TROPICAL DOLBEAULT COHOMOLOGY OF NON-ARCHIMEDEAN SPACES

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Abstract. In this survey article, we discuss some recent progress on tropical Dolbeault cohomology of varieties over non-Archimedean fields, a new cohomology theory based on real forms defined by Chambert-Loir and Ducros.

Contents

We discuss some recent results on tropical Dolbeault cohomology of varieties over non-Archimedean fields, a new cohomology theory based on real forms defined by Chambert-Loir and Ducros.

In this article, by a non-Archimedean field, we mean a complete topological field with respect to a nontrivial non-Archimedean valuation of rank one. We fix a finite field $\mathbf{F}$ throughout the article. Denote by $\mathbf{Z}_{\mathbf{F}}$ the ring of Witt vectors in $\mathbf{F}$ and $\mathbf{Q}_{\mathbf{F}}$ the field of fractions of $\mathbf{Z}_{\mathbf{F}}$. Then $\mathbf{Q}_{\mathbf{F}}$ is naturally a non-Archimedean field, which is locally compact. Moreover, we fix a complete algebraic closure $\mathbf{C}_{\mathbf{F}}$ of $\mathbf{Q}_{\mathbf{F}}$, which is also a non-Archimedean field. We say that a non-Archimedean field is *arithmetic* if it is isomorphic, as a topological field, to a complete subfield of $\mathbf{C}_{\mathbf{F}}$ for some finite field $\mathbf{F}$. For example, locally compact non-Archimedean fields of characteristic zero are arithmetic.

For a non-Archimedean field K with the valuation $|\ |_K$, we put

$$K^\circ = \{x \in K \mid |x|_K \leq 1\}, \quad K^{\circ\circ} = \{x \in K \mid |x|_K < 1\} \subseteq K^\circ,$$

and $\widetilde{K} = K^\circ/K^{\circ\circ}$ which is known as the *residue field* of K. If K is arithmetic, then $\widetilde{K}$ is algebraic over a finite field. Finally, we denote by K^{a} the algebraic closure of K, and $\widehat{K^{\mathrm{a}}}$ the completion of K^{a}.

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1. Tropical Dolbeault cohomology and cycle class map

In [Sha17], the notion of superforms on polyhedral complexes have been defined. Let V be a finite dimensional real vector space and $P \subseteq V$ a polyhedral complex. Then we have a bicomplex $(\mathscr{A}_P^{\bullet,\bullet}, \mathrm{d}', \mathrm{d}'')$ of real sheaves on (the underlying topological space of) P, concentrated in the first quadrant. In particular, when $P = V$, we have

$$\mathscr{A}_V^{p,q} = \mathscr{C}_V^\infty \otimes_{\mathbf{R}} \wedge^p \mathrm{T}_V^* \otimes_{\mathbf{R}} \wedge^q \mathrm{T}_V^*$$

for $p, q \geq 0$, where $\mathscr{C}_V^\infty$ is the sheaf of smooth real valued functions on V, and T_V is the tangent space of V.

Now let K be a non-Archimedean field. For every analytic space X over K, we have a similar bicomplex $(\mathscr{A}_X^{\bullet,\bullet}, \mathrm{d}', \mathrm{d}'')$ of real sheaves on (the underlying topological space of) X, defined in [CLD12]. We recall the construction: A *tropical chart* of X is given by a moment map $f\colon X \to T$ to a torus T over K and a compact polyhedral complex P of T_{trop} that contains $f_{\mathrm{trop}}(X)$. Here T_{trop} is the tropicalization of T, which is a real vector space of finite dimension, and $f_{\mathrm{trop}}\colon X \to T \to T_{\mathrm{trop}}$ is the composite map. For every open subset U of X, denote by $\mathscr{A}_{\mathrm{pre}}^{p,q}(U)$ the inductive limit of $\mathscr{A}_P^{p,q}(P)$ for all tropical charts $(f\colon U \to T, P)$ of U. The sheaf of (p,q)-forms on X is defined as the sheafification of the presheaf $U \mapsto \mathscr{A}_{\mathrm{pre}}^{p,q}(U)$, denoted by $\mathscr{A}_X^{p,q}$. One can regard this bicomplex as the non-Archimedean analogue of the bicomplex of differential (p,q)-forms in complex geometry. Moreover, if the dimension of X is n, then we have an integration map [CLD12]:

$$\int_X \colon \mathscr{A}_X^{n,n}(X)_c \to \mathbf{R}$$

where $\mathscr{A}_X^{n,n}(X)_c$ is the subset of $\mathscr{A}_X^{n,n}(X)$ of global sections of compact support.

For a fixed integer $p \geq 0$, we have the complex

$$(\mathscr{A}_X^{p,\bullet}, \mathrm{d}'')\colon \mathscr{A}_X^{p,0} \xrightarrow{\mathrm{d}''} \mathscr{A}_X^{p,1} \xrightarrow{\mathrm{d}''} \mathscr{A}_X^{p,2} \to \cdots .$$

It is a resolution of $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$ by [Jel16a, Corollary 4.6]; and the resolution is fine if X is a paracompact good K-analytic space by [CLD12, Corollaire 3.3.7].

