_m(\mathbb{A})}$$

$$\sigma(F_n)(g^{-1}[\zeta]\mathbf{h}g^\tau)F_m(g^{-1})\left(\omega^\dagger_{\overline{\psi},\overline{\mu}}(\mathbf{h}g)\Phi^\dagger\right)(x, y)\underline{\psi}(\mathbf{h})|\det\mathbf{h}|_{\mathbb{A}}^{s-\frac{1}{2}}|\det g|_{\mathbb{A}}^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}\mathbf{h}. \tag{5.8}$$

We denote $\mathrm{S}_{n,m}(k')_{\mathrm{reg}}$ the set of regular k'-elements, a notion that will be defined in Definition 5.9. In particular, the $\mathbf{H}$-stabilizer $\mathrm{Stab}^{\mathbf{H}}_{[\zeta,x,y]}$ is trivial for $[\zeta, x, y]$ regular, and the corresponding term

$$\mathcal{O}_\mu(s; F_n\otimes F_m\otimes\Phi, [\zeta, x, y]) := \mathfrak{J}^\mu_{[\zeta,x,y]}(s; F_n\otimes F_m\otimes\Phi)$$

is an orbital integral. If $F_n = \otimes_{v'} F_{n,v'}$, $F_m = \otimes_{v'} F_{m,v'}$ and $\Phi = \otimes_{v'} \Phi_{v'}$ are factorizable, then

$$\mathcal{O}_\mu(s; F_n \otimes F_m \otimes \Phi, [\zeta, x, y]) = \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}_{\mu_{v'}}(s; F_{n,v'} \otimes F_{m,v'} \otimes \Phi_{v'}, [\zeta, x, y]),$$

where the local orbital integrals $\mathcal{O}_{\mu_{v'}}$ are defined similarly as in (5.8) with $\mathbb{A}'$ (resp. $\mathbb{A}$) replaced by $k'_{v'}$ (resp. $k_{v'}$). In particular, if the function

$$[\zeta, x, y] \mapsto \int_{\mathrm{GL}_m(\mathbb{A})} \sigma(F_n)(g^{-1}\zeta g^\tau) F_m(g^{-1}) \left(\omega^\dagger_{\underline{\psi},\overline{\mu}}(g)\Phi^\dagger\right)(x, y) |\det g|_{\mathbb{A}}^{s-\frac{1}{2}} \mathrm{d}g$$

supports only on regular elements, then

$$\begin{aligned}
\mathcal{J}^\mu(s; F_n \otimes F_m \otimes \Phi) &= \mathcal{J}^\mu_{\mathrm{reg}}(F_n \otimes F_m \otimes \Phi) \\
&= \sum_{[\zeta,x,y] \in [\mathrm{S}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}(k')} \mathcal{J}^\mu_{[\zeta,x,y]}(s; F_n \otimes F_m \otimes \Phi) \\
&= \sum_{[\zeta,x,y] \in [\mathrm{S}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}(k')} \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}_{\mu_{v'}}(s; F_{n,v'} \otimes F_{m,v'} \otimes \Phi_{v'}, [\zeta, x, y]).
\end{aligned}$$

When $s = \frac{1}{2}$, we suppress s in notation. In particular, if $[\zeta, x, y]$ is regular,

$$\begin{aligned}
\mathcal{J}^\mu_{[\zeta,x,y]}(F_n \otimes F_m \otimes \Phi) &= \mathcal{O}_\mu(F_n \otimes F_m \otimes \Phi, [\zeta, x, y]) \\
&= \int_{\mathbf{H}(\mathbb{A}')} \int_{\mathrm{GL}_m(\mathbb{A})} \sigma(F_n)(g^{-1}[\zeta]\mathbf{h}g^\tau) F_m(g^{-1}) \\
&\qquad \left(\omega^\dagger_{\underline{\psi},\overline{\mu}}(\mathbf{h}g)\Phi^\dagger\right)(x, y)\underline{\psi}(\mathbf{h})\mathrm{d}g\mathrm{d}\mathbf{h}. \qquad (5.9)
\end{aligned}$$

Now we describe the relative trace formula on unitary groups. Take $f_n \in \mathcal{S}(\mathrm{U}_n(\mathbb{A}'))$, $f_m \in \mathcal{S}(\mathrm{U}_m(\mathbb{A}'))$ and $\phi_\alpha \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$ for $\alpha = 1, 2$. As before, we have the kernel function $\mathcal{K}_{f_n \otimes f_m}$ (4.2). Put

$$\begin{aligned}
&\mathcal{J}^\mu(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) \\
&= \iint_{(H'(k')\backslash H'(\mathbb{A}'))^2} \mathcal{K}_{f_n \otimes f_m}(\varepsilon(h'_1), \kappa(h'_1); \varepsilon(h'_2), \kappa(h'_2)) \theta_{\widetilde{\nu'}}(h'_1, \phi_1) \theta_{\nu'}\left(h'_2, \phi_2\right) \mathrm{d}h'_1 \mathrm{d}h'_2.
\end{aligned}$$

(5.10)

Such distribution formally computes

$$\sum_{\pi,\sigma} \sum \mathcal{F}\mathcal{J}^{\nu'}_r(\rho(f_n)\varphi_\pi, \rho(f_m)\varphi_\sigma; \phi_1) \mathcal{F}\mathcal{J}^{\widetilde{\nu'}}_r(\overline{\varphi_\pi}, \overline{\varphi_\sigma}; \phi_2),$$

where the inner sum is taken over orthonormal bases of $\mathcal{A}_\pi$ and $\mathcal{A}_\sigma$. Collapse the summation over ξ' and make a change of variable as $g_2'^{-1}g_1' \mapsto g_1'$. We have

$$(5.10) = \int_{\mathrm{U}_m(k')\backslash \mathrm{U}_m(\mathbb{A}')} \iint_{(U'_{1^r,m}(k')\backslash U'_{1^r,m}(\mathbb{A}'))^2} \int_{\mathrm{U}_m(\mathbb{A}')} \sum_{\zeta' \in \mathrm{U}_n(k')} f_n(g_1'^{-1}g_2'^{-1}\underline{u}_1'^{-1}\zeta'\underline{u}_2'g_2') f_m(g_1'^{-1})\theta_{\widetilde{\nu'}}\left(\underline{u}_1'g_2', \omega'_{\overline{\psi'},\overline{\mu}}(g_1')\phi_1\right) \theta_{\nu'}\left(\underline{u}_2'g_2', \phi_2\right) \mathrm{d}g_1'\mathrm{d}\underline{u}_1'\mathrm{d}\underline{u}_2'\mathrm{d}g_2'. \tag{5.11}$$

Put $\mathrm{U}_{n,m} := \mathrm{U}_n \times \mathrm{Res}_{k/k'} \mathrm{Mat}_{1,m,k}$. Recall that we have defined an k'-algebraic group $\mathbf{H}'$ in Sect. 4.2, which acts on $\mathrm{U}_{n,m}$ from right in the following way. For a k'-algebra R, $\mathbf{h}' = \mathbf{h}'(\underline{u}_1', \underline{u}_2', g') \in \mathbf{H}'(R)$ and $[g, z] \in \mathrm{U}_n(R) \times \mathrm{Mat}_{1,m}(R \otimes k)$, the right action is given by $[g, z]\mathbf{h}' = [g'^{-1}\underline{u}_1'^{-1}g\underline{u}_2'g', zg']$. We also put

$$\underline{\psi}'(\mathbf{h}') = \underline{\psi}'\left(\mathbf{h}'(\underline{u}_1', \underline{u}_2'; g')\right) = \underline{\psi}'(\underline{u}_1'^{-1}\underline{u}_2'),$$

and $\det \mathbf{h}' = \det g'$. Introduce the group

$$\mathbf{H}^\ddagger = H^\ddagger \underset{\mathrm{U}_m}{\times} H^\ddagger \text{ (resp. } \mathrm{U}_m)$$

if $r > 0$ (resp. $r = 0$), whose elements are denoted by $\mathbf{h}^\ddagger = \mathbf{h}^\ddagger(\underline{u}_1^\ddagger, \underline{u}_2^\ddagger; g')$ where where

$$\underline{u}_i^\ddagger = \underline{u}_i^\ddagger(n_i^\ddagger, b_i^\ddagger) = \begin{bmatrix} 1 & n_i^\ddagger & b_i^\ddagger - \frac{n_i^\ddagger \beta^{-1}\, {}^t n_i^{\ddagger,\tau}}{2} \\ & \mathbf{1}_m & -\beta^{-1}\, {}^t n_i^{\ddagger,\tau} \\ & & 1 \end{bmatrix}$$

for $n_i^\ddagger \in W^\vee(\mathbb{A})$ and $b_i^\ddagger \in \mathbb{A}^-$. We have a natural quotient homomorphism $\mathbf{H}' \to \mathbf{H}^\ddagger$.

