TROPICAL CYCLE CLASSES FOR NON-ARCHIMEDEAN SPACES AND WEIGHT DECOMPOSITION OF DE RHAM COHOMOLOGY SHEAVES

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Abstract. This article has three major goals. First, we define tropical cycle class maps for smooth varieties over non-Archimedean fields, valued in the Dolbeault cohomology defined in terms of real forms introduced by Chambert-Loir and Ducros. Second, we construct a functorial decomposition of de Rham cohomology sheaves, called weight decomposition, for smooth analytic spaces over certain non-Archimedean fields of characteristic zero, which generalizes a construction of Berkovich and solves a question raised by himself. Third, we reveal a connection between the tropical theory and the algebraic de Rham theory. As an application, we show that algebraic cycles that are trivial in the algebraic de Rham cohomology are trivial as currents for Dolbeault cohomology as well.

Contents

1. Introduction

This article has three major goals. First, we define tropical cycle class maps for smooth varieties over non-Archimedean fields, valued in the Dolbeault cohomology defined in terms of real forms introduced by Chambert-Loir and Ducros [CLD12]. Second, we construct a functorial decomposition of de Rham cohomology sheaves, called weight decomposition, for smooth analytic spaces over non-Archimedean fields embeddable into $\mathbf{C}_{\mathbf{F}}$ (see below), which generalizes a construction of Berkovich and solves a question raised by himself in [Ber07]. Third, we reveal a connection between the tropical theory and the algebraic de Rham theory. As an application, we show that algebraic cycles that are trivial in the algebraic de Rham cohomology are trivial as currents for Dolbeault cohomology as well.

In this article, by a non-Archimedean field, we mean a complete topological field with respect to a nontrivial non-Archimedean valuation of rank one. We fix a finite field $\mathbf{F}$ throughout the article. Denote by $\mathbf{Z}_{\mathbf{F}}$ the ring of Witt vectors in $\mathbf{F}$ and $\mathbf{Q}_{\mathbf{F}}$ the field of fractions of $\mathbf{Z}_{\mathbf{F}}$. Then $\mathbf{Q}_{\mathbf{F}}$ is naturally a non-Archimedean field, which is locally compact.

Date: July 7, 2017.
2010 *Mathematics Subject Classification.* 14G22.

Moreover, we fix a complete algebraic closure $\mathbf{C}_\mathbf{F}$ of $\mathbf{Q}_\mathbf{F}$, which is also a non-Archimedean field.

1.1. **Tropical cycle class map.** Let K be a non-Archimedean field. In [CLD12], Chambert-Loir and Ducros define, for every K-analytic (Berkovich) space[1] X, a bicomplex $(\mathscr{A}_X^{\bullet,\bullet},\mathrm{d}',\mathrm{d}'')$ of sheaves of real vector spaces on X concentrated in the first quadrant. It is a non-Archimedean analogue of the bicomplex of (p,q)-forms on complex manifolds. In particular, we may define analogously the *Dolbeault cohomology* (Definition 3.1) of X to be

$$\mathrm{H}^{p,q}(X) \coloneqq \frac{\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,q}(X)\to\mathscr{A}_X^{p,q+1}(X))}{\mathrm{im}(\mathrm{d}''\colon \mathscr{A}_X^{p,q-1}(X)\to\mathscr{A}_X^{p,q}(X))}.$$

Moreover, we have an integration map

$$\int_X\colon \mathscr{A}_X^{n,n}(X)_c\to\mathbf{R}$$

for $n=\dim(X)$, where $\mathscr{A}_X^{n,n}(X)_c$ is the space of (n,n)-forms on X whose support is compact and disjoint from the boundary of X.

By [Jel16] and [CLD12], we know that for every $p\geq 0$, the complex $(\mathscr{A}_X^{p,\bullet},\mathrm{d}'')$ is a fine resolution of the sheaf $\ker(\mathrm{d}''\colon\mathscr{A}_X^{p,0}\to\mathscr{A}_X^{p,1})$. In §3, we will construct a canonical $\mathbf{Q}$-subsheaf $\mathscr{T}_X^p$ of $\ker(\mathrm{d}''\colon\mathscr{A}_X^{p,0}\to\mathscr{A}_X^{p,1})$ such that the induced map

$$\mathscr{T}_X^p\otimes_\mathbf{Q}\mathbf{R}\to\ker(\mathrm{d}''\colon\mathscr{A}_X^{p,0}\to\mathscr{A}_X^{p,1})$$

is an isomorphism. In particular, we have a canonical isomorphism

$$\mathrm{H}^q(X,\mathscr{T}_X^p)\otimes_\mathbf{Q}\mathbf{R}\cong\mathrm{H}^{p,q}(X)$$

for every $p,q\geq 0$.

Recall that in the complex world, for a smooth complex algebraic variety $\mathcal{X}$, we have a cycle class map from $\mathrm{CH}^p(\mathcal{X})$ to the classical Dolbeault cohomology $\mathrm{H}^{p,p}_{\overline{\partial}}(\mathcal{X}^{\mathrm{an}})$ of the associated complex manifold $\mathcal{X}^{\mathrm{an}}$. Over a non-Archimedean field K, we may associate a separated scheme $\mathcal{X}$ of finite type over K a K-analytic space $\mathcal{X}^{\mathrm{an}}$ [Ber93, §2.6]. We put

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(\mathcal{X})\coloneqq\mathrm{H}^q(\mathcal{X}^{\mathrm{an}},\mathscr{T}^p_{\mathcal{X}^{\mathrm{an}}}).$$

The following theorem is an analogue of the above cycle class map in the non-Archimedean setup.

Theorem 1.1 (Definition 3.6, Theorem 3.7, Corollary 3.10)**.** *Let K be a non-Archimedean field and $\mathcal{X}$ a separated smooth scheme of finite type over K of dimension n. Then there is a* tropical cycle class map

$$\mathrm{cl}_{\mathrm{trop}}\colon\mathrm{CH}^p(\mathcal{X})\to\mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X}),$$

functorial in $\mathcal{X}$ and K, such that for every algebraic cycle $\mathcal{Z}$ of $\mathcal{X}$ of codimension p,

$$\int_{\mathcal{X}^{\mathrm{an}}}\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z})\wedge\omega=\int_{\mathcal{Z}^{\mathrm{an}}}\omega \tag{1.1}$$

for every d''-closed form $\omega\in\mathscr{A}_{\mathcal{X}^{\mathrm{an}}}^{n-p,n-p}(\mathcal{X}^{\mathrm{an}})$ with compact support.

In particular, if $\mathcal{X}$ is proper and $\mathcal{Z}$ is of dimension 0, then

$$\int_{\mathcal{X}^{\mathrm{an}}}\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z})=\deg\mathcal{Z}.$$

[1] In this article, we assume that all K-analytic spaces are good, paracompact, Hausdorff and strictly K-analytic. See §1.5.

The above theorem has the following corollary.

Corollary 1.2 (Corollary 3.11)**.** *Let K be a non-Archimedean field and $\mathcal{X}$ a proper smooth scheme over K. Let $\mathrm{NS}^p(\mathcal{X})$ be the quotient of $\mathrm{CH}^p(\mathcal{X})$ modulo numerical equivalence. Then we have*

$$\dim \mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X}) \geq \dim \mathrm{NS}^p(\mathcal{X}) \otimes \mathbf{Q}$$

for every $p \geq 0$.

Remark 1.3. Let the situation be as in Theorem 1.1.

(1) The tropical cycle class respects products on both sides. More precisely, for $\mathcal{Z}_i \in \mathrm{CH}^{p_i}(\mathcal{X})$ with $i = 1, 2$, we have

$$\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}_1 \cdot \mathcal{Z}_2) = \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}_1) \wedge \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}_2),$$

where we have used the natural pairing

$$\wedge \colon \mathrm{H}^{p_1,q_1}_{\mathrm{trop}}(\mathcal{X}) \times \mathrm{H}^{p_2,q_2}_{\mathrm{trop}}(\mathcal{X}) \to \mathrm{H}^{p_1+p_2,q_1+q_2}_{\mathrm{trop}}(\mathcal{X}).$$

(2) We may regard the formula (1.1) as a tropical version of the Cauchy formula in multi-variable complex analysis.

(3) Based on this theorem, we will give a counterexample of the Künneth decomposition for the cohomology theory $\mathrm{H}^{\bullet,\bullet}_{\mathrm{trop}}$ in Example 3.12.

1.2. **Weight decomposition.** Suppose that K is of characteristic zero. We have the following complex of $\mathfrak{c}_X$-modules in either analytic or étale topology:

$$\Omega^\bullet_X \colon \mathcal{O}_X = \Omega^0_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X \xrightarrow{\mathrm{d}} \cdots,$$

known as the *de Rham complex*, where $\mathfrak{c}_X \coloneqq \ker(\mathrm{d} \colon \mathcal{O}_X \to \Omega^1_X)$ is the sheaf of constants. It is *not* exact from the term Ω^1_X if $\dim(X) \geq 1$. The cohomology sheaves of the de Rham complex $\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X$ are called *de Rham cohomology sheaves*.

For $p \geq 0$, denote by Υ^p_X the subsheaf of $\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X$ generated by sections of the form

$$\sum_\alpha c_\alpha \frac{\mathrm{d}f_{\alpha 1}}{f_{\alpha 1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{\alpha p}}{f_{\alpha p}}$$

where the sum is finite, c_α are sections of $\mathfrak{c}_X$, and $f_{\alpha i}$ are sections of $\mathcal{O}^*_X$. In particular, we have $\Upsilon^0_X = \mathfrak{c}_X$, and that Υ^1_X is simply the sheaf Υ_X defined in [Ber07, §4.3] in the case of étale topology.

Theorem 1.4. *Suppose that K is embeddable*[2] *into $\mathbf{C}_{\mathbf{F}}$. Let X be a smooth K-analytic space. Then for every $p \geq 0$, we have a decomposition*

$$\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X = \bigoplus_{w \in \mathbf{Z}} (\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X)_w$$

of $\mathfrak{c}_X$-modules in either analytic or étale topology. It satisfies the following:

(i) $(\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X)_w = 0$ *unless* $p \leq w \leq 2p$;

(ii) $\Upsilon^p_X \subset (\Omega^{p,\mathrm{cl}}_X / \mathrm{d}\Omega^{p-1}_X)_{2p}$, *and they are equal if* $p = 1$;

(iii) $(\Omega^{1,\mathrm{cl}}_X / \mathrm{d}\mathcal{O}_X)_1$ *coincides with the sheaf* Ψ_X *defined in* [Ber07, §4.5] *in the case of étale topology.*

Such decomposition is stable under base change, cup product, and functorial in X.

[2] However, we do not choose any such embedding.

The proof of this theorem will be given at the end of Section 5. We call the decomposition in the above theorem the *weight decomposition of de Rham cohomology sheaves*.

Corollary 1.5. *Suppose that K is embeddable into $\mathbf{C}_\mathbf{F}$. Then for every smooth K-analytic space X, we have $\Omega_X^{1,\mathrm{cl}}/\mathrm{d}\mathcal{O}_X = \Upsilon_X \oplus \Psi_X$ in étale topology. This answers the question in* [Ber07, Remark 4.5.5] *for such K.*

Remark 1.6. We expect that Theorem 1.4 and thus Corollary 1.5 hold by only requiring that the residue field of K is algebraic over a finite field and K is of characteristic zero. In Theorem 1.4 (ii), it is probably not true that the inclusion $\Upsilon_X^p \subset (\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_{2p}$ is an isomorphism when $p \geq 2$.

1.3. **Connection between tropical and de Rham theories.** The following theorem reveals a connection between $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$ and the algebraic de Rham cohomology sheaves of X (in the analytic topology).

Theorem 1.7 (Proposition 6.2, Theorem 7.1)**.** *Let K be a non-Archimedean field embeddable into $\mathbf{C}_\mathbf{F}$, and X a smooth K-analytic space. Let $\mathscr{L}_X^p$ be the subsheaf on X of $\mathbf{Q}$-vector spaces of $\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$ generated by sections of the form $\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_p}{f_p}$ where f_i's are sections of $\mathcal{O}_X^*$. Then*

(1) the canonical map $\mathscr{L}_X^p \otimes_\mathbf{Q} \mathfrak{c}_X \to \Upsilon_X^p$ is an isomorphism of sheaves on X;
(2) there is a canonical isomorphism $\mathscr{L}_X^p \simeq \mathscr{T}_X^p$ of sheaves on X.

For a proper smooth scheme $\mathcal{X}$ of dimension n over a general field K of characteristic zero, we have a cycle class map $\mathrm{cl}_{\mathrm{dR}}\colon \mathrm{CH}^p(\mathcal{X}) \to \mathrm{H}^{2p}_{\mathrm{dR}}(\mathcal{X}/K)$ with values in the algebraic de Rham cohomology. It is known that when $K=\mathbf{C}$, the kernel of $\mathrm{cl}_{\mathrm{dR}}$ coincides with the kernel of the cycle class map valued in Dolbeault cohomology. In particular, if $\mathrm{cl}_{\mathrm{dR}}(\mathcal{Z})=0$, then $\int_{\mathcal{Z}^{\mathrm{an}}}\omega=0$ for every $\overline{\partial}$-closed $(n-p,n-p)$-form ω on $\mathcal{X}^{\mathrm{an}}$. In the next theorem, we prove that the same conclusion holds in the non-Archimedean setting as well, with mild restriction on the field K.

Theorem 1.8. *Let $k \subset \mathbf{C}_\mathbf{F}$ be a finite extension of $\mathbf{Q}_\mathbf{F}$ and $\mathcal{X}$ a proper smooth scheme over k of dimension n. Let $\mathcal{Z}$ be an algebraic cycle of $\mathcal{X}$ of codimension p such that $\mathrm{cl}_{\mathrm{dR}}(\mathcal{Z})=0$. If we put $\mathcal{X}_\mathrm{a} = \mathcal{X}\otimes_k \mathbf{C}_\mathbf{F}$ and $\mathcal{Z}_\mathrm{a} = \mathcal{Z}\otimes_k\mathbf{C}_\mathbf{F}$, then*

$$\int_{\mathcal{Z}_\mathrm{a}^{\mathrm{an}}}\omega = 0$$

for every d''-closed form $\omega \in \mathscr{A}^{n-p,n-p}_{\mathcal{X}_\mathrm{a}^{\mathrm{an}}}(\mathcal{X}_\mathrm{a}^{\mathrm{an}})$.

We remark that in the above theorem, we do not know whether $\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z})=0$ or not. If we know the Poincaré duality for $\mathrm{H}^{\bullet,\bullet}(\mathcal{X}^{\mathrm{an}})$, then $\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z})=0$. Nevertheless, we have the following result for lower degree.

Theorem 1.9. *(Theorem 7.3) Let $\mathcal{X}$ be a proper smooth scheme over $\mathbf{C}_\mathbf{F}$. Then*

(1) $\mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X})$ is of finite dimension.
(2) For a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathrm{cl}_{\mathrm{dR}}(\mathcal{L})=0$, we have $\mathrm{cl}_{\mathrm{trop}}(\mathcal{L})=0$.

To the best of our knowledge, the first conclusion in the above theorem is the only known case of the finiteness of $\dim\mathrm{H}^{p,q}_{\mathrm{trop}}(\mathcal{X})$ when both p,q are positive and $\mathcal{X}$ is of general dimension. Note that in the above theorem, we do not require that $\mathcal{X}$ can be defined over a finite extension of $\mathbf{Q}_\mathbf{F}$.

Remark 1.10. We can interpret Theorem 1.8 in the following way. Let k be a number field. Let $\mathcal{X}$ be a proper smooth scheme over k of dimension n, and $\mathcal{Z}$ an algebraic cycle of $\mathcal{X}$ of codimension p. Suppose that there exists *one* embedding $\iota_\infty\colon k\hookrightarrow \mathbf{C}$ such that

$$\int_{(\mathcal{Z}\otimes_{k,\iota_\infty}\mathbf{C})^{\mathrm{an}}}\omega=0$$

for every $\overline{\partial}$-closed $(n-p,n-p)$-form ω on $(\mathcal{X}\otimes_{k,\iota_\infty}\mathbf{C})^{\mathrm{an}}$. Then for *every* finite field $\mathbf{F}$ and *every* embedding $\iota_{\mathbf{F}}\colon k\hookrightarrow \mathbf{C}_{\mathbf{F}}$, we have

$$\int_{(\mathcal{Z}\otimes_{k,\iota_{\mathbf{F}}}\mathbf{C}_{\mathbf{F}})^{\mathrm{an}}}\omega=0$$

for every d''-closed $(n-p,n-p)$-form ω on $(\mathcal{X}\otimes_{k,\iota_{\mathbf{F}}}\mathbf{C}_{\mathbf{F}})^{\mathrm{an}}$.

1.4. **Structure of the article.** We start in §2 by reviewing the relation between Chow cohomology and sheaves of Milnor K-theory, in order to define tropical cycle class maps later.

In §3, we introduce the Dolbeault cohomology for non-Archimedean analytic spaces and construct the tropical cycle class map for smooth varieties over non-Archimedean fields. We show an integration formula for tropical cycle classes in Theorem 3.7, which can be regarded as a tropical version of the Cauchy formula in multi-variable complex analysis. As a corollary, we give a lower bound for the dimension of certain Dolbeault cohomology in terms of algebraic cycles modulo numerical equivalence.

In §4, we review the theory of rigid cohomology, and compute some logarithmic differential forms on strictly semistable schemes in terms of rigid cohomology. These will be used later.

In §5, we prove Theorem 1.4. We work mainly in the étale topology, by using de Jong's alteration and the theory of rigid cohomology reviewed in the previous section. In particular, our terminology "weight decomposition" comes from the notion of weights in rigid cohomology. The theorem for analytic topology will be deduced from the one for étale topology not difficultly.

In §6, we construct the so-called log-differential cycle class map, through $\mathbf{Q}$-subsheaves of de Rham cohomology sheaves spanned by logarithmic differential forms. The main result we show is Theorem 6.6, an analogue of Theorem 1.8 in this context.

In the last section §7, we first reveal a connection between the tropical theory and the algebraic de Rham theory in Theorem 7.1. In particular, this implies that the tropical cycle class map and the log-differential cycle class map are essentially the same one. Based on this, we deduce Theorems 1.8 & 1.9.

1.5. Conventions and Notation.

General conventions and notation:

- For a product $\cdots\times W\times\cdots$ in a suitable category, we denote by
$$\mathrm{pr}_W\colon\ \cdots\times W\times\cdots\to W$$
the canonical projection morphism to the factor W.
- If $\mathcal{X}$ is a scheme over an affine scheme $\operatorname{Spec} A$ and B is an A-algebra, then we put $\mathcal{X}_B=\mathcal{X}\times_{\operatorname{Spec} A}\operatorname{Spec} B$. Such abbreviation will be applied only to schemes, neither formal schemes nor analytic spaces. If $\mathcal{X}$ is a scheme over $\operatorname{Spec} K^\circ$ for a non-Archimedean field K, then we write $\mathcal{X}_s$ for $\mathcal{X}_{\widetilde{K}}$.

- Let X be a site. Whenever we have a suitable notion of de Rham complex $(\Omega^\bullet_X, \mathrm{d})$ on X, we denote by $\mathrm{H}^\bullet_{\mathrm{dR}}(X) := \mathrm{H}^\bullet(X, \Omega^\bullet_X)$ the corresponding de Rham cohomology of X, as the hypercohomology of the de Rham complex.
- Let X be a site and $\mathscr{K}$ a complex of sheaves on X. We denote by $\underline{\mathrm{H}}^\bullet(X, \mathscr{K})$ the sheaf on X associated to the presheaf $U \mapsto \mathrm{H}^\bullet(U, \mathscr{K}_U) := \mathrm{h}^\bullet\mathrm{R}\Gamma(U, \mathscr{K}_U)$.

Conventions and notation for non-Archimedean geometry:

- Throughout the article, by a *non-Archimedean field*, we mean a complete topological field whose topology is induced by a nontrivial non-Archimedean valuation $|\ |$ of rank 1. If the valuation is discrete, then we say that it is a *discrete non-Archimedean field* by abuse of terminology. In the main text, discrete non-Archimedean fields are usually denoted by lower-case letters like k, k', etc. In addition, ϖ will always be a uniformizer of a discrete non-Archimedean field, though we will still remind readers of this.
- Throughout the article, we fix a finite field $\mathbf{F}$. Denote by $\mathbf{Z}_\mathbf{F}$ the ring of Witt vectors in $\mathbf{F}$ and $\mathbf{Q}_\mathbf{F}$ the field of fractions of $\mathbf{Z}_\mathbf{F}$. We fix a complete algebraic closure $\mathbf{C}_\mathbf{F}$ of $\mathbf{Q}_\mathbf{F}$, both being non-Archimedean fields.
- For a non-Archimedean field K with valuation $|\ |$, put
$$K^\circ = \{x \in K \mid |x| \leq 1\}, \quad K^{\circ\circ} = \{x \in K \mid |x| < 1\}, \quad \widetilde{K} = K^\circ/K^{\circ\circ}.$$
Denote by K^a an algebraic closure of K and $\widehat{K^\mathrm{a}}$ its completion. A *residually algebraic* extension of K is an extension K'/K of non-Archimedean fields such that the induced extension $\widetilde{K'}/\widetilde{K}$ is algebraic.
- Let K be a non-Archimedean field. All K-analytic (Berkovich) spaces are assumed to be *good* [Ber93, 1.2.16], *paracompact*, *Hausdorff* and *strictly K-analytic* [Ber93, 1.2.15].
- Let K be a non-Archimedean field, and A an affinoid K-algebra. We then have the K-analytic space $\mathcal{M}(A)$. Denote by A° the subring of power-bounded elements, which is a K°-algebra. Put $A_s = A^\circ \otimes_{K^\circ} \widetilde{K}$. We say that A is *integrally smooth* if A is strictly K-affinoid and $\operatorname{Spf} A^\circ$ is a smooth formal K°-scheme.
- Let K be a non-Archimedean field. For a real number $r > 0$, we denote by $D(0; r)_K$ the open disc over K with center at zero of radius r as a K-analytic space. For real numbers $R > r > 0$, we denote by $B(0; r, R)_K$ the open annulus over K with center at zero of inner radius r and outer radius R as a K-analytic space. An *open poly-disc (of dimension n)* over K is the product of finitely many open discs $D(0; r_i)_K$ (of number n) over K.
- Suppose that K'/K is an extension of non-Archimedean fields. For a K-analytic space X and a K'-analytic space Y, we put
$$X \widehat{\otimes}_K K' = X \times_{\mathcal{M}(K)} \mathcal{M}(K'), \quad Y \times_K X = Y \times_{\mathcal{M}(K')} (X \widehat{\otimes}_K K').$$
For a formal K°-scheme $\mathfrak{X}$ and a formal K'°-scheme $\mathfrak{Y}$, we put
$$\mathfrak{X} \widehat{\otimes}_{K^\circ} K'^\circ = \mathfrak{X} \times_{\operatorname{Spf} K^\circ} \operatorname{Spf} K'^\circ, \quad \mathfrak{Y} \times_{K^\circ} \mathfrak{X} = \mathfrak{Y} \times_{\operatorname{Spf} K'^\circ} (\mathfrak{X} \widehat{\otimes}_{K^\circ} K'^\circ).$$
- If k is a discrete non-Archimedean field and $\mathfrak{X}$ is a special formal k°-scheme in the sense of [Ber96], then we have the notion $\mathfrak{X}_\eta$, the generic fiber of $\mathfrak{X}$, which is a k-analytic space; and $\mathfrak{X}_s$, the closed fiber of $\mathfrak{X}$, which is a scheme locally of finite type over $\widetilde{k}$; and a reduction map $\pi \colon \mathfrak{X}_\eta \to \mathfrak{X}_s$. For a general non-Archimedean field K, we

say that a formal K°-scheme $\mathfrak{X}$ is special if there exist a discrete non-Archimedean field $k \subset K$ and a special formal k°-scheme $\mathfrak{X}'$ such that $\mathfrak{X} \simeq \mathfrak{X}'\widehat{\otimes}_{k^\circ} K^\circ$. For a special formal K°-scheme, we have similar notion $\pi\colon \mathfrak{X}_\eta \to \mathfrak{X}_s$ which is canonically defined. In this article, all formal K°-schemes will be special. Note that if $\mathcal{Z}$ is a subscheme of $\mathfrak{X}_s$, then $\pi^{-1}\mathcal{Z}$ is usually denoted as $]\mathcal{Z}[_{\mathfrak{X}_\eta}$ in rigid analytic geometry.

- Let K be a non-Archimedean field and X a K-analytic space. For a point $x \in X$, one may associate nonnegative integers $s_K(x), t_K(x)$ as in [Ber90, §9.1]. For readers' convenience, we recall the definition. The number $s_K(x)$ is equal to the transcendence degree of $\widetilde{\mathcal{H}(x)}$ over $\widetilde{K}$, and the number $t_K(x)$ is equal to to the dimension of the $\mathbf{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|K^*|}$, where $\mathcal{H}(x)$ is the completed residue field of x. In the text, the field K will always be clear so will be suppressed in the notation $s_K(x), t_K(x)$.

Acknowledgements. The author is partially supported by NSF grant DMS–1602149. He thanks Walter Gubler, Phillipp Jell, Klaus Künnemann, and Weizhe Zheng for helpful discussions. He also thanks the anonymous referee for very careful reading and numerous helpful comments on the article.

2. Chow cohomology revisited

In this section, we study Chow cohomology of a smooth scheme via sheaves of Milnor K-theory.