Let X be a paracompact good K-analytic space.

Definition 1.1 (Dolbeault cohomology, [Liua]). We define the *Dolbeault cohomology* to be

$$\mathrm{H}^{p,q}(X) \coloneqq \frac{\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,q}(X) \to \mathscr{A}_X^{p+1}(X))}{\mathrm{im}(\mathrm{d}''\colon \mathscr{A}_X^{p,q-1}(X) \to \mathscr{A}_X^{p,q}(X))}.$$

We have a canonical isomorphism $\mathrm{H}^{p,q}(X) \cong \mathrm{H}^q(X, \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1}))$.

Let $\mathscr{O}_X$ be the structure sheaf of X. For $p \geq 0$, let $\mathscr{O}_X^{(p)}$ be the sheaf such that for every open subset U of X, $\mathscr{O}_X^{(p)}(U)$ is the $\mathbf{Q}$-vector space spanned by symbols $\{f_1, \dots, f_p\}$ with $f_i \in \mathscr{O}_X^*(U)$. For $p \geq 0$, we have a natural map

$$\tau_X^p\colon \mathscr{O}_X^{(p)} \to \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$$

of $\mathbf{Q}$-sheaves on X. Let $\mathscr{T}_X^p$ be its image sheaf.

We recall the definition of τ_X^p. For an open subset U of X and $f_1, \dots, f_p \in \mathscr{O}_X^*(U)$, we have a moment map $f = (f_1, \dots, f_p)\colon U \to T = (\mathbf{G}_m^{\mathrm{an}})^p$. Let $\{x_1, \dots, x_p\}$ be the standard coordinates of $T_{\mathrm{trop}} = \mathbf{R}^p$. Then $\tau_X^p(\{f_1, \dots, f_p\})$ is defined as $\mathrm{d}'x_1 \wedge \cdots \wedge \mathrm{d}'x_p$, regarded as an element in $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0}(U) \to \mathscr{A}_X^{p,1}(U))$.

It is proved in [Liua] that the natural map

$$\mathscr{T}_X^p \otimes_{\mathbf{Q}} \mathbf{R} \to \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$$

is an isomorphism. Therefore, we obtain a canonical isomorphism

$$\mathrm{H}^{p,q}(X) = \mathrm{H}^q(X, \mathscr{T}_X^p) \otimes_{\mathbf{Q}} \mathbf{R}. \tag{1.1}$$

Remark 1.2. We have the canonical isomorphism $\mathrm{H}^{0,q}(X) = \mathrm{H}^q(X, \mathbf{R})$. In other words, $\mathrm{H}^{0,q}(X)$ canonically computes the singular cohomology of the underlying topological space of X of real coefficients.

It is easy to see that the map τ_X^p satisfies the following properties:

- $\tau_X^p(\{f_1, \dots, f_i f_i', \dots, f_p\}) = \tau_X^p(\{f_1, \dots, f_i, \dots, f_p\}) + \tau_X^p(\{f_1, \dots, f_i', \dots, f_p\})$ for $f_1, \dots, f_i, f_i', \dots, f_p \in \mathscr{O}_X^*(U)$;
- $\tau_X^p(\{f_1, \dots, f_i, \dots, f_j, \dots, f_p\}) = 0$ for $f_1, \dots, f_i, \dots, f_j, \dots, f_p \in \mathscr{O}_X^*(U)$ with $f_i + f_j = 1$.

Therefore, the map τ_X^p factors through the *sheaf of rational Milnor K-theory* $\mathscr{K}_X^p$ of the ringed space $(X, \mathscr{O}_X)$. More precisely, $\mathscr{K}_X^p$ is the sheaf associated to the presheaf that assigns every open subset $U \subseteq X$ the rational Milnor K-group $K_p^M(\mathscr{O}_X(U)) \otimes_{\mathbf{Z}} \mathbf{Q}$. See [Liua] for more details. From now on, we will regard τ_X^p as map

$$\tau_X^p\colon \mathscr{K}_X^p \to \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$$

with image $\mathscr{T}_X^p$. This observation is crucial for the later definition of cycle class maps.

Now we move to the algebraic setup. Let X be a separated scheme of finite type over K. We can associate to X an analytic space X^{an}, called the *(Berkovich) analytification* of X [Ber93], which is a Hausdorff paracompact good strictly K-analytic space. For example, if X is the affine line, then X^{an} is the union of affinoid discs with center 0 and radius r for all $r > 0$.