Lemma 5.3. *Let $\omega^\ddagger_{\overline{\psi'}}$ be the representation of $\mathbf{H}^\ddagger$ on the space $\mathcal{S}(W^\vee(\mathbb{A}))$ defined by the formula*

$$\begin{aligned}\left(\omega^\ddagger_{\overline{\psi'}}(\mathbf{h}^\ddagger)\phi\right)(z) &= \left(\omega^\ddagger_{\overline{\psi'}}\left(\mathbf{h}^\ddagger(\underline{u}_1^\ddagger, \underline{u}_2^\ddagger; g')\right)\phi\right)(z) \\ &= \overline{\psi}\left(b_1^\ddagger - b_2^\ddagger + z\beta^{-1}\left({}^t n_2^{\ddagger,\tau} - {}^t n_1^{\ddagger,\tau}\right)\right. \\ &\quad \left. + n_1^\ddagger\beta^{-1}\, {}^t n_2^{\ddagger,\tau}\right)\phi\left(\left(z + n_1^\ddagger + n_2^\ddagger\right)g'\right),\end{aligned}$$

which we also regard as a representation of $\mathbf{H}'$ via the inflation $\mathbf{H}' \to \mathbf{H}^\ddagger$. Then (the complete projective tensor product) $\omega'_{\overline{\psi'},\overline{\mu}} \otimes \omega'_{\psi',\mu}$, viewed as a representation of $\mathbf{H}'$, is isomorphic to $\omega^\ddagger_{\overline{\psi'}}$, under which we have

$$\theta_{\widetilde{\nu'}}\left(\underline{u}_1'g', \phi_1\right)\theta_{\nu'}\left(\underline{u}_2'g', \phi_2\right) = \underline{\psi}'(\mathbf{h}') \sum_{z \in W^\vee(k)} \left(\omega^\ddagger_{\overline{\psi'}}(\mathbf{h}')\left(\phi_1 \otimes \phi_2\right)^\ddagger\right)(z)$$

for $\mathbf{h}' = \mathbf{h}'(\underline{u}_1', \underline{u}_2', g') \in \mathbf{H}'$ via restriction, where $(\phi_1 \otimes \phi_2)^\ddagger$ is the image of $\phi_1 \otimes \phi_2$ under such isomorphism.

Proof. The isomorphism is given by [14, Proposition 2.2 (i), (ii)], which comes from a (partial) Fourier transform, and the last formula follows from the Poisson summation formula. □

By the above lemma and repeating the processes in (5.6), (5.7), (5.8), we have

$$(5.11) = \mathcal{J}^{\mu}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) := \sum_{[\zeta', z] \in [\mathrm{U}_{n,m}(k')]/\mathbf{H}'(k')} \mathcal{J}^{\mu}_{[\zeta', z]}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2),$$

where

$$\mathcal{J}^{\mu}_{[\zeta', z]}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) = \int_{\mathrm{Stab}^{\mathbf{H}'}_{[\zeta', z]}(k') \backslash \mathbf{H}'(\mathbb{A}')} \int_{\mathrm{U}_m(\mathbb{A}')} f_n(g'^{-1}[\zeta']\mathbf{h}') f_m(g'^{-1}) \left(\omega^{\ddagger}_{\overline{\psi'}}(\mathbf{h}') \left(\omega'_{\overline{\psi'}, \overline{\mu}}(g')\phi_1 \otimes \phi_2\right)^{\ddagger}\right)(z) \underline{\psi}'(\mathbf{h}') \mathrm{d}g' \mathrm{d}\mathbf{h}'. \tag{5.12}$$

We denote $\mathrm{U}_{n,m}(k')_{\mathrm{reg}}$ the set of regular k'-elements, a notion that will be defined in Definition 5.9. In particular, the $\mathbf{H}'$-stabilizer $\mathrm{Stab}^{\mathbf{H}'}_{[\zeta', z]}$ is trivial for $[\zeta', z]$ regular, and the corresponding term

$$\mathcal{O}_{\mu}(f_n \otimes f_m \otimes \phi \otimes \phi_2, [\zeta', z]) := \mathcal{J}^{\mu}_{[\zeta', z]}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2)$$

is an orbital integral. If $f_n = \otimes_{v'} f_{n,v'}$, $f_m = \otimes_{v'} f_{m,v'}$ and $\phi_i = \otimes_{v'} \phi_{i,v'}$ are factorizable, then

$$\mathcal{O}_{\mu}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2, [\zeta', z]) = \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}_{\mu_{v'}}(f_{n,v'} \otimes f_{m,v'} \otimes \phi_{1,v'} \otimes \phi_{2,v'}, [\zeta', z]),$$

where the local orbital integrals $\mathcal{O}_{\mu_{v'}}$ are defined similarly as in (5.12) with $\mathbb{A}'$ replaced by $k'_{v'}$. In particular, if the function

$$[\zeta', z] \mapsto \int_{\mathrm{U}_m(\mathbb{A}')} f_n(g'^{-1}\zeta') f_m(g'^{-1}) \left(\omega'_{\overline{\psi'}, \overline{\mu}}(g')\phi_1 \otimes \phi_2\right)^{\ddagger}(z) \mathrm{d}g'$$

supports only on regular elements, then

$$\begin{aligned} \mathcal{J}^{\mu}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) &= \mathcal{J}^{\mu}_{\mathrm{reg}}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) \\ &= \sum_{[\zeta', z] \in [\mathrm{U}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}'(k')} \mathcal{J}^{\mu}_{[\zeta', z]}(f_n \otimes f_m \otimes \phi_1 \otimes \phi_2) \\ &= \sum_{[\zeta', z] \in [\mathrm{U}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}'(k')} \prod_{v' \in \mathscr{M}_{k'}} \mathcal{O}_{\mu_{v'}}(f_{n,v'} \otimes f_{m,v'} \otimes \phi_{1,v'} \otimes \phi_{2,v'}, [\zeta', z]). \end{aligned}$$

Remark 5.4. Take a place v' of k' and let $[\zeta, x, y]$ be a regular element. The orbital integral $\mathcal{O}_{\mu_{v'}}(-, [\zeta, x, y])$ introduced above actually factors through (the algebraic tensor product) $\mathcal{S}(\mathrm{S}_n(k'_{v'})) \otimes \mathcal{S}(\mathrm{GL}_m(k_{v'})) \otimes \mathcal{S}(W^\vee(k_{v'}))$. Note that the (Fréchet) space $\mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'}))$ has a natural topology under which the subspace $\mathcal{S}(\mathrm{S}_n(k'_{v'}))\otimes\mathcal{S}(\mathrm{GL}_m(k_{v'}))\otimes\mathcal{S}(W^\vee(k_{v'}))$ is dense (which is the whole space if and only if v' is non-archimedean). In fact, the linear functional $\mathcal{O}_{\mu_{v'}}(-, [\zeta, x, y])$ is continuous and hence extends uniquely to a continuous linear functional on $\mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'}))$. Similar, for a regular element $[\zeta', z]$, the linear functional $\mathcal{O}_{\mu_{v'}}(-, [\zeta', z])$ extends uniquely to $\mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$. If v' splits in k, we will give a direct construction of such extensions in Proposition 5.11.

5.3. Matching of orbits and functions

Suppose we are in either local or global situations.

Definition 5.5. (*Regular element*). Put $\mathfrak{M}_m = \mathrm{Mat}_m \times \mathrm{Mat}_{1,m} \times \mathrm{Mat}_{m,1}$. An element $[\xi, x, y] \in \mathfrak{M}_m(k)$ is called *regular* if it satisfies

- ξ is regular semisimple as an element of $\mathrm{Mat}_m(k)$;
- the vectors $\left\{x, x\xi, \dots, x\xi^{m-1}\right\}$ span the k-vector space $\mathrm{Mat}_{1,m}(k)$;
- the vectors $\left\{y, \xi y, \dots, \xi^{m-1} y\right\}$ span the k-vector space $\mathrm{Mat}_{m,1}(k)$.

For $[\xi, x, y] \in \mathfrak{M}_m(k)$, put

- $a_i([\xi, x, y]) = \mathrm{Tr} \bigwedge^i \xi$ for $1 \le i \le m$;
- $b_i([\xi, x, y]) = x\xi^i y$ for $0 \le i \le m-1$;
-
$$\mathbf{T}_{[\xi,x,y]} = \det \begin{bmatrix} x \\ x\xi \\ \vdots \\ x\xi^{m-1} \end{bmatrix};$$
- $\mathbf{D}_{[\xi,x,y]}$ to be the matrix $[x\xi^{i+j-2}y]_{i,j=1}^m$; and
- $\Delta_{[\xi,x,y]} = \det \mathbf{D}_{[\xi,x,y]}$.

It is clear that $\Delta_{[\xi,x,y]} \neq 0$ if $[\xi, x, y]$ is regular. The group GL_m acts on $\mathfrak{M}_m$ from right by $[\xi, x, y]g = [g^{-1}\xi g, xg, g^{-1}y]$, under which a_i, b_i, $\mathbf{D}_{[\xi,x,y]}$ and $\Delta_{[\xi,x,y]}$ are invariant. We denote $\mathfrak{M}_m(k)_{\mathrm{reg}}$ the set of regular elements.