Definition 2.1 (Sheaf of rational Milnor K-theory)**.** Let $(X, \mathcal{O}_X)$ be a ringed site. We define the *p-th sheaf of rational Milnor K-theory* $\mathscr{K}_X^p$ for $(X, \mathcal{O}_X)$ to be the sheaf associated to the presheaf assigning an open U in X to $K_p^M(\mathcal{O}_X(U)) \otimes \mathbf{Q}$ ([Sou85, §6.1]). Here for a (commutative) ring R and an integer $p \geq 1$, $K_p^M(R)$ is the abelian group generated by the symbols $\{f_1, \dots, f_p\}$ with $f_1, \dots, f_p \in R^*$, modulo the two relations

(1) $\{f_1, \dots, f_i f_i', \dots, f_p\} = \{f_1, \dots, f_i, \dots, f_p\} + \{f_1, \dots, f_i', \dots, f_p\}$,
(2) $\{f_1, \dots, f, \dots, 1-f, \dots, f_p\} = 0$.

We also adopt the convention that $K_0^M(R) = \mathbf{Z}$ and $K_p^M(R) = 0$ if $p < 0$.

Let k be a field. Let X be a smooth scheme of finite type over k. For every integer $p \geq 0$, denote by $X^{(p)} \subset X$ the subset of points of codimension p. For $x \in X$, denote by $k(x)$ the residue field at x and $i_x\colon \operatorname{Spec} k(x) \to X$ the induced morphism. By [Sou85, Théorème 5], we have a functorial exact sequence

$$0 \to \mathscr{K}_X^p \to \mathscr{K}_X^{p,0} \xrightarrow{\mathrm{d}} \mathscr{K}_X^{p,1} \xrightarrow{\mathrm{d}} \mathscr{K}_X^{p,2} \xrightarrow{\mathrm{d}} \cdots \tag{2.1}$$

of $\mathbf{Q}$-sheaves on X (in the Zariski topology) for every $p \geq 0$, where

$$\mathscr{K}_X^{p,q} = \bigoplus_{x \in X^{(i)}} i_{x*} K_{p-q}^M(k(x)) \otimes \mathbf{Q}.$$

The differential map $\mathrm{d}\colon \mathscr{K}_X^{p,q} \to \mathscr{K}_X^{p,q+1}$ in (2.1) is given by residue homomorphisms (see for example [Ros96, §1]).

The sheaf $\mathscr{K}_X^{p,q}$ is obviously flasque. Therefore, we have a canonical isomorphism

$$\mathrm{H}^q(X, \mathscr{K}_X^p) \cong \frac{\ker(\mathrm{d}\colon \mathscr{K}_X^{p,q}(X) \to \mathscr{K}_X^{p,q+1}(X))}{\mathrm{im}(\mathrm{d}\colon \mathscr{K}_X^{p,q-1}(X) \to \mathscr{K}_X^{p,q}(X))}. \tag{2.2}$$

In particular, we obtain the following.

Lemma 2.2 ([Sou85]). *Let X be a smooth scheme of finite type over a field k. Then*

(1) For every $p \geq 0$, we have a canonical isomorphism

$$\mathrm{H}^p(X, \mathscr{K}_X^p) \simeq \mathrm{CH}^p(X)_{\mathbf{Q}} \coloneqq \mathrm{CH}^p(X) \otimes \mathbf{Q}.$$

(2) For $q > p \geq 0$, we have $\mathrm{H}^q(X, \mathscr{K}_X^p) = 0$.

(3) For an irreducible closed subset Z of X of codimension p, we have

$$\mathrm{H}^q(X, i_{Z!}i_Z^!\mathscr{K}_X^p) = 0$$

if $q \neq p$, and

$$\mathrm{H}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \cong \mathbf{Q},$$

where $i_Z \colon Z \to X$ is the closed embedding.

Proof. Both (1) and (2) are direct consequences of (2.2). For (3), let $U \coloneqq X \setminus Z$ be the complement. Then we have the Gysin exact sequence

$$\cdots \to \mathrm{H}^q(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \to \mathrm{H}^q(X, \mathscr{K}_X^p) \to \mathrm{H}^q(U, \mathscr{K}_U^p) \to \mathrm{H}^{q+1}(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \to \cdots$$

as $\mathscr{K}_X^p|_U \cong \mathscr{K}_U^p$. By (2.2), we know that the restriction map $\mathrm{H}^q(X, \mathscr{K}_X^p) \to \mathrm{H}^q(U, \mathscr{K}_U^p)$ is an isomorphism if $q \notin \{p-1, p\}$, and is surjective if $q = p$. Thus, $\mathrm{H}^q(X, i_{Z!}i_Z^!\mathscr{K}_X^p)$ if $q \neq p$. When $q = p$, we have two cases:

- if Z is nontrivial in $\mathrm{CH}^p(X) \otimes \mathbf{Q}$, then we have canonical isomorphisms
$$\mathrm{H}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \cong \ker(\mathrm{H}^p(X, \mathscr{K}_X^p) \to \mathrm{H}^p(U, \mathscr{K}_U^p)) \cong K_0^M(k(Z)) \otimes \mathbf{Q} \cong \mathbf{Q};$$
- if Z is trivial in $\mathrm{CH}^p(X) \otimes \mathbf{Q}$, then we have canonical isomorphisms
$$\mathrm{H}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \cong \mathrm{coker}(\mathrm{H}^{p-1}(X, \mathscr{K}_X^p) \to \mathrm{H}^{p-1}(U, \mathscr{K}_U^p)) \cong K_0^M(k(Z)) \otimes \mathbf{Q} \cong \mathbf{Q}.$$

The lemma is proved. □

In what follows, we will denote by

$$\mathrm{cl}_{\mathrm{univ}} \colon \mathrm{CH}^p(X)_{\mathbf{Q}} \to \mathrm{H}^p(X, \mathscr{K}_X^p)$$

the map obtained in Lemma 2.2 (1).

Let Z be a smooth closed subscheme of X of codimension p. For later use, we provide an explicit description of $\mathrm{cl}_{\mathrm{univ}}(Z)$ as follows: Choose a finite affine open covering U_α of X and a regular sequence $f_{\alpha 1}, \dots, f_{\alpha p} \in \mathcal{O}_X(U_\alpha)$ such that $Z \cap U_\alpha$ is defined by the ideal $(f_{\alpha 1}, \dots, f_{\alpha p})$. Let $U_{\alpha i}$ be the nonvanishing locus of $f_{\alpha i}$. Then $\{U_{\alpha i} \mid 1 \leq i \leq p\}$ forms an open covering of $U_\alpha \backslash Z$. Thus the element $\{f_{\alpha 1}, \dots, f_{\alpha p}\} \in K_p^M(\mathcal{O}_X(\bigcap_{i=1}^p U_{\alpha i}))$ gives rise to an element in $\mathrm{H}^{p-1}(U_\alpha \backslash Z, \mathscr{K}_X^p)$ and hence an element $c(Z)_\alpha$ in $\mathrm{H}^p(U_\alpha, i_{Z!}i_Z^!\mathscr{K}_X^p)$. We have the following lemma.

Lemma 2.3. *Let the notation be as above. Assume that $Z \cap U_\alpha$ is nonempty. Then under the isomorphism in Lemma 2.2 (3), the class $c(Z)_\alpha \in \mathrm{H}^p(U_\alpha, i_{Z!}i_Z^!\mathscr{K}_X^p)$ equals 1. In particular, it does not depend on the choice of $f_{\alpha 1}, \dots, f_{\alpha p}$.*

Proof. We may assume that $Z \cap U_\alpha$ is irreducible for simplicity. To ease notation, we suppress the subscript α in the proof.

We first construct a Dolbeault representative $\theta \in \mathscr{K}_U^{p,p-1}(U \backslash Z)$ of the (alternative) closed Čech cocycle $\{f_1, \dots, f_p\} \in \mathscr{K}_U^p(\bigcap_{i=1}^p U_i)$, where we have used the ordered covering $\underline{U} \coloneqq$

$\{U_1, \ldots, U_p\}$ of $U\backslash Z$. For $I \subset \{1, \ldots, p\}$, put $U_I = \bigcap_{i\in I} U_i$. We inductively construct elements $\theta_i \in \mathrm{H}^{p-i-1}(U\backslash Z, \mathscr{K}_U^{p,i})$ represented by an (alternative) closed Čech cocycle

$$\theta_i = \{\theta_{i,I} \in \mathscr{K}_U^{p,i}(U_I) \mid |I| = p - i\}$$

for $i = 0, \ldots, p-1$. The class θ_0 is simply

$$\{\theta_{0,\{1,\ldots,p\}} = \{f_1, \ldots, f_p\} \in \mathscr{K}_U^{p,0}(U_{\{1,\ldots,p\}})\}.$$

Suppose that we have θ_{i-1} for some $1 \le i \le p-1$. As $\mathscr{K}_U^{p,i-1}$ is acyclic, the Čech cohomology $\mathrm{H}^{p-i}(\underline{U}, \mathscr{K}_U^{p,i-1})$ is trivial. Thus there exists $\vartheta_i = \{\vartheta_{i,J} \in \mathscr{K}_U^{p,i-1}(U_J) \mid |J| = p - i\}$ with $\delta_{\underline{U}}\vartheta_i = \theta_{i-1}$, where $\delta_{\underline{U}}$ denotes the Čech differential for the covering $\underline{U}$. Now we set $\theta_i = \mathrm{d}\vartheta_i \coloneqq \{\mathrm{d}\vartheta_{i,J} \in \mathscr{K}_U^{p,i}(U_J) \mid |J| = p - i\}$. The last closed Čech cocycle $\theta_{p-1} = \{\theta_{p-1,\{i\}} \in \mathscr{K}_U^{p,p-1}(U_i) \mid i = 1, \ldots, p\}$ can be glued to a Dolbeault representative $\theta \in \mathscr{K}_U^{p,p-1}(U\backslash Z)$ we are looking for.

In practice, we may take

$$\vartheta_{1,J} = \begin{cases} \{f_1, \ldots, f_p\} \in K_p^M(k(\eta)) \otimes \mathbf{Q} \subset \mathscr{K}_U^{p,0}(U_J), & J = \{2, \ldots, p\}, \\ 0, & J \neq \{2, \ldots, p\}, \end{cases}$$

where η is the generic point of $U_{\{2,\ldots,p\}}$. Since $f_1, \ldots, f_p$ is a regular sequence, the intersection $\{f_1 = 0\} \cap U_{\{2,\ldots,p\}}$ if of pure codimension 1. Denote by Γ^1 the set of generic points of $\{f_1 = 0\} \cap U_{\{2,\ldots,p\}}$, which is a finite subset of $U^{(1)}_{\{2,\ldots,p\}}$. Then we have a function $\mu^1 \colon \Gamma^1 \to \mathbf{Z}$ given by the vanishing order of f_1. For $j > 1$, let $f_j^{(1)}$ be the residue of f_j to the set Γ^1. Thus, we have

$$\theta_{1,I} = \begin{cases} \mu^1 \cdot \{f_2^{(1)}, \ldots, f_p^{(1)}\} \in \mathscr{K}_U^{p,1}(U_I), & I = \{2, \ldots, p\}, \\ 0, & I \neq \{2, \ldots, p\}. \end{cases}$$

By induction, we have

$$\theta_{i,I} = \begin{cases} \mu^i \cdot \{f_{i+1}^{(i)}, \ldots, f_p^{(i)}\} \in \mathscr{K}_U^{p,i}(U_I), & I = \{i+1, \ldots, p\}, \\ 0, & I \neq \{i+1, \ldots, p\}. \end{cases}$$

for $i = 2, \ldots, p-1$. We explain the notation inductively: Γ^i is the set of generic points of $\{f_1 = \cdots = f_i = 0\} \cap U_{\{i+1,\ldots,p\}}$, which is a finite subset of $U^{(i)}_{\{i+1,\ldots,p\}}$; $\mu^i \colon \Gamma^i \to \mathbf{Z}$ is given by the vanishing order of $f_i^{(i-1)}$; and $f_j^{(i)}$ is the residue of $f_j^{(i-1)}$ to the set Γ^i for $j > i$. In particular, we may glue $\{\theta_{p-1,\{1\}}, \ldots, \theta_{p-1,\{p\}}\}$ together to give an element $\theta \in \mathscr{K}_U^{p,p-1}(U\backslash Z)$. Then the lemma follows as $\{f_1 = \cdots = f_p = 0\}$ coincides with $Z \cap U$, and $\mu^{p-1} \cdot f_p^{(p)}$ has vanishing order 1 along it. □

By the above lemma, we can glue the collection $\{c(Z)_\alpha\}$ hence obtain an element $c(Z) \in \mathrm{H}^0(X, \underline{\mathrm{H}}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p))$. By Lemma 2.2 (3) and the local to global spectral sequence, we obtain a canonical isomorphism $\mathrm{H}^0(X, \underline{\mathrm{H}}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p)) \cong \mathrm{H}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p)$.

Lemma 2.4. *Let the notation be as above. The image of the class $c(Z)$ under the map $\mathrm{H}^p(X, i_{Z!}i_Z^!\mathscr{K}_X^p) \to \mathrm{H}^p(X, \mathscr{K}_X^p)$ coincides with $\mathrm{cl}_{\mathrm{univ}}(Z)$.*

Proof. The follows directly from Lemma 2.3 and Lemma 2.2. □

Remark 2.5. Suppose that k is of characteristic zero and X is a smooth scheme of finite type over k. We have the algebraic de Rham complex

$$\Omega^\bullet_X\colon \mathscr{O}_X=\Omega^0_X\xrightarrow{\mathrm{d}}\Omega^1_X\xrightarrow{\mathrm{d}}\Omega^2_X\xrightarrow{\mathrm{d}}\cdots,$$

and the algebraic de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X/k)\coloneqq\mathrm{H}^i(X,\Omega^\bullet_X)$. Moreover, we have the algebraic de Rham cycle class map

$$\mathrm{cl}_{\mathrm{dR}}\colon\ \mathrm{CH}^p(X)_{\mathbf{Q}}\to\mathrm{H}^{2p}_{\mathrm{dR}}(X/k).$$

One can construct $\mathrm{cl}_{\mathrm{dR}}$ via $\mathrm{cl}_{\mathrm{univ}}$ as follows: we have a ($\mathbf{Q}$-linear) map of complexes

$$\mathscr{K}^p_X[-p]\to\Omega^\bullet_X$$

sending a section $\{f_1,\dots,f_p\}\in\mathscr{K}^p_X(U)$ to

$$\frac{\mathrm{d}f_1}{f_1}\wedge\cdots\wedge\frac{\mathrm{d}f_p}{f_p}\in\Omega^{p,\mathrm{cl}}_X(U).$$

It induces the map

$$\mathrm{H}^p(X,\mathscr{K}^p)=\mathrm{H}^{2p}(X,\mathscr{K}^p_X[-p])\to\mathrm{H}^{2p}(X,\Omega^\bullet_X)=\mathrm{H}^{2p}_{\mathrm{dR}}(X/k).$$

Then $\mathrm{cl}_{\mathrm{dR}}$ is simply the composition of $\mathrm{cl}_{\mathrm{univ}}$ with the above map.

3. Tropical cycle class map

In this section, we introduce the Dolbeault cohomology for non-Archimedean analytic spaces, define the tropical cycle class map for separated smooth schemes over a non-Archimedean field, and study some basic properties.

First we recall some facts from the theory of real forms on non-Archimedean analytic spaces developed by Chambert-Loir and Ducros in [CLD12]. (See also [Gub13] for a slightly different formulation.) Let K be a non-Archimedean field, and X a K-analytic space. There is a bicomplex $(\mathscr{A}^{\bullet,\bullet}_X,\mathrm{d}',\mathrm{d}'')$ of sheaves of real vector spaces on (the underlying topological space of) X, where $\mathscr{A}^{p,q}_X$ is the *sheaf of* (p,q)*-forms* ([CLD12, §3.1]). It is known that $\mathscr{A}^{p,q}_X=0$ unless $0\le p,q\le\dim(X)$. We have an integration functor

$$\int_X\colon\ \mathscr{A}^{n,n}_X(X)_c\to\mathbf{R}$$

for $n=\dim(X)$ and $\mathscr{A}^{n,n}_X(X)_c$ denotes the set of (n,n)-forms with compact support.

Definition 3.1 (Dolbeault cohomology)**.** Let X be a K-analytic space. We define the *Dolbeault cohomology* of X to be

$$\mathrm{H}^{p,q}(X)\coloneqq\frac{\ker(\mathrm{d}''\colon\mathscr{A}^{p,q}_X(X)\to\mathscr{A}^{p,q+1}_X(X))}{\mathrm{im}(\mathrm{d}''\colon\mathscr{A}^{p,q-1}_X(X)\to\mathscr{A}^{p,q}_X(X))}.$$

By [Jel16, Corollary 4.6] and [CLD12, Corollaire 3.3.7], the complex $(\mathscr{A}^{p,\bullet}_X,\mathrm{d}'')$ is a fine resolution[3] of $\ker(\mathrm{d}''\colon\mathscr{A}^{p,0}_X\to\mathscr{A}^{p,1}_X)$ for every $p\ge0$. In particular, we have a canonical isomorphism

$$\mathrm{H}^\bullet(X,\ker(\mathrm{d}''\colon\mathscr{A}^{p,0}_X\to\mathscr{A}^{p,1}_X))\simeq\mathrm{H}^{p,\bullet}(X).$$

On the other hand, we have the sheaf of rational Milnor K-theory $\mathscr{K}^\bullet_X$ for the ringed topological space $(X,\mathscr{O}_X)$ from Definition 2.1.

[3] Recall that we have assumed that all K-analytic spaces are good and paracompact in §1.5.

Definition 3.2. Let X be a K-analytic space. For every $p \geq 0$, we define a ($\mathbf{Q}$-linear) map of sheaves

$$\tau_X^p\colon \mathscr{K}_X^p \to \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$$

as follows. For a symbol $\{f_1, \ldots, f_p\} \in \mathscr{K}_X^p(U)$ with $f_1, \ldots, f_p \in \mathcal{O}_X^*(U)$, we have the induced moment morphism $(f_1, \ldots, f_p)\colon U \to (\mathbf{G}_{m,K}^{\mathrm{an}})^p$. Composing with the evaluation map $-\log|\;|\colon (\mathbf{G}_{m,K}^{\mathrm{an}})^p \to \mathbf{R}^p$, we obtain a continuous map

$$\mathrm{trop}_{\{f_1,\ldots,f_p\}}\colon U \to \mathbf{R}^p.$$

If we endow the target with coordinates $x_1, \ldots, x_p$ where $x_i = -\log|f_i|$, then we define

$$\tau_X^p(\{f_1, \ldots, f_p\}) = \mathrm{d}'x_1 \wedge \cdots \wedge \mathrm{d}'x_p \in \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0}(U) \to \mathscr{A}_X^{p,1}(U)).$$

It is easy to see that τ_X^p factors through the relations of Milnor K-theories, and thus induces a map of corresponding sheaves. See also Remark 3.3. Finally, put

$$\mathscr{T}_X^p = \mathscr{K}_X^p / \ker \tau_X^p.$$

It is canonically a subsheaf of $\ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$.

Remark 3.3. In Definition 3.2, the factorization of the first Milnor relation is due to the additivity of differential forms; and the factorization of the second Milnor relation is due to the fact that the image $\mathrm{trop}_{\{f_1,\ldots,f_p\}}(U)$ has dimension at most $p-1$.

In fact, the factorization of Milnor relations is not relevant in Definition 3.2. We include this observation in the definition simply for the preparation of defining cycle class maps in Definition 3.6 later.

Proposition 3.4. *Let K be a non-Archimedean field and X a K-analytic space. Then τ_X^p induces an isomorphism*

$$\mathscr{T}_X^p \otimes_{\mathbf{Q}} \mathbf{R} \simeq \ker(\mathrm{d}''\colon \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1}).$$

Proof. It suffices to show that the induced map

$$\tau_x\colon \mathscr{T}_{X,x}^p \otimes_{\mathbf{Q}} \mathbf{R} \to \ker(\mathrm{d}''\colon \mathscr{A}_{X,x}^{p,0} \to \mathscr{A}_{X,x}^{p,1}) \tag{3.1}$$

on stalks is an isomorphism for every point $x \in X$.

We fix a point $x \in X$. For every open neighborhood U of x in X and $f_1, \ldots, f_N \in \mathcal{O}_X^*(U)$, we have the induced tropicalization map

$$\mathrm{trop}_U\colon U \xrightarrow{f_1,\ldots,f_N} (\mathbf{G}_{m,K}^{\mathrm{an}})^N \xrightarrow{-\log|\;|} T_N \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^N$$

where T_N is the cocharacter lattice of $\mathbf{G}_m^N$. It is proved by Berkovich that there is an integral polyhedral complex[4] $\mathcal{C}_U$ such that $\mathrm{trop}_U(U)$ is an open subset of $\mathcal{C}_U$. We may shrink U so that one can choose $\mathcal{C}_U$ such that it contains only one minimal polyhedron, denoted by σ_U, and σ_U contains $\mathrm{trop}_U(x)$. Such data $(U; f_1, \ldots, f_N)$ will be called a basic chart at x. Let τ be an arbitrary polyhedron of $\mathcal{C}_U$ under a basic chart. Then σ_U is a face of τ, and we denote by $\mathbb{L}(\tau)$ the underlying $\mathbf{Q}$-linear subspace of $T_{N,\mathbf{Q}} := T_N \otimes_{\mathbf{Z}} \mathbf{Q}$ of τ. We have an inclusion

$$\sum_{\tau \in \mathcal{C}_U} \bigwedge^p \mathbb{L}(\tau) \subset \bigwedge^p T_{N,\mathbf{Q}}$$

[4]In this article, a *polyhedron* in $T_N \otimes_{\mathbf{Z}} \mathbf{R}$ is a subset defined by finitely many non-strict inequalities. A polyhedral complex in $T_N \otimes_{\mathbf{Z}} \mathbf{R}$ is *integral* if it consists of polyhedra with slopes in T_N.

of $\mathbf{Q}$-vector spaces, and thus a surjective map

$$r_U\colon \operatorname{Hom}_{\mathbf{Q}}\left(\bigwedge^p T_{N,\mathbf{Q}},\mathbf{R}\right)\to \operatorname{Hom}_{\mathbf{Q}}\left(\sum_{\tau\in\mathcal{C}_U}\bigwedge^p \mathbb{L}(\tau),\mathbf{R}\right).$$

By [JSS15, Proposition 3.20], the canonical map

$$\tau_U\colon \operatorname{Hom}_{\mathbf{Q}}\left(\bigwedge^p T_{N,\mathbf{Q}},\mathbf{R}\right)\to \ker\left(\mathrm{d}''\colon \mathscr{A}^{p,0}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\to \mathscr{A}^{p,1}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\right)$$

factors through r_U and induces an isomorphism

$$\operatorname{Hom}_{\mathbf{Q}}\left(\sum_{\tau\in\mathcal{C}_U}\bigwedge^p \mathbb{L}(\tau),\mathbf{R}\right)\xrightarrow{\sim}\ker\left(\mathrm{d}''\colon \mathscr{A}^{p,0}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\to \mathscr{A}^{p,1}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\right).$$

Here, $\mathscr{A}^{p,q}_{\mathcal{C}_U}$ is the sheaf of (p,q)-forms on $\mathcal{C}_U$ (see [JSS15] for more details).

On the other hand, we have a $\mathbf{Q}$-linear subspace $\mathscr{T}^p_X(U)_{f_1,\dots,f_N}$ of $\mathscr{T}^p_X(U)$ generated by $\tau^p_X(\{f_{i_1},\dots,f_{i_p}\})$ where $\{f_{i_1},\dots,f_{i_p}\}$ is a subset of $\{f_1,\dots,f_N\}$. Then the map

$$\tau^p_X\colon \mathscr{T}^p_X(U)_{f_1,\dots,f_N}\to \ker\left(\mathrm{d}''\colon \mathscr{A}^{p,0}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\to \mathscr{A}^{p,1}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\right)$$

factors through $\operatorname{Hom}_{\mathbf{Q}}\left(\sum_{\tau\in\mathcal{C}_U}\bigwedge^p \mathbb{L}(\tau),\mathbf{R}\right)$ and induces isomorphisms

$$\mathscr{T}^p_X(U)_{f_1,\dots,f_N}\xrightarrow{\sim}\operatorname{Hom}_{\mathbf{Q}}\left(\sum_{\tau\in\mathcal{C}_U}\bigwedge^p \mathbb{L}(\tau),\mathbf{Q}\right)$$

hence

$$\mathscr{T}^p_X(U)_{f_1,\dots,f_N}\otimes_{\mathbf{Q}}\mathbf{R}\xrightarrow{\sim}\ker\left(\mathrm{d}''\colon \mathscr{A}^{p,0}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\to \mathscr{A}^{p,1}_{\mathcal{C}_U}(\mathrm{trop}_U(U))\right).$$

Taking colimit over all basic charts at x, we conclude that τ_x (3.1) is an isomorphism. The proposition follows. ☐

Proposition 3.4 has the following corollary, which provide the Dolbeault cohomology $\mathrm{H}^{p,q}(X)$ a canonical rational structure.

Corollary 3.5. *Let K be a non-Archimedean field and X a K-analytic space. Then for all $p,q\geq 0$, we have a canonical isomorphism*

$$\mathrm{H}^q(X,\mathscr{T}^p_X)\otimes_{\mathbf{Q}}\mathbf{R}\simeq \mathrm{H}^{p,q}(X).$$

In particular, the real vector space $\mathrm{H}^{p,q}(X)$ has a canonical rational structure.

In what follows, we will always regard $\mathrm{H}^q(X,\mathscr{T}^p_X)$ as a subspace of $\mathrm{H}^{p,q}(X)$. Now we are ready to define the tropical cycle class map.