Definition 1.3 (Tropical Dolbeault cohomology)**.** We define the *tropical Dolbeault cohomology* of X to be

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(X) \coloneqq \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}),$$

so $\mathrm{H}^{p,q}_{\mathrm{trop}}(X)_{\mathbf{R}} \coloneqq \mathrm{H}^{p,q}_{\mathrm{trop}}(X) \otimes_{\mathbf{Q}} \mathbf{R}$ is canonically isomorphic to $\mathrm{H}^{p,q}(X^{\mathrm{an}})$. We define the corresponding *tropical Hodge number* of X to be

$$\mathrm{h}^{p,q}_{\mathrm{trop}}(X) \coloneqq \dim_{\mathbf{Q}} \mathrm{H}^{p,q}_{\mathrm{trop}}(X).$$

It could be infinity in general.

Similar to the case of analytic space, we have the *sheaf of rational Milnor K-theory* $\mathscr{K}_X^p$ of the ringed space $(X, \mathscr{O}_X)$. Moreover, we have a comparison map

$$\mathrm{H}^q(X, \mathscr{K}_X^p) \to \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{K}^p_{X^{\mathrm{an}}}).$$

Now suppose that X is smooth. Then by a theorem in [Sou85], we have a canonical isomorphism

$$\mathrm{CH}^p(X)_{\mathbf{Q}} \cong \mathrm{H}^p(X, \mathscr{K}_X^p).$$

Definition 1.4 (Tropical cycle class map, [Liua]). Let X be a smooth separated scheme of finite type over K. We define the tropical cycle class map

$$\mathrm{cl}_{\mathrm{trop}}\colon \mathrm{CH}^p(X)_{\mathbf{Q}} \to \mathrm{H}^{p,p}_{\mathrm{trop}}(X)$$

to be the composition

$$\mathrm{CH}^p(X)_{\mathbf{Q}} \xrightarrow{\simeq} \mathrm{H}^p(X, \mathscr{K}^p_X) \to \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{K}^p_{X^{\mathrm{an}}}) \to \mathrm{H}^p(X^{\mathrm{an}}, \mathscr{T}^p_{X^{\mathrm{an}}}) = \mathrm{H}^{p,p}_{\mathrm{trop}}(X)$$

in which the third map is induced by $\tau^p_{X^{\mathrm{an}}}$.

We have the following fundamental result on the compatibility of tropical cycle classes and integration.

Theorem 1.5 ([Liua]). *Let X be a separated smooth scheme of finite type over K of dimension n. Let Z be an algebraic cycle on X of codimension p. Then we have*

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) \wedge \omega = \int_{Z^{\mathrm{an}}} \omega$$

for every d''-closed form $\omega \in \mathscr{A}^{n-p,n-p}_{X^{\mathrm{an}}}(X^{\mathrm{an}})_c$ with compact support.

The theorem has the following corollary, which says that the tropical Dolbeault cohomology essentially captures all information about algebraic cycles up to the numerical equivalence.

Corollary 1.6 ([Liua]). *Let X be a proper smooth scheme over K. For every $p \geq 0$, denote by $\mathrm{NS}^p(X)$ the quotient group of $\mathrm{CH}^p(X)$ modulo elements that are numerical equivalent to zero. Then we have*

$$\mathrm{h}^{p,p}_{\mathrm{trop}}(X) \geq \dim \mathrm{NS}^p(X) \otimes \mathbf{Q}.$$

Using the above corollary, we can produce a counterexample of the Künneth formula when K is algebraically closed and *arithmetic*, as in the following example.

Example 1.7. Let X be an irreducible proper smooth curve over K of genus $g \geq 1$, such that X has smooth reduction. In particular, X^{an} is contractible hence $\mathrm{h}^{0,0}_{\mathrm{trop}}(X) = 1$ and $\mathrm{h}^{0,1}_{\mathrm{trop}}(X) = 0$ by Remark 1.2. By Theorem 2.3 (2), we have $\mathrm{h}^{1,0}_{\mathrm{trop}}(X) = 0$. Finally by Theorem 3.4 (2), we have $\mathrm{h}^{1,1}_{\mathrm{trop}}(X) = 1$. If the Künneth formula holds for the product $X \times_K X$, then we should have $\mathrm{h}^{1,1}_{\mathrm{trop}}(X \times_K X) = 2$. However, by the above corollary, we get $\mathrm{h}^{1,1}_{\mathrm{trop}}(X \times_K X) \geq \dim_{\mathbf{Q}} \mathrm{NS}^1(X \times_K X)$ with $\dim_{\mathbf{Q}} \mathrm{NS}^1(X \times_K X) = 3$ as $g \geq 1$.

2. Monodromy map and Hodge numbers

The goal of this section is to introduce a map $\mathrm{N}_X\colon \mathrm{H}^{p,q}(X) \to \mathrm{H}^{p-1,q+1}(X)$ ($p \geq 1$), called *monodromy map*, for every K-analytic space X[1]. In fact, N_X is induced from a map of sheaves $\mathrm{N}_X\colon \mathscr{A}^{p,q}_X \to \mathscr{A}^{p-1,q+1}_X$ that commutes with d''.