Lemma 5.6. *Two regular elements $[\xi, x, y]$ and $[\xi', x', y']$ are in the same $\mathrm{GL}_m(k)$-orbit if and only if $a_i([\xi, x, y]) = a_i([\xi', x', y'])$ $(1 \le i \le m)$ and $b_i([\xi, x, y]) = b_i([\xi', x', y'])$ $(0 \le i \le m-1)$. The GL_m-stabilizer of a regular element is trivial.*

Proof. See [32, Proposition 6.2 and Theorem 6.1] for the first and second statements, respectively. □

We define two more spaces

$$\mathfrak{S}_m = \{[\xi, x, y] \in \mathfrak{M}_m(k)_{\mathrm{reg}} \mid \xi \in \mathrm{S}_m(k'), x \in \mathrm{Mat}_{1,m}(k'), y \in \mathrm{Mat}_{m,1}(k')\};$$
$$\mathfrak{U}_m^\natural = \{[\beta; \xi^\beta, z, z^*] \mid \beta \in \mathrm{Her}_m^\times(k'), [\xi^\beta, z, z^*] \in \mathfrak{M}_m(k)_{\mathrm{reg}}, \xi^\beta \in \mathrm{U}_m^\beta(k'), z^* = \beta^{-1}\, {}^t z^\tau\},$$

where $\mathrm{U}_m^\beta = \mathrm{U}(W^\beta)$. For $\mathfrak{U}_m^\natural$, we also define a right $\mathrm{GL}_m(k)$-action by $[\beta; \xi^\beta, z, z^*]g = [{}^t g^\tau \beta g; g^{-1}\xi g, zg, g^{-1}z^*]$. For $[\xi, x, y] \in \mathfrak{S}_m$, we write $[\xi, x, y] \Leftrightarrow [\beta; \xi^\beta, z, z^*]$ if there exists $g \in \mathrm{GL}_m(k)$ such that $[\xi, x, y] = [\xi^\beta, z, z^*]g$. We have the following lemma which is similar to Lemma 4.11.

Lemma 5.7. *For $[\xi, x, y] \in \mathfrak{S}_m$, there exists an element $[\beta; \xi^\beta, z, z^*] \in \mathfrak{U}_m^\natural$, unique up to the previous $\mathrm{GL}_m(k)$-action, such that $[\xi, x, y] \Leftrightarrow [\beta; \xi^\beta, z, z^*]$. Conversely, for $[\beta; \xi^\beta, z, z^*] \in \mathfrak{U}_m^\natural$, there exists an element $[\xi, x, y] \in \mathfrak{S}_m$, unique up to the $\mathrm{GL}_m(k')$-action, such that $[\xi, x, y] \Leftrightarrow [\beta; \xi^\beta, z, z^*]$.*

Proof. We first point out that two elements $[\xi, x, y]$ and $[\xi', x', y']$ of $\mathfrak{S}_m$ are conjugate under $\mathrm{GL}_m(k)$ if and only if they are conjugate under $\mathrm{GL}_m(k')$. In fact, assume $[\xi, x, y]g = [\xi', x', y']$. Then $g^{-1}\xi g = \xi'$ implies that $g^{\tau,-1}\xi g^\tau = \xi'$; $xg = x'$ implies that $xg^\tau = x'$; $g^{-1}y = y'$ implies that $g^{\tau,-1}y = y'$. Therefore, $g = g^\tau$.

It is easy to see that for $[\xi, x, y] \in \mathfrak{M}(k)_{\mathrm{reg}}$, $[\xi, x, y]$ and $[{}^t\xi, {}^t y, {}^t x]$ have the same invariants. Therefore, there is a unique $g \in \mathrm{GL}_m(k)$ such that $g^{-1}\xi g = {}^t\xi$, $xg = {}^t y$, $g^{-1}y = {}^t x$. Now if $[\xi, x, y] \in \mathfrak{S}_m$, then $[{}^t\xi, {}^t y, {}^t x] \in \mathfrak{S}_m$, which implies that $g = g^\tau$. Moreover, we have $g = {}^t g$ and ${}^t\xi^\tau g^{-1}\xi = g^{-1}$. Therefore, $g^{-1} \in \mathrm{Her}_m^\times(k')$ and $\xi \in \mathrm{U}_m^{g^{-1}}(k')$. We also have $y = (g^{-1})^{-1}\, {}^t x^\tau$ which means that $[g^{-1}; \xi, x, y] \in \mathfrak{U}_m^\natural$ and $[\xi, x, y] \Leftrightarrow [\jmath g^{-1}; \xi, x, y]$.

Conversely, given $[\beta; \xi, z, z^*] \in \mathfrak{U}_m^\natural$, since ${}^t\xi\beta^\tau\xi^\tau = \beta^\tau$, $\beta^{\tau,-1}\, {}^t\xi\beta^\tau = \xi^{\tau,-1}$, $z^* = \beta^{-1}\, {}^t z^\tau$, we have $[{}^t\xi, {}^t z^*, {}^t z]\beta^\tau = [\xi^{\tau,-1}, z^\tau, (z^*)^\tau]$. Moreover, since $[\xi, z, z^*]$ and $[{}^t\xi, {}^t z^*, {}^t z]$ have the same invariants, there exists $\gamma \in \mathrm{GL}_m(k)$ such that $[\xi, z, z^*]\gamma = [{}^t\xi, {}^t z^*, {}^t z]$. Therefore, $[\xi, z, z^*](\gamma\beta^\tau) = [\xi^{\tau,-1}, z^\tau, (z^*)^\tau]$, which implies that $\gamma\beta^\tau \in \mathrm{S}_m(k')$; $\gamma\beta^\tau = gg^{\tau,-1}$ for some $g \in \mathrm{GL}_m(k)$. Then $(g^{-1}\xi g)(g^{-1}\xi g)^\tau = \mathbf{1}_m$. Moreover, $zgg^{\tau,-1} = z^\tau$ implies $zg = (zg)^\tau$; $g^\tau g^{-1}z^* = (z^*)^\tau$ implies $g^{-1}z^* = (g^{-1}z^*)^\tau$. In all, $[\xi, z, z^*]g = [g^{-1}\xi g, zg, g^{-1}z^*] \in \mathfrak{S}_m$. The uniqueness of g is obvious. □

For $\beta \in [\mathrm{Her}_m^\times(k')]$, we write $W^\beta = W$ and $V^\beta = V = W \oplus X \oplus X^*$ if the matrix representing the hermitian form on W is in the class β. Write U_m^β (resp. U_n^β, $\mathrm{U}_{n,m}^\beta$, $\mathbf{H}^\beta$) for $\mathrm{U}(W^\beta)$ (resp. $\mathrm{U}(V^\beta)$, $\mathrm{U}(V^\beta) \times \mathrm{Res}_{k/k'} \mathrm{Mat}_{1,m,k}$, $\mathbf{H}'$).

Definition 5.8. (*Pre-regular element*). An element $[\zeta, x, y] \in \mathrm{S}_{n,m}(k')$ (resp. $[\zeta^\beta, z] \in \mathrm{U}_{n,m}^\beta(k')$) is called *pre-regular* if the stabilizer of ζ (resp. ζ^β) under the action of $\mathrm{Res}_{k/k'} U_{1^r,m,1^r}$ (resp. $(U'_{1^r,m})^2$) is trivial.

Applying Lemma 4.6 to $\mathbf{P} = P_{1^r,m,1^r}$, the $U_{1^r,m,1^r}(k)$-orbit of ζ for which $[\zeta, x, y]$ is pre-regular necessarily contains a unique element of the form (4.14) with $t_i(\zeta) \in k^\times$ and $\mathrm{Pr}(\zeta) \in \mathrm{S}_m(k')$. It is called the *normal form* of $[\zeta, x, y]$, and we say $[\zeta, x, y]$ is *normal* if it is of such form. Applying Lemma 4.7 to $\mathbf{P}' = P_{1^r,m}^\beta$, the $(U'_{1^r,m})^2$-orbit of ζ^β for which $[\zeta^\beta, z]$ is pre-regular necessarily contains a unique element of the form (4.16) with $t_i(\zeta^\beta) \in k^\times$ and $\mathrm{Pr}(\zeta^\beta) \in \mathrm{U}^\beta_m(k')$. It is called the *normal form* of $[\zeta^\beta, z]$, and we say $[\zeta^\beta, z]$ is *normal* if it is of such form.

Definition 5.9. (Regular element). An element $[\zeta, x, y] \in \mathrm{S}_{n,m}(k')$ (resp. $[\zeta^\beta, z] \in \mathrm{U}^\beta_{n,m}(k')$) is called *regular* if it is pre-regular and $[\mathrm{Pr}(\zeta), x, y] \in \mathfrak{S}_m$ (resp. $[\beta; \mathrm{Pr}(\zeta^\beta), z, z^*] \in \mathfrak{U}^\natural_m$). We have the notions $\mathrm{S}_{n,m}(k')_{\mathrm{reg}}$, $\mathrm{U}^\beta_{n,m}(k')_{\mathrm{reg}}$ for the sets of regular elements.

As before, we have the following proposition whose proof we omit.

Proposition 5.10. *Let the notation be as above.*

(1) There is a natural bijection

$$[\mathrm{S}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}(k') \xleftrightarrow{\mathbf{N}} \coprod_{\beta \in [\mathrm{Her}^\times_m(k')]} [\mathrm{U}^\beta_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k').$$

If $\mathbf{N}[\zeta^\beta, z] = [\zeta, x, y]$, *we say that they* match *and denote by* $[\zeta, x, y] \leftrightarrow [\zeta^\beta, z]$.