Definition 3.6 (Tropical Dolbeault cohomology and tropical cycle class map)**.** Let K be a non-Archimedean field and $\mathcal{X}$ a separated scheme of finite type over K.

(1) For $p,q\geq 0$, we define the *tropical Dolbeault cohomology* of $\mathcal{X}$ to be

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(\mathcal{X}):=\mathrm{H}^q(\mathcal{X}^{\mathrm{an}},\mathscr{T}^p_{\mathcal{X}^{\mathrm{an}}}).$$

It is a rational subspace of $\mathrm{H}^{p,q}(\mathcal{X}^{\mathrm{an}})$.

(2) If $\mathcal{X}$ is moreover smooth, then we define the *tropical cycle class map* $\mathrm{cl}_{\mathrm{trop}}$ to be the composition

$$\mathrm{CH}^p(\mathcal{X})_{\mathbf{Q}}\xrightarrow{\mathrm{cl}_{\mathrm{univ}}}\mathrm{H}^p(\mathcal{X},\mathscr{K}^p_{\mathcal{X}})\to \mathrm{H}^p(\mathcal{X}^{\mathrm{an}},\mathscr{K}^p_{\mathcal{X}^{\mathrm{an}}})\xrightarrow{\mathrm{H}^p(\mathcal{X}^{\mathrm{an}},\tau^p_{\mathcal{X}^{\mathrm{an}}})}\mathrm{H}^p(\mathcal{X}^{\mathrm{an}},\mathscr{T}^p_{\mathcal{X}^{\mathrm{an}}})=\mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X}).$$

The following theorem can be regarded as a tropical version of the Cauchy formula in multi-variable complex analysis.

Theorem 3.7. *Let K be a non-Archimedean field and $\mathcal{X}$ a separated smooth scheme of finite type over K of dimension n. Then for every algebraic cycle $\mathcal{Z}$ of $\mathcal{X}$ of codimension p, we have the equality*

$$\int_{\mathcal{X}^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}) \wedge \omega = \int_{\mathcal{Z}^{\mathrm{an}}} \omega$$

for every d''-closed form $\omega \in \mathscr{A}_{\mathcal{X}^{\mathrm{an}}}^{n-p,n-p}(\mathcal{X}^{\mathrm{an}})_c$ with compact support. Here, if we write $\mathcal{Z} = \sum_i a_i \mathcal{Z}_i$ where $\mathcal{Z}_i$'s are prime, then we define

$$\int_{\mathcal{Z}^{\mathrm{an}}} \omega \coloneqq \sum_i a_i \int_{\mathcal{Z}_i^{\mathrm{an}}} \omega.$$

Proof. By linearity, we may assume that $\mathcal{Z}$ is prime, that is, a reduced irreducible closed subscheme of $\mathcal{X}$ of codimension p. Let $\mathcal{Z}_{\mathrm{sing}} \subset \mathcal{Z}$ be the singular locus, which is a closed subscheme of $\mathcal{X}$ of codimension $> p$. Put $\mathcal{U} = \mathcal{X} \backslash \mathcal{Z}_{\mathrm{sing}}$, $\mathcal{Z}_{\mathrm{sm}} = \mathcal{Z} \backslash \mathcal{Z}_{\mathrm{sing}}$, $X = \mathcal{X}^{\mathrm{an}}$, $U = \mathcal{U}^{\mathrm{an}}$, and $Z = \mathcal{Z}_{\mathrm{sm}}^{\mathrm{an}}$. In particular, $i_Z \colon Z \hookrightarrow U$ is a Zariski closed subset.

In [CLD12], we have the bicomplex $(\mathscr{D}_U^{\bullet,\bullet}, \mathrm{d}', \mathrm{d}'')$ of currents and a map $\mathscr{A}_U^{\bullet,\bullet} \to \mathscr{D}_U^{\bullet,\bullet}$ of bicomplexes. In particular, we have a map $\mathscr{T}_U^p \to \mathscr{D}_U^{p,\bullet}$ of complexes for every $p \geq 0$ hence the induced map

$$\mathrm{H}^p(U, \mathscr{T}_U^p) \to \mathrm{H}^p(U, \mathscr{D}_U^{p,\bullet}).$$

Denote by $\mathrm{cl}'_{\mathrm{trop}} \colon \mathrm{CH}^p(\mathcal{U})_{\mathbf{Q}} \to \mathrm{H}^p(U, \mathscr{D}_U^{p,\bullet})$ the composition of $\mathrm{cl}_{\mathrm{trop}}$ with the above map.

To ease notation, put

$$\mathscr{A}_?^{p,q,\mathrm{cl}} = \ker(\mathrm{d}'' \colon \mathscr{A}_?^{p,q} \to \mathscr{A}_?^{p,q+1}), \qquad \mathscr{D}_?^{p,q,\mathrm{cl}} = \ker(\mathrm{d}'' \colon \mathscr{D}_?^{p,q} \to \mathscr{D}_?^{p,q+1}).$$

We fix a form $\omega \in \mathscr{A}_X^{n-p,n-p,\mathrm{cl}}(X)_c$. By [CLD12, Lemme 3.2.5], ω belongs to $\mathscr{A}_X^{n-p,n-p,\mathrm{cl}}(U)_c$.

The proof consists of two parts. We first reduce the theorem to an explicit formula (3.2) of local integration. Then we establish it.

Step 1. We describe explicitly the class $\mathrm{cl}'_{\mathrm{trop}}(\mathcal{Z}_{\mathrm{sm}}) \in \mathrm{H}^p(U, \mathscr{D}_U^{p,\bullet})$ in terms of local cohomology. We choose a finite affine open covering $\mathcal{U}_\alpha$ of $\mathcal{U}$ and $f_{\alpha 1}, \dots, f_{\alpha p} \in \mathcal{O}_{\mathcal{U}}(\mathcal{U}_\alpha)$ such that $\mathcal{Z}_{\mathrm{sm}} \cap \mathcal{U}_\alpha$ is defined by the ideal $(f_{\alpha 1}, \dots, f_{\alpha p})$ such that the induced morphism $(f_{\alpha 1}, \dots, f_{\alpha p}) \colon \mathcal{U}_\alpha \to \mathbf{A}_K^p$ is smooth. Let $\mathcal{U}_{\alpha i}$ be the nonvanishing locus of $f_{\alpha i}$. Put $U_\alpha = \mathcal{U}_\alpha^{\mathrm{an}}$ and $U_{\alpha i} = \mathcal{U}_{\alpha i}^{\mathrm{an}}$. Then $\{U_{\alpha i} \mid i = 1, \dots, p\}$ is an open covering of $U_\alpha \backslash Z$. Thus the element $\tau_U^p(\{f_{\alpha 1}, \dots, f_{\alpha p}\})$ gives rise to an element in $\mathrm{H}^{p-1}(U_\alpha \backslash Z, \mathscr{A}_U^{p,0,\mathrm{cl}}) \simeq \mathrm{H}^{p-1}(U_\alpha \backslash Z, \mathscr{A}_U^{p,\bullet})$, and we denote its image under the composite map

$$\mathrm{H}^{p-1}(U_\alpha \backslash Z, \mathscr{A}_U^{p,\bullet}) \to \mathrm{H}^{p-1}(U_\alpha \backslash Z, \mathscr{D}_U^{p,\bullet}) \xrightarrow{\delta''} \mathrm{H}^p(U_\alpha, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet})$$

by $c(Z)_\alpha$, where δ'' is the coboundary map in the corresponding Gysin exact sequence. By Lemma 2.3, $c(Z)_\alpha$ does not depend on the choice of $f_{\alpha 1}, \dots, f_{\alpha p}$. Therefore, $\{c(Z)_\alpha\}$ gives rise to an element $c(Z) \in \mathrm{H}^0(U, \underline{\mathrm{H}}^p(U, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet}))$. By Lemma 3.8 below, we know that $\underline{\mathrm{H}}^q(U, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet}) = 0$ for $q < p$, hence we have an isomorphism $\mathrm{H}^p(U, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet}) \simeq \mathrm{H}^0(U, \underline{\mathrm{H}}^p(U, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet}))$ obtained from the local to global spectral sequence. By Lemma 2.4 and construction, the image of $c(Z)$ in $\mathrm{H}^p(U, \mathscr{D}_U^{p,\bullet})$ coincides with $\mathrm{cl}'_{\mathrm{trop}}(\mathcal{Z}_{\mathrm{sm}})$.

Step 2. By Lemma 3.8 below, we have a canonical isomorphism

$$\mathrm{H}^p(U_\alpha, i_{Z!} i_Z^! \mathscr{D}_U^{p,\bullet}) \simeq \ker(\mathscr{D}_U^{p,p,\mathrm{cl}}(U_\alpha) \to \mathscr{D}_U^{p,p}(U_\alpha \backslash Z)) \subset \mathscr{D}_U^{p,p,\mathrm{cl}}(U_\alpha).$$

Let $\theta_\alpha \in \mathscr{A}_U^{p,p-1,\mathrm{cl}}(U_\alpha\backslash Z)$ be a Dolbeault representative of $\tau_U^p(\{f_{\alpha 1},\dots,f_{\alpha p}\})$ as a cohomology class in $\mathrm{H}^{p-1}(U_\alpha\backslash Z,\mathscr{A}_U^{p,0,\mathrm{cl}})$, with induced class $[\theta_\alpha]\in\mathrm{H}^{p-1}(U_\alpha\backslash Z,\mathscr{D}_U^{p,\bullet})$. By partition of unity, we may write $\omega=\sum_\alpha\omega_\alpha$ with $\omega_\alpha\in\mathscr{A}_U^{n-p,n-p}(U_\alpha)_c$. Note that

$$\int_X \mathrm{cl}_{\mathrm{trop}}(\mathscr{Z})\wedge\omega=\langle\mathrm{cl}'_{\mathrm{trop}}(\mathscr{Z}_{\mathrm{sm}}),\omega\rangle_U=\sum_\alpha\langle\mathrm{cl}'_{\mathrm{trop}}(\mathscr{Z}_{\mathrm{sm}}),\omega_\alpha\rangle_U=\sum_\alpha\langle\delta''([\theta_\alpha]),\omega_\alpha\rangle_{U_\alpha}$$

and

$$\int_{\mathscr{Z}^{\mathrm{an}}}\omega=\sum_\alpha\int_{U_\alpha\cap Z}\omega_\alpha.$$

To prove the theorem, it suffices to show that

$$\langle\delta''([\theta_\alpha]),\omega_\alpha\rangle_{U_\alpha}=\int_{U_\alpha\cap Z}\omega_\alpha$$

for every α. Here, $\langle\ ,\ \rangle$ denotes the pairing between currents and forms.

Step 3. In what follows, we suppress the subscript α. We summarize our data as follows:

- an affine smooth scheme $\mathcal{U}$ over K of dimension n; put $U=\mathcal{U}^{\mathrm{an}}$,
- a smooth irreducible closed subscheme $\mathscr{Z}$ of codimension p defined by the ideal $(f_1,\dots,f_p)$ where $f_1,\dots,f_p\in\mathscr{O}_{\mathcal{U}}(\mathcal{U})$ such that the induced morphism $(f_1,\dots,f_p)\colon\mathcal{U}\to\mathbf{A}_K^p$ is smooth; put $Z=\mathscr{Z}^{\mathrm{an}}$,
- $\theta\in\mathscr{A}_U^{p,p-1,\mathrm{cl}}(U\backslash Z)$ a Dolbeault representative of $\tau_U^p(\{f_1,\dots,f_p\})$ as a cohomology class in $\mathrm{H}^{p-1}(U\backslash Z,\mathscr{A}_U^{p,\bullet})$, and
- $\omega\in\mathscr{A}_U^{n-p,n-p}(U)_c$.

Our goal is to show that

$$\langle\delta''([\theta]),\omega\rangle_U=\int_Z\omega. \tag{3.2}$$

Here we recall that $[\theta]\in\mathrm{H}^{p-1}(U\backslash Z,\mathscr{D}_U^{p,\bullet})$ is the class induced by θ, and

$$\delta''\colon\mathrm{H}^{p-1}(U\backslash Z,\mathscr{D}_U^{p,\bullet})\to\mathrm{H}^p(U,i_{Z!}i_Z^!\mathscr{D}_U^{p,\bullet})$$

is the coboundary map, in which the target $\mathrm{H}^p(U_\alpha,i_{Z!}i_Z^!\mathscr{D}_U^{p,\bullet})$ is canonically a subspace of $\mathscr{D}_U^{p,p,\mathrm{cl}}(U)$. As Z is a closed Zariski subset of U of codimension p, the image of $\mathscr{A}_U^{n-p,n-p}(U)_c$ under d'' is in $\mathscr{A}_U^{n-p,n-p+1,\mathrm{cl}}(U\backslash Z)_c$. By definition, the following diagram

$$\begin{array}{ccccc}
\mathrm{H}^{p-1}(U\backslash Z,\mathscr{D}_U^{p,\bullet}) & \times & \mathscr{A}_U^{n-p,n-p+1,\mathrm{cl}}(U\backslash Z)_c & \xrightarrow{\langle\ ,\ \rangle_U} & \mathbf{R}\\
\big\downarrow{\scriptstyle\delta''} & & \big\uparrow{\scriptstyle\mathrm{d}''} & & \big\|\\
\mathscr{D}_U^{p,p,\mathrm{cl}}(U) & \times & \mathscr{A}_U^{n-p,n-p}(U)_c & \xrightarrow{\langle\ ,\ \rangle_U} & \mathbf{R}
\end{array}$$

is commutative. Therefore, we have

$$\langle\delta''([\theta]),\omega\rangle_U=\int_{U\backslash Z}\theta\wedge\mathrm{d}''\omega.$$

Thus it suffices to show that

$$\int_{U\backslash Z}\theta\wedge\mathrm{d}''\omega=\int_Z\omega.$$

Obviously, the equality does not depend on the choice of the Dolbeault representative.

Step 4. Let $\mathcal{U}_i\subset\mathcal{U}$ be the nonvanishing locus of f_i. Then we have an open covering $\underline{U}=\{U_i\}$ of $U\backslash Z$, where $U_i=\mathcal{U}_i^{\mathrm{an}}$. For $I\subset\{1,\dots,p\}$, put $U_I=\bigcap_{i\in I}U_i$.

Let us recall the construction of a Dolbeault representative θ. We inductively construct elements $\theta_i \in \mathrm{H}^{p-i-1}(U\backslash Z, \mathscr{A}_U^{p,i,\mathrm{cl}})$ represented by an (alternative) closed Čech cocycle

$$\theta_i = \{\theta_{i,I} \in \mathscr{A}_U^{p,i,\mathrm{cl}}(U_I) \mid |I| = p-i\}$$

for $i = 0, \dots, p-1$. The class θ_0 is simply

$$\{\theta_{0,\{1,\dots,p\}} = \tau_U^p(\{f_1, \dots, f_p\}) \in \mathscr{A}_U^{p,0,\mathrm{cl}}(U_{\{1,\dots,p\}})\}.$$

Suppose that we have θ_{i-1} for some $1 \le i \le p-1$. As $\mathscr{A}_U^{p,i-1}$ is acyclic, the Čech cohomology $\mathrm{H}^{p-i}(\underline{U}, \mathscr{A}_U^{p,i-1})$ is trivial. Thus there exists $\vartheta_i = \{\vartheta_{i,J} \in \mathscr{A}_U^{p,i-1}(U_J) \mid |J| = p-i\}$ with $\delta_{\underline{U}}\vartheta_i = \theta_{i-1}$, where $\delta_{\underline{U}}$ denotes the Čech differential for the covering $\underline{U}$. Now we set

$$\theta_i = \mathrm{d}''\vartheta_i := \{\mathrm{d}''\vartheta_{i,J} \in \mathscr{A}_U^{p,i,\mathrm{cl}}(U_J) \mid |J| = p-i\}.$$

The last closed Čech cocycle $\theta_{p-1} = \{\theta_{p-1,\{i\}} \in \mathscr{A}_U^{p,p-1,\mathrm{cl}}(U_i) \mid i = 1, \dots, p\}$ is simply a Dolbeault representative of $\tau_U^p(\{f_1, \dots, f_p\}) \in \mathrm{H}^{p-1}(U\backslash Z, \mathscr{A}_U^{p,\bullet})$.

For $\varepsilon > 0$ and $I \subset \{1, \dots, p\}$, put

$$V_\varepsilon^I = \{x \in U \mid f_i(x) \in \partial\overline{D(0,\varepsilon)}, i \in I;\; f_j(x) \in \overline{D(0,\varepsilon)}, j \notin I\},$$

and $U_\varepsilon = \overline{U\backslash V_\varepsilon^\emptyset}$. Here, $\overline{D(0,\varepsilon)}$ is the closed disc of radius ε with center at zero, and $\overline{U\backslash V_\varepsilon^\emptyset}$ is the closure of $U\backslash V_\varepsilon^\emptyset$ in U. As $\mathrm{d}''\omega \in \mathscr{A}_U^{n-p,n-p+1}(U\backslash Z)_c$, there is a real number $\varepsilon_0 > 0$ such that $\mathrm{d}''\omega = 0$ on $V_{\varepsilon_0}^\emptyset$. Thus for every $0 < \varepsilon < \varepsilon_0$, we have

$$\int_{U\backslash Z} \theta \wedge \mathrm{d}''\omega = \int_{\overline{U\backslash V_\varepsilon^\emptyset}} \theta \wedge \mathrm{d}''\omega = -\int_{\overline{U\backslash V_\varepsilon^\emptyset}} \mathrm{d}''(\theta \wedge \omega) = -\int_{U_\varepsilon} \mathrm{d}''(\theta_{p-1} \wedge \omega). \tag{3.3}$$

Since U_ε is a closed subset of U, the forms ω and hence $\theta \wedge \omega$ have compact support on U_ε.

Step 5. Now we have to use integration on boundaries V_ε^I and the corresponding Stokes' formula, for some fixed $0 < \varepsilon < \varepsilon_0$. We use the formulation of boundary integration through contraction as in [Gub13, §2]. We consider first a tropical chart

$$\mathrm{trop}_W \colon W \to (\mathbf{G}_{m,K}^{\mathrm{an}})^N \xrightarrow{-\log|\,|} \mathbf{R}^N,$$

where W is an open subset of U_ε. Since V_ε^I is a K_ε^I-analytic space of dimension $n - |I|$ for some extension K_ε^I/K of non-Archimedean fields, the codimension of $\mathrm{trop}_W(W \cap V_\varepsilon^I)$ in $\mathrm{trop}_W(W)$ is at least $|I|$. We denote by σ_I the union of closed faces in $\mathrm{trop}_W(W \cap V_\varepsilon^I)$ of codimension exactly $|I|$. For every $i \in I$, we choose a tangent vector ω_i for the closed face $\sigma_{\{i\}}$ of $\sigma_\emptyset$ of codimension 1, as defined in [Gub13, 2.8].

Suppose that $I = \{m_1, \dots, m_j\}$ where $1 \le m_1 \le \dots \le m_j \le p$. If α is an $(n, n-i)$-superform on $\mathbf{R}^N$ with compact support, then we define

$$\int_{\sigma_I} \alpha := \int_{\sigma_I} \langle \alpha; -\omega_{m_1}, \dots, -\omega_{m_j}\rangle_{\{1,\dots,j\}}.$$

It is easy too see that the above integral does not depend on the choice of ω_i; however, it does depend on the order. We may patch the above integral to define the integral $\int_{V_\varepsilon^I} \alpha$ for an $(n, n-|I|)$-form α on V_ε^I with compact support. The negative signs for ω_i ensure that we have the following Stokes' formula

$$\int_{V_\varepsilon^I} \mathrm{d}''\alpha = \sum_{j \notin I} (-1)^{(j, I\cup\{j\})} \int_{V_\varepsilon^{I\cup\{j\}}} \alpha$$

for an $(n, n-|I|-1)$-form α on V_ε^I with compact support, for $|I| \geq 1$. Here, (j, J) is the position from the rear of the index j when J is ordered in the usual manner. However, for the initial Stokes' formula, we have

$$\int_{U_\varepsilon} \mathrm{d}''\alpha = \int_{\partial U_\varepsilon} \alpha = -\sum_{|I|=1} \int_{V_\varepsilon^I} \alpha$$

for an $(n, n-1)$-form α on U_ε with compact support. Here, the inclusion $\partial U_\varepsilon \subset \bigcup_{|I|=1} V_\varepsilon^I$ is obvious. The other inclusion $\bigcup_{|I|=1} V_\varepsilon^I \subset \partial U_\varepsilon$ is due to the compatibility of relative interior under composition [Ber93, 1.5.5 (ii)]. In particular, we have

$$-\int_{U_\varepsilon} \mathrm{d}''(\theta_{p-1} \wedge \omega) = -\int_{\partial U_\varepsilon} \theta_{p-1} \wedge \omega = \sum_{|I|=1} \int_{V_\varepsilon^I} \theta_{p-1,I} \wedge \omega. \tag{3.4}$$

In general, for $1 \leq i \leq p-1$, we have

$$\begin{aligned}
\sum_{|I|=i} \int_{V_\varepsilon^I} \theta_{p-i,I} \wedge \omega &= \sum_{|I|=i} \int_{V_\varepsilon^I} \mathrm{d}''\vartheta_{p-i,I} \wedge \omega \\
&= \sum_{|I|=i} \int_{V_\varepsilon^I} \mathrm{d}''(\vartheta_{p-i,I} \wedge \omega) \\
&= \sum_{|I|=i} \sum_{j \notin I} (-1)^{(j, I \cup \{j\})} \int_{V_\varepsilon^{I \cup \{j\}}} \vartheta_{p-i,I} \wedge \omega \\
&= \sum_{|J|=i+1} \int_{V_\varepsilon^J} \sum_{j \in J} (-1)^{(j,J)} \vartheta_{p-i, J \setminus \{j\}} \wedge \omega \\
&= \sum_{|J|=i+1} \int_{V_\varepsilon^J} (\delta_{\underline{U}} \vartheta_{p-i})_J \wedge \omega \\
&= \sum_{|J|=i+1} \int_{V_\varepsilon^J} \theta_{p-(i+1),J} \wedge \omega.
\end{aligned}$$

Combining with (3.3), (3.4), we have

$$\int_{U \setminus Z} \theta \wedge \mathrm{d}''\omega = \int_{V_\varepsilon^{\{1,\ldots,p\}}} \theta_{0,\{1,\ldots,p\}} \wedge \omega = \int_{V_\varepsilon^{\{1,\ldots,p\}}} \tau_U^p(\{f_1, \ldots, f_p\}) \wedge \omega \tag{3.5}$$

for every $0 < \varepsilon < \varepsilon_0$.

Step 6. By (3.5), the theorem is reduced to the formula

$$\int_{V_\varepsilon^{\{1,\ldots,p\}}} \tau_U^p(\{f_1, \ldots, f_p\}) \wedge \omega = \int_Z \omega \tag{3.6}$$

for sufficiently small $\varepsilon > 0$. Note that $\partial \overline{D(0,\varepsilon)}$ is actually a point, which we denote by $\eta(\varepsilon)$, we have a morphism $f_1\colon U \to \mathbf{A}_K^{1,\mathrm{an}}$. Denote by $U_{\{1\}} := U_{\eta(\varepsilon)}$ the fiber of U over $\eta(\varepsilon)$. It is a smooth K_1-analytic space purely of dimension $n-1$ for some non-Archimedean field extension K_1/K determined by ε. Then we have the induced map $f_2\colon U_{\{1\}} \to \mathbf{A}_{K_1}^{1,\mathrm{an}}$ hence the fiber $U_{\{1,2\}} := (U_{\{1\}})_{\eta(\varepsilon)}$. Inductively, we have $U_{\{1,\ldots,p\}}$, which is a smooth K_p-analytic space purely of dimension $n-p$ for some non-Archimedean field extension K_p/K determined by ε. The underlying topological space $U_{\{1,\ldots,p\}}$, which is canonically a subspace of U, coincides with $V_\varepsilon^{\{1,\ldots,p\}}$ (see [Ber93, §1.4]). Moreover, from definition, we have

$$\int_{V_\varepsilon^{\{1,\ldots,p\}}} \tau_U^p(\{f_1, \ldots, f_p\}) \wedge \omega = \int_{U_{\{1,\ldots,p\}}} \omega.$$

Then (3.6) follows from successively application of Lemma 3.9. ☐

Lemma 3.8. *Let U be a K-analytic space without boundary, $i_Z\colon Z \hookrightarrow U$ a Zariski closed subset of codimension at least p for some $p \geq 0$. Then $i_Z^!\mathscr{D}_U^{p,q} = 0$ for $q < p$, and we have a canonical isomorphism*

$$\mathrm{H}^p(U, i_{Z!}i_Z^!\mathscr{D}_U^{p,\bullet}) \cong \ker(\mathscr{D}_U^{p,p}(U) \to \mathscr{D}_U^{p,p+1}(U) \oplus \mathscr{D}_U^{p,p}(U\backslash Z)),$$

where the map in the kernel is induced by the differential d'' to the first factor and the restriction to the second factor.

Proof. Let $j\colon U\backslash Z \to U$ be the open compliment. Then $j^*\mathscr{D}_U^{p,q} = \mathscr{D}_{U\backslash Z}^{p,q}$. By the partition of unity, we know that both $\mathscr{D}_U^{p,q}$ and $\mathscr{D}_{U\backslash Z}^{p,i}$ are acyclic. Thus $\mathrm{R}j_*j^*\mathscr{D}_U^{p,q} = j_*j^*\mathscr{D}_U^{p,q}$, which is also acyclic. In particular, we have $i_Z^!\mathscr{D}_U^{p,q} = \ker(\mathscr{D}_U^{p,q} \to j_*j^*\mathscr{D}_U^{p,q})$ which is acyclic. By [CLD12, Lemme 3.2.5], we have $i_Z^!\mathscr{D}_U^{p,q} = 0$ for $q < p$.