Let V be a finite dimensional real vector space, and $U \subseteq V$ an open subset. Let $p \geq 1$ be an integer. Define the map

$$\mathrm{N}\colon \mathscr{A}^{p,q}_V(U) \to \mathscr{A}^{p-1,q+1}_V(U) \tag{2.1}$$

[1]We will now assume that all K-analytic spaces are Hausdorff, paracompact, good, and strictly K-analytic.

to be the composite map

$$
\begin{aligned}
&\mathscr{C}^\infty(U)\otimes_{\mathbf{R}}\wedge^p\mathrm{T}_V^*\otimes_{\mathbf{R}}\wedge^q\mathrm{T}_V^*\\
&\xrightarrow{\sim}\mathscr{C}^\infty(U)\otimes_{\mathbf{R}}\wedge^p\mathrm{T}_V^*\otimes_{\mathbf{R}}\mathbf{R}\otimes_{\mathbf{R}}\wedge^q\mathrm{T}_V^*\\
&\to\mathscr{C}^\infty(U)\otimes_{\mathbf{R}}\wedge^p\mathrm{T}_V^*\otimes_{\mathbf{R}}(\mathrm{T}_V\otimes_{\mathbf{R}}\mathrm{T}_V^*)\otimes_{\mathbf{R}}\wedge^q\mathrm{T}_V^*\\
&\xrightarrow{\sim}\mathscr{C}^\infty(U)\otimes_{\mathbf{R}}(\wedge^p\mathrm{T}_V^*\otimes_{\mathbf{R}}\mathrm{T}_V)\otimes_{\mathbf{R}}(\mathrm{T}_V^*\otimes_{\mathbf{R}}\wedge^q\mathrm{T}_V^*)\\
&\to\mathscr{C}^\infty(U)\otimes_{\mathbf{R}}\wedge^{p-1}\mathrm{T}_V^*\otimes_{\mathbf{R}}\wedge^{q+1}\mathrm{T}_V^*,
\end{aligned}
$$

where the second map is given by the coevaluation map for T_V (see the remark below), and the last map is given by the contraction map and the wedge product. If we choose a coordinate system $\{x_1,\dots,x_n\}$ of V, then for

$$
\omega=\sum_{I=\{i_1<\cdots<i_p\},J=\{j_1<\cdots<j_q\}}\omega_{I,J}(x)\mathrm{d}'x_{i_1}\wedge\cdots\wedge\mathrm{d}'x_{i_p}\wedge\mathrm{d}''x_{j_1}\wedge\cdots\wedge\mathrm{d}''x_{j_q}
$$

with $p\geq 1$, we have

$$
\begin{aligned}
\mathrm{N}\omega&=\sum_{k=1}^{p}\sum_{I,J}(-1)^{p-k}\omega_{I,J}(x)\mathrm{d}'x_{i_1}\wedge\cdots\wedge\widehat{\mathrm{d}'x_{i_k}}\wedge\cdots\wedge\mathrm{d}'x_{i_p}\wedge\mathrm{d}''x_{i_k}\wedge\mathrm{d}''x_{j_1}\wedge\cdots\wedge\mathrm{d}''x_{j_q}\\
&=\sum_{k=1}^{p}\sum_{I,J}(-1)^{p-k}\omega_{I,J}(x)\mathrm{d}'x_{I\setminus\{i_k\}}\wedge\mathrm{d}''x_{i_k}\wedge\mathrm{d}''x_J.
\end{aligned}
$$

Moreover, it is straightforward, by the above formula, to check that N commutes with d''.

Remark 2.1. Let W be an arbitrary finite dimensional real vector space with W^* its dual space. We have a canonical evaluation map

$$
\mathrm{ev}\colon W^*\otimes_{\mathbf{R}}W\to\mathbf{R}.
$$

We also have the *coevaluation map*, which is the unique linear map

$$
\mathrm{coev}\colon \mathbf{R}\to W\otimes_{\mathbf{R}}W^*
$$

such that both composite maps

$$
\begin{gathered}
W^*\xrightarrow{1_{W^*}\otimes\mathrm{coev}}W^*\otimes_{\mathbf{R}}(W\otimes_{\mathbf{R}}W^*)\xrightarrow{\sim}(W^*\otimes_{\mathbf{R}}W)\otimes_{\mathbf{R}}W^*\xrightarrow{\mathrm{ev}\otimes 1_{W^*}}W^*\\
W\xrightarrow{\mathrm{coev}\otimes 1_W}(W\otimes_{\mathbf{R}}W^*)\otimes_{\mathbf{R}}W\xrightarrow{\sim}W\otimes_{\mathbf{R}}(W^*\otimes_{\mathbf{R}}W)\xrightarrow{1_W\otimes\mathrm{ev}}W
\end{gathered}
$$

are identity maps.

The map (2.1) is canonical. From this, it is not hard to see that it induces, after several steps, a map $\mathrm{N}_X\colon \mathscr{A}_X^{p,q}\to\mathscr{A}_X^{p-1,q+1}$ that commutes with d''. See [Liub] for more details.