(2) The set $\mathrm{S}_{n,m}(k')_{\mathrm{reg}}$ *(resp.* $\mathrm{U}^\beta_{n,m}(k')_{\mathrm{reg}}$*) is non-empty and Zariski open in* $\mathrm{S}_{n,m}$ *(resp.* $\mathrm{U}^\beta_{n,m}$*). Moreover, the* $\mathbf{H}$*-stabilizer (resp.* $\mathbf{H}^\beta$*-stabilizer) of regular* $[\zeta, x, y]$ *(resp.* $[\zeta^\beta, z]$*) is trivial.*

It is clear that the regular orbit $[\zeta, x, y] \in [\mathrm{S}_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}(k')$ (resp. $[\zeta^\beta, z] \in [\mathrm{U}^\beta_{n,m}(k')_{\mathrm{reg}}]/\mathbf{H}^\beta(k')$) is determined by its invariants $t_i(\zeta)$ (resp. $t_i(\zeta^\beta)$) $(i = 1, \dots, r)$, $a_i([\zeta, x, y]) := a_i([\mathrm{Pr}(\zeta), x, y])$ (resp. $a_i([\zeta^\beta, z]) := a_i([\mathrm{Pr}(\zeta^\beta), z, z^*])$) $(i = 1, \dots, m)$, and $b_i([\zeta, x, y]) := b_i([\mathrm{Pr}(\zeta), x, y])$ (resp. $b_i([\zeta^\beta, z]) := b_i([\mathrm{Pr}(\zeta^\beta), z, z^*])$) $(i = 0, \dots, m-1)$. We have $[\zeta, x, y] \leftrightarrow [\zeta^\beta, z]$ if and only if they have the same invariants. For simplicity, we put $\mathbf{T}_{[\zeta,x,y]} := \mathbf{T}_{[\mathrm{Pr}(\zeta),x,y]}$, $\mathbf{D}_{[\zeta,x,y]} := \mathbf{D}_{[\mathrm{Pr}(\zeta),x,y]}$ and $\Delta_{[\zeta,x,y]} := \Delta_{[\mathrm{Pr}(\zeta),x,y]}$.

Proposition 5.11. *(Smooth matching at a split place). Let* v' *be a place of* k' *which splits into two places* $v_\bullet$ *and* $v_\circ$ *of* k*. Then* $[\mathrm{Her}^\times_m(k')]$ *is a singleton and we suppress* β *in notation.*

(1) For $[\zeta, x, y] \in \mathrm{S}_{n,m}(k')_{\mathrm{reg}}$*, we may extend the local orbital integral* $\mathcal{O}_{\mu_{v'}}(-, [\zeta, x, y])$ *uniquely to a continuous linear functional on* $\mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'}))$.

(2) For $[\zeta', z] \in \mathrm{U}_{n,m}(k')_{\mathrm{reg}}$*, we may extend the local orbital integral* $\mathcal{O}_{\mu_{v'}}(-, [\zeta', z])$ *uniquely to a continuous linear functional on* $\mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$.

(3) There is a surjective continuous linear map

$$\mathrm{SM}\colon \mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'})) \to \mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$$

such that for all $\mathbf{F}_{v'}$ *in the former space,*

$$\mathcal{O}_{\mu_{v'}}(\mathbf{F}_{v'}, [\zeta, x, y]) = \mathcal{O}_{\mu_{v'}}(\mathrm{SM}(\mathbf{F}_{v'}), [\zeta', z]), \tag{5.13}$$

for every pair of normal elements $[\zeta, x, y]$ *and* $[\zeta', z]$ *that match.*

Proof. Since v' is split, we may take $\beta = \mathbf{1}_m$. As in the proof of Proposition 4.14, we identify $\mathrm{S}_{n,v'}$ (resp. $\mathrm{U}_{n,v'}$) with $\mathrm{GL}_{n,v'}$ (resp. $\mathrm{GL}_{n,v'}$), and moreover $\mathrm{S}_{n,m,v'}$ (resp. $\mathrm{U}_{n,m,v'}$) with $\mathrm{GL}_{n,v'} \times \mathrm{Mat}_{1,m,v'} \times \mathrm{Mat}_{m,1,v'}$ (resp. $\mathrm{GL}_{n,v'} \times (\mathrm{Mat}_{1,m,v'})^2$). Then $[\zeta, x, y] \leftrightarrow [\zeta, (x, {}^t y)]$.

For (1), consider $F_{n,v'} \otimes F_{m,v'} \otimes \Phi_{v'} \in \mathcal{S}(\mathrm{S}_n(k'_{v'})) \otimes \mathcal{S}(\mathrm{GL}_m(k_{v'})) \otimes \mathcal{S}(W^\vee(k_{v'}))$. By (5.4) and (5.9), we have

$$\begin{aligned}
&\mathcal{O}_{\mu_{v'}}(F_{n,v'} \otimes F_{m,v'} \otimes \Phi_{v'}, [\zeta, x, y]) \\
&= \int\limits_{\mathrm{GL}_{m,v'}} \iint\limits_{(U_{1^r,m,1^r,v'})^2} \iint\limits_{(\mathrm{GL}_{m,v'})^2} F_{n,v'}(g_\bullet^{-1} g^{-1} \underline{u}_\bullet^{-1} \zeta \underline{u}_\circ g g_\circ) F_{m,v'}((g_\bullet, g_\circ)^{-1}) \\
&\quad \left(\omega_{\overline{\psi},\overline{\mu}}(g_\bullet, g_\circ)\Phi_{v'}\right)^\dagger \left(\left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2}\right) g, g^{-1}\left(y - \frac{(n_r^*)_\bullet + (n_r^*)_\circ}{2}\right)\right) \\
&\quad \underline{\underline{\psi}}'\left(\underline{u}_\bullet^{-1}\underline{u}_\circ\right) \overline{\psi'}\Bigg(j\Bigg(\left(b_{r,r}\right)_\bullet - \left(b_{r,r}\right)_\circ + x\frac{(n_r^*)_\bullet - (n_r^*)_\circ}{2} + \frac{(n_r)_\bullet - (n_r)_\circ}{2} y \\
&\quad + (n_r)_\circ (n_r^*)_\bullet - (n_r)_\bullet (n_r^*)_\circ\Bigg)\Bigg) \mathrm{d}g_\bullet \mathrm{d}g_\circ \mathrm{d}\underline{u}_\bullet \mathrm{d}\underline{u}_\circ \mathrm{d}g,
\end{aligned}$$

where

$$\underline{\underline{\psi}}'(\underline{u}) = \psi'\left(j\left(u_{1,2} + \cdots + u_{r-1,r} + u^*_{r,r-1} + \cdots + u^*_{2,1}\right)\right); \quad \jmath = (j, -j),$$

and

$$\underline{u} = \begin{bmatrix}
1 & u_{1,2} & \ddots & n_{1,1} & \cdots & n_{1,m} & b_{1,r} & \cdots & b_{1,1} \\
 & \ddots & u_{r-1,r} & \vdots & & \vdots & \vdots & & \vdots \\
 & & 1 & n_{r,1} & \cdots & n_{r,m} & b_{r,r} & \cdots & b_{r,1} \\
 & & & 1 & & & n^*_{1,r} & \cdots & n^*_{1,1} \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & 1 & n^*_{r,r} & \cdots & n^*_{r,1} \\
 & & & & & & 1 & u^*_{r,r-1} & \ddots \\
 & & & & & & & \ddots & u^*_{2,1} \\
 & & & & & & & & 1
\end{bmatrix};$$

$$n_r = \begin{bmatrix} n_{r,1} & \cdots & n_{r,r} \end{bmatrix}; \quad n_r^* = {}^t\begin{bmatrix} n^*_{1,r} & \cdots & n^*_{r,r} \end{bmatrix}.$$

By (5.3) and (5.4),

$$\begin{aligned}
\left(\omega_{\overline{\psi},\overline{\mu}}(g_\bullet, g_\circ)\Phi_{v'}\right)^\dagger (x, y) &= \mu(\det g_\bullet^{-1} g_\circ) |\det g_\bullet g_\circ|_{v'}^{\frac{1}{2}} \\
&\int\limits_{\mathrm{Mat}_{1,m,v'}} \Phi_{v'}\left((x+z)g_\bullet, (x-z)g_\circ\right) \psi'(jzy)\mathrm{d}z,
\end{aligned}$$

where $\mu = (\mu, \mu^{-1})$ by abuse of notation. Therefore, the functional assigning $\mathbf{F}_{v'} \in \mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'}))$ to

$$\begin{aligned}
&\mathcal{O}_{\mu_{v'}}(\mathbf{F}_{v'}, [\zeta, x, y]) = \int\limits_{\mathrm{GL}_{m,v'}} \iint\limits_{(U_{1^r,m,1^r,v'})^2} \iint\limits_{(\mathrm{GL}_{m,v'})^2} \int\limits_{\mathrm{Mat}_{1,m,v'}} \\
&\mathbf{F}_{v'}(g_\bullet^{-1} g^{-1} \underline{u}_\bullet^{-1} \zeta \underline{u}_\circ g g_\circ, (g_\bullet, g_\circ)^{-1}, w^\vee) \\
&\overline{\psi'}\left(j\left(\left(b_{r,r}\right)_\bullet - \left(b_{r,r}\right)_\circ + x\frac{\left(n_r^*\right)_\bullet - \left(n_r^*\right)_\circ}{2} + \frac{(n_r)_\bullet - (n_r)_\circ}{2} y\right.\right. \\
&\left.\left. + (n_r)_\circ (n_r^*)_\bullet - (n_r)_\bullet (n_r^*)_\circ\right)\right) \\
&\underline{\underline{\psi'}}\left(\underline{u}_\bullet^{-1} \underline{u}_\circ\right) \psi'\left(jzg^{-1}\left(y - \frac{\left(n_r^*\right)_\bullet + \left(n_r^*\right)_\circ}{2}\right)\right) \mu(\det g_\bullet^{-1} g_\circ) \\
&|\det g_\bullet g_\circ|_{v'}^{\frac{1}{2}} \mathrm{d}z \mathrm{d}g_\bullet \mathrm{d}g_\circ \mathrm{d}\underline{u}_\bullet \mathrm{d}\underline{u}_\circ \mathrm{d}g, \qquad (5.14)
\end{aligned}$$

where

$$w^\vee = \left(\left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2} + zg^{-1}\right) g g_\bullet, \left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2} - zg^{-1}\right) g g_\circ\right),$$

is the desired extension which is apparently unique.