Moreover, we have

$$\mathrm{H}^p(U, i_{Z!}i_Z^!\mathscr{D}_U^{p,\bullet}) = \ker(i_Z^!\mathscr{D}_U^{p,p}(U) \xrightarrow{\mathrm{d}''} i_Z^!\mathscr{D}_U^{p,p+1}(U)),$$

which is canonically isomorphic to $\ker(\mathscr{D}_U^{p,p}(U) \to \mathscr{D}_U^{p,p+1}(U) \oplus \mathscr{D}_U^{p,p}(U\backslash Z))$. The lemma is proved. ☐

Lemma 3.9. *Let X be a smooth K-analytic space purely of dimension n, and $f\colon X \to \mathbf{A}_K^{1,\mathrm{an}}$ a smooth morphism. For every $\varepsilon > 0$, let $\eta(\varepsilon) \in \mathbf{A}_K^{1,\mathrm{an}}$ be the boundary point of the closed disc $\overline{D(0,\varepsilon)}$. Then for every compactly supported form $\omega \in \mathscr{A}_X^{n-1,n-1}(X)_c$, we have for sufficiently small ε,*

$$\int_{X_0} \omega = \int_{X_{\eta(\varepsilon)}} \omega.$$

Here, X_0 (resp. $X_{\eta(\varepsilon)}$) denotes the fiber of X over 0 (resp. $\eta(\varepsilon)$).

Proof. Since f is smooth, we have

$$\int_{X_0} \omega = \langle \delta_{\mathrm{div}(f)}, \omega\rangle_X. \tag{3.7}$$

By the Poincaré–Lelong formula [CLD12, Théorème 4.6.5], we have

$$(3.7) = \langle \mathrm{d}'\mathrm{d}'' \log|f|, \omega\rangle_X = \int_X \log|f| \wedge \mathrm{d}'\mathrm{d}''\omega \tag{3.8}$$

by [CLD12, (4.4.3.3)]. Since $\mathrm{d}''\omega$ has compact support on $X \setminus X_0$, there exists $\varepsilon_0 > 0$ such that $\mathrm{d}''\omega = 0$ on $X \setminus X_{\varepsilon_0}$ where $X_\varepsilon \coloneqq \{x \in X \mid |f(x)| \geq \varepsilon\}$. Now for every $0 < \varepsilon < \varepsilon_0$, we have

$$(3.8) = \int_{X_\varepsilon} \log|f| \wedge \mathrm{d}'\mathrm{d}''\omega = -\int_{X_\varepsilon} \mathrm{d}'\log|f| \wedge \mathrm{d}''\omega = \int_{X_\varepsilon} \mathrm{d}''(\mathrm{d}'\log|f| \wedge \omega) \tag{3.9}$$

by the Stokes' formula. By the Stokes' formula again, we have

$$(3.9) = \int_{\partial X_\varepsilon} \mathrm{d}'\log|f| \wedge \omega,$$

which is nothing but

$$\int_{X_{\eta(\varepsilon)}} \omega.$$

The lemma follows. ☐

Theorem 3.7 has following corollaries.

Corollary 3.10. *Let K be a non-Archimedean field and $\mathcal{X}$ a proper smooth scheme over K of dimension n. Then for every algebraic cycle $\mathcal{Z}$ of $\mathcal{X}$ of dimension 0, we have*

$$\int_{\mathcal{X}^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}) = \deg \mathcal{Z}.$$

Proof. Since $\mathcal{X}$ is proper, we may take ω to be the constant function 1 on $\mathcal{X}^{\mathrm{an}}$ in Theorem 3.10. Then the conclusion follows as

$$\int_{\mathcal{Z}^{\mathrm{an}}} 1 = \deg \mathcal{Z}.$$

□

Corollary 3.11. *Let K be a non-Archimedean field and $\mathcal{X}$ a proper smooth scheme over K. Let $\mathrm{NS}^p(\mathcal{X})$ be the quotient of $\mathrm{CH}^p(\mathcal{X})$ modulo numerical equivalence. Then we have*

$$\dim \mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X}) \geq \dim \mathrm{NS}^p(\mathcal{X}) \otimes \mathbf{Q}$$

for every $p \geq 0$.

Proof. Without lost of generality, we may assume that $\mathcal{X}$ has dimension $n \geq p$. It suffices to show that for algebraic cycles $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ on $\mathcal{X}$ of codimension p, if they are linearly independent in $\mathrm{NS}^p(\mathcal{X})$, then they are linearly independent in $\mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X})$. Suppose that $\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}_1), \dots, \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}_r)$ are linearly dependent in $\mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X})$. Then there are $c_1, \dots, c_r \in \mathbf{Q}$, not all zero, such that $\mathcal{Z} \coloneqq c_1\mathcal{Z}_1 + \cdots + c_r\mathcal{Z}_r$ is zero in $\mathrm{H}^{p,p}_{\mathrm{trop}}(\mathcal{X})$, that is, $\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}) = 0$. As $\mathcal{Z}$ is nonzero in $\mathrm{NS}^p(\mathcal{X})$, there is an algebraic cycle $\mathcal{Z}'$ on $\mathcal{X}$ of codimension $n-p$ such that $\deg(\mathcal{Z} \cdot \mathcal{Z}') \neq 0$, where $\mathcal{Z} \cdot \mathcal{Z}' \in \mathrm{CH}^n(\mathcal{X})$ is the intersection product of $\mathcal{Z}$ and $\mathcal{Z}'$. Since the tropical cycle class map preserves ring structure, we have

$$\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}) \wedge \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}') = \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z} \cdot \mathcal{Z}').$$

Thus, $\mathrm{cl}_{\mathrm{trop}}(\mathcal{Z} \cdot \mathcal{Z}') = 0$. However, by Corollary 3.10, we have

$$\int_{\mathcal{X}^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z} \cdot \mathcal{Z}') = \int_{(\mathcal{Z} \cdot \mathcal{Z}')^{\mathrm{an}}} 1 = \deg(\mathcal{Z} \cdot \mathcal{Z}'),$$

which is a contradiction. The corollary follows. □

We finish this section by exhibiting a counterexample of the Künneth decomposition for $\mathrm{H}^{\bullet,\bullet}_{\mathrm{trop}}$.

Example 3.12. Let $\mathcal{X}_1, \mathcal{X}_2$ be two separated schemes of finite type over a non-Archimedean field K. Put $\mathcal{X} = \mathcal{X}_1 \times_{\mathrm{Spec}\, K} \mathcal{X}_2$. We have a canonical map

$$\bigoplus_{p_1+p_2=p,\, q_1+q_2=q} \mathrm{H}^{p_1,q_1}_{\mathrm{trop}}(\mathcal{X}_1) \otimes_{\mathbf{Q}} \mathrm{H}^{p_2,q_2}_{\mathrm{trop}}(\mathcal{X}_2) \to \mathrm{H}^{p,q}_{\mathrm{trop}}(\mathcal{X}). \tag{3.10}$$

We say that the *Künneth decomposition* holds for $(\mathcal{X}_1, \mathcal{X}_2)$ if the above map is an isomorphism for every (p, q).

We now show an example that the Künneth decomposition *fails* for certain projective smooth schemes over $K = \mathbf{C}_{\mathbf{F}}$. Take $\mathcal{X}_1 = \mathcal{X}_2$ to be an irreducible projective smooth curve over $\mathbf{C}_{\mathbf{F}}$ of genus $g \geq 1$ with good reduction. Then we have $\dim \mathrm{H}^{0,0}_{\mathrm{trop}}(\mathcal{X}_i) = 1$; $\dim \mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X}_i) = 1$ by Theorem 7.3 (2); $\dim \mathrm{H}^{0,1}_{\mathrm{trop}}(\mathcal{X}_i) = \dim \mathrm{H}^{1,0}_{\mathrm{trop}}(\mathcal{X}_i) = b^1(\mathcal{X}_i^{\mathrm{an}}) = 0$ by Remark 7.4 and the assumption that $\mathcal{X}_i$ has good reduction. Therefore, for $(p, q) = (1, 1)$, the left-hand side of (3.10) has dimension 2. However by Corollary 3.11, the right-hand side of (3.10) has dimension as least $\dim \mathrm{NS}^1(\mathcal{X}) \otimes \mathbf{Q}$, which is 3 as $g \geq 1$.

4. Rigid cohomology and logarithmic differential forms

In this section, we review the theory of rigid cohomology developed in, for example, [Bert97] and [LS07]. Then we study the behavior of logarithmic differential forms in the rigid cohomology.

Let $\mathfrak{R}$ be the category of triples (K, X, Z) where K is a non-Archimedean field of characteristic zero; X is a scheme of finite type over a subfield of $\widetilde{K}$; and Z is a Zariski closed subset of X. A morphism from (K', X', Z') to (K, X, Z) consists of a field extension K'/K and a morphism $X' \to X \otimes_{\widetilde{K}} \widetilde{K'}$ whose restriction to Z' factors through $Z \otimes_{\widetilde{K}} \widetilde{K'}$. Let $\mathfrak{V}$ be the category of pairs $(K, V^\bullet)$ where K is a field and $V^\bullet$ is a graded K-vector space. A morphism from $(K, V^\bullet)$ to $(K', V'^\bullet)$ consists of a field extension K'/K and a graded linear map $V^\bullet \otimes_K K' \to V'^\bullet$.

We have a functor of *rigid cohomology with support*: $\mathfrak{R}^{\mathrm{opp}} \to \mathfrak{V}$ sending (K, X, Z) to $\mathrm{H}^\bullet_{Z,\mathrm{rig}}(X/K)$. Put $\mathrm{H}^\bullet_{\mathrm{rig}}(X/K) = \mathrm{H}^\bullet_{X,\mathrm{rig}}(X/K)$ for simplicity. The following lemma summarizes properties which will be used extensively in this article

Lemma 4.1. *Let notation be as above.*

(1) *Suppose that we have a morphism $(K', X', Z') \to (K, X, Z)$ with $X' \simeq X \otimes_{\widetilde{K}} \widetilde{K'}$ and $Z' \simeq Z \otimes_{\widetilde{K}} \widetilde{K'}$. Then the induced map $\mathrm{H}^\bullet_{Z,\mathrm{rig}}(X/K) \otimes_K K' \to \mathrm{H}^\bullet_{Z',\mathrm{rig}}(X'/K')$ is an isomorphism of finite dimensional graded K'-vector spaces.*

(2) *For $Y = X \backslash Z$, we have a long exact sequence:*

$$\cdots \to \mathrm{H}^i_{Z,\mathrm{rig}}(X/K) \to \mathrm{H}^i_{\mathrm{rig}}(X/K) \to \mathrm{H}^i_{\mathrm{rig}}(Y/K) \to \mathrm{H}^{i+1}_{Z,\mathrm{rig}}(X/K) \to \cdots . \tag{4.1}$$

(3) *If both X, Z are smooth, and Z is of codimension r in X, then we have a Gysin isomorphism $\mathrm{H}^i_{Z,\mathrm{rig}}(X/K) \simeq \mathrm{H}^{i-2r}_{\mathrm{rig}}(Z/K)$.*

(4) *Suppose that K is residually algebraic over $\mathbf{Q}_{\mathbf{F}}$. Then both X, Z can be defined over a finite extension $\mathbf{F}'$ of $\mathbf{F}$, and the sequence (4.1) is equipped with the absolute Frobenius action of degree $|\mathbf{F}'|$ by functoriality. In particular, each item V in (4.1) admits a direct sum decomposition $V = \bigoplus_{w \in \mathbf{Z}} V_w$ where V_w is the maximal subspace of V such that all of its generalized eigenvalues under the previous Frobenius action are Weil $|\mathbf{F}'|^{w/2}$-numbers; it is clear that V_w is independent of the choice of $\mathbf{F}'$.*

(5) *Suppose that X is smooth and Z is of codimension r, then $\mathrm{H}^i_{Z,\mathrm{rig}}(X/K)_w = 0$ unless $i \le w \le 2(i - r)$. If X is moreover proper, then $\mathrm{H}^i(X/K)$ is of pure weight i.*

Proof. Part (1) is due to [GK02, Corollary 3.8] and [Ber07, Corollary 5.5.2]. Parts (2) is a standard fact in rigid cohomology; see for example, [Bert97] and [LS07]. Part (3) is due to [Tsu99]. Part (4) is due to [Chi98, §1 & §2]. Part (5) is due to [Chi98, Theorem 2.3]. ☐

We will extensively use the notion of K-analytic germs ([Ber07, §5.1]), rather than K-dagger spaces, for a non-Archimedean field K. Roughly speaking, a K-analytic germ is a pair (X, S) where X is a K-analytic space and $S \subset X$ is a subset. We say that (X, S) is a K-affinoid germ if S is an affinoid domain. We say that (X, S) is smooth if X is smooth in an open neighborhood of S. We have the structure sheaf $\mathcal{O}_{(X,S)}$, and the de Rham complex $\Omega^\bullet_{(X,S)}$ when (X, S) is smooth. (See [Ber07, §5.2] for details.) In particular, we have the de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X, S)$ when (X, S) is smooth. For a smooth K-analytic germ (X, S) where $S = \mathcal{M}(A)$ for an integrally smooth K-affinoid algebra A, we have a canonical functorial isomorphism $\mathrm{H}^\bullet_{\mathrm{dR}}(X, S) \simeq \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_s/K)$ (see [Bert97, Proposition 1.10], whose proof actually works for general K).

The following lemma generalizes the construction in [GK02, Lemma 2].

Lemma 4.2. *Let K be a non-Archimedean field of characteristic zero. Let (X_1, Y_1) and (X_2, Y_2) be two smooth K-affinoid germs. Then for every morphism $\phi\colon Y_2 \to Y_1$ of K-affinoid domains, one can associate in a functorial way a pullback map*

$$\phi^*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(X_1, Y_1) \to \mathrm{H}^\bullet_{\mathrm{dR}}(X_2, Y_2).$$

It satisfies the following conditions:

(i) if ϕ extends to a morphism $(X_2, Y_2) \to (X_1, Y_1)$ of germs, then ϕ^* coincides with the usual pullback;

(ii) for a finite extension K' of K, if we write X'_i (resp. Y'_i) for $X_i\widehat{\otimes}_K K'$ (resp. $Y_i\widehat{\otimes}_K K'$) for $i = 1, 2$ and ϕ' for $\phi\widehat{\otimes}_K K'$, then ϕ'^* coincides with the scalar extension of ϕ^*, in which we identify $\mathrm{H}^\bullet_{\mathrm{dR}}(X'_i, Y'_i)$ with $\mathrm{H}^\bullet_{\mathrm{dR}}(X_i, Y_i) \otimes_K K'$ for $i = 1, 2$;

(iii) if $Y_1 = \mathcal{M}(A_1)$ and $Y_2 = \mathcal{M}(A_2)$ for some integrally smooth K-affinoid algebras A_1 and A_2, then ϕ^* coincides with $\phi_s^*\colon \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_{1,s}/K) \to \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_{2,s}/K)$ under the canonical isomorphism $\mathrm{H}^\bullet_{\mathrm{dR}}(X_i, Y_i) \simeq \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_{i,s}/K)$ for $i = 1, 2$, where $\phi_s\colon \mathrm{Spec}\, A_{2,s} \to \mathrm{Spec}\, A_{1,s}$ is the induced morphism.

Proof. Put $X = X_1 \times_K X_2$, $Y = Y_1 \times_K Y_2$, and $\Delta \subseteq Y$ the graph of ϕ, which is isomorphic to Y_2 via the projection to the second factor. Denote by $a_i\colon X \to X_i$ the projection morphism. We have maps

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X_1, Y_1) \xrightarrow{a_1^*} \varinjlim_V \mathrm{H}^\bullet_{\mathrm{dR}}(V) \xleftarrow{a_2^*} \mathrm{H}^\bullet_{\mathrm{dR}}(X_2, Y_2),$$

where V runs through open neighborhoods of Δ in X. We show that a_2^* is an isomorphism. Then we define ϕ^* as $(a_2^*)^{-1} \circ a_1^*$. The functoriality of ϕ^* is straightforward but tedious to check; we will leave it to readers.

The proof of properties is similar to that of [GK02, Lemma 2]. The claim that a_2^* is an isomorphism is local on (X_2, Y_2) by the Mayer–Vietoris sequence. Thus we may assume that there are elements $t_1, \dots, t_m \in \mathscr{O}_{X_1}(X_1)$ such that $\mathrm{d}t_1, \dots, \mathrm{d}t_m$ form a basis of $\Omega^1(X_1, Y_1)$ over $\mathscr{O}(X_1, Y_1)$, and there exist a K-affinoid neighborhood $U_\varepsilon \subset X$ of Δ with an element $\varepsilon \in |K^\times|$, and an isomorphism

$$U_\varepsilon \cap Y \xrightarrow{\sim} \mathcal{M}(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle) \times_K \Delta,$$

in which Z_i is sent to $t_i \otimes 1 - 1 \otimes \phi^* t_i$. Note that $K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle$ is an integrally smooth K-affinoid algebra, and $\mathrm{Spec}\, K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle_s$ is isomorphic to $\mathbf{A}^m_{\widetilde{K}}$. Thus by [GK02, Lemma 2], the restriction map $\mathrm{H}^\bullet_{\mathrm{dR}}(X_2, Y_2) \to \mathrm{H}^\bullet_{\mathrm{dR}}(X, U_\varepsilon \cap Y)$ is an isomorphism. We may choose a sequence of such U_ε with $\bigcap_\varepsilon U_\varepsilon = \Delta$. Then $\varinjlim_\varepsilon \mathrm{H}^\bullet_{\mathrm{dR}}(X, U_\varepsilon \cap Y) \simeq \varinjlim_V \mathrm{H}^\bullet_{\mathrm{dR}}(V)$ and thus a_2^* is an isomorphism.

Properties (i) and (ii) follow easily from the construction. We now check Property (iii), as it is important for our later argument. The induced projection morphism

$$\mathcal{M}(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle) \times_K \Delta \simeq U_\varepsilon \cap Y \to Y_i$$

extends canonically to a morphism of formal K°-schemes

$$\mathrm{Spf}(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle \widehat{\otimes}_K A_\Delta)^\circ \to \mathrm{Spf}\, A_i^\circ,$$

where A_Δ is the coordinate K-affinoid algebra of Δ which is isomorphic to A_2. Therefore, the restriction map $\mathrm{H}^\bullet_{\mathrm{dR}}(X_i, Y_i) \to \mathrm{H}^\bullet_{\mathrm{dR}}(X, U_\varepsilon \cap Y)$ coincides with the map

$$a_{i,s}^*\colon \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_{i,s}/K) \to \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m\rangle \widehat{\otimes}_K A_\Delta)_s/K)$$

induced from the homomorphism $A_{i,s} \to (K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m \rangle \widehat{\otimes}_K A_\Delta)_s$ of $\widetilde{K}$-algebras. Note that $a^*_{2,s}$ is an isomorphism, and that $(a^*_{2,s})^{-1}$ coincides with the restriction map

$$\mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m \rangle \widehat{\otimes}_K A_\Delta)_s/K) \to \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, A_{2,s}/K)$$

induced from the homomorphism $(K\langle \varepsilon^{-1}Z_1, \dots \varepsilon^{-1}Z_m \rangle \widehat{\otimes}_K A_\Delta)_s \to A_{2,s}$ sending $\varepsilon^{-1}Z_i$ to 0 for all i. Property (iii) follows. □

The following example will be used in the computation later.

Example 4.3. Let K be a non-Archimedean field of characteristic zero. For an integer $t \geq 0$ and an element $\varpi \in K$, define the formal K°-scheme

$$\mathfrak{E}^t_{K,\varpi} \coloneqq \mathrm{Spf}\, K^\circ[[T_0, \dots, T_t]]/(T_0 \cdots T_t - \varpi)$$

and let $\mathbf{E}^t_{K,\varpi}$ be its generic fiber. Let $E^t_{K,\varpi}$ be the K-affinoid algebra

$$K\langle |\varpi|^{-\frac{1}{t+1}}T_0, \cdots, |\varpi|^{-\frac{1}{t+1}}T_t, |\varpi|^{\frac{1}{t+1}}T_0^{-1}, \cdots, |\varpi|^{\frac{1}{t+1}}T_t^{-1} \rangle/(T_0 \cdots T_t - \varpi),$$

which is integrally smooth, as $\mathrm{Spf}(E^t_{K,\varpi})^\circ$ is the formal completion along a smooth open in some semistable scheme over K°. Moreover, $\mathcal{M}(E^t_{K,\varpi})$ is canonically a K-affinoid domain in $\mathbf{E}^t_{K,\varpi}$, and the restriction map

$$\mathrm{H}^\bullet_{\mathrm{dR}}(\mathbf{E}^t_{K,\varpi}) \to \mathrm{H}^\bullet_{\mathrm{dR}}(\mathbf{E}^t_{K,\varpi}, \mathcal{M}(E^t_{K,\varpi})) \simeq \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}(E^t_{K,\varpi})_s/K)$$

is an isomorphism by [GK02, Lemma 3]. If K is residually algebraic over $\mathbf{Q_F}$, then we have

$$\mathrm{H}^p_{\mathrm{rig}}(\mathrm{Spec}(E^t_{K,\varpi})_s/K) = \mathrm{H}^p_{\mathrm{rig}}(\mathrm{Spec}(E^t_{K,\varpi})_s/K)_{2p}$$

for every $p \geq 0$.

Now we study the behavior of logarithmic differential forms in the rigid cohomology. We first review the notion of strictly semistable schemes.

Definition 4.4 (Strictly semistable scheme)**.** Let k be a discrete non-Archimedean field. We say that a scheme $\mathcal{X}$ over k° is *strictly semistable of dimension* n if $\mathcal{X}$ is locally of finite presentation, Zariski locally étale over $\mathrm{Spec}\, K^\circ[T_0, \dots, T_n]/(T_0 \cdots T_r - \varpi)$ for some uniformizer ϖ of k.

For every integer $0 \leq r \leq n$, denote by $\mathcal{X}^{[r]}_s$ the union of intersection of $r+1$ distinct irreducible components of $\mathcal{X}_s$. It is a closed subscheme of $\mathcal{X}_s$ whose irreducible components with their reduced structure are smooth.

Let k be a finite extension of $\mathbf{Q_F}$. Let $\mathcal{S}$ be a proper strictly semistable scheme over k° of dimension s such that every irreducible component of $\mathcal{S}^{[r]}_s$ is geometrically irreducible for every $r \geq 0$. We fix an irreducible component $\mathcal{E}$ of $\mathcal{S}_s$ and let $\mathcal{E}_1, \dots, \mathcal{E}_M$ be all other irreducible components that intersect $\mathcal{E}$. For a subset $I \subset \{1, \dots, M\}$, put $\mathcal{E}_I = (\bigcap_{i \in I} \mathcal{E}_i) \cap \mathcal{E}$ (in particular, $\mathcal{E}_\emptyset = \mathcal{E}$) and $\mathcal{E}^\heartsuit_I = \mathcal{E}_I \backslash \mathcal{S}^{[|I|+1]}_s$. For two subsets I, J of $\{1, \dots, M\}$, we write $I \prec J$ if $I \subset J$ and numbers in $J \backslash I$ are all greater than those in I.

For $I \subset \{1, \dots, M\}$, we have the open immersion $\mathcal{E}^\heartsuit_I \subset \mathcal{E}_I \backslash \mathcal{S}^{[|I|+2]}_s$, whose compliment is $\coprod_{I \subset J, |J|=|I|+1} \mathcal{E}^\heartsuit_J$. Thus we have maps

$$\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}^\heartsuit_I/k) \to \bigoplus_{I \subset J, |J|=|I|+1} \mathrm{H}^{\bullet+1}_{\mathcal{E}^\heartsuit_J, \mathrm{rig}}(\mathcal{E}_I \backslash \mathcal{S}^{[|I|+2]}_s/k) \xrightarrow{\sim} \bigoplus_{I \subset J, |J|=|I|+1} \mathrm{H}^{\bullet-1}_{\mathrm{rig}}(\mathcal{E}^\heartsuit_J/k),$$

where the second map is the Gysin isomorphism. In the above composite map, denote by ξ_J^I the induced map from $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}_I^\heartsuit/k)$ to the component $\mathrm{H}^{\bullet-1}_{\mathrm{rig}}(\mathcal{E}_J^\heartsuit/k)$ if $I \prec J$, and the zero map if not.

In general, for $I \prec J$, there is a unique strictly increasing sequence $I = I_0 \prec I_1 \prec \cdots \prec I_{|J\setminus I|} = J$ and we define

$$\xi_J^I \coloneqq \xi_J^{I_{|J\setminus I|-1}} \circ \cdots \circ \xi_{I_1}^I \colon \mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}_I^\heartsuit/k) \to \mathrm{H}^{\bullet-|J\setminus I|}_{\mathrm{rig}}(\mathcal{E}_J^\heartsuit/k),$$

and $\xi_J^I = 0$ if $I \prec J$ does not hold. Together, for $i \leq j$, they induce a map

$$\xi_j^i \colon \bigoplus_{|I|=i} \mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}_I^\heartsuit/k) \to \bigoplus_{|J|=j} \mathrm{H}^{\bullet+i-j}_{\mathrm{rig}}(\mathcal{E}_J^\heartsuit/k),$$

such that $\xi_j^i|_{\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}_I^\heartsuit/k)}$ is the direct sum of ξ_J^I for all J with $|J| = j$. First, we have the following lemma.