Now let X be a paracompact good K-analytic space. Taking Dolbeault cohomology, we obtain a map

$$
\mathrm{N}_X\colon \mathrm{H}^{p,q}(X)\to\mathrm{H}^{p-1,q+1}(X).
$$

In the algebraic setting, if X is a separated scheme of finite type over K, then we have the *monodromy map*

$$
\mathrm{N}_X\colon \mathrm{H}^{p,q}_{\mathrm{trop}}(X)_{\mathbf{R}}\to\mathrm{H}^{p-1,q+1}_{\mathrm{trop}}(X)_{\mathbf{R}}
$$

for $p \geq 1$ for tropical Dolbeault cohomology after tensoring with $\mathbf{R}$. We propose the following conjecture.

Conjecture 2.2 (Hodge isomorphism, [Liub])**.** *Suppose that K is an algebraically closed non-Archimedean field such that $\widetilde{K}$ is algebraic over a finite field. Let X be a proper smooth scheme over K. Then for $p \geq q \geq 0$, the (iterated) monodromy map*

$$\mathrm{N}_X^{p-q} \colon \mathrm{H}^{p,q}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{q,p}_{\mathrm{trop}}(X)_{\mathbf{R}}$$

is an isomorphism.

We prove in [Liub] the following theorem as evidence toward the above conjecture.

Theorem 2.3. *Let X_0 be a proper smooth scheme over a non-Archimedean field K_0. Let K be a closed subfield of $\widehat{K_0^{\mathrm{a}}}$ containing K_0. Put $X = X_0 \otimes_{K_0} K$.*

(1) Suppose that K_0 is isomorphic to $k(\!(t)\!)$ for k either a finite field or a field of characteristic zero. Then the (iterated) monodromy map

$$\mathrm{N}_X^{p} \colon \mathrm{H}^{p,0}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{0,p}_{\mathrm{trop}}(X)_{\mathbf{R}}$$

is injective for every $p \geq 0$. In particular, $\mathrm{H}^{p,0}_{\mathrm{trop}}(X)$ is of finite dimension.

(2) Suppose that K_0 is locally compact, $K = \widehat{K_0^{\mathrm{a}}}$, and X_0 admits a proper strictly semistable model (see [dJ96]) over K_0°. Then the monodromy map

$$\mathrm{N}_X \colon \mathrm{H}^{1,0}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{0,1}_{\mathrm{trop}}(X)_{\mathbf{R}}$$

is an isomorphism.

Remark 2.4. Let K be an algebraically closed non-Archimedean field.

(1) In his thesis, Jell proved that for a proper smooth scheme X over K of dimension n, the map $\mathrm{N}_X^{p} \colon \mathrm{H}^{p,0}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{0,p}_{\mathrm{trop}}(X)_{\mathbf{R}}$ is injective for $p = 0, 1, n$ [Jel16b, Proposition 3.4.11].

(2) In [JW16], Jell and Wanner proved that for X either $\mathbf{P}^1_K$ or a (proper smooth) Mumford curve over K, the map $\mathrm{N}_X \colon \mathrm{H}^{1,0}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{0,1}_{\mathrm{trop}}(X)_{\mathbf{R}}$ is an isomorphism.

(3) In fact, in the above two results, the map $\mathrm{H}^{p,0}_{\mathrm{trop}}(X)_{\mathbf{R}} \to \mathrm{H}^{0,p}_{\mathrm{trop}}(X)_{\mathbf{R}}$ the authors considered is induced by "flipping $(p,0)$-forms to $(0,p)$-forms". However, one can easily check that this agrees with our map N_X^{p} up to a factor of $p!$.

Conjecture 2.2 could be wrong if $\widetilde{K}$ is not algebraic over a finite field, as seen in the following example.

Example 2.5. Put $K_0 = \mathbf{C}(\!(t)\!)$ and $K \coloneqq \widehat{K_0^{\mathrm{a}}} = \mathbf{C}\{\{t\}\}$ the field of Puiseux series. In particular, $\widetilde{K} = \mathbf{C}$ is not algebraic over a finite field. Let Y_0 (resp. Y_1) be a genus zero (resp. one) curve over $\mathbf{C}$, and let A, B, C be three closed points on Y_1 such that $A - B$ and $A - C$ are $\mathbf{Q}$-linearly independent degree zero divisors on Y_1. There is a projective strictly semistable curve $\mathcal{X}_0$ over K_0° such that its special fiber is $Y_0 \cup Y_1$ with $Y_0 \cap Y_1 = \{A, B, C\}$ in Y_1. This example was constructed in [BGS95] for other purpose. However, we will now explain that for $X \coloneqq \mathcal{X}_0 \otimes_{K_0^{\circ}} K$, we have $\mathrm{h}^{1,0}_{\mathrm{trop}}(X) = 0$ but $\mathrm{h}^{0,1}_{\mathrm{trop}}(X) = 2$.