For (2), consider $f_{n,v'} \otimes f_{m,v'} \otimes \phi_{1,v'} \otimes \phi_{2,v'} \in \mathcal{S}(\mathrm{U}_n(k'_{v'})) \otimes \mathcal{S}(\mathrm{U}_m(k'_{v'})) \otimes \mathcal{S}(\mathbf{L}(k'_{v'})) \otimes \mathcal{S}(\mathbf{L}(k'_{v'}))$. By Lemma 5.3(1) and (5.12), we have

$$\begin{aligned}
&\mathcal{O}_{\mu_{v'}}(f_{n,v'} \otimes f_{m,v'} \otimes \phi_{1,v'} \otimes \phi_{2,v'}, [\zeta, (x, {}^t y)]) \\
&= \int\limits_{\mathrm{GL}_{m,v'}} \iint\limits_{(U_{1^r,m,1^r,v'})^2} \int\limits_{\mathrm{GL}_{m,v'}} f_{n,v'}(g_\bullet^{-1} g^{-1} \underline{u}_\bullet^{-1} \zeta \underline{u}_\circ g) f_{m,v'}(g_\bullet^{-1}) \\
&\times \left(\left(\omega'_{\overline{\psi'}, \overline{\mu}}(g_\bullet)\phi_{1,v'}\right) \otimes \phi_{2,v'}\right)^\ddagger \left(\left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2}\right) g,\right. \\
&\times \left. {}^t\!\left(y - \frac{\left(n_r^*\right)_\bullet + \left(n_r^*\right)_\circ}{2}\right) {}^t g^{-1}\right) \\
&\overline{\psi'}\left(j\left(\left(b_{r,r}\right)_\bullet - \left(b_{r,r}\right)_\circ + x\frac{\left(n_r^*\right)_\bullet - \left(n_r^*\right)_\circ}{2} + \frac{(n_r)_\bullet - (n_r)_\circ}{2} y\right.\right. \\
&\left.\left. + (n_r)_\circ (n_r^*)_\bullet - (n_r)_\bullet (n_r^*)_\circ\right)\right) \\
&\underline{\underline{\psi'}}\left(\underline{u}_\bullet^{-1} \underline{u}_\circ\right) \mathrm{d}g_\bullet \mathrm{d}\underline{u}_\bullet \mathrm{d}\underline{u}_\circ \mathrm{d}g.
\end{aligned}$$

By Lemma 5.3, we have

$$\begin{aligned}
&\left(\left(\omega'_{\overline{\psi'}, \overline{\mu}}(g_\bullet)\phi_{1,v'}\right) \otimes \phi_{2,v'}\right)^\ddagger (x, {}^t y) \\
&= \mu(\det g_\bullet^{-1}) |\det g_\bullet|_{v'}^{\frac{1}{2}} \int\limits_{\mathrm{Mat}_{1,m,v'}} \phi_{1,v'}\left((x+z) g_\bullet\right) \phi_{2,v'}(x-z) \psi'(jzy) \mathrm{d}z.
\end{aligned}$$

Therefore, the functional assigning $\mathbf{f}_{v'} \in \mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$ to

$$\mathcal{O}_{\mu_{v'}}(\mathbf{f}_{v'}, [\zeta', z]) = \int\limits_{\mathrm{GL}_{m,v'}} \iint\limits_{(U_{1^r,m,1^r,v'})^2} \int\limits_{\mathrm{GL}_{m,v'}} \int\limits_{\mathrm{Mat}_{1,m,v'}} \mathbf{f}_{v'}(g_\bullet^{-1} g^{-1} \underline{u}_\bullet^{-1} \zeta \underline{u}_\circ g, g_\bullet^{-1}, l)$$
$$\overline{\psi'}\left(j\left(\left(b_{r,r}\right)_\bullet - \left(b_{r,r}\right)_\circ + x\frac{\left(n_r^*\right)_\bullet - \left(n_r^*\right)_\circ}{2} + \frac{(n_r)_\bullet - (n_r)_\circ}{2} y\right.\right.$$
$$\left.\left. +(n_r)_\circ (n_r^*)_\bullet - (n_r)_\bullet (n_r^*)_\circ\right)\right)$$
$$\underline{\underline{\psi'}}\left(\underline{u}_\bullet^{-1} \underline{u}_\circ\right) \psi'\left(jzg^{-1}\left(y - \frac{\left(n_r^*\right)_\bullet + \left(n_r^*\right)_\circ}{2}\right)\right) \mu(\det g_\bullet^{-1})$$
$$|\det g_\bullet|_{v'}^{\frac{1}{2}} \mathrm{d}z \mathrm{d}g_\bullet \mathrm{d}\underline{u}_\bullet \mathrm{d}\underline{u}_\circ \mathrm{d}g, \tag{5.15}$$

where

$$l = \left(\left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2} + zg^{-1}\right) gg_\bullet, \left(x + \frac{(n_r)_\bullet + (n_r)_\circ}{2} - zg^{-1}\right) g\right),$$

is the desired extension which is apparently unique.

For (3), we put

$$\mathrm{SM}(\mathbf{F}_{v'})(h, g_\bullet, (x, y)) = \int\limits_{\mathrm{GL}_{m,v'}} \mathbf{F}_{v'}(hg_\circ, (g_\bullet, g_\circ^{-1}), x, yg_\circ) \mu(\det g_\circ) |\det g_\circ|_{v'}^{\frac{1}{2}} \mathrm{d}g_\circ,$$

where we have naturally identified $\mathrm{S}_n(k'_{v'})$ (resp. $\mathrm{GL}_m(k_{v_\bullet})$ and $W^\vee(k_{v'})$) with $\mathrm{U}_n(k'_{v'})$ (resp. $\mathrm{U}_m(k'_{v'})$ and $\mathbf{L}(k'_{v'})^{\oplus 2}$). Then (5.13) follows by (5.14) and (5.15). The continuity of SM is clear. We only need to show the subjectivity. Consider $\mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$ as a smooth Fréchet representation $\rho_\circ$ of $\mathrm{GL}_m(k_{v_\circ})$ via the action

$$(\rho_\circ(g_\circ)\mathbf{f}_{v'})(h, g_\bullet, (x, y)) = \mathbf{f}_{v'}(hg_\circ, g_\bullet, (x, yg_\circ))\mu(\det g_\circ)|\det g_\circ|_{v'}^{\frac{1}{2}}.$$

By Dixmier–Malliavin theorem [5], there exist finitely many functions $\varphi^{(i)} \in \mathcal{S}(\mathrm{GL}_m(k_{v_\circ}))$ and $\mathbf{f}_{v'}^{(i)} \in \mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$ such that $\mathbf{f}_{v'} = \sum_i \rho_\circ(\varphi^{(i)})\mathbf{f}_{v'}^{(i)}$. Put

$$\mathbf{F}_{v'}(h, (g_\bullet, g_\circ), x, y) = \sum_i \varphi^{(i)}(g_\circ^{-1})\mathbf{f}_{v'}^{(i)}(h, g_\bullet, x, y).$$

Then $\mathrm{SM}(\mathbf{F}_{v'}) = \mathbf{f}_{v'}$ by construction. □

Remark 5.12. For almost all split places v' where everything is unramified, if we take the test functions to be the characteristic functions on corresponding hyperspecial maximal compact subgroups (or the image in the symmetric space) and lattices, then the two orbital integrals are equal. More precisely, the one for the unitary group is the image of the one for the general linear group under SM.

Inspired by the split case in Proposition 5.11, we conjecture that, similar to Conjecture 4.13, the smooth matching of functions in $\mathcal{S}(\mathrm{S}_n(k'_{v'}) \times \mathrm{GL}_m(k_{v'}) \times W^\vee(k_{v'}))$ and $\mathcal{S}(\mathrm{U}_n(k'_{v'}) \times \mathrm{U}_m(k'_{v'}) \times \mathbf{L}(k'_{v'})^{\oplus 2})$ holds for all places v'. We omit the explicit form of this conjecture in the current case.

5.4. The fundamental lemma

We now state the fundamental lemma for Fourier–Jacobi periods. We use the notation in the beginning of Sect. 4.4, except that we have

$$\mathbf{t}([\zeta, x, y]) = \begin{cases} (-1)^{\mathrm{val}(\mathbf{T}_{[\zeta,x,y]}\cdot\prod_{i=1}^{r} t_i(\zeta))} & m \text{ is even} \\ (-1)^{\mathrm{val}(\mathbf{T}_{[\zeta,x,y]})} & m \text{ is odd.} \end{cases} \tag{5.16}$$

For simplicity, we only consider the fundamental lemma for unit elements.