Lemma 4.5. *Let notation be as above. For every $0 \leq p \leq s$, the restriction of*

$$\xi_p^0 \colon \mathrm{H}^p_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k) \to \bigoplus_{|J|=p} \mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}_J^\heartsuit/k)$$

to $\mathrm{H}^p_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)_{2p}$ is injective.

Proof. By the long exact sequence of cohomology with support (4.1), the kernel of the map ξ_p^0 is a weight preserving extension of k-vector spaces $\mathrm{H}^{p+|I|}_{\mathcal{E}_I,\mathrm{rig}}(\mathcal{E}/k)$ for $|I| < p$. Therefore, the lemma follows since $\mathrm{H}^{p+|I|}_{\mathcal{E}_I,\mathrm{rig}}(\mathcal{E}/k)$ is pure of weight $p + |I| < 2p$ by [Tsu99, Theorems 5.2.1 & 6.2.5] (with constant coefficients). □

Denote by $Z^r(\mathcal{E})^\heartsuit$ the abelian group generated by irreducible components of $\mathcal{E}_I$ with $|I| = r$. Denote by $[\mathcal{Z}]$ the generator corresponding to a component $\mathcal{Z}$ of $\mathcal{E}_I$. Note that the set I is determined by $\mathcal{Z}$, which we denote by $I(\mathcal{Z})$. Put $Z(\mathcal{E})^\heartsuit = \bigoplus_{r=0}^M Z^r(\mathcal{E})^\heartsuit$. We define a wedge product

$$\wedge \colon Z(\mathcal{E})^\heartsuit \otimes Z(\mathcal{E})^\heartsuit \to Z(\mathcal{E})^\heartsuit,$$

which is group homomorphism uniquely determined by the following conditions:

- $Z_1 \wedge Z_2 = (-1)^{r_1 r_2} Z_2 \wedge Z_1$, if $Z_1 \in Z^{r_1}(\mathcal{E})^\heartsuit$ and $Z_2 \in Z^{r_2}(\mathcal{E})^\heartsuit$;
- $[\mathcal{Z}_1] \wedge [\mathcal{Z}_2] = 0$ if $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$;
- $[\mathcal{Z}_1] \wedge [\mathcal{Z}_2] = [\mathcal{Z}_1 \cap \mathcal{Z}_2]$ if $\mathcal{Z}_1 \cap \mathcal{Z}_2 \neq \emptyset$ and $I(\mathcal{Z}_1) \prec I(\mathcal{Z}_1) \cup I(\mathcal{Z}_2)$.

It is easy to see that $\wedge$ is associative and maps $Z^{r_1}(\mathcal{E})^\heartsuit \otimes Z^{r_2}(\mathcal{E})^\heartsuit$ into $Z^{r_1+r_2}(\mathcal{E})^\heartsuit$. We have an (injective) class map

$$\mathrm{cl}^\heartsuit \colon Z(\mathcal{E})^\heartsuit \to \bigoplus_I \mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}_I^\heartsuit/k) \simeq \bigoplus_I k^{\pi_0(\mathcal{E}_I)}$$

sending $[\mathcal{Z}]$ to 1 corresponding to the irreducible component of $\mathcal{E}_{I(\mathcal{Z})}$.

For an element $f \in \mathcal{O}^*(\mathcal{S}_k^{\mathrm{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$, that is, an invertible function on some open neighborhood of $\pi^{-1}\mathcal{E}^\heartsuit$ in $\mathcal{S}_k^{\mathrm{an}}$, we can associate canonically an element $\mathrm{div}(f) \in Z^1(\mathcal{E})^\heartsuit$. In fact, there exists an element $c \in k^\times$ such that $|cf| = 1$ on $\pi^{-1}\mathcal{E}^\heartsuit$. Thus the reduction $\widetilde{cf}$ is an element in $\mathcal{O}^*_{\mathcal{E}}(\mathcal{E}^\heartsuit)$, and we define $\mathrm{div}(f)$ to be the associated divisor of $\widetilde{cf}$, which is an element in $Z^1(\mathcal{E})^\heartsuit$. Obviously, it does not depend on the choice of c. Finally, note that by the definition of rigid cohomology, we have a canonical isomorphism $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathcal{S}_k^{\mathrm{an}}, \pi^{-1}\mathcal{E}^\heartsuit) \simeq \mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)$.

Proposition 4.6. *Let notation be as above. Given $f_1, \dots, f_p \in \mathcal{O}^*(\mathcal{S}_k^{\mathrm{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$, if we regard $\frac{\mathrm{d}f_1}{f_1} \wedge \dots \wedge \frac{\mathrm{d}f_p}{f_p}$ as an element in $\mathrm{H}^p_{\mathrm{dR}}(\mathcal{S}_k^{\mathrm{an}}, \pi^{-1}\mathcal{E}^\heartsuit) \simeq \mathrm{H}^p_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)$, then we have*

$$\xi_p^0 \left(\frac{\mathrm{d}f_1}{f_1} \wedge \dots \wedge \frac{\mathrm{d}f_p}{f_p} \right) = \mathrm{cl}^\heartsuit \left(\mathrm{div}(f_1) \wedge \dots \wedge \mathrm{div}(f_p) \right). \tag{4.2}$$

Proof. The question is local around the generic point of every irreducible component of $\mathcal{E}_I$ with $|I| = p$. So we fix such an irreducible component $\mathcal{Z}$ (with $|I(\mathcal{Z})| = p$) and take an affine open neighborhood $\mathcal{S}_0$ of the generic point of $\mathcal{Z}$ in $\mathcal{S}$ such that it admits a smooth morphism

$$f \colon \mathcal{S}_0 \to \operatorname{Spec} k^\circ[T_0, \dots, T_p]/(T_0 \cdots T_p - \varpi)$$

where ϖ is a uniformizer of k, satisfying

- $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{E}_i$ $(i = 1, \dots, p)$ are all the irreducible components of $\mathcal{T}_s$ that intersect $\mathcal{E}$, where $\mathcal{E}_i$ is defined by the ideal (f^*T_i, ϖ);
- $\mathcal{E}_I \coloneqq \bigcap_{i \in I} \mathcal{E}_i$ is irreducible and nonempty for $I \subset \{1, \dots, p\}$.

Note that

- $\frac{\mathrm{d}(cf)}{cf} = \frac{\mathrm{d}f}{f}$;
- both sides of (4.2) are multi-linear in $f_1, \dots, f_p \in \mathcal{O}^*(\mathcal{S}_{0,k}^{\mathrm{an}}, \pi^{-1}\mathcal{E}^\heartsuit)$; and
- $\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}f'}{f'}$ in $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)$ if $|f| = |f'| = 1$ on $\pi^{-1}\mathcal{E}^\heartsuit$ and $\widetilde{f} = \widetilde{f}'$.

Thus we may assume that $f_i = f^*T_i$. Then as both sides of (4.2) are functorial in f under pullback, we may assume that $\mathcal{S}_0 = \operatorname{Spec} k^\circ[T_0, \dots, T_p]/(T_0 \cdots T_p - \varpi)$ and $f_i = T_i$.

Put $\mathcal{S}' = \operatorname{Spec} k^\circ[T_1, \dots, T_p]$ and let $g \colon \mathcal{S}_0 \to \mathcal{S}'$ be the morphism sending T_i to T_i $(1 \le i \le p)$. For $I \subset \{1, \dots, p\}$, let $\mathcal{E}'_I$ be the closed subscheme of $\mathcal{S}'_s$ defined by the ideal $(\varpi, T_i | i \in I)$. Then g induces an isomorphism $\mathcal{E}_I \simeq \mathcal{E}'_I$. Similarly, we have maps

$$\xi'^I_J \colon \mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_I/k) \to \mathrm{H}^{\bullet+i-j}_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_J/k)$$

for $I \subset J$ and ξ'^i_j for $i \le j$, where $\mathcal{E}'^\heartsuit_I = g(\mathcal{E}^\heartsuit_I)$. It is easy to see that $\xi'^I_J = \xi^I_J$ if we identify $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_I/k)$ with $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}^\heartsuit_I/k)$ through g^*. Therefore, it suffices to show the equality (4.2) for $\mathcal{S}'$, that is,

$$\xi'^0_p \left(\frac{\mathrm{d}T_1}{T_1} \wedge \dots \wedge \frac{\mathrm{d}T_p}{T_p} \right) = 1 \in \mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}'_{\{1,\dots,p\}}/k) \simeq k.$$

However, $\mathcal{S}'$, which is isomorphic to $\mathbf{A}^p_{k^\circ}$ can be canonically embedded into the proper smooth scheme $\mathbf{P}^p_{k^\circ}$ over k°. Thus, the rigid cohomology $\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_I/k)$ and the map ξ'^i_j can be computed on $\mathbf{P}^{p,\mathrm{an}}_k$. On the generic fiber $\mathcal{S}'_k$, we similarly define $\mathcal{T}_I$ to be the closed subscheme $\operatorname{Spec} k[T_1, \dots, T_p]/(T_i \mid i \in I)$ of $\mathcal{S}'_k$ for $I \subset \{1, \dots, p\}$, and $\mathcal{T}^\heartsuit_I = \mathcal{T}_I \setminus \bigcup_{I \subsetneq J} \mathcal{T}_J$. We may similarly define maps $\alpha^I_J \colon \mathrm{H}^\bullet_{\mathrm{dR}}(\mathcal{T}^\heartsuit_I) \to \mathrm{H}^{\bullet-|J\setminus I|}_{\mathrm{dR}}(\mathcal{T}^\heartsuit_J)$ and α^i_j via algebraic de Rham cohomology theory. Then we have canonical vertical isomorphisms rendering the diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{dR}}(\mathcal{T}^\heartsuit_I) & \xrightarrow{\alpha^I_J} & \mathrm{H}^{\bullet-|J\setminus I|}_{\mathrm{dR}}(\mathcal{T}^\heartsuit_J) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ \mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_I/k) & \xrightarrow{\xi'^I_J} & \mathrm{H}^{\bullet-|J\setminus I|}_{\mathrm{rig}}(\mathcal{E}'^\heartsuit_J/k) \end{array}$$

commutative. From the standard computation in algebraic de Rham cohomology, we have

$$\alpha_p^0\left(\frac{\mathrm{d}T_1}{T_1}\wedge\cdots\wedge\frac{\mathrm{d}T_p}{T_p}\right)=1\in \mathrm{H}^0_{\mathrm{dR}}(\mathcal{T}_{\{1,\dots,p\}}),$$

where $\mathcal{T}_{\{1,\dots,p\}}$ is just the point of origin. Thus, the proposition is proved. □

5. Weight decomposition of de Rham cohomology sheaves

In this section, we prove Theorem 1.4 for étale topology, and then deduce the one for analytic topology. In particular, sheaves like $\mathcal{O}_X$, $\mathfrak{c}_X$, and the de Rham complex $(\Omega^\bullet_X,\mathrm{d})$ are understood in the étale topology, until we say otherwise.

Definition 5.1 (Marked pair)**.** Let k be a discrete non-Archimedean field. A *marked k-pair $(\mathcal{X},\mathcal{D})$ of dimension n and depth t* consists of an affine strictly semistable scheme $\mathcal{X}$ over k° of dimension n (Definition 4.4), and an irreducible component $\mathcal{D}$ of $\mathcal{X}_s^{[t]}$ that is geometrically irreducible.

We start from the following lemma, which generalizes [Ber07, Lemma 2.1.2].

Lemma 5.2. *Suppose that K is embeddable into $\widehat{k^{\mathrm{a}}}$ for some discrete non-Archimedean field k. Let X be a smooth K-analytic space, and x a point of X with $s(x)+t(x)=\dim_x(X)$. Given a morphism of K-analytic spaces $X\to\mathfrak{Y}_\eta$, where $\mathfrak{Y}$ is a special formal K°-scheme, there exist*

- *a finite extension K' of K, a finite extension k' of k contained in K',*
- *a marked k'-pair $(\mathcal{X},\mathcal{D})$ of dimension $\dim_x(X)$ and depth $t(x)$,*
- *an open neighborhood U of $(\widehat{\mathcal{X}_{/\mathcal{D}}})_\eta\widehat{\otimes}_{k'}K'$ in $\mathcal{X}^{\mathrm{an}}_{K'}$,*
- *a point $x'\in(\widehat{\mathcal{X}_{/\mathcal{D}}})_\eta\widehat{\otimes}_{k'}K'$,*
- *a morphism of K-analytic spaces $\varphi\colon U\to X$, and*
- *a morphism of formal K°-schemes $\widehat{\mathcal{X}_{/\mathcal{D}}}\widehat{\otimes}_{k'^\circ}K'^\circ\to\mathfrak{Y}$,*

such that the following are true:

(i) φ is étale and $\varphi(x')=x$;
(ii) the induced morphism $(\widehat{\mathcal{X}_{/\mathcal{D}}})_\eta\widehat{\otimes}_{k'}K'\to\mathfrak{Y}_\eta$ coincides with the composition

$$(\widehat{\mathcal{X}_{/\mathcal{D}}})_\eta\widehat{\otimes}_{k'}K'\hookrightarrow U\xrightarrow{\varphi}X\to\mathfrak{Y}_\eta.$$

Proof. Put $t=t(x)$, $s=s(x)$, and $n=t+s$. By [Ber07, Proposition 2.3.1], by possibly replacing k and K by their finite extensions, we may replace X by $(B\times_k Y)\widehat{\otimes}_k K$, where $B=\prod_{j=1}^t B(0;r_j,R_j)_k$ for some $0<r_j<R_j$ and Y is a smooth k-analytic space of dimension s, and x projects to $b\in B$ with $t(b)=t$ and $y\in Y$ with $s(y)=s$. Denote by $\mathcal{P}$ the k°-scheme $\mathbf{P}^1_{k^\circ}$ with the point 0 on the special fiber blown up, and by $\mathfrak{P}$ the formal completion of $\mathcal{P}$ along the open subscheme $\mathcal{P}_s\backslash\{\pi(0),\pi(\infty)\}$, which is isomorphic to $\mathrm{Spf}\,k^\circ\langle X,Y\rangle/(XY-\varpi)$ for some uniformizer ϖ of k. By further replacing k and K by their finite extensions such that $|\varpi|<R_j-r_j$ for every j, we may assume that there is an embedding $\prod_t\mathfrak{P}_\eta\subset B$ whose image contains b such that $\pi(b)=\mathbf{0}$, where $\mathbf{0}$ is the closed point in $\prod_t\mathcal{P}_s$ that is nodal in every component. In particular $\prod_t\mathfrak{P}_\eta$ is a neighborhood of b.

For Y, we proceed exactly as in the Step 1 of the proof of [Ber07, Lemma 2.1.2]. We obtain, by shrinking Y if necessary, two k-affinoid domains $Z'\subset W'\subset Y$ such that W' is a neighborhood of Z'. We follow Berkovich's construction in the beginning of Step 3 of the proof of [Ber07, Lemma 2.1.2] after Raynaud to obtain an integral scheme $\mathcal{Y}'$ proper and flat over k° with an embedding $Y \subset \mathcal{Y}'^{\mathrm{an}}_\eta$, open subschemes $\mathcal{Z}' \subset \mathcal{W}' \subset \mathcal{Y}'_s$, such that $Z' = \pi^{-1}\mathcal{Z}'$ and $W' = \pi^{-1}\mathcal{W}'$.

Now we put two parts together. Define $\mathcal{Y} = \prod_t \mathcal{P} \times \mathcal{Y}'$ where the fiber product is taken over k°, and $\mathcal{W} = \prod_t \mathfrak{P}_s \times \mathcal{W}'$ where the fiber product is taken over $\widetilde{k}$. Then $W := \prod_t \mathfrak{P}_\eta \times W'$ coincides with $\pi^{-1}\mathcal{W}$ in $\mathcal{Y}^{\mathrm{an}}_k$. Moreover, W_K is a neighborhood of x where W_K denotes the inverse image of W in $X = (B \times_k Y)\widehat{\otimes}_k K$. Note that we have the induced map $\alpha\colon W_K \to \mathfrak{Y}_\eta$ of K-analytic spaces by restriction. By the same argument in Step 2 of the proof of [Ber07, Lemma 2.1.2], we have finitely many open affine subschemes $\mathfrak{Y}^i$ for $1 \leq i \leq l$ of $\mathfrak{Y}$ such that $W_{K,i} := \alpha^{-1}\mathfrak{Y}^i_\eta$ is an affinoid subdomain of W_K and $W_K = \bigcup_{i=1}^l W_{K,i}$. By [Ber07, Lemma 2.1.3 (ii)], we may assume that $W_{K,i}$ can be descent to an k-affinoid subdomain W_i of W for every i by replacing k and K by their finite extensions if necessary. Making a finite number of additional blow-ups, we may also assume that there are open subschemes $\mathcal{W}_i \subset \mathcal{W}$ with $W_i = \pi^{-1}\mathcal{W}_i$ and $\mathcal{W} = \bigcup_{i=1}^l \mathcal{W}_i$.

Now we proceed as in Step 4 of the proof of [Ber07, Lemma 2.1.2]. Take an alteration $\phi\colon \mathcal{X}' \to \mathcal{Y}$ from a strictly semistable scheme $\mathcal{X}'$ over k° after further replacing k and K by their finite extensions, and a point $x' \in \mathcal{X}'^{\mathrm{an}}_K$ such that $\phi(x') = x$. By a similar argument, one can show that $\pi(x') \in \mathcal{X}'_s \otimes_{\widetilde{k}} \widetilde{K}$ has dimension at least s. On the other hand, we have $s(x') \geq s$ and $t(x') \geq t$. Thus, $s(x') = s$ and $t(x') = t$. Denote by $\mathcal{C}$ the Zariski closure of $\pi(x')$ in $\mathcal{X}'_s$, equipped with the reduced induced scheme structure. Suppose that $\mathcal{C}$ is contained in $t' + 1$ distinct irreducible components of $\mathcal{X}'_s$. Then $t' \leq t$ as the codimension in $\mathcal{X}'_s$ of the intersection of $t' + 1$ distinct irreducible components is t'. We take an affine open subscheme $\mathcal{U}'$ of $\mathcal{X}'$ satisfying: $\mathcal{D}' := \mathcal{U}' \cap \mathcal{C}$ is open dense in $\mathcal{C}$; $\phi(\mathcal{D}')$ is contained in $\mathcal{W}$; $\mathcal{U}'$ is étale over $\operatorname{Spec} k^\circ[T_0, \dots, T_n]/(T_0 \cdots T_{t'} - \pi)$ for some uniformizer ϖ of k such that $\mathcal{D}'$ is the zero locus of the ideal generated by $(T_0, \dots, T_t, \pi)$. Now we blow up the ideal generated by $(T_{t'+1}, \varpi)$, and then the strict transform of the ideal generated by $(T_{t'+2}, \varpi)$, and continue to obtain an affine strictly semistable scheme $\mathcal{X}$ over k° such that the strict transform $\mathcal{D}$ of $\mathcal{D}'$ is an irreducible component of $\mathcal{X}^{[t]}_s$. Possibly after further replacing k and K by their finite extensions, and replacing $\mathcal{X}$ by an affine open subscheme such that $\mathcal{X}_s \cap \mathcal{D}$ is dense in $\mathcal{D}$, we obtain a marked k-pair $(\mathcal{X}, \mathcal{D})$ of dimension n and depth t such that $\phi\colon \mathcal{X} \to \mathcal{Y}$ is étale on the generic fiber. Here, the further finite extension of ground fields is to ensure that $\mathcal{D}$ is geometrically irreducible. Note that $(\phi^{\mathrm{an}}_K)^{-1}W_K$ is a neighborhood of x' containing $\pi^{-1}\mathcal{D}$ as $\phi(\mathcal{D}) \subseteq \mathcal{W}$. Here, $x' \in \mathcal{X}^{\mathrm{an}}_K$ is an arbitrary preimage of the original $x' \in \mathcal{X}'^{\mathrm{an}}_K$, which exists by construction.

We take U to be an arbitrary open neighborhood of $\pi^{-1}\mathcal{D}$, and φ to be $\phi^{\mathrm{an}}_K|_U$. By the same argument in Step 5 of the proof of [Ber07, Lemma 2.1.2], ϕ induces a morphism of K°-formal schemes $\widehat{\mathcal{X}}_{/\phi^{-1}\mathcal{W}}\widehat{\otimes}_{k^\circ}K^\circ \to \mathfrak{Y}$ and thus a morphism $\widehat{\mathcal{X}}_{/\mathcal{D}}\widehat{\otimes}_{k^\circ}K^\circ \to \mathfrak{Y}$. The conclusions of the lemma are all satisfied by the construction. □

From now on to the end of this section, we assume that K is a residually algebraic extension of $\mathbf{Q}_{\mathbf{F}}$.

Definition 5.3 (Fundamental chart)**.** Let X be a K-analytic space and $x \in X$ a point. A *fundamental chart* of $(X; x)$ consists of data $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ where

- $(\mathcal{Y}, \mathcal{D})$ is a marked k-pair of dimension $t(x) + s(x)$ and depth $t(x)$, where k is a finite extension of $\mathbf{Q}_{\mathbf{F}}$;
- $\mathbf{D}_L$ is an open poly-disc over L of dimension $\dim_x(X) - t(x) - s(x)$, where L is simultaneously a finite extension of K and a (residually algebraic) extension of k;

- D is an integrally smooth affinoid k-algebra, and

$$\delta\colon \operatorname{Spf} D^\circ[[T_0,\dots,T_t]]/(T_0\cdots T_t-\varpi)\xrightarrow{\sim}\widehat{\mathcal{Y}}_{/\mathcal{D}} \tag{5.1}$$

  is an isomorphism of formal k°-schemes, where ϖ is a uniformizer of k;
- W is an open neighborhood of $(\widehat{\mathcal{Y}}_{/\mathcal{D}})_\eta\widehat{\otimes}_k L=(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L$ in $\mathcal{Y}_L^{\mathrm{an}}$;
- y is a point in $\mathbf{D}_L\times_L W$ with $t(y)=t(\mathrm{pr}_W(y))=t(x)$, $s(y)=s(\mathrm{pr}_W(y))=s(x)$, and such that $\mathrm{pr}_W(y)$ belongs to $(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L$ whose reduction is the generic point of $\mathcal{D}_{\widetilde{L}}$;
- $\alpha\colon \mathbf{D}_L\times_L W\to X$ is an étale morphism of K-analytic spaces such that $\alpha(y)=x$.

Note that the field k will be implicit from the notation (as it is not important).

The isomorphism (5.1) induces an isomorphism $\operatorname{Spec} D_s\simeq\mathcal{D}$ of $\widetilde{k}$-schemes, and an isomorphism

$$\delta^*\colon \mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))\xrightarrow{\sim}\bigoplus_{j=0}^{p}\mathrm{H}^j_{\mathrm{rig}}(\mathcal{D}/k)\otimes_k\mathrm{H}^{p-j}_{\mathrm{dR}}(\mathbf{E}^t_{k,\varpi})\otimes_k L \tag{5.2}$$

of L-vector spaces [GK02, Lemmas 2 & 3] and [Ber07, Corollary 5.5.2]. Here, $\mathbf{E}^t_{k,\varpi}$ is the k-analytic space defined in Example 4.3. Denote by $\mathrm{H}^p_w(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$ the subspace of the left-hand side of (5.2) corresponding to the subspace

$$\bigoplus_{j=0}^{p}\mathrm{H}^j_{\mathrm{rig}}(\mathcal{D}/k)_{w-2(p-j)}\otimes_k\mathrm{H}^{p-j}_{\mathrm{dR}}(\mathbf{E}^t_{k,\varpi})\otimes_k L$$

on the right-hand side. In particular, all elements in $\mathrm{H}^{p-j}_{\mathrm{dR}}(\mathbf{E}^t_{k,\varpi})$ are regarded to be of weight $2(p-j)$. Then we have a direct sum decomposition

$$\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))=\bigoplus_{w\in\mathbf{Z}}\mathrm{H}^p_w(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W). \tag{5.3}$$

Finally, we denote by $\mathrm{H}^p_{(w)}(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$ the subspace of $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L W)$ as the inverse image of $\mathrm{H}^p_w(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$ under the restriction map

$$\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L W)\to\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L)).$$

Remark 5.4. Note that the composite morphism of formal k°-schemes

$$\operatorname{Spf}((E^t_{k,\varpi})^\circ\widehat{\otimes}_{k^\circ}D^\circ)\to\operatorname{Spf} D^\circ[[T_1,\dots,T_t]]/(T_1\cdots T_t-\varpi)\xrightarrow{\delta}\widehat{\mathcal{Y}}_{/\mathcal{D}}$$

induces an isomorphism

$$\mathrm{H}^p_{\mathrm{dR}}(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L)\simeq\mathrm{H}^p_{\mathrm{rig}}((E^t_{k,\varpi})_s\otimes_{\widetilde{k}}\mathcal{D}/L).$$

Therefore, an element $\omega\in\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L W)$ belongs to $\mathrm{H}^p_{(w)}(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$ if and only if the restriction of ω to $\{0\}\times_L W$ belongs to $\mathrm{H}^p_{\mathrm{rig}}((E^t_{k,\varpi})_s\otimes_{\widetilde{k}}\mathcal{D}/L)_w$ under the above isomorphism, by Example 4.3.

Remark 5.5. Note that $\mathrm{H}^p_w(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)=0$ unless $p\leq w\leq 2p$, and the decomposition (5.3) is stable under base change along a residually algebraic extension of K (and L accordingly). We warn that the decomposition (5.3) depends on all of the data $(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$, not just the L-analytic germ $\mathbf{D}_L\times_L(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L)$.