Let Γ be the graph that has two vertices indexed by $\{0,1\}$ and three edges indexed by $\{a,b,c\}$, all connecting 0 and 1; it is the reduction graph of $\mathcal{X}_0$. We know that Γ is a deformation retract of X^{an}; thus $\mathrm{H}^{0,1}(X^{\mathrm{an}}) \cong \mathrm{H}^1(\Gamma,\mathbf{R}) \cong \mathbf{R}^{\oplus 2}$ hence $\mathrm{h}^{0,1}_{\mathrm{trop}}(X)=2$. To show that $\mathrm{h}^{1,0}_{\mathrm{trop}}(X)=0$, it suffices to show that for every finite open covering $\{U_i\}$ of X^{an} and $f_i \in \mathscr{O}^*_{X^{\mathrm{an}}}(U_i)$ such that $|f_i|=|f_j|$ on $U_i\cap U_j$, we must have that $|f_i|$ is a constant for every i. We can assume that both $\{U_i\}$ and $\{f_i\}$ descend to a finite base change X_n^{an} where $X_n \coloneqq \mathcal{X}_0 \otimes_{K_0^\circ} K_n$ with $K_n=\mathbf{C}(\!(t^{1/n})\!)$ for some $n\geq 1$. After possibly enlarging n, we have a strictly semistable model $\mathcal{X}_n$ of X_n by blowing up $\mathcal{X}_0\otimes_{K_0^\circ} K_n^\circ$ such that for every irreducible component Y of $\mathcal{X}_n\otimes_{K_n^\circ}\widetilde{K_n^\circ}$, $\pi_n^{-1}Y$ is contained in some U_i, where $\pi_n\colon X_n^{\mathrm{an}}\to\mathcal{X}_n\otimes_{K_n^\circ}\widetilde{K_n^\circ}$ is the reduction map. The collection $\{f_i\}$ induce a divisor D_Y on each Y. Note that Y induces canonically a point η_Y in X^{an}. If η_Y does not belong to Γ, then we can show that D_Y has to be trivial. Therefore, if Y dominates Y_1, then D_Y must support on $\{A,B,C\}$, which is again trivial by our assumption. One can further deduce that all D_Y should be trivial. Thus $|f_i|$ is a constant for every i.

Remark 2.6. In the setup of tropical spaces, Mikhalkin and Zharkov in [MZ13] defined a similar map $\mathrm{H}_{p,q}(X)_{\mathbf{R}}\to\mathrm{H}_{p+1,q-1}(X)_{\mathbf{R}}$ for the topical homology of a compact tropical space X via combinatorial construction. In view of the work [JSS15], one can modify our construction to define a map $\mathrm{H}^{p,q}(X)_{\mathbf{R}}\to\mathrm{H}^{p-1,q+1}(X)_{\mathbf{R}}$ for the topical cohomology of an arbitrary tropical space X. We expect that the two maps are closely related.

3. Relation to algebraic de Rham cohomology over arithmetic fields

In this section, we assume that K is an arithmetic non-Archimedean field. Let X be a smooth K-analytic space. We have the de Rham complex

$$(\Omega^\bullet_X,\mathrm{d})\colon \mathscr{O}_X\xrightarrow{\mathrm{d}}\Omega^1_X\xrightarrow{\mathrm{d}}\Omega^2_X\to\cdots.$$

It is a complex of $\mathfrak{c}_X$-modules and is *not* exact if $\dim(X)\geq 1$, where

$$\mathfrak{c}_X\coloneqq\ker(\mathrm{d}\colon\mathscr{O}_X\to\Omega^1_X)$$

is the sheaf of constants.

For $p\geq 0$, we have a natural map

$$\lambda^p_X\colon\mathscr{O}^{(p)}_X\to\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$$

of $\mathbf{Q}$-sheaves on X. It is defined as follows: For an open subset U of X and $f_1,\dots,f_p\in\mathscr{O}^*_X(U)$, we put $\lambda^p_X(\{f_1,\dots,f_p\})$ to be the image of the closed differential form

$$\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_p}{f_p}$$

in $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)(U)$. It is also clear that λ^p_X factors through the quotient sheaf $\mathscr{K}^p_X$. Let $\mathscr{L}^p_X$ be the image sheaf of λ^p_X. We have the following theorem that relates τ^p_X with λ^p_X.

Theorem 3.1 ([Liua]). *Let X be a smooth K-analytic space. Let $p\geq 0$ be an integer. Then $\ker\tau^p_X$ coincides with $\ker\lambda^p_X$. In particular, we have a canonical isomorphism*

$$\mathscr{T}^p_X\cong\mathscr{L}^p_X$$

of $\mathbf{Q}$*-sheaves on* X*.*

The above theorem actually identifies a $\mathbf{Q}$-subsheaf of the $\mathbf{R}$-sheaf $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$ with a $\mathbf{Q}$-subsheaf of the K-sheaf $\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$. Therefore, it is worth studying the sheaf $\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$ in order to understand the tropical Dolbeault cohomology. In fact, in [Liua], we obtain a canonical decomposition of $\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$, which we call *weight decomposition*. It generalize a result of Berkovich [Ber07] for curves.