Conjecture 5.13. (The fundamental lemma) *Assume that k/k', ψ', μ are all unramified and $\jmath \in \mathfrak{o}$. Then we have*

$$\begin{aligned}&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}, [\zeta, x, y]) \\ &= \begin{cases} \mathbf{t}([\zeta, x, y])\mathcal{O}_\mu(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})}, [\zeta^+, z]) & [\zeta, x, y] \leftrightarrow [\zeta^+, z] \in \mathrm{U}^+_{n,m}(k') \\ 0 & [\zeta, x, y] \leftrightarrow [\zeta^-, z] \in \mathrm{U}^-_{n,m}(k'), \end{cases}\end{aligned}$$

where $[\zeta, x, y]$, $[\zeta^+, z]$ are normal, and

$$\begin{aligned}&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}, [\zeta, x, y]) \\ &= \int_{\mathbf{H}(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}([\zeta]\mathbf{h}) \left(\omega^\dagger_{\psi,\overline{\mu}}(\mathbf{h}) \left(\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}\right)\right)(x, y)\underline{\psi}(\mathbf{h})\mathrm{d}\mathbf{h}; \\ &\mathcal{O}_\mu(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})}, [\zeta^+, z]) \\ &= \int_{\mathbf{H}^+(k')} \mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}([\zeta^+]\mathbf{h}') \left(\omega^\ddagger_{\psi',\overline{\mu}}(\mathbf{h}')\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o})}\right)(z)\underline{\psi}'(\mathbf{h}')\mathrm{d}\mathbf{h}'.\end{aligned}$$

When $n = m$, the above orbital integrals become the following ones

$$\begin{aligned}&\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{n,1}(\mathfrak{o}')}, [\zeta, x, y]) \\ &= \int_{\mathrm{GL}_n(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(g^{-1}\zeta g)\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o}')}(xg)\mathbb{1}_{\mathrm{Mat}_{n,1}(\mathfrak{o}')}(g^{-1}y)\eta(\det g)\mathrm{d}g; \\ &\mathcal{O}_\mu(\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o})}, [\zeta^+, z]) = \int_{\mathrm{U}_n^+(k')} \mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}(g'^{-1}\zeta^+ g')\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o})}(zg')\mathrm{d}g',\end{aligned}$$

which are much simpler.

It is easy to see that $[\zeta, x, y]$ matches some element $[\zeta^+, z] \in \mathrm{U}^+_{n,m}(k')$ (resp. $[\zeta^-, z] \in \mathrm{U}^-_{n,m}(k')$) if and only if $\mathrm{val}(\Delta_{[\zeta,x,y]})$ is even (resp. odd).

Proposition 5.14. *If* $\mathrm{val}(\Delta_{[\zeta,x,y]})$ *is odd, then*

$$\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}, [\zeta, x, y]) = 0.$$

Proof. The proof is similar to Proposition 4.16. We first assume that $r > 0$. Put

$$\mathsf{w} = \begin{bmatrix} & & \mathsf{w}_r \\ & \mathbf{1}_m & \\ \mathsf{w}_r & & \end{bmatrix}.$$

Expanding all the definitions, we have

$$\begin{aligned} &\mathcal{O}_\mu(\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}; \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')} \otimes \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}, [\zeta, x, y]) = \int_{U_{1^r,m,1^r}(k)} \int_{\mathrm{GL}_m(k')} \\ &\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}\left(g^{-1}\underline{u}^{-1}\zeta\underline{u}^\tau g\right) \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\left((x+n^+)g\right) \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}\left(g^{-1}(y-n_+)\right) \\ &\overline{\psi}\left(b^\dagger + xn_- + n^-y + \frac{n^+n_- - n^-n_+}{2}\right)\underline{\psi}(\underline{u}^{-1})\eta(\det g)\mathrm{d}g\mathrm{d}\underline{u}, \end{aligned} \tag{5.17}$$

where $\underline{u}^\dagger(n^+, n^-, n_+, n_-, b^\dagger)$ is the image of $\underline{u}$ under the projection $H \to H^\dagger$. By the following identities

$$\begin{aligned} &\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(s) = \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(\mathsf{w}\,{}^t s\mathsf{w}), \quad \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}(x) = \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}({}^t x), \\ &\mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}(y) = \mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}({}^t y), \end{aligned}$$

we have

$$\begin{aligned} (5.17) = &\int_{U_{1^r,m,1^r}(k)} \int_{\mathrm{GL}_m(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}\left({}^t g(\mathsf{w}\,{}^t\underline{u}^\tau\mathsf{w})(\mathsf{w}\,{}^t\zeta\mathsf{w})(\mathsf{w}\,{}^t\underline{u}^{-1}\mathsf{w})\,{}^t g^{-1}\right) \\ &\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\left(({}^t y - {}^t n_+)\,{}^t g^{-1}\right) \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}\left({}^t g({}^t x + {}^t n^+)\right) \\ &\overline{\psi}\left(b^\dagger + xn_- + n^-y + \frac{n^+n_- - n^-n_+}{2}\right)\underline{\psi}(\underline{u}^{-1})\eta(\det g)\mathrm{d}g\mathrm{d}\underline{u}. \end{aligned} \tag{5.18}$$

Since $[\zeta, x, y]$ and $[\mathsf{w}\,{}^t\zeta\mathsf{w}, {}^t y, {}^t x]$ have the same invariants, as we see in the proof of Lemma 5.7, there exists $h \in \mathrm{GL}_m(k')$ such that $\mathsf{w}\,{}^t\zeta\mathsf{w} = h^{-1}\zeta h$, ${}^t y = xh$, ${}^t x = h^{-1}y$, ${}^t h = h$, and $\eta(\det h) = -1$. Plugging h, we have

$$\begin{aligned} (5.18) = &\int_{U_{1^r,m,1^r}(k)} \int_{\mathrm{GL}_m(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}\Big((h\,{}^t g^{-1})^{-1}(h\mathsf{w}\,{}^t\underline{u}^{\tau,-1}\mathsf{w}h^{-1})^{-1} \\ &\zeta(h\mathsf{w}\,{}^t\underline{u}^{\tau,-1}\mathsf{w}h^{-1})^\tau\left(h\,{}^t g^{-1}\right)\Big) \\ &\mathbb{1}_{\mathrm{Mat}_{1,m}(\mathfrak{o}')}\left((x + (-{}^t n_+ h^{-1}))(h\,{}^t g^{-1})\right) \mathbb{1}_{\mathrm{Mat}_{m,1}(\mathfrak{o}')}\left((h\,{}^t g^{-1})^{-1}(y - (-h\,{}^t n^+))\right) \\ &\overline{\psi}\left(b^\dagger + xn_- + n^-y + \frac{n^+n_- - n^-n_+}{2}\right)\underline{\psi}(\underline{u}^{-1})\eta(\det g)\mathrm{d}g\mathrm{d}\underline{u}. \end{aligned} \tag{5.19}$$

Note that $\underline{\psi}(\underline{u}^{-1}) = \underline{\psi}\left(\left(h\mathsf{w}\,{}^t\underline{u}^{\tau,-1}\mathsf{w}h^{-1}\right)^{-1}\right)$, and

$$\begin{aligned}
&\overline{\psi}\left(b^\dagger + xn_- + n^- y + \frac{n^+n_- - n^-n_+}{2}\right)\\
&= \overline{\psi}\left(b^\dagger + x\left(h\,{}^tn^-\right) + \left({}^tn_-h^{-1}\right)y + \frac{n^+n_- - n^-n_+}{2}\right)\\
&= \overline{\psi}\left(-b^{\dagger,\tau} + x\left(h\,{}^tn^-\right) + \left({}^tn_-h^{-1}\right)y\right.\\
&\quad\left. + \frac{\left(-{}^tn_+h^{-1}\right)\left(h\,{}^tn^-\right) - \left({}^tn_-h^{-1}\right)\left(-h\,{}^tn^+\right)}{2}\right).
\end{aligned}$$

If we make the following change of variables: $h\mathsf{w}\,{}^t\underline{u}^{\tau,-1}\mathsf{w}h^{-1} \mapsto \underline{u}$, $h\,{}^tg^{-1} \mapsto g$, then

$$-{}^tn_+h^{-1} \mapsto n^+;\quad {}^tn_-h^{-1} \mapsto n^-;\quad -h\,{}^tn^+ \mapsto n_+;\quad h\,{}^tn^- \mapsto n_-;\quad -b^{\dagger,\tau} \mapsto b^\dagger.$$

Therefore, (5.19) $= \eta(\det h)\times$ (5.19) $= -$ (5.19), which confirms the proposition. The case $r = 0$ follows from a similar, but much simpler argument. □

Theorem 5.15. *(Fundamental lemma for* $\mathrm{U}_n \times \mathrm{U}_n$*). Assume* $\mathrm{char}(k) > n$*, or* $\mathrm{char}(k) = 0$ *and the residue characteristic is sufficiently large with respect to* n*. Then the fundamental lemma holds for* $\mathrm{U}_n \times \mathrm{U}_n$*, that is, we have the following equality*

$$\begin{aligned}
&\int\limits_{\mathrm{GL}_n(k')} \mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(g^{-1}\zeta g)\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o}')}(xg)\mathbb{1}_{\mathrm{Mat}_{n,1}(\mathfrak{o}')}(g^{-1}y)\eta(\det g)\mathrm{d}g\\
&= \int\limits_{\mathrm{U}_n^+(k')} \mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}(g'^{-1}\zeta^+g')\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o})}(zg')\mathrm{d}g'
\end{aligned}$$

when $[\zeta, x, y]$ *and* $[\zeta^+, z]$ *match with* $\zeta^+ \in \mathrm{U}_n^+(k')$*.*

The proof uses the Cayley transform to reduce the statement to the Lie algebra version of the fundamental lemma for $\mathrm{U}_{n+1} \times \mathrm{U}_n$, which is proved in [37]. The idea of using Cayley transform is inspired by the work of Zhang [38]. In what follows, we fix a basis of $V^+ = W^+$ such that β^+ is simply the matrix $\mathbf{1}_n$. We first recall the following well-known lemma.