Lemma 5.6. *For part of the data* $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ *from Definition 5.3 and* $f \in \mathcal{O}^*(\mathbf{D}_L \times_L W)$*, we have*

$$\frac{\mathrm{d}f}{f} \in \mathrm{H}^1_{(2)}(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W).$$

Here, we regard $\frac{\mathrm{d}f}{f}$*, a priori a closed* 1*-form on* $\mathbf{D}_L \times_L W$*, as an element in* $\mathrm{H}^1_{\mathrm{dR}}(\mathbf{D}_L \times_L W)$*.*

Proof. We may assume $L = K$ without lost of generality. The function f restricts to a morphism of formal K°-schemes, denoted again by

$$f\colon \operatorname{Spf}((E^t_{k,\varpi})^\circ \widehat{\otimes}_{k^\circ} D^\circ \widehat{\otimes}_{k^\circ} K^\circ) \to \operatorname{Spf} K^\circ[[T]].$$

On the generic fiber, since the Shilov boundary of $\mathcal{M}(E^t_{k,\varpi} \widehat{\otimes}_k D \widehat{\otimes}_k K)$ consists of one point and the image of the induced morphism $\mathcal{M}(E^t_{k,\varpi} \widehat{\otimes}_k D \widehat{\otimes}_k K) \to D(0;1)_K$ does not contain 0, the pullback of T on $\mathcal{M}(E^t_{k,\varpi} \widehat{\otimes}_k D \widehat{\otimes}_k K)$ has constant norm. In other words, the morphism $\mathcal{M}(E^t_{k,\varpi} \widehat{\otimes}_k D \widehat{\otimes}_k K) \to D(0;1)_K$ induced by f factors through a morphism $\mathcal{M}(E^t_{k,\varpi} \widehat{\otimes}_k D \widehat{\otimes}_k K) \to \mathcal{M}(K\langle r^{-1}T, rT^{-1}\rangle)$ for a unique number $r < 1$ in $\sqrt{|K^\times|}$. By replacing K by a finite extension, we may assume $r \in |K^{\circ\circ}|$. Then $K\langle r^{-1}T, rT^{-1}\rangle$ is integrally smooth, and we have $\operatorname{Spec} K\langle r^{-1}T, rT^{-1}\rangle_s \simeq (\mathbf{G}_m)_{\widetilde{K}}$. Moreover,

$$\mathrm{H}^1_{\mathrm{rig}}(\mathbf{G}_m/K)_2 = \mathrm{H}^1_{\mathrm{rig}}(\mathbf{G}_m/K) \simeq \mathrm{H}^1_{\mathrm{dR}}(D(0,1)_K, \mathcal{M}(K\langle r^{-1}T, rT^{-1}\rangle)) = K\{\frac{\mathrm{d}T}{T}\}.$$

Thus Lemma 4.2 and Remark 5.4 imply the lemma. □

Definition 5.7. Let X be a K-analytic space and $x \in X$ a point.

(1) Let $\mathrm{f\acute{E}t}(X;x)$ be the category whose objects are fundamental charts of $(X;x)$, and a morphism

$$\phi\colon (\mathbf{D}_{L_2}, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2, \alpha_2; y_2) \to (\mathbf{D}_{L_1}, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1, \alpha_1; y_1)$$

consists of extensions $K \subset L_1 \subset L_2$[5] such that $k_1 \subset k_2$, and a morphism

$$\Phi(\phi)\colon \mathbf{D}_{L_2} \times_{L_2} W_2 \to \mathbf{D}_{L_1} \times_{L_1} W_1$$

of L_1-analytic spaces sending y_2 to y_1, and such that

$$\Phi(\phi)^*\mathrm{H}^p_{(w)}(\mathbf{D}_{L_1}, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1) \subset \mathrm{H}^p_{(w)}(\mathbf{D}_{L_2}, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2)$$

for all $p, w \in \mathbf{Z}$. Note that $\Phi(\phi)$ needs *not* to respect each factor.

(2) Let $\mathrm{\acute{E}t}(X;x)$ be the category of étale neighborhoods of $(X;x)$. Recall that its objects are triples $(Y, \alpha; y)$ where $\alpha\colon Y \to X$ is an étale morphism sending $y \in Y$ to x, and morphisms are defined in the obvious way. In the notation $(Y, \alpha; y)$, the morphism α will be suppressed if it is not relevant. For a presheaf $\mathscr{F}$ on $X_{\acute{e}t}$, the stalk of $\mathscr{F}$ at x is defined to be $\mathscr{F}_x \coloneqq \varinjlim_{(Y,\alpha;y)} \mathscr{F}(Y)$ where the colimit is taken over the category $\mathrm{\acute{E}t}(X;x)$.

(3) We have a functor $\Phi\colon \mathrm{f\acute{E}t}(X;x) \to \mathrm{\acute{E}t}(X;x)$ sending an object

$$(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$$

of $\mathrm{f\acute{E}t}(X;x)$ to $(\mathbf{D}_L \times_L W, \alpha; y)$, and a morphism ϕ to $\Phi(\phi)$.

The following lemma generalizes [Ber07, Proposition 2.1.1].

[5] Even when $L_1 = L_2$, we regard $\mathbf{D}_{L_1}$ and $\mathbf{D}_{L_2}$ as two different poly-discs.

Lemma 5.8. *Suppose that K is embeddable into $\mathbf{C_F}$ and X is a smooth K-analytic space. Fix an arbitrary point $x \in X$ and let $(Y, \alpha_0; y_0)$ be an object of $\acute{\mathrm{E}}\mathrm{t}(X; x)$. Then*

(1) *there exists an object $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \mathrm{f}\acute{\mathrm{E}}\mathrm{t}(X; x)$ such that its image under Φ admits a morphism to $(Y, \alpha_0; y_0)$;*

(2) *given two morphisms $\beta_i \colon \Phi(\mathbf{D}_{L_i}, (\mathcal{Y}_i, \mathcal{D}_i), (D_i, \delta_i), W_i, \alpha_i; y_i) \to (Y, \alpha_0; y_0)$ in $\acute{\mathrm{E}}\mathrm{t}(X; x)$ for $i = 1, 2$, there exists an object $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \mathrm{f}\acute{\mathrm{E}}\mathrm{t}(X; x)$ together with morphisms ϕ_i to $(\mathbf{D}_i, (\mathcal{Y}_{L_i}, \mathcal{D}_i), (D_i, \delta_i), W_i, \alpha_i; y_i)$ in $\mathrm{f}\acute{\mathrm{E}}\mathrm{t}(X; x)$ for $i = 1, 2$ such that the following diagram*

$$\begin{array}{ccccc}
 & & \Phi(\mathbf{D}_{L_1}, (\mathcal{Y}_1, \mathcal{D}_1), (D_1, \delta_1), W_1, \alpha_1; y_1) & & \\
 & \overset{\Phi(\phi_1)}{\nearrow} & & \overset{\beta_1}{\searrow} & \\
\Phi(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) & & & & (Y, \alpha_0; y_0) \\
 & \underset{\Phi(\phi_2)}{\searrow} & & \underset{\beta_2}{\nearrow} & \\
 & & \Phi(\mathbf{D}_{L_2}, (\mathcal{Y}_2, \mathcal{D}_2), (D_2, \delta_2), W_2, \alpha_2; y_2) & &
\end{array}$$

commutes.

Proof. Let n be the dimension of X at x; we may assume by arguing locally that X is of dimension n. Put $t = t(x)$ and $s = s(x)$.

For (1), by [Ber07, Proposition 2.3.1], after replacing K by a finite extension, we may assume $Y = \mathbf{D}_K \times_K X'$ and $y_0 \in Y$ such that $t(\mathrm{pr}_{X'}(y_0)) = t$ and $s(\mathrm{pr}_{X'}(y_0)) = s$, where X' is a smooth K-analytic space of dimension $s + t$. Now we only need to apply Lemma 5.2 to $\mathfrak{Y} = \mathrm{Spf}\, K^\circ$, the pair $(X'; x')$, and the structure morphism $X' \to \mathfrak{Y}_\eta = \mathcal{M}(K)$. The existence of (D, δ) is due to the argument in Part (iv) of the proof of [GK02, Theorem 2.3].

For (2), we put $\mathbf{D}_i = \mathbf{D}_{L_i}$, and may assume $K = L_1 = L_2$. For $i = 1, 2$, we choose a relative compactification $\mathcal{Y}_i \hookrightarrow \overline{\mathcal{Y}_i}$ over k_i°, where $\overline{\mathcal{Y}_i}$ is proper. Such compactification is needed to construct a formal scheme $\mathfrak{Y}$ (see below) in order to apply Lemma 5.2. Then W_i is open in $\mathfrak{Y}_{i\eta}$, where $\mathfrak{Y}_i = \widehat{\overline{\mathcal{Y}_i}} \widehat{\otimes}_{k_i^\circ} K^\circ$. Consider the étale morphism

$$\alpha_0' \colon Y' \coloneqq (\mathbf{D}_1 \times_K W_1) \times_Y (\mathbf{D}_2 \times_K W_2) \to Y,$$

and a point $y_0' \in Y'$ projecting to y_1 (resp. y_2) in the first (resp. second) factor. Again by [Ber07, Proposition 2.3.1], we may find an object of the form $(\mathbf{D}_K \times_K X', \alpha'; y')$ in $\acute{\mathrm{E}}\mathrm{t}(X; x)$ such that $t(\mathrm{pr}_{X'}(y')) = t$ and $s(\mathrm{pr}_{X'}(y')) = s$ as in (1) with a morphism to $(Y', \alpha_0'; y_0')$. Now we apply Lemma 5.2 to X', the point $\mathrm{pr}_{X'}(y')$, $\mathfrak{Y} = \mathfrak{Y}_1 \times_{K^\circ} \mathfrak{Y}_2$, the morphism

$$X' \xrightarrow{(\beta_1, \beta_2)} W_1 \times_K W_2 \subset \mathfrak{Y}_\eta,$$

where β_i equals the composition

$$X' \simeq \{0\} \times_K X' \subset \mathbf{D}_K \times_K X' \to \mathbf{D}_i \times_K W_i \to W_i \quad (i = 1, 2)$$

with the last arrow being the projection. We obtain a marked k-pair $(\mathcal{Y}, \mathcal{D})$ of dimension $s + t$ and depth t, for some discrete non-Archimedean field k containing k_1, k_2 and contained in (possibly a finite extension of) K; an open neighborhood W of $(\widehat{\mathcal{Y}}_{/\mathcal{D}})_\eta \widehat{\otimes}_k K$ in $\mathcal{Y}_K^{\mathrm{an}}$, a point $u \in W$, an étale morphism of K-analytic spaces $\varphi \colon W \to X'$ such that $\varphi(u) = \mathrm{pr}_{X'}(y')$, and a morphism of formal K°-schemes $\psi = (\psi_1, \psi_2) \colon \widehat{\mathcal{Y}}_{/\mathcal{D}} \widehat{\otimes}_{k^\circ} K^\circ \to \mathfrak{Y}_1 \times_{K^\circ} \mathfrak{Y}_2$ compatible with φ. As ψ_i maps the generic point of $\mathcal{D}_{\widetilde{K}}$ to the generic point of $(\mathcal{D}_i)_{\widetilde{K}}$, we may replace

$(\mathcal{Y},\mathcal{D})$ by an affine open such that $\psi_i(\mathcal{D}_{\widetilde{K}})\subset(\mathcal{D}_i)_{\widetilde{K}}$ for $i=1,2$. In particular, we have morphisms $\psi_i\colon \widehat{\mathcal{Y}}_{/\mathcal{D}}\widehat{\otimes}_{k^\circ}K^\circ\to\widehat{\mathcal{Y}}_{i/\mathcal{D}_i}\widehat{\otimes}_{k_i^\circ}K^\circ$. Note that ψ_i does not necessarily descent to any finite extension of k. By the proof of [GK02, Theorem 2.3], there is an integrally smooth k-affinoid algebra D and an isomorphism δ as in (5.1). Finally, we take a point $y\in\mathbf{D}_K\times_K W$ above y' such that $\mathrm{pr}_W(y)=u$.

Now the object $(\mathbf{D}_K,(\mathcal{Y},\mathcal{D}),(D,\delta),W,\alpha;y)$ has been constructed with the obvious α (with $L=K$ possibly a finite extension of the starting one). Let $\Phi(\phi_i)$ be the composite morphism $\mathbf{D}_K\times_K W\to\mathbf{D}_K\times_K X'\to\mathbf{D}_i\times_K W_i$ for $i=1,2$. It remains to show that

(i) For $i=1,2$, every p, every w, and an element $\omega\in\mathrm{H}^p_{\mathrm{rig}}(\mathcal{D}_i/k_i)_w$, we have

$$(\beta_i\circ\varphi)^*(\delta_i^*)^{-1}\omega\in\mathrm{H}^p_{(w)}((\mathcal{Y},\mathcal{D}),(D,\delta),W).$$

(ii) For $i=1,2$ and an arbitrary coordinate T of $\mathbf{E}^t_{k_i,\varpi_i}$ (where ϖ_i is a uniformizer of k_i), we have

$$(\beta_i\circ\varphi)^*(\delta_i^*)^{-1}\frac{\mathrm{d}T}{T}\in\mathrm{H}^1_{(2)}((\mathcal{Y},\mathcal{D}),(D,\delta),W).$$

For (i), as we have morphisms of formal K°-schemes

$$\mathrm{Spf}((E^t_{k,\varpi})^\circ\widehat{\otimes}_{k^\circ}D^\circ\widehat{\otimes}_{k^\circ}K^\circ)\to\widehat{\mathcal{Y}}_{/\mathcal{D}}\widehat{\otimes}_{k^\circ}K^\circ\xrightarrow{\psi_i}\widehat{\mathcal{Y}}_{i/\mathcal{D}_i}\widehat{\otimes}_{k_i^\circ}K^\circ\to\mathrm{Spf}\,D_i^\circ\widehat{\otimes}_{k_i^\circ}K^\circ,$$

Lemma 4.2 implies that $(\beta_i\circ\varphi)^*(\delta_i^*)^{-1}\omega$ coincides with $\varphi_i^*\omega$ in

$$\mathrm{H}^p_{\mathrm{dR}}(W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k K)\simeq\mathrm{H}^p_{\mathrm{rig}}((E^t_{k,\varpi})_s\otimes_{\widetilde{k}}\mathcal{D}/K)$$

in view of Remark 5.4, where

$$\varphi_i\colon(E^t_{k,\varpi})_s\otimes_{\widetilde{k}}\mathcal{D}_{\widetilde{K}}\to(\mathcal{D}_i)_{\widetilde{K}}$$

is the induced morphism of (affine smooth) $\widetilde{K}$-schemes. Thus (i) follows from weight preservation of rigid cohomology.

For (ii), we only need to apply Lemma 5.6 to the function T. □

Now we are ready to define the desired direct summand $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_w$ in the weight decomposition of de Rham cohomology sheaves.

Definition 5.9 (De Rham cohomology sheaves with weights)**.** Suppose that K is residually algebraic over $\mathbf{Q}_\mathbf{F}$ and X is a smooth K-analytic space. Let $p\geq 1$ be an integer.

For every object U of $X_{\text{ét}}$, define $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)(U)^{\mathrm{pre}}_w\subset(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)(U)$ to be the image of elements $\omega\in\Omega^{p,\mathrm{cl}}_X(U)$ such that for every point $u\in U$, there exists a fundamental chart $(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W,\alpha;y)$ of $(U;u)$ such that $\alpha^*\omega$, regarded as an element in $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L\times_L W)$, belongs to $\mathrm{H}^p_{(w)}(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$. The assignment $U\mapsto(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)(U)^{\mathrm{pre}}_w$ defines a sub-presheaf $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)^{\mathrm{pre}}_w$ of $\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$.

We define $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_w$ to be the sheafification of $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)^{\mathrm{pre}}_w$, which is canonically a subsheaf of $\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$.

The following lemma can be proved by the same way as for [Ber07, Corollary 5.5.3].

Lemma 5.10. *Let K'/K be an extension such that K' is embeddable into $\mathbf{C}_\mathbf{F}$. Let X be a smooth K-analytic space and $\varsigma\colon X':=X\widehat{\otimes}_K K'\to X$ the canonical projection. Then the canonical map of sheaves on $X'_{\text{ét}}$*

$$\varsigma^{-1}(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)\otimes_L K'\to\Omega^{p,\mathrm{cl}}_{X'}/\mathrm{d}\Omega^{p-1}_{X'}$$

is an isomorphism, where L is the algebraic closure of K in K'.

The following theorem establishes the functorial weight decomposition of de Rham cohomology sheaves in Theorem 1.4 in the case of étale topology.

Theorem 5.11. *If K is embeddable into $\mathbf{C_F}$ and X is a smooth K-analytic space, then the following hold:*

(1) under the situation of Lemma 5.10, we have

$$\varsigma^{-1}(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w \otimes_L K' = (\Omega_{X'}^{p,\mathrm{cl}}/\mathrm{d}\Omega_{X'}^{p-1})_w,$$

for every $w \in \mathbf{Z}$;

(2) the image of the composite map

$$(\Omega_X^{p_1,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_1-1})_{w_1} \otimes (\Omega_X^{p_2,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_2-1})_{w_2} \to \Omega_X^{p_1,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_1-1} \otimes \Omega_X^{p_2,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_2-1} \xrightarrow{\wedge} \Omega_X^{p_1+p_2,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_1+p_2-1}$$

is contained in the subsheaf $(\Omega_X^{p_1+p_2,\mathrm{cl}}/\mathrm{d}\Omega_X^{p_1+p_2-1})_{w_1+w_2}$;

(3) the sheaf $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ is zero unless $p \leq w \leq 2p$;

(4) the canonical map

$$\bigoplus_{w\in\mathbf{Z}}(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w \to \Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$$

is an isomorphism;

(5) for every morphism $f\colon Y \to X$ of smooth K-analytic spaces, we have

$$f^{\#}(f^{-1}(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w) \subset (\Omega_Y^{p,\mathrm{cl}}/\mathrm{d}\Omega_Y^{p-1})_w$$

for every $w \in \mathbf{Z}$. Here, $f^{\#}$ denotes the canonical map $f^{-1}\Omega_X^\bullet \to \Omega_Y^\bullet$ and induced maps of cohomology sheaves.

Proof. Part (1) follows from the definition and Remark 5.5. Part (2) follows from definition and Lemma 5.8 (2).

For the remaining parts, it suffices to work on stalks. Thus we fix a point $x \in X$ with $t = t(x)$ and $s = s(x)$.

For (3), take an element $[\omega]$ in the stalk of $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ at x for some $w < p$ or $w > 2p$. We may assume that it has a representative $\omega \in \Omega_X^{p,\mathrm{cl}}(U)$ for some étale neighborhoods $(U;u)$ of $(X;x)$. By definition, we have a fundamental chart $(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W,\alpha;y)$ of $(U;u)$ such that $\alpha^*\omega = 0$ in $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L (W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))$ by Remark 5.5. Then there exists an open neighborhood W^- of $(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L$ in W, such that $\alpha^*\omega = 0$ in $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L W^-)$. In other words, $[\omega] = 0$ in the stalk of $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ at x as α is étale and we are working with differential sheaves in étale topology.

For (4), we first show that the map is injective. Let $[\omega]$ be an element in the stalk $\Omega_{X,x}^{p,\mathrm{cl}}/\mathrm{d}\Omega_{X,x}^{p-1}$. Suppose that we have $[\omega] = \sum[\omega]_w^1 = \sum[\omega]_w^2$ in which both $[\omega]_w^1$ and $[\omega]_w^2$ are in the stalk of $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ at x. We may choose an object $(U;u) \in \mathrm{\acute{E}t}(X,x)$ such that $[\omega]_w^i$ has a representative $\omega_w^i \in (\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})(U)_w^{\mathrm{pre}}$ for $i = 1,2$ and every $w \in \mathbf{Z}$, and $\sum\omega_w^1 = \sum\omega_w^2$. In particular, $[\omega]$ has a representative $\omega \coloneqq \sum\omega_w^1 = \sum\omega_w^2$ on $(U;u)$. Fix a weight $w \in \mathbf{Z}$. It suffices to show that $[\omega_w^1] = [\omega_w^2]$ in the stalk at x. By Definition 5.9, there exist two fundamental charts $(\mathbf{D}_{L_i},(\mathcal{Y}_i,\mathcal{D}_i),(D_i,\delta_i),W_i,\alpha_i;y_i)$ of $(U;u)$ such that $\alpha_i^*\omega_w^i$ belongs to $\mathrm{H}^p_{(w)}(\mathbf{D}_{L_i},(\mathcal{Y}_i,\mathcal{D}_i),(D_i,\delta_i),W_i)$ for $i = 1,2$. By Lemma 5.8, we may find another fundamental chart $(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W,\alpha;y) \in \mathrm{f\acute{E}t}(U;u)$ as in that lemma. Then we have $\Phi(\phi_i)^*\omega_w^i \in \mathrm{H}^p_{(w)}(\mathbf{D}_L,(\mathcal{Y},\mathcal{D}),(D,\delta),W)$ for both $i = 1,2$. However, $\Phi(\phi_1)^*\omega_w^1$ and $\Phi(\phi_2)^*\omega_w^2$, after restriction to $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L (W,(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))$, must be equal, as they are both the weight

w component of $\alpha^*\omega$ in $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L (W, (\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))$ under the decomposition (5.3). As the map $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L W) \to (\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_x$ factors through $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D}_L \times_L (W, (\pi^{-1}\mathcal{D})\widehat{\otimes}_k L))$, we have $[\omega]^1_w = [\omega]^2_w$. Finally, Lemma 5.12 below implies that the map in (4) is surjective as well.

For (5), we take a point $y \in Y$ such that $f(y) = x$. We may take a fundamental chart $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y)$ of $(X; x)$ and replace X by $\mathbf{D}_L \times_L W$ and x by y. By the same proof of Lemma 5.8 (2), we may find a fundamental chart $(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W', \alpha'; y')$ of $(Y; y)$ such that $(f \circ \alpha')^*\mathrm{H}^p_{(w)}(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W) \subset \mathrm{H}^p_{(w)}(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$. This confirms Part (5) since $\mathrm{H}^p_{(w)}(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$ (resp. $\mathrm{H}^p_{(w)}(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$) restricts to the weight w part in the stalk of $\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}$ (resp. $\Omega_Y^{p,\mathrm{cl}}/\mathrm{d}\Omega_Y^{p-1}$) at x (resp. y), by Lemma 5.12 below. □

The following lemma is the most crucial and difficult part in the proof of the weight decomposition.

Lemma 5.12. *Let the assumptions be as in Theorem 5.11. We take a point $x \in X$. For any fixed weight w, an object $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \alpha; y) \in \mathrm{f\acute{E}t}(X, x)$, and an element $\omega \in \mathrm{H}^p_{(w)}(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$, the induced class $[\omega] \in \Omega^{p,\mathrm{cl}}_{X,x}/\mathrm{d}\Omega^{p-1}_{X,x}$ belongs to the stalk of $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})_w$ at x.*

We first explain why the lemma is not immediate. For simplicity, let us assume that $\mathbf{D}_L$ is trivial. To prove the lemma, we have to find an étale neighborhood U of x such that ω lies in $(\Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})(U)^{\mathrm{pre}}_w$. However, no matter what U we take, there are always points u of U such that its image in W is not in $(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L$. On $W\backslash(\pi^{-1}\mathcal{D})\widehat{\otimes}_k L$, there is a priori no obvious control of the form ω, so it is a question to show that one can choose a fundamental chart at u such that the induced form has weight w.

Before the proof, let us explain the main steps. Again, we assume that $\mathbf{D}_L$ is trivial at this moment. In Steps 1 & 2, we make the germ $(W, (\pi^{-1}\mathcal{D})\widehat{\otimes}_k L)$ a better shape. We will construct a strictly semistable scheme $\mathcal{S}$ over k° together with an irreducible component $\mathcal{E}$ on the special fiber, and an étale neighborhood $(W^\natural, y^\natural)$ of x above (W, y) equipped with a morphism to $(\pi^{-1}\mathcal{E}, \pi^{-1}\mathcal{E}^\natural)$ where $\mathcal{E}^\natural = \mathcal{E}\backslash\mathcal{S}^{[1]}_s$. In Step 3, we study some basic structure of the boundary part $\pi^{-1}\mathcal{E}\backslash\pi^{-1}\mathcal{E}^\natural$. In Step 4, we choose an appropriate covering of $\pi^{-1}\mathcal{E}$ with nice behavior of ω on some open neighborhood of $\pi^{-1}\mathcal{E}^\natural$ in every member of the cover. In Step 5, we show that ω has weight w (in certain sense) even on the boundary part $\pi^{-1}\mathcal{E}\backslash\pi^{-1}\mathcal{E}^\natural$. Finally, we conclude the lemma via functoriality in rigid cohomology in Step 6.

Proof. We may assume $L = K$ and write $\mathbf{D} = \mathbf{D}_L$. To simplify notation, we denote by V the K-affinoid domain $(\pi^{-1}\mathcal{D})\widehat{\otimes}_k K$ in W. By Example 4.3 and Lemma 5.6, we may assume that the image of ω in $\mathrm{H}^p_{\mathrm{dR}}(\mathbf{D} \times_K (W, V))$ is in $\mathrm{H}^p_{\mathrm{rig}}(\mathcal{D}/K)_w$ in view of the decomposition (5.2).