Theorem 3.2 ([Liua]). *Let* X *be a smooth* K*-analytic space. Then for every* $p \geq 0$*, we have a decomposition*

$$\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1} = \bigoplus_{w\in\mathbf{Z}} (\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$$

of $\mathfrak{c}_X$*-modules. It satisfies that*

(1) $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w = 0$ *unless* $p \leq w \leq 2p$*;*

(2) *the wedge product of forms restricts to a map*

$$\wedge\colon (\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w \times (\Omega_X^{p',\mathrm{cl}}/\mathrm{d}\Omega_X^{p'-1})_{w'} \to (\Omega_X^{p+p',\mathrm{cl}}/\mathrm{d}\Omega_X^{p+p'-1})_{w+w'};$$

(3) *the image of the natural map*

$$\mathscr{T}_X^p \otimes_{\mathbf{Q}} \mathfrak{c}_X \to \Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$$

is contained in $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_{2p}$*;*

(4) *the natural map* $\mathscr{T}_X^1 \otimes_{\mathbf{Q}} \mathfrak{c}_X \to (\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X)_2$ *is an isomorphism.*

Moreover, such decomposition is stable under base change and functorial in X*.*

In general, the definition of the subsheaf $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ is quite complicated. We will look at one special example to help understand the nature of this decomposition. Assume that K is algebraically closed. Let $\mathcal{X}$ be a projective smooth curve over K° of genus g, and put $X = (\mathcal{X}\otimes_{K^\circ}K)^{\mathrm{an}}$. We study the quotient sheaf $\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X$. The special fiber of $\mathcal{X}$ induces a point $\eta \in X$ which is a type II point. In fact, it is a deformation retract of X. If $x \in X$ is a point of type I or IV, then we know that $(\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X)|_x = 0$. If $x \in X$ is a point of type II or III other than η, then $(\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X)|_x$ is generated by $\frac{\mathrm{d}f}{f}$ for $f \in \mathscr{O}_{X,x}^*$. For $x = \eta$, the stalk $(\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X)|_\eta$ may contain elements other than $\frac{\mathrm{d}f}{f}$. In fact, we have

$$(\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathscr{O}_X)|_\eta = \varinjlim_U \mathrm{H}^1_{\mathrm{dR}}(U) \tag{3.1}$$

where U runs over all open neighborhoods of η. Let $\pi\colon X \to \mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}$ be the reduction map. Then every open neighborhood U contains $\pi^{-1}V$ for some nonempty Zariski open subset V of $\mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}$. Thus, one may write (3.1) as

$$\varinjlim_V \varinjlim_{\pi^{-1}V\subseteq U} \mathrm{H}^1_{\mathrm{dR}}(U).$$

However, for every fixed V, the colimit

$$\varinjlim_{\pi^{-1}V\subseteq U} \mathrm{H}^1_{\mathrm{dR}}(U)$$

is nothing but the rigid cohomology $\mathrm{H}^1_{\mathrm{rig}}(V/K)$ of V (over K). By the Gysin exact sequence from the theory of rigid cohomology, we have a canonical injective map

$$\mathrm{H}^1_{\mathrm{rig}}(\mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}/K)\hookrightarrow \mathrm{H}^1_{\mathrm{rig}}(V/K)$$

compatible with changing of V. As $\mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}$ is projective smooth, we have the comparison isomorphism

$$\mathrm{H}^1_{\mathrm{rig}}(\mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}/K)\cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{X}/K)\cong K^{\oplus 2g}.$$

One can show that in the stalk $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathscr{O}_X)|_\eta$, the subspace $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{X}\otimes_{K^\circ}\widetilde{K^\circ}/K)$ and the subspace spanned by $\frac{\mathrm{d}f}{f}$ form a direct sum. In fact, the former is the stalk of our summand $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathscr{O}_X)_1$ at η and the latter is the stalk of $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathscr{O}_X)_2$ at η. More generally, if X is a smooth analytic curve over K (assumed to be algebraically closed for simplicity), then $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathscr{O}_X)_1$ is only supported on type II points; and for every such x, the stalk $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathscr{O}_X)_1$ at x is isomorphic to $K^{2g(x)}$ where $g(x)$ is the intrinsic genus of x.

The main result we proved in [Liua] is that the tropical current defined by a de Rham cohomologically trivial cycle is trivial. More precisely, we have the following theorem.

Theorem 3.3 ([Liua]). *Let K be a locally compact non-Archimedean field of characteristic zero, X a proper smooth scheme over K of dimension n. Let Z be an algebraic cycle of X of codimension p such that the cycle class of Z in the algebraic de Rham cohomology $\mathrm{H}^{2p}_{\mathrm{dR}}(X/K)$ is zero. Then we have*

$$\int_{(Z\otimes_K\widehat{K^{\mathrm{a}}})^{\mathrm{an}}}\omega=0$$

for every d''-closed form $\omega\in\mathscr{A}^{n-p,n-p}_{(X\otimes_K\widehat{K^{\mathrm{a}}})^{\mathrm{an}}}((X\otimes_K\widehat{K^{\mathrm{a}}})^{\mathrm{an}})$. Moreover when $p=1$, we have the stronger conclusion that $\mathrm{cl}_{\mathrm{trop}}(Z\otimes_K\widehat{K^{\mathrm{a}}})=0$.