Lemma 5.16. (Cayley transform) *Let* $\mathfrak{s}_n(k')$ *(resp.* $\mathfrak{u}_n(k')$*) be the subset of* $\mathrm{Mat}_n(k)$ *consisting of matrices* A *such that* $A^\tau = -A$ *(resp.* ${}^tA^\tau = -A$*). Let* $\mathfrak{s}_n(\mathfrak{o}')$ *(resp.* $\mathfrak{u}_n(\mathfrak{o}')$*) be the intersection of* $\mathfrak{s}_n(k')$ *(resp.* $\mathfrak{u}_n(k')$*) with* $\mathrm{Mat}_n(\mathfrak{o})$*.*

(1) If A *is in* $\mathfrak{s}_n(k')$*, then* $\mathbf{1}_n + A$ *is invertible and* $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ *is in* $\mathrm{S}_n(k')$*; conversely, if* A *is in* $\mathrm{S}_n(k')$ *for which* -1 *is not an eigenvalue, then* $\mathbf{1}_n + A$ *is invertible and* $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ *is in* $\mathfrak{s}_n(k')$*. Moreover,* A *belongs to* $\mathfrak{s}_n(\mathfrak{o}')$ *if and only if* $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ *belongs to* $\mathrm{S}_n(\mathfrak{o}')$*;*

(2) *If A is in $\mathfrak{u}_n(k')$, then $\mathbf{1}_n + A$ is invertible and $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ is in $\mathrm{U}_n^+(k')$; conversely, if A is in $\mathrm{U}_n^+(k')$ for which -1 is not an eigenvalue, then $\mathbf{1}_n + A$ is invertible and $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ is in $\mathfrak{u}_n(k')$. Moreover, A belongs to $\mathfrak{u}_n(\mathfrak{o}')$ if and only if $(\mathbf{1}_n - A)(\mathbf{1}_n + A)^{-1}$ belongs to $\mathrm{U}_n^+(\mathfrak{o}')$.*

The following lemma in linear algebra will be used shortly.

Lemma 5.17. *Let A be a matrix in $\mathfrak{u}_n(k')$ that is regular semisimple. Then for every element $z \in \mathrm{Mat}_{1,n}(k)$ there exists an element $d \in \mathfrak{o} \cap k^-$ such that*

$$\begin{bmatrix} A & \jmath\,{}^t z^\tau \\ \jmath z & -d \end{bmatrix}, \tag{5.20}$$

which is a matrix in $\mathfrak{u}_{n+1}(k')$, is also regular semisimple.

Proof. Let $P_d(\lambda)$ be the characteristic polynomial of (5.20). Then

$$P_d(\lambda) = (\lambda + d)P_A(\lambda) + Q(\lambda),$$

where P_A is the characteristic polynomial of A, and Q is a polynomial of degree $n-1$, whose leading coefficient is $\jmath^2 z\,{}^t z^\tau$. The matrix (5.20) is regular semisimple if and only if the resultant $\mathrm{Res}(P_d, P_d')$ of P_d and P_d' is nonzero. Since $\mathrm{Res}(P_d, P_d') = \det S(P_d, P_d')$, where $S(P_d, P_d')$ is the Sylvester matrix of P_d and P_d', which is a $(2n+1) \times (2n+1)$ matrix here. A simple calculation shows that $\det S(P_d, P_d')$ is a polynomial in d whose highest degree term is $(-1)^n \mathrm{Res}(P_A, P_A')d^{2n}$. Since A is regular semisimple, $\mathrm{Res}(P_A, P_A') \neq 0$. Therefore, there are only finitely many d such that (5.20) is not regular semisimple. The lemma follows immediately. □

Proof of Theorem 5.15. By multiplying a scalar $\alpha \in k^{\times,1}$, we may assume that ζ^+ does not have -1 as an eigenvalue. By Lemma 5.16(1), $A^+ = (\mathbf{1}_n - \zeta^+)(\mathbf{1}_n + \zeta^+)^{-1}$ is in $\mathfrak{u}_n(k')$, and $\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}(g'^{-1}\zeta^+ g') = \mathbb{1}_{\mathfrak{u}_n(\mathfrak{o}')}(g'^{-1}A^+ g')$. Applying Lemma 5.17 to the matrix A^+ and z, we may choose an element $d \in \mathfrak{o} \cap k^-$ such that

$$\tilde{A}^+ = \begin{bmatrix} A & \jmath\,{}^t z^\tau \\ \jmath z & -d \end{bmatrix}$$

is regular semisimple. Then

$$\mathbb{1}_{\mathrm{U}_n^+(\mathfrak{o}')}(g'^{-1}\zeta^+ g')\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o})}(zg') = \mathbb{1}_{\mathfrak{u}_{n+1}(\mathfrak{o}')}(g'^{-1}\tilde{A}^+ g'),$$

where U_n^+ embeds into U_{n+1}^+ via

$$g' \mapsto \begin{bmatrix} g' & \\ & 1 \end{bmatrix}.$$

We do the same process to the symmetric space. Since $[\zeta, x, y]$ and $[\zeta^+, z]$ match, ζ does not have -1 as an eigenvalue. By Lemma 5.16(2), $A = (\mathbf{1}_n - \zeta)(\mathbf{1}_n + \zeta)^{-1}$ is in $\mathfrak{s}_n(k')$. Put

$$\tilde{A} = \begin{bmatrix} A & \jmath\,{}^t y \\ \jmath x & -d \end{bmatrix},$$

which is in $\mathfrak{s}_{n+1}(k')$. Then

$$\mathbb{1}_{\mathrm{S}_n(\mathfrak{o}')}(g^{-1}\zeta g)\mathbb{1}_{\mathrm{Mat}_{1,n}(\mathfrak{o}')}(xg)\mathbb{1}_{\mathrm{Mat}_{n,1}(\mathfrak{o}')}(g^{-1}y) = \mathbb{1}_{\mathfrak{s}_{n+1}(\mathfrak{o}')}(g^{-1}\tilde{A}g),$$

where GL_n embeds into GL_{n+1} via

$$g \mapsto \begin{bmatrix} g & \\ & 1 \end{bmatrix}.$$

By definition, $\tilde{A}^+$ is *strongly regular semisimple* in the sense of [37, Definition 2.2.1]. Moreover, $\tilde{A}$ and $\tilde{A}^+$ match in the sense of [37, Definition 2.5.1] (and hence $\tilde{A}$ is strongly regular semisimple as well). By the main theorem of [37] (under the assumption on the characteristic), we have

$$\int_{\mathrm{GL}_n(k')} \mathbb{1}_{\mathfrak{s}_{n+1}(\mathfrak{o}')}(g^{-1}\tilde{A}g)\eta(\det g)\mathrm{d}g = \int_{\mathrm{U}_n^+(k')} \mathbb{1}_{\mathfrak{u}_{n+1}(\mathfrak{o}')}(g'^{-1}\tilde{A}^+g')\mathrm{d}g',$$

which is equivalent to the identity in the theorem.

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A. Appendix: A brief summary on local Whittaker integrals

We collect some facts about integrals of local Whittaker functions that are used in this article. All the results here can be found in [4,18,21,23–25].

Let k be a local field, $\psi : k \to \mathbb{C}^\times$ a nontrivial character. We write $|\ | = |\ |_k$, and $\mathrm{Mat}_{r,s} = \mathrm{Mat}_{r,s}(k)$ for simplicity. Let π (resp. σ) be an irreducible admissible representation of $\mathrm{GL}_n = \mathrm{GL}_n(k)$ (resp. $\mathrm{GL}_m = \mathrm{GL}_m(k)$). Let $\mathcal{W}(\psi) = \mathrm{Ind}_{U_{1^n}}^{\mathrm{GL}_n}(\psi)$ be the space of all smooth functions $W(g)$ on GL_n satisfying $W(ug) = \underline{\psi}(u)W(g) := \psi(u_{1,2} + \cdots + u_{n-1,n})W(g)$ for all $u = (u_{ij}) \in U_{1^n}$, the group of upper-triangular unipotent matrices. It is a smooth representation of GL_n by right translation. Let V_π be the space where π realizes. If k is archimedean, then we take V_π as the canonical Casselman–Wallach completion of the corresponding Harish–Chandra module of π. A fundamental theorem of Gelfand–Kazhdan and Shalika states that there is at most one GL_n-equivariant map, up to a constant multiple, from V_π to $\mathcal{W}(\psi)$. If it exists, then we say π is *generic*. Being generic is independent of ψ we choose. Same arguments apply to σ. In what follows, we will assume that π and σ are generic. We denote by $\mathcal{W}(\pi, \psi)$ (resp. $\mathcal{W}(\sigma, \overline{\psi})$) the nontrivial image of V_π (resp. V_σ) in $\mathcal{W}(\psi)$ (resp. $\mathcal{W}(\overline{\psi})$). Moreover, we have

$$\mathcal{W}(\widetilde{\pi}, \overline{\psi}) = \left\{ \widetilde{W}(g) := W(\mathsf{w}_n\, {}^t g^{-1}) \mid W \in \mathcal{W}(\pi, \psi) \right\}.$$

Put $e_m = [0, \ldots, 0, 1] \in \mathrm{Mat}_{1,m}$. For $W \in \mathcal{W}(\pi, \sigma)$, $W^- \in \mathcal{W}(\sigma, \overline{\psi})$ and $\Phi \in \mathcal{S}(\mathrm{Mat}_{1,m})$, we consider following integrals