Step 1. We choose a smooth k°-algebra $D^\natural$ (of relative dimension s) such that its ϖ-adic completion is D°, where we recall that ϖ is a uniformizer of the discrete non-Archimedean field $k \subset K$. In particular, we may identify $(\operatorname{Spec} D^\natural)_s$ with $\mathcal{D}$, and $\mathcal{M}(D)$ with a k-affinoid domain in $(\operatorname{Spec} D^\natural)^{\mathrm{an}}_k$. As in Lemma 4.2, we have germs (W, V) and $((\operatorname{Spec} D^\natural)^{\mathrm{an}}_k, \mathcal{M}(D))$ and a morphism $V \to \mathcal{M}(D)\widehat{\otimes}_k K$ induced from δ. We choose a neighborhood U_ε of the graph of the previous morphism in $W \times_k (\operatorname{Spec} D^\natural)^{\mathrm{an}}_k$ as in the proof of Lemma 4.2, such that the induced map

$$\mathrm{H}^\bullet_{\mathrm{dR}}(W,V)\to \mathrm{H}^\bullet_{\mathrm{dR}}(W\times_k(\operatorname{Spec}D^\natural)^{\mathrm{an}}_k, U_\varepsilon\cap(V\times_k\mathcal{M}(D)))$$

is an isomorphism. By a similar argument in the proof of [GK02, Lemma 2], we may replace W by a smaller open neighborhood of V such that there is a morphism $W\to U_\varepsilon$ sending V into $U_\varepsilon\cap(V\times_k\mathcal{M}(D))$ whose induced map

$$\mathrm{H}^\bullet_{\mathrm{dR}}(W\times_k(\operatorname{Spec}D^\natural)^{\mathrm{an}}_k, U_\varepsilon\cap(V\times_k\mathcal{M}(D)))\to\mathrm{H}^\bullet_{\mathrm{dR}}(W,V)$$

is the inverse of the previous isomorphism. In other words, we have a morphism $\delta'\colon W\to(\operatorname{Spec}D^\natural)^{\mathrm{an}}_K$ sending V into $\mathcal{M}(D)\widehat{\otimes}_kK$ such that, although $\delta'|_V$ might not coincide with the original morphism $V\to\mathcal{M}(D)\widehat{\otimes}_kK$ induced from δ, we still have that the induced map

$$\mathrm{H}^\bullet_{\mathrm{rig}}(\mathcal{D}/K)\simeq\mathrm{H}^\bullet_{\mathrm{dR}}((\operatorname{Spec}D^\natural)^{\mathrm{an}}_k,\mathcal{M}(D))\otimes_kK\xrightarrow{\delta'^*}\mathrm{H}^\bullet_{\mathrm{dR}}(W,V)\simeq\mathrm{H}^\bullet_{\mathrm{rig}}((E^t_{k,\varpi})_s\otimes_{\widetilde{k}}\mathcal{D}/K)$$

coincides with the pullback map.

Step 2. We choose a compactification $(\operatorname{Spec}D^\natural)_k\hookrightarrow\mathcal{T}_k$ over k, and define $\mathcal{T}$ to be the k°-scheme $\mathcal{T}_k\coprod_{(\operatorname{Spec}D^\natural)_k}\operatorname{Spec}D^\natural$. Apply [dJ96, Theorem 8.2] to the k°-variety $\mathcal{T}$ and $Z=\emptyset$. We obtain a finite extension k'/k, an alteration $\mathcal{S}^\natural\to\mathcal{T}_{k'^\circ}$ and a k'°-compactification $\mathcal{S}^\natural\hookrightarrow\mathcal{S}$ where $\mathcal{S}$ is a projective strictly semistable scheme over k'° such that $\mathcal{S}\backslash\mathcal{S}^\natural$ is a strict normal crossing divisor of $\mathcal{S}$ (concentrated on the special fiber). We may further assume that every irreducible component of $\mathcal{S}^{[r]}_s$ is geometrically irreducible for every $r\geq0$. To ease notation, we replace k by k' and possibly K by a finite extension. We may fix an irreducible component $\mathcal{E}$ of $\mathcal{S}_s$ such that its generic point belongs to $\mathcal{S}^\natural_s$ and maps to the generic point of $\mathcal{T}_s\simeq\mathcal{D}$. Note that the complement of $\mathcal{E}^\natural:=\mathcal{E}\cap\mathcal{S}^\natural_s$ in $\mathcal{E}$ is exactly $\mathcal{S}^{[1]}_s\cap\mathcal{E}$. Denote by $\sigma_{\mathcal{E}}$ the unique point in $\mathcal{S}^{\mathrm{an}}_K$ who reduction is the generic point of $\mathcal{E}_{\widetilde{K}}$. Then $\pi^{-1}\mathcal{E}_{\widetilde{K}}$ is an open neighborhood of $\sigma_{\mathcal{E}}$.

We have open subsets $(\mathcal{S}^\natural)^{\mathrm{an}}_K$ and $\pi^{-1}\mathcal{E}_{\widetilde{K}}$ of $\mathcal{S}^{\mathrm{an}}_K$. Define $W^\natural$ via the following Cartesian diagram:

$$\begin{array}{ccc} W^\natural & \longrightarrow & (\mathcal{S}^\natural)^{\mathrm{an}}_K\cap\pi^{-1}\mathcal{E}_{\widetilde{K}} \\ \downarrow & & \downarrow \\ W & \xrightarrow{\delta'} & \mathcal{T}^{\mathrm{an}}_K, \end{array}$$

and let $\delta^\natural\colon W^\natural\to\mathfrak{S}_\eta\widehat{\otimes}_kK=(\pi^{-1}\mathcal{E})\widehat{\otimes}_kK$ the induced morphism, where $\mathfrak{S}:=\widehat{\mathcal{S}}_{/\mathcal{E}}$. We choose a point $y^\natural\in\mathbf{D}\times_KW^\natural$ which lifts y and such that $\delta^\natural(\mathrm{pr}_{W^\natural}(y^\natural))=\sigma_{\mathcal{E}}$. The image of the form ω in $\mathrm{H}^p_{\mathrm{rig}}(\mathcal{D}/K)_w$ induces a class $[\omega^\natural]\in\mathrm{H}^p_{\mathrm{rig}}(\mathcal{E}^\natural/K)_w$ via restriction along the alteration. Therefore, there exists a monic polynomial $P\in k[X]$ whose roots are all Weil $|\widetilde{k}|^{w/2}$-numbers such that $P(\mathrm{Fr}^*)[\omega^\natural]=0$ where Fr denotes the relative Frobenius of $\mathcal{E}^\natural/\widetilde{k}$. We fix an open neighborhood U of $\pi^{-1}\mathcal{E}^\natural$ in $\mathfrak{S}_\eta$ such that $[\omega^\natural]$ has a representative $\omega^\natural\in\mathrm{H}^p_{\mathrm{dR}}(U\widehat{\otimes}_kK)$.

By construction, we may remove a Zariski closed subset of $W^\natural$ of dimension at most $s+t-1$ such that the morphism $W^\natural\to W$ hence the composition $\mathbf{D}\times_KW^\natural\to\mathbf{D}\times_KW\xrightarrow{\alpha}X$ are both étale. In particular, $(\mathbf{D}\times_KW^\natural;y^\natural)$ is an object of $\acute{\mathrm{E}}\mathrm{t}(X;x)$.

Step 3. Now we define a continuous function

$$\mathrm{d}_{\mathcal{E}}\colon\mathfrak{S}_\eta\to\mathbf{R}$$

valued in $[0,1]$ such that $\mathrm{d}_{\mathcal{E}}(x)=0$ if and only if $\pi(x)\in\mathcal{E}^\natural$, as follows: For a point $x\in\mathfrak{S}_\eta$, choose an open neighborhood $\mathfrak{S}_x$ of $\pi(x)$ in $\mathfrak{S}$ such that $\mathfrak{S}_x$ is étale over

$$\operatorname{Spf} k^\circ[[t_0]]\langle t_1,\dots,t_r,t_{r+1},t_{r+1}^{-1},\dots,t_s,t_s^{-1}\rangle/(t_0\cdots t_r-\varpi)$$

for some $0\le r\le s$; and if we write f_j for the image of t_j in $\mathfrak{S}_x$, then $\mathcal{E}^\natural\cap\mathfrak{S}_x$ is defined by the equations $f_0=0$ and $f_1\cdots f_r\neq 0$. Then we define

$$\mathrm{d}_{\mathcal{E}}(x)=1-\frac{\log|f_0(x)|}{\log|\varpi|}.$$

It is independent of the choice of the étale coordinates hence is continuous. Moreover, we have $|\varpi|\le|f_0(x)|<1$ and the equality holds if and only if $\pi(x)\in\mathcal{E}^\natural$.

Let $\mathcal{E}_1$ be an open subscheme of $\mathcal{E}$ and put $\mathcal{E}_1^\natural=\mathcal{E}^\natural\cap\mathcal{E}_1$. For $\varepsilon>0$, put

$$(\pi^{-1}\mathcal{E}_1)^{<\epsilon}\coloneqq\{x\in\pi^{-1}\mathcal{E}_1\mid \mathrm{d}_{\mathcal{E}}(x)<\varepsilon\}.$$

Then $(\pi^{-1}\mathcal{E}_1)^{<\epsilon}$ form a fundamental system of open neighborhoods of $\pi^{-1}\mathcal{E}_1^\natural$ in $\pi^{-1}\mathcal{E}_1$. In fact, let $U_1\subset\pi^{-1}\mathcal{E}_1$ be an open neighborhood of $\pi^{-1}\mathcal{E}_1^\natural$. Consider $(\pi^{-1}\mathcal{E}_1)^{\le 1/2}\coloneqq\{x\in\pi^{-1}\mathcal{E}_1\mid\mathrm{d}_{\mathcal{E}}(x)\le 1/2\}$, which is a compact subset of $\pi^{-1}\mathcal{E}_1$. Thus the image of the function $\mathrm{d}_{\mathcal{E}}$ on $(\pi^{-1}\mathcal{E}_1)^{\le 1/2}\backslash U_1$ is a compact subset of $(0,1/2]$. Thus, there exists $\varepsilon>0$ such that $\mathrm{d}_{\mathcal{E}}(x)>\varepsilon$ if $x\notin U_1$. In other words, U_1 contains $(\pi^{-1}\mathcal{E}_1)^{<\varepsilon}$.

Step 4. For every point $e\in\mathcal{E}$, we fix an affine open neighborhood $\mathfrak{S}_e$ of e in $\mathfrak{S}$ together with an étale morphism to

$$\operatorname{Spf} k^\circ[[t_0]]\langle t_1,\dots,t_r,t_{r+1},t_{r+1}^{-1},\dots,t_s,t_s^{-1}\rangle/(t_0\cdots t_r-\varpi)$$

where $r=r(e)\ge 0$ is the unique integer such that $e\in\mathcal{S}_s^{[r]}\backslash\mathcal{S}_s^{[r+1]}$, such that

- if we denote $f_{e,i}$ the image of t_i in $\mathfrak{S}_e$, then $\mathcal{E}_e^\natural\coloneqq\mathcal{E}^\natural\cap\mathfrak{S}_e$ is defined by the equations $f_{e,0}=0$ and $f_{e,1}\cdots f_{e,r}\neq 0$;
- if we denote by $\mathcal{F}_e$ the subscheme of $\mathfrak{S}_e$ defined by $f_{e,0}=\cdots=f_{e,r}=0$, then there exist an integrally smooth k-affinoid algebra F_e together with an isomorphism, which we fix,

$$\operatorname{Spf} F_e^\circ[[t_{e,0},\dots,t_{e,r}]]/(t_{e,0}\cdots t_{e,r}-\varpi)\simeq\widehat{\mathfrak{S}}_{e/\mathcal{F}_e} \tag{5.4}$$

of formal k°-schemes, sending $t_{e,i}$ to $f_{e,i}$ for $0\le i\le r$.

Note that $\mathcal{F}_e$ necessarily contains e by our choice of r. For every point $e\in\mathcal{E}$, we further fix an open neighborhood U_e of $\pi^{-1}\mathcal{E}_e^\natural$ in $\mathfrak{S}_{e,\eta}$ contained in U, together with an absolute Frobenius lifting $\phi_e\colon U_e\to U$ satisfying properties

(a) $\phi_e^*f_{e,i}=f_{e,i}^{|\widetilde{k}|}$ for $1\le i\le r$ (as in [Chi98, Lemma 3.1.1]);
(b) $|(\phi_e^*g-g^{|\widetilde{k}|})(x)|<1$ for all regular functions g on $\widehat{\mathfrak{S}}_{e/\mathcal{E}_e^\natural}$ and all $x\in U_e$ at which both g and ϕ_e^*g are defined (as [Ber07, Lemma 6.1.1]);
(c) $P(\phi_e^*)\omega^\natural=0$ in $\mathrm{H}^p_{\mathrm{dR}}(U_e\widehat{\otimes}_kK)$.

Since $U_e\cap\pi^{-1}\mathcal{E}_e$, where $\mathcal{E}_e\coloneqq\mathcal{E}\cap\mathfrak{S}_e$, is an open neighborhood of $\pi^{-1}\mathcal{E}_e^\natural$ in $\pi^{-1}\mathcal{E}_e$, there exists $\varepsilon_e>0$ such that U_e contains $(\pi^{-1}\mathcal{E}_e)^{<\varepsilon_e}$ by Step 3.

By [GK02, Lemma 3], we know that the restriction map

$$\mathrm{H}^\bullet_{\mathrm{dR}}(\mathfrak{S}_\eta,\pi^{-1}\mathcal{F}_e)\to\mathrm{H}^\bullet_{\mathrm{dR}}(U_e,U_e\cap\pi^{-1}\mathcal{F}_e)$$

is an isomorphism, which is isomorphic to $\mathrm{H}^\bullet_{\mathrm{rig}}(\operatorname{Spec}F_{e,s}/k)\otimes_k\mathrm{H}^\bullet_{\mathrm{dR}}(\mathbf{E}^r_{k,\varpi})$ induced by (5.4). In particular, we have the notion of weights on $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathfrak{S}_\eta,\pi^{-1}\mathcal{F}_e)\otimes_kK$.

Step 5. For a point $e \in \mathcal{E}$, the element $\omega^\natural \in \mathrm{H}^p_{\mathrm{dR}}(U\widehat{\otimes}_k K)$ induces an element $\omega_e \in \mathrm{H}^p_{\mathrm{dR}}(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}_e) \otimes_k K$. We show that ω_e has weight w for every $e \in \mathcal{E}$.

Without lost of generality, we assume $\omega_e \in \mathrm{H}^p_{\mathrm{dR}}(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}_e)$. If $r(e) = 0$, that is, $e \in \mathcal{E}^\natural$, then it is trivial as $[\omega^\natural] \in \mathrm{H}^p_{\mathrm{rig}}(\mathcal{E}^\natural/K)_w$. Now we assume that $r > 0$.

To compute the weight, we use the Frobenius lifting $\phi_e\colon U'_e \to U_e$ where $U'_e \subset U_e$ is a smaller open neighborhood of $\pi^{-1}\mathcal{E}^\natural_e$. Assume that U'_e contains $(\pi^{-1}\mathcal{E}_e)^{<\varepsilon'_e}$ for some $0 < \varepsilon'_e < \varepsilon_e$. We introduce more notations as follows: Fix an integer $N > |\widetilde{k}|(\varepsilon'_e)^{-1}$. Take a totally ramified extension k_+/k with an element $\varpi_+ \in k_+^\circ$ such that $\varpi_+^{rN} = \varpi$. We consider the following k_+-affinoid algebras

$$F_0 = F_e\widehat{\otimes}_k k_+\langle \tau_1, \tau_1^{-1}, \dots, \tau_r, \tau_r^{-1}\rangle,$$
$$F_1 = F_e\widehat{\otimes}_k k_+\left\langle \frac{t_{e,0}}{\varpi_+^{rN-r}}, \frac{\varpi_+^{rN-r}}{t_{e,0}}, \frac{t_{e,1}}{\varpi_+}, \frac{\varpi_+}{t_{e,1}}, \dots, \frac{t_{e,r}}{\varpi_+}, \frac{\varpi_+}{t_{e,r}}\right\rangle/(t_{e,0}\cdots t_{e,r} - \varpi),$$
$$F_2 = F_e\widehat{\otimes}_k k_+\left\langle \frac{t_{e,0}}{\varpi_+^{rN-r|\widetilde{k}|}}, \frac{\varpi_+^{rN-r|\widetilde{k}|}}{t_{e,0}}, \frac{t_{e,1}}{\varpi_+^{|\widetilde{k}|}}, \frac{\varpi_+^{|\widetilde{k}|}}{t_{e,1}}, \dots, \frac{t_{e,r}}{\varpi_+^{|\widetilde{k}|}}, \frac{\varpi_+^{|\widetilde{k}|}}{t_{e,r}}\right\rangle/(t_{e,0}\cdots t_{e,r} - \varpi).$$

Note that F_0 is integrally smooth. We have natural isomorphisms

$$\rho_1\colon F_1 \xrightarrow{\sim} F_0, \quad t_{e,i} \mapsto \varpi_+\tau_i, 1 \le i \le r,\ t_{e,0} \mapsto \varpi_+^{rN-r}\prod_{i=1}^r \tau_i^{-1};$$
$$\rho_2\colon F_2 \xrightarrow{\sim} F_0, \quad t_{e,i} \mapsto \varpi_+^{|\widetilde{k}|}\tau_i, 1 \le i \le r,\ t_{e,0} \mapsto \varpi_+^{rN-r|\widetilde{k}|}\prod_{i=1}^r \tau_i^{-1}.$$

For $\alpha = 1, 2$, we define a formal k_+°-scheme $\mathfrak{S}_\alpha$ via the following pullback diagram

$$\begin{array}{ccc} \mathfrak{S}_\alpha & \longrightarrow & \widehat{\mathfrak{S}}_{e/\mathcal{F}_e}\widehat{\otimes}_{k^\circ} k_+^\circ \\ \downarrow & & \downarrow {\scriptstyle (5.4)} \\ \mathrm{Spf}\, F_\alpha^\circ & \longrightarrow & \mathrm{Spf}\, F_e^\circ[[t_{e,0}, \dots, t_{e,r}]]/(t_{e,0}\cdots t_{e,r} - \varpi) \otimes_{k^\circ} k_+^\circ \end{array}$$

so that $\mathfrak{S}_{\alpha,\eta}$ is canonically a k_+-affinoid domain in $U'_\alpha\widehat{\otimes}_k k_+$ by our choice of N. Moreover, ρ_α induces an isomorphism, denoted again by ρ_α,

$$\rho_\alpha\colon \mathrm{Spf}\, F_0^\circ \xrightarrow{\sim} \mathfrak{S}_\alpha$$

of formal k_+°-schemes. Properties (a) and (b) of the Frobenius lifting ϕ_e implies that it induces by restriction a morphism $\phi_e\colon \mathfrak{S}_{1,\eta} \to \mathfrak{S}_{2,\eta}$, and the composition

$$\rho_{2,\eta}^{-1} \circ \phi_e \circ \rho_{1,\eta}\colon \mathcal{M}(F_0) \to \mathcal{M}(F_0)$$

is a Frobenius lifting. By Lemma 4.2, we have induced isomorphisms

$$\rho_1^*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(U'_e\widehat{\otimes}_k k_+, \mathfrak{S}_{1,\eta}) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, F_{0,s}/k_+),$$
$$(\rho_2^{-1})^*\colon \mathrm{H}^\bullet_{\mathrm{rig}}(\mathrm{Spec}\, F_{0,s}/k_+) \to \mathrm{H}^\bullet_{\mathrm{dR}}(U'_e\widehat{\otimes}_k k_+, \mathfrak{S}_{2,\eta})$$

On the other hand, by [GK02, Lemma 3], we have isomorphisms

$$r_\alpha\colon \mathrm{H}^\bullet_{\mathrm{dR}}(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}) \otimes_k k_+ \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{dR}}(U'_e\widehat{\otimes}_k k_+, \mathfrak{S}_{\alpha,\eta})$$

for $\alpha = 1, 2$ via restriction satisfying $r_2 = (\rho_2^{-1})^* \circ \rho_1^* \circ r_1$. In particular, we may equip $\mathrm{H}^\bullet_{\mathrm{dR}}(U'_e\widehat{\otimes}_k k_+, \mathfrak{S}_{\alpha,\eta})$ with a weight decomposition inherited from that of $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F})$.

Recall that we have $\omega_e \in \mathrm{H}^p_{\mathrm{dR}}(\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}_e)$ obtained from $\omega^\natural$ by restriction. Let ω_0 be the unique element in $\mathrm{H}^p_{\mathrm{rig}}(\operatorname{Spec} F_{0,s}/k_+)$ such that $(\rho_2^{-1})^*\omega_0 = \omega_e$. By Property (c) of the Frobenius lifting ϕ_e, we have that $P(\rho_1^* \circ \phi_e^* \circ (\rho_2^{-1})^*)\omega_0 = 0$. However, $\rho_2^{-1} \circ \phi_e \circ \rho_1 \colon \operatorname{Spf} F_0^\circ \to \operatorname{Spf} F_0^\circ$ is a Frobenius lifting of the Frobenius endomorphism of $\operatorname{Spec} F_{0,s}$ over $\widetilde{k_+} = \widetilde{k}$. Therefore, ω_0 and hence ω_e have weight w.

Step 6. To conclude the lemma, it remains to show that for *every* point $u \in W^\natural$, there exists a fundamental chart $(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W', \alpha'; y')$ of $(W^\natural; u)$ such that $\alpha'^*(\delta^\natural)^*\omega^\natural$ belongs to $\mathrm{H}^p_{(w)}(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$.

Note that $\delta^\natural(u)$ belongs to $\mathfrak{S}_\eta \widehat{\otimes}_k K$. Let e be the image of $\pi(\delta^\natural(u))$ in $\mathcal{E}$. Then we can find a fundamental chart $(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W', \alpha'; y')$ of $(W^\natural; u)$ with the induced morphism

$$(W', (\pi^{-1}\mathcal{D}')\widehat{\otimes}_{k'} L') \to (\mathfrak{S}_\eta, \pi^{-1}\mathcal{F}_e)\widehat{\otimes}_k L'.$$

Then the claim that $\alpha'^*(\delta^\natural)^*\omega^\natural$ belongs to $\mathrm{H}^p_{(w)}(\mathbf{D}_{L'}, (\mathcal{Y}', \mathcal{D}'), (D', \delta'), W')$ follows from Remark 5.4, Lemma 5.6, and the claim in Step 5. ☐

Remark 5.13. From the proof of Theorem 5.11, we know that the support of $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_w$ is contained in the subset $\{x \in X \mid s(x) \geq 2p - w, s(x) + t(x) \geq p\}$.

Now we are ready to prove Theorem 1.4. We begin with the case of étale cohomology and then the case of analytic topology.

Proof of Theorem 1.4 in étale topology. Recall that sheaves like $\mathcal{O}_X$, $\mathfrak{c}_X$, and the de Rham complex $(\Omega^\bullet_X, \mathrm{d})$ are understood in the étale topology.

The direct sum decomposition has been proved in Theorem 5.11 (4). Property (i) follows from Theorem 5.11 (3).

For Property (ii), the inclusion $\Upsilon^p_X \subset (\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_{2p}$ follows from Theorem 5.11 (2) and Lemma 5.6. Now we show that $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)_2 \subset \Upsilon^1_X$. We check the inclusion on stalks. Take a point $x \in X$ with $s = s(x)$ and $t = t(x)$. For every class $[\omega]$ in the stalk of $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)_2$ at x, we may find a fundamental chart $(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W, \mathcal{Z}, \alpha; y)$ of $(X; x)$ such that $[\omega]$ has a representative $\omega \in \mathrm{H}^1_{(2)}(\mathbf{D}_L, (\mathcal{Y}, \mathcal{D}), (D, \delta), W)$. Note that the decomposition (5.2) specializes to the decomposition

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbf{D}_L \times_L (W, (\pi^{-1}\mathcal{D}) \otimes_k L)) = \mathrm{H}^1_{\mathrm{rig}}(\mathcal{D}/L) \oplus \mathrm{H}^1_{\mathrm{dR}}(\mathbf{E}^t_{k,\varpi}\widehat{\otimes}_k L).$$

If the restriction of ω to $\mathrm{H}^1_{\mathrm{dR}}(\mathbf{D} \times_L (W, (\pi^{-1}\mathcal{D}) \otimes_k L))$ belongs to $\mathrm{H}^1_{\mathrm{dR}}(\mathbf{E}^t_{k,\varpi}\widehat{\otimes}_k L)$, then we are done. Otherwise, ω restricts to $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{D}/L)_2$. It suffices to show that classes in $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{D}/L)_2$ can be represented by logarithmic differential of invertible functions étale locally, up to a constant multiple.

We repeat certain process in Step 2 of the proof of Lemma 5.12 as follows. Choose a smooth k°-algebra $D^\natural$ (of dimension s) such that its ϖ-adic completion is D°, a compactification $(\operatorname{Spec} D^\natural)_k \hookrightarrow \overline{\mathcal{S}}_k$ over k, and define $\overline{\mathcal{S}}$ to be the k°-scheme $\overline{\mathcal{S}}_k \coprod_{(\operatorname{Spec} D^\natural)_k} \operatorname{Spec} D^\natural$. Then we obtain a finite extension k'/k, an alteration $\mathcal{S}^\natural \to \overline{\mathcal{S}}_{k'^\circ}$ and a k'°-compactification $\mathcal{S}^\natural \hookrightarrow \mathcal{S}$ where $\mathcal{S}$ is a projective strictly semistable scheme over k'° such that $\mathcal{S}\backslash\mathcal{S}^\natural$ is a strict normal crossing divisor of $\mathcal{S}$. We may further assume that all irreducible components of $\mathcal{S}^{[r]}_s$ are geometrically irreducible for every $r \geq 0$. To ease notation, we replace k by k' and possibly L by a finite extension. We may fix an irreducible component $\mathcal{E}$ of $\mathcal{S}_s$ such that its generic point belongs to $\mathcal{S}^\natural_s$ and maps to the generic point of $\overline{\mathcal{S}}_s \simeq \mathcal{D}$. Thus there is a unique point $\sigma_{\mathcal{E}} \in (\widehat{\mathcal{S}^\natural})_\eta$ such that $\pi(\sigma_{\mathcal{E}})$ is the generic point of $\mathcal{E}$.

Now we apply the setup in the beginning of this section to $\mathcal{S}$ and $\mathcal{E}$. Note that $\mathcal{E}\cap\mathcal{S}_s^\natural$ coincides with $\mathcal{E}^\heartsuit$. It suffices to show that every class in $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)_2$ can be represented by the logarithmic differential of an invertible function on some étale neighborhood of $\sigma_\mathcal{E}$. Put $\mathcal{E}^{[r]}=\mathcal{E}\cap\mathcal{S}_s^{[r]}$ for $r\geq 1$. We have $\mathcal{E}^{[1]}\backslash\mathcal{E}^{[2]}=\coprod_{i=1}^M\mathcal{E}^\heartsuit_{\{i\}}$. Consider the following Gysin exact sequence

$$\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)\to\mathrm{H}^2_{\mathcal{E}^{[1]},\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^2_{\mathrm{rig}}(\mathcal{E}/k). \tag{5.5}$$

We claim that the restriction map $\mathrm{H}^2_{\mathcal{E}^{[1]},\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^2_{\mathcal{E}^{[1]}\backslash\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}\backslash\mathcal{E}^{[2]}/k)$ is an isomorphism. In fact, it fits into another Gysin exact sequence

$$\mathrm{H}^2_{\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^2_{\mathcal{E}^{[1]},\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^2_{\mathcal{E}^{[1]}\backslash\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}\backslash\mathcal{E}^{[2]}/k)\to\mathrm{H}^3_{\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}/k)$$

in which $\mathrm{H}^2_{\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}/k)=\mathrm{H}^3_{\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}/k)=0$ by the semi-purity theorem (see [Fuj02][6]) as the codimension of $\mathcal{E}^{[2]}$ in $\mathcal{E}$ is at least 2. By the purity theorem (Lemma 4.1 (3)), we have a canonical isomorphism

$$\mathrm{H}^2_{\mathcal{E}^{[1]}\backslash\mathcal{E}^{[2]},\mathrm{rig}}(\mathcal{E}\backslash\mathcal{E}^{[2]}/k)\simeq\mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}^{[1]}\backslash\mathcal{E}^{[2]}/k)\simeq\bigoplus_{i=1}^M\mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}^\heartsuit_{\{i\}}/k),$$

where the right-hand side is canonically isomorphic to $Z^1(\mathcal{E})^\heartsuit\otimes k$. Therefore, we may rewrite (5.5) as

$$\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}/k)\to\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)\to Z^1(\mathcal{E})^\heartsuit\otimes k\to\mathrm{H}^2_{\mathrm{rig}}(\mathcal{E}/k).$$

in which the last map $Z^1(\mathcal{E})^\heartsuit\otimes k\to\mathrm{H}^2_{\mathrm{rig}}(\mathcal{E}/k)$ is nothing but the Chern class map (see [Pet03, §6]). As $\mathrm{H}^i_{\mathrm{rig}}(\mathcal{E}/k)$ is of pure weight i by Lemma 4.1 (5), we have the isomorphism

$$\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)_2\xrightarrow{\sim}\ker(Z^1(\mathcal{E})^\heartsuit\otimes k\to\mathrm{H}^2_{\mathrm{rig}}(\mathcal{E}/k)). \tag{5.6}$$

Now take a divisor $D=\sum_{i=1}^M c_i[\mathcal{E}_{\{i\}}]$ with $c_i\in\mathbf{Z}$ such that its Chern class in $\mathrm{H}^2_{\mathrm{rig}}(\mathcal{E}/k)\simeq\mathrm{H}^2_{\mathrm{cris}}(\mathcal{E}/k)$ is trivial. Then by [Pet03, §5] and [Del81, Remarque 3.5] there exists some integer $\mu>0$ such that μD is algebraically equivalent to zero, and in particular $\mathcal{O}_\mathcal{E}(\mu D)$ is an element in $\mathrm{Pic}^0_{\mathcal{E}/\widetilde{k}}(\widetilde{k})$. Since $\mathrm{Pic}^0_{\mathcal{E}/\widetilde{k}}$ is a projective scheme over the finite field $\widetilde{k}$, one may replace μ by some multiple such that $\mathcal{O}_\mathcal{E}(\mu D)$ is a trivial line bundle. Therefore, there exists a function $\tilde{f}\in\mathcal{O}^*_\mathcal{E}(\mathcal{E}^\heartsuit)$ with $\mathrm{div}(\tilde{f})=\mu D$. We may choose a finite set of affine open subschemes $\mathrm{Spec}\,R_i$ of $\mathcal{S}$ such that $(\mathrm{Spec}\,R_i)_s$ form an open covering of $\mathcal{E}^\heartsuit$ and $\tilde{f}|_{(\mathrm{Spec}\,R_i)_s}$ lifts to an invertible function $f_i\in R_i^*$. Note that for every i, $U_i\coloneqq(\mathrm{Spec}\,R_i)^{\mathrm{an}}\cap\pi^{-1}\mathcal{E}$ is an open neighborhood of $\sigma_\mathcal{E}$. Since on $U_i\cap U_j$ we have that $|f_i/f_j|$ is constant, the collection $\{\frac{\mathrm{d}f_i}{f_i}\}$ gives rise to a section of $\Upsilon^1_{\mathcal{S}^{\mathrm{an}}}$ in an open neighborhood of $\pi^{-1}\mathcal{E}^\heartsuit$, whose image is μD under the map $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{E}^\heartsuit/k)\to\mathrm{H}^2_{\mathcal{E}^{[1]},\mathrm{rig}}(\mathcal{E}/k)\simeq Z^1(\mathcal{E})^\heartsuit\otimes k$ by Proposition 4.6. Thus, (ii) is proved.

For Property (iii), when X has dimension 1, it follows from (the proof of) [Ber07, Theorem 4.3.1]. In general, it suffices to show that $(\Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)_1\subset\Psi_X$ by [Ber07, Theorem 4.5.1 (i)] and Theorem 5.11 (4). However, this follows from the definition of Ψ_X, Theorem 5.11 (5), and the case of curves. □

Proof of Theorem 1.4 in analytic topology. Now sheaves like $\mathcal{O}_X$, $\mathfrak{c}_X$, Υ^p_X, and the de Rham complex $(\Omega^\bullet_X,\mathrm{d})$ are understood in the analytic topology. The corresponding objects in the étale topology will be denoted by $\mathcal{O}_{X_{\text{ét}}}$, $\mathfrak{c}_{X_{\text{ét}}}$, $\Upsilon^p_{X_{\text{ét}}}$, and $(\Omega^\bullet_{X_{\text{ét}}},\mathrm{d})$.

[6]Although in [Fuj02], the author did not prove the semi-purity theorem for the rigid cohomology. But it is a formal argument from the purity theorem, which is known for the rigid cohomology.

Note that we have a canonical morphism $\nu\colon X_{\text{ét}} \to X$ of sites, and $\mathcal{O}_X = \nu_*\mathcal{O}_{X_{\text{ét}}}$, $\mathfrak{c}_X = \nu_*\mathfrak{c}_{X_{\text{ét}}}$, $\Omega^p_X = \nu_*\Omega^p_{X_{\text{ét}}}$ for every $p \geq 0$. We claim that the canonical map

$$\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X \to \nu_*(\Omega^{p,\mathrm{cl}}_{X_{\text{ét}}}/\mathrm{d}\Omega^{p-1}_{X_{\text{ét}}})$$

is an isomorphism. It will follow from:

(a) $\Omega^{p,\mathrm{cl}}_X = \nu_*\Omega^{p,\mathrm{cl}}_{X_{\text{ét}}}$ as subsheaves of Ω^p_X;

(b) $\mathrm{d}\Omega^{p-1}_X = \nu_*\mathrm{d}\Omega^{p-1}_{X_{\text{ét}}}$ as subsheaves of Ω^p_X; and

(c) $\mathrm{R}^1\nu_*\mathrm{d}\Omega^{p-1}_{X_{\text{ét}}} = 0$.

Assertion (a) is obvious. Both (b) and (c) will follow from the general fact that $\mathrm{R}^i\nu_*\mathscr{F} = 0$ for $i > 0$ and any sheaf of $\mathbf{Q}$-vector spaces $\mathscr{F}$ on $X_{\text{ét}}$. In fact for every $x \in X$, we have $(\mathrm{R}^i\nu_*\mathscr{F})_x = \mathrm{H}^i(\mathcal{H}(x), i_x^{-1}\mathscr{F})$, where $\mathcal{H}(x)$ is the completed residue field of x and $i_x\colon \mathcal{M}(\mathcal{H}(x)) \to X$ is the canonical morphism, and we know that the profinite Galois cohomology $\mathrm{H}^i(\mathcal{H}(x), i_x^{-1}\mathscr{F})$ is torsion hence trivial for $i > 0$.

Now for $w \in \mathbf{Z}$, we define $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_w = \nu_*(\Omega^{p,\mathrm{cl}}_{X_{\text{ét}}}/\mathrm{d}\Omega^{p-1}_{X_{\text{ét}}})_w$. Then we have a decomposition

$$\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X = \bigoplus_{w\in\mathbf{Z}}(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_w,$$

stable under base change and functorial in X and satisfying Property (i).

For Property (ii), we have the inclusion $\Upsilon^p_X \subset \nu_*\Upsilon^p_{X_{\text{ét}}}$ as subsheaves of $\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$, which is canonically isomorphic to $\nu_*(\Omega^{p,\mathrm{cl}}_{X_{\text{ét}}}/\mathrm{d}\Omega^{p-1}_{X_{\text{ét}}})$. Thus, we have the inclusion of sheaves $\Upsilon^p_X \subset (\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)_{2p}$. When $p = 1$, we have to show that $\nu_*\Upsilon^1_{X_{\text{ét}}} \subset \Upsilon^1_X$. We check this on the stalk at an arbitrary point $x \in X$. Take an element $[\omega]$ in $(\nu_*\Upsilon^1_{X_{\text{ét}}})_x$. We may assume that it has a representative $\omega \in \Omega^1_X(U)$ for some open neighborhood U of x that satisfies $\alpha^*\omega = \frac{\mathrm{d}f'}{f'} + \mathrm{d}g'$ for some finite étale surjective morphism $\alpha\colon U' \to U$ and $f' \in \mathcal{O}^*(U')$, $g' \in \mathcal{O}(U')$. Then we have $\omega = \deg(\alpha)^{-1}\frac{\mathrm{d}f}{f} + \mathrm{d}g$ where f (resp. g) is the multiplicative (resp. additive) trace of f' (resp. g') along α. □

6. Cohomological triviality before tropicalization

In this section, we study the sheaf Υ^p_X (in analytic topology) in more details. We show that it also has a canonical rational structure. Then we construct the so-called log-differential cycle class map and prove Theorem 6.6.

Definition 6.1. Let K be a non-Archimedean field of characteristic zero. Let X be a smooth K-analytic space. For every $p \geq 0$, we define a ($\mathbf{Q}$-linear) map

$$\lambda^p_X\colon \mathscr{K}^p_X \to \Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$$

of sheaves on X as follows. For a symbol $\{f_1, \dots, f_p\} \in \mathscr{K}^p_X(U)$ with $f_1, \dots, f_p \in \mathcal{O}^*_X(U)$, we put

$$\lambda^p_X(\{f_1, \dots, f_p\}) = \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_p}{f_p},$$

where the right-hand side is regarded as an element in $\Omega^{p,\mathrm{cl}}_X(U)$ and hence in $(\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X)(U)$. It is easy to see that λ^p_X factors through the relations of Milnor K-theories, and thus induces a map of corresponding sheaves. Finally, put

$$\mathscr{L}^p_X = \mathscr{K}^p_X/\ker\lambda^p_X.$$

It is canonically a subsheaf of $\Omega^{p,\mathrm{cl}}_X/\mathrm{d}\Omega^{p-1}_X$.

Proposition 6.2. *Let K be a non-Archimedean field embeddable into $\mathbf{C}_\mathbf{F}$, and X a smooth K-analytic space. Then the canonical map $\mathscr{L}^p_X\otimes_\mathbf{Q}\mathfrak{c}_X\to\Upsilon^p_X$ is an isomorphism of sheaves on X for every $p\geq 0$.*

Proof. By definition, it suffices to show that the map $\mathscr{L}^p_X\otimes_\mathbf{Q}\mathfrak{c}_X\to\Upsilon^p_X$ is injective on stalks. Thus we fix a point $x\in X$ with $s=s(x)$ and $t=t(x)$. Take an element

$$F=\sum_{l=1}^M b_l\lambda^p_X(F^l)\in\mathfrak{c}_X(U)\otimes_\mathbf{Q}\mathscr{L}^p_X(U)$$

such that $F=0$ in $\Upsilon^p_X(U)$, where U is a connected open neighborhood of x, and $b_l\in\mathfrak{c}_X(U)$, $F^l\in\mathscr{K}^p_X(U)$. It suffices to show that possibly after shrinking U, the elements $\lambda^p_X(F^l)$ are linearly dependent in $\Upsilon^p_X(U)$ over $\mathbf{Q}$.

Write $F^l=\sum_{\alpha=1}^{N_l}c^l_\alpha\{f^l_{\alpha1},\dots,f^l_{\alpha p}\}$ where $c^l_\alpha\in\mathbf{Q}$ and $f^l_{\alpha\beta}\in\mathscr{O}^*_X(U)$. We apply Lemma 6.3 to the finite collection $\{f^l_{\alpha\beta}\}$ and adopt the notation there. Then by Künneth decomposition, Example 4.3, and Proposition 4.6, we have for every subset $\Gamma\subset\{1,\dots,t\}$ with $|\Gamma|\leq p$ that

$$\sum_{l=1}^M b_l\sum_{\alpha=1}^{N_l}c^l_\alpha\sum_{\iota\colon\Gamma\hookrightarrow\{1,\dots,p\}}\epsilon(\iota)\prod_{\gamma\in\Gamma}d^l_{\alpha\iota(\gamma)\gamma}\mathrm{cl}^\heartsuit\left(\bigwedge_{\beta\notin\iota(\Gamma)}\mathrm{div}(g^l_{\alpha\beta})\right)\in\bigoplus_{I,|I|=p-|\Gamma|}\mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}^\heartsuit_I/L)\tag{6.1}$$

vanishes, for some finite extension of non-Archimedean fields $L/\mathfrak{c}_X(U)$. Here, $\epsilon(\iota)\in\{\pm1\}$ is certain sign determined by ι, and the multiple wedge product is taken in the increasing order of the relevant indices.

Note that $\mathrm{H}^0_{\mathrm{rig}}(\mathcal{E}^\heartsuit_I/L)$ is canonically isomorphic to $\mathbf{Q}^{\oplus\pi_0(\mathcal{E}^\heartsuit_I)}\otimes_\mathbf{Q}L$, and for every l,

$$\sum_{i=1}^{N_l}c^l_\alpha\sum_\iota\epsilon(\iota)\prod_{\gamma\in\Gamma}d^l_{\alpha\iota(\gamma)\gamma}\mathrm{cl}^\heartsuit\left(\bigwedge_{\beta\notin\iota(\Gamma)}\mathrm{div}(g^l_{\alpha\beta})\right)\in\bigoplus_{I,|I|=p-|\Gamma|}\mathbf{Q}^{\oplus\pi_0(\mathcal{E}^\heartsuit_I)}.$$

Thus, there exist $b'_l\in\mathbf{Q}$, not all being zero, such that (6.1) vanishes for every Γ if we replace b_l by b'_l.

This implies that there is an open neighborhood V' of y contained in V such that $\mu^*\lambda^p_X(F')$ is exact where $F'=\sum_{l=1}^M b'_l\lambda^p_X(F^l)$, by Lemma 4.5. By shrinking V', we may assume that the restricted morphism $\mu\colon V'\to U'$ is finite étale, where $U':=\mu(V')$ is an open neighborhood of x in U. Write $\mu^*\lambda^p_X(F')|_{V'}=\mathrm{d}\omega'$ for some $(p-1)$-form ω' on V'. Then $\lambda^p_X(F')|_{U'}=(\deg\mu)^{-1}\mathrm{d}\omega$, where ω is the trace of ω' along $\mu\colon V'\to U'$. The proposition follows. ☐

Lemma 6.3. *Let K be a non-Archimedean field embeddable into $\mathbf{C}_\mathbf{F}$, and X a smooth K-analytic space. Let $x\in X$ be a point with $s=s(x)$ and $t=t(x)$. For finitely many elements $f_\alpha\in\mathscr{O}^*_X(U)$ where U is an open neighborhood of x in X, there exist*

- *a proper strictly semistable scheme $\mathcal{S}$ over k° of dimension s such that every irreducible component of $\mathcal{S}^{[r]}_s$ is geometrically irreducible for every $r\geq 0$, where k is a finite extension of $\mathbf{Q}_\mathbf{F}$,*
- *an irreducible component $\mathcal{E}$ of $\mathcal{S}_s$,*
- *an open neighborhood W of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_kL$ in $\mathcal{S}^{\mathrm{an}}_L$ where L is a finite extension of K containing k, and $\mathcal{E}^\heartsuit=\mathcal{E}\backslash\mathcal{S}^{[1]}_s$,*
- *a closed subset $\mathcal{Z}$ of $\mathcal{S}_k$ of dimension at most $s-1$,*

- *a point* $y \in V := \mathbf{D}_L \times_L \prod_{\gamma=1}^{t} B(0; r_\gamma, R_\gamma)_L \times_L W$ *which projects to* $\sigma_{\mathcal{E}}$ *in* W*, where* $\mathbf{D}_L$ *is a poly-disc of dimension* s *over* L*, and* $\sigma_{\mathcal{E}} \in W$ *is the unique point whose reduction is the generic point of* $\mathcal{E}_{\widetilde{L}}$*,*
- *a morphism* $\mu\colon V \to U$ *that is étale away from* $\mathbf{D}_L \times_L \prod_{\gamma=1}^{t} B(0; r_\gamma, R_\gamma)_L \times_L (W \cap \mathcal{Z}_L^{\mathrm{an}})$*, such that* $\mu(y) = x$*,*
- *integers* $d_{\alpha 1}, \dots, d_{\alpha t}$ *and functions* $g_\alpha \in \mathcal{O}^*(W)$*, such that*
$$\mu^* \frac{\mathrm{d} f_\alpha}{f_\alpha} - \frac{\mathrm{d}\left(\nu^* g_\alpha \prod_{\gamma=1}^{t} T_\gamma^{d_{\alpha\gamma}}\right)}{\nu^* g_\alpha \prod_{\gamma=1}^{t} T_\gamma^{d_{\alpha\gamma}}}$$
is an exact 1*-form on* V *for every* α*. Here,* T_γ *is the coordinate function on* $B(0; r_\gamma, R_\gamma)_L$ *for* $1 \leq \gamma \leq t$*, which will be regarded as a function in* $\mathcal{O}^*(V)$ *via the obvious pullback; and* $\nu\colon V \to W$ *is the projection morphism.*

In particular, $|\mu^* f_\alpha \cdot (\nu^* g_\alpha \prod_{\gamma=1}^{t} T_\gamma^{d_{\alpha\gamma}})^{-1}|$ *is a constant on* V*.*

Proof. The existence of the data except for the last part follows from [Ber07, Propositions 2.1.1, 2.3.1]. The existence of integers $d_{\alpha 1}, \dots, d_{\alpha t}$ and functions $g_\alpha \in \mathcal{O}^*(W)$ (after possibly shrinking W) is fulfilled by (5.2), Example 4.3 and Theorem 1.4 (ii).

The last assertion is due to [Ber07, Theorem 4.3.1 (i)]. □

Definition 6.4 (Log-differential cycle class map)**.** Let K be a non-Archimedean field of characteristic zero and $\mathcal{X}$ a smooth scheme of finite type over K. We define the *log-differential cycle class map* $\mathrm{cl}_{\log}$ to be the composition

$$\mathrm{CH}^p(\mathcal{X})_{\mathbf{Q}} \xrightarrow{\mathrm{cl}_{\mathrm{univ}}} \mathrm{H}^p(\mathcal{X}, \mathscr{K}^p_{\mathcal{X}}) \to \mathrm{H}^p(\mathcal{X}^{\mathrm{an}}, \mathscr{K}^p_{\mathcal{X}^{\mathrm{an}}}) \xrightarrow{\mathrm{H}^p(\mathcal{X}^{\mathrm{an}}, \lambda^p_{\mathcal{X}^{\mathrm{an}}})} \mathrm{H}^p(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^p_{\mathcal{X}^{\mathrm{an}}}).$$

Suppose that $\mathcal{X}$ is a geometrically connected proper smooth scheme over K of dimension n. We have a pairing

$$\langle\ ,\ \rangle_{\mathcal{X}}\colon \mathrm{H}^p(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^p_{\mathcal{X}^{\mathrm{an}}}) \times \mathrm{H}^{n-p}(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^{n-p}_{\mathcal{X}^{\mathrm{an}}}) \to K \tag{6.2}$$

coming from the composite map

$$\mathrm{H}^p(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^p_{\mathcal{X}^{\mathrm{an}}}) \times \mathrm{H}^{n-p}(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^{n-p}_{\mathcal{X}^{\mathrm{an}}}) \to \mathrm{H}^n(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^n_{\mathcal{X}^{\mathrm{an}}}) \to \mathrm{H}^n(\mathcal{X}^{\mathrm{an}}, \Omega^n_{\mathcal{X}^{\mathrm{an}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{X}^{\mathrm{an}}}) \simeq K$$

in which the last isomorphism comes from the following lemma.

Lemma 6.5. *Let* K *be a non-Archimedean field of characteristic zero. Let* $\mathcal{X}$ *be a geometrically connected proper smooth scheme over* K *of dimension* n*. Then we have a canonical isomorphism* $\mathrm{H}^n(\mathcal{X}^{\mathrm{an}}, \Omega^n_{\mathcal{X}^{\mathrm{an}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{X}^{\mathrm{an}}}) \simeq K$*.*

Proof. Put $X = \mathcal{X}^{\mathrm{an}}$. By the spectral sequence $\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \Omega_X^{q,\mathrm{cl}}/\mathrm{d}\Omega_X^{q-1}) \Rightarrow \mathrm{H}^{p+q}(X, \Omega_X^\bullet)$ and the GAGA comparison isomorphism $\mathrm{H}^\bullet_{\mathrm{dR}}(X) \simeq \mathrm{H}^\bullet_{\mathrm{dR}}(\mathcal{X}/K)$, it suffices to show that $\mathrm{H}^i(X, \mathscr{F}) = 0$ for $i > n$ and every abelian sheaf $\mathscr{F}$ on X, which holds by [Ber93, Proposition 1.2.18]. Then we have canonical isomorphisms $\mathrm{H}^n(X, \Omega^n_X/\mathrm{d}\Omega^{n-1}_X) \simeq \mathrm{H}^{2n}_{\mathrm{dR}}(\mathcal{X}/K) \simeq K$. □

The following theorem is an analogous version of Theorem 1.8 for $\mathrm{cl}_{\log}$. We will eventually reduce Theorem 1.8 to this one.

Theorem 6.6. *Let* k *be a finite extension of* $\mathbf{Q}_{\mathbf{F}}$ *and* $\mathcal{X}$ *a geometrically connected proper smooth scheme over* k *of dimension* n*. Let* $\mathcal{Z}$ *be an algebraic cycle of* $\mathcal{X}$ *of codimension* p *such that* $\mathrm{cl}_{\mathrm{dR}}(\mathcal{Z}) = 0$*. Then*
$$\langle \mathrm{cl}_{\log}(\mathcal{Z}), \omega \rangle_{\mathcal{X}} = 0$$
for every $\omega \in \mathrm{H}^{n-p}(\mathcal{X}^{\mathrm{an}}, \mathscr{L}^{n-p}_{\mathcal{X}^{\mathrm{an}}})$*.*

Before the proof, we review some facts about cup products from [SP, 01FP]. Let X be a topological space, k a field, $n \geq 0$ an integer. Let Ω be a sheaf of k-vector spaces on X. Suppose that we have two bounded complexes $\mathscr{F}^\bullet, \mathscr{G}^\bullet$ of sheaves of k-vector spaces on X, with a map of complexes of sheaves of k-vector spaces

$$\chi\colon \operatorname{Tot}(\mathscr{F}^\bullet \otimes_k \mathscr{G}^\bullet) \to \Omega[n].$$

Then we have a bilinear pairing

$$\cup_\chi\colon \mathrm{H}^i(X, \mathscr{F}^\bullet) \times \mathrm{H}^{2n-i}(X, \mathscr{G}^\bullet) \to \mathrm{H}^{2n}(X, \Omega[n]) = \mathrm{H}^n(X, \Omega)$$

for every $i \in \mathbf{Z}$. Now suppose that we have four bounded complexes $\mathscr{F}_1^\bullet, \mathscr{F}_2^\bullet, \mathscr{G}_1^\bullet, \mathscr{G}_2^\bullet$ of sheaves of k-vector spaces on X, maps $\alpha_1\colon \mathscr{F}_1^\bullet \to \mathscr{F}_2^\bullet$, $\alpha_2\colon \mathscr{G}_2^\bullet \to \mathscr{G}_1^\bullet$, and

$$\chi_1\colon \operatorname{Tot}(\mathscr{F}_1^\bullet \otimes_k \mathscr{G}_1^\bullet) \to \Omega[n], \qquad \chi_2\colon \operatorname{Tot}(\mathscr{F}_2^\bullet \otimes_k \mathscr{G}_2^\bullet) \to