The proof of the above theorem substantially uses Theorem 3.2. To get some flavor of the argument, we will prove the following theorem as an easy exercise.

Theorem 3.4. *Let K be an algebraically closed arithmetic non-Archimedean field. Let X be an irreducible proper smooth scheme over K. We have*

(1) $\mathrm{h}^{1,1}_{\mathrm{trop}}(X)$ is finite;
(2) if $\dim(X)=1$, then $\mathrm{h}^{1,1}_{\mathrm{trop}}(X)=1$.

Proof. By (1.1) and Theorem 3.1, it suffices to show that: (1) $\mathrm{H}^1(X^{\mathrm{an}},\mathscr{L}^1_{X^{\mathrm{an}}})$ is finite dimensional; and (2) $\mathrm{H}^1(X^{\mathrm{an}},\mathscr{L}^1_{X^{\mathrm{an}}})$ has dimension 1 if $\dim(X)=1$.

Since K is algebraically closed, the sheaf $\mathfrak{c}_{X^{\mathrm{an}}}$ is simply the constant sheaf K. By Theorem 3.2 (4), we have an isomorphism

$$\mathrm{H}^1(X^{\mathrm{an}},\mathscr{L}^1_{X^{\mathrm{an}}})\otimes_{\mathbf{Q}}K\cong \mathrm{H}^1(X^{\mathrm{an}},(\Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}})_2),$$

which is a direct summand of $\mathrm{H}^1(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}})$. We have a spectral sequence $\mathrm{H}^p(X, \Omega^{q,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\Omega^{q-1}_{X^{\mathrm{an}}}) \Rightarrow \mathrm{H}^{p+q}(X^{\mathrm{an}}, \Omega^{\bullet}_{X^{\mathrm{an}}})$ whose second page is

$$\begin{array}{cccc}
\mathrm{H}^0(X^{\mathrm{an}}, K) & \mathrm{H}^0(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}}) & \mathrm{H}^0(X^{\mathrm{an}}, \Omega^{2,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\Omega^{1}_{X^{\mathrm{an}}}) & \cdots \\
\mathrm{H}^1(X^{\mathrm{an}}, K) & \mathrm{H}^1(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}}) & \mathrm{H}^1(X^{\mathrm{an}}, \Omega^{2,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\Omega^{1}_{X^{\mathrm{an}}}) & \cdots \\
\mathrm{H}^2(X^{\mathrm{an}}, K) & \mathrm{H}^2(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}}) & \mathrm{H}^2(X^{\mathrm{an}}, \Omega^{2,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\Omega^{1}_{X^{\mathrm{an}}}) & \cdots \\
\mathrm{H}^3(X^{\mathrm{an}}, K) & \mathrm{H}^3(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}}) & \mathrm{H}^3(X^{\mathrm{an}}, \Omega^{2,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\Omega^{1}_{X^{\mathrm{an}}}) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

In particular, to show that $\mathrm{H}^1(X^{\mathrm{an}}, \Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}})$ has finite dimension, it suffices to show that both $\mathrm{H}^3(X^{\mathrm{an}}, K)$ and $\mathrm{H}^1(X^{\mathrm{an}}, \Omega^{\bullet}_{X^{\mathrm{an}}})$ have finite dimension. As X^{an} is homotopy equivalent to a finite CW complex, $\mathrm{H}^3(X^{\mathrm{an}}, K)$ is of finite dimension over K by [HL16]. By GAGA, $\mathrm{H}^1(X^{\mathrm{an}}, \Omega^{\bullet}_{X^{\mathrm{an}}})$ is canonically isomorphic to the algebraic de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(X/K)$ hence is of finite dimension over K. Therefore, (1) follows.

For (2), as we have $\mathrm{H}^3(X^{\mathrm{an}}, K) = 0$ and that $\mathrm{H}^1_{\mathrm{dR}}(X/K)$ has dimension 1, the dimension of $\mathrm{H}^1(X^{\mathrm{an}}, (\Omega^{1,\mathrm{cl}}_{X^{\mathrm{an}}}/\mathrm{d}\mathscr{O}_{X^{\mathrm{an}}})_2)$ is at most 1. Thus $\mathrm{h}^{1,1}_{\mathrm{trop}}(X) \le 1$. However, it is easy to write down a $(1,1)$-form ω on X^{an} such that

$$\int_{X^{\mathrm{an}}} \omega \neq 0.$$

Therefore, $\mathrm{H}^{1,1}(X^{\mathrm{an}})$ does not vanish hence (2) follows. □

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