(1) For $n > m$ and $0 \le r \le n-m-1$,

$$\Psi_r(s; W, W^-) = \int_{U_{1^m} \backslash \mathrm{GL}_m} \int_{\mathrm{Mat}_{r,m}} W\left(\begin{bmatrix} g & 0 & 0 \\ x & \mathbf{1}_r & 0 \\ 0 & 0 & \mathbf{1}_{n-m-r} \end{bmatrix}\right) W^-(g)$$
$$|\det g|^{s-\frac{n-m}{2}} \mathrm{d}x \mathrm{d}g. \tag{6.1}$$

(2) For $n > m$ and $1 \le r \le n-m$,

$$\Psi_r(s; W, W^-; \Phi) =$$
$$\int_{U_{1^m} \backslash \mathrm{GL}_m} \int_{\mathrm{Mat}_{r-1,m}} \int_{\mathrm{Mat}_{1,m}} W\left(\begin{bmatrix} g & 0 & 0 & 0 \\ x & \mathbf{1}_{r-1} & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-m-r} \end{bmatrix}\right) W^-(g)\Phi(y)$$
$$|\det g|^{s-\frac{n-m}{2}} \mathrm{d}y \mathrm{d}x \mathrm{d}g. \tag{6.2}$$

(3) For $n \ge m$,

$$\Psi_0(s; W, W^-; \Phi) = \int_{U_{1^m} \backslash \mathrm{GL}_m} W\left(\begin{bmatrix} g & 0 \\ 0 & \mathbf{1}_{n-m} \end{bmatrix}\right) W^-(g)\Phi(e_m g)$$
$$|\det g|^{s-\frac{n-m}{2}} \mathrm{d}g. \tag{6.3}$$

We put

$$\mathfrak{J}_r(\pi \times \sigma) = \left\{\Psi_r(s; W, W^-) \mid W \in \mathcal{W}(\pi, \sigma),\ W^- \in \mathcal{W}(\sigma, \overline{\psi})\right\}$$

for $0 \le r \le n-m-1$, and

$$\mathfrak{J}'_r(\pi \times \sigma) = \left\{\Psi_r(s; W, W^-; \Phi) \mid W \in \mathcal{W}(\pi, \sigma),\ W^- \in \mathcal{W}(\sigma, \overline{\psi}),\ \Phi \in \mathcal{S}(\mathrm{Mat}_{1,m})\right\}$$

for $0 \le r \le n-m$, which are complex vector spaces.

The following proposition is proved in [4,18,21,25]. For simplicity, we state the result for k non-archimedean only, while the statement for k archimedean can be found, for example, in [18, Sect. 2].

Proposition 6.1. *Let π and σ be as above, and ω_σ the central character of σ. Then*

(1) Each element in $\mathfrak{J}_r(\pi \times \sigma)$ and $\mathfrak{J}'_r(\pi \times \sigma)$ is absolutely convergent for $\mathrm{Re}\, s \gg 0$, and has a meromorphic continuation to the entire complex plane.

(2) There exists a unique function $L(s, \pi \times \sigma)$ of the form $P(q^{-s})^{-1}$, where $P \in \mathbb{C}[X]$ and q is the cardinality of the residue field of k, such that

$$\mathfrak{J}_r(\pi \times \sigma) = \mathfrak{J}'_r(\pi \times \sigma) = L(s, \pi \times \sigma) \cdot \mathbb{C}[q^{-s}, q^s]$$

for every possible r. In particular, for every $s_0 \in \mathbb{C}$, there exist W, W^- (resp. and Φ) such that

$$\frac{\Psi_r(s; W, W^-)}{L(s, \pi \times \sigma)}\Big|_{s=s_0} \quad \textit{resp.} \quad \frac{\Psi_r(s; W, W^-; \Phi)}{L(s, \pi \times \sigma)}\Big|_{s=s_0}$$

is nonzero.

(3) *There is a factor* $\epsilon(s, \pi \times \sigma, \psi)$*, depending only on* π*,* σ *and* ψ*, of the form* cq^{-fs} *such that*

$$\frac{\Psi_{n-m-1-r}(1-s; \widetilde{W}, \widetilde{W^-})}{L(1-s, \widetilde{\pi} \times \widetilde{\sigma})} = \omega_\sigma(-1)^{n-1}\epsilon(s, \pi \times \sigma, \psi)\frac{\Psi_r(s; W, W^-)}{L(s, \pi \times \sigma)}$$

for $n > m$*, and*

$$\frac{\Psi_{n-m-r}(1-s; \widetilde{W}, \widetilde{W^-}; \widehat{\Phi})}{L(1-s, \widetilde{\pi} \times \widetilde{\sigma})} = \omega_\sigma(-1)^{n-1}\epsilon(s, \pi \times \sigma, \psi)\frac{\Psi_r(s; W, W^-; \Phi)}{L(s, \pi \times \sigma)}$$

for $n \geq m$*, where* $\widehat{\Phi}$ *is the* ψ*-Fourier transform of* Φ*, that is,*

$$\widehat{\Phi}(y) = \int_{\mathrm{Mat}_{1,m}} \Phi(x)\psi(x\,{}^t y)\mathrm{d}x,$$

in which the measure $\mathrm{d}x$ *is selfdual.*

Proof. The proof of these statements can be found in the literature mentioned previously, except for $\mathcal{J}'_r(\pi \times \sigma)$ when $n > m$. We provide a proof for the latter case, following [21].

By the functional equation, we can assume that $0 \leq r < n - m$. Put

$$W_1 = \int_{\mathrm{Mat}_{m,1}} \rho\left(\begin{bmatrix} \mathbf{1}_m & & u & \\ & \mathbf{1}_r & & \\ & & 1 & \\ & & & \mathbf{1}_{n-m-r-1} \end{bmatrix}\right) W\widehat{\Phi}(-{}^t u)\mathrm{d}u,$$

which is in $\mathcal{W}(\pi, \psi)$. Then $\Psi_r(s; W_1, W^-) = \Psi_r(s; W, W^-; \Phi)$. To conclude, we only need to show that $\Psi_{n-m-1-r}(s; \rho(\mathsf{w}_{n,m})\widetilde{W_1}, \widetilde{W^-}) = \Psi_{n-m-r}(s; \rho(\mathsf{w}_{n,m})\widetilde{W}, \widetilde{W^-}; \widehat{\Phi})$, which is implied by the Fourier inversion formula.

Put

$$\Psi_r^\natural(W, W^-) = \frac{\Psi_r(s; W, W^-)}{L(s, \pi \times \sigma)}|_{s=\frac{n-m}{2}},$$
$$\Psi_r^\natural(W, W^-; \Phi) = \frac{\Psi_r(s; W, W^-; \Phi)}{L(s, \pi \times \sigma)}|_{s=\frac{n-m}{2}}. \tag{6.4}$$

Corollary 6.2. *Assume* π *and* σ *are both generic.*

(1) *When* $n > m$*, the Whittaker integral* $\Psi_r^\natural(W, W^-)$ (6.4) *defines a nonzero element in* $\mathrm{Hom}_H(\pi \otimes \sigma, \nu)$ *by choosing a suitable basis as in Sect.* 2.2.
(2) *When* $n \geq m$*, the Whittaker integral* $\Psi_r^\natural(W, W^-; \Phi)$ (6.4) *(with* σ *replaced by* $\sigma \otimes \mu^{-1}$*) defines a nonzero element in* $\mathrm{Hom}_H(\pi \otimes \sigma \otimes \widetilde{\nu_\mu}, \mathbb{C})$ *by choosing a suitable basis as in Sect.* 3.2.

When k is archimedean, or the representations π and σ are unramified, we can use the Langlands parameter to define the representation $\pi \boxtimes \sigma$, its L-factor $L(s, \pi \boxtimes \sigma)$ and ϵ-factor $\epsilon(s, \pi \boxtimes \sigma, \psi)$. We have the following proposition.

Proposition 6.3. [4,18,23–25] *Let the notation be as above.*

(1) *If k is archimedean, then $L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma)$ and $\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \pi \boxtimes \sigma, \psi)$.*

(2) *If k is non-archimedean with $\mathfrak{o}$ its ring of integers, take ψ to be unramified. Let π (resp. σ) be an unramified representation associated to a semisimple conjugacy class $A_\pi \in \mathrm{GL}_n(\mathbb{C})$ (resp. $A_\sigma \in \mathrm{GL}_m(\mathbb{C})$). Let $W_\circ$ (resp. $W_\circ^-$) be the unique $\mathrm{GL}_n(\mathfrak{o})$- (resp. $\mathrm{GL}_m(\mathfrak{o})$-) fixed Whittaker functions such that $W(\mathbf{1}_n) = 1$ (resp. $W^-(\mathbf{1}_m) = 1$), and $\Phi_\circ$ the characteristic function of $\mathrm{Mat}_{1,m}(\mathfrak{o})$. Then*

$$\Psi_r(s; W_\circ, W_\circ^-) = \Psi_r(s; W_\circ, W_\circ^-; \Phi_\circ) = \det(1 - q^{-s} A_\pi \otimes A_\sigma)^{-1}$$
$$= L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma)$$

for every possible r.

Proof. In (2), the proof for the integral $\Psi_0(s; W_\circ, W_\circ^-)$ when $n > m$, and for the integral $\Psi_0(s; W_\circ, W_\circ^-; \Phi_\circ)$ when $n = m$ can be found in [23]. The remaining part follows easily as in Proposition 6.1. □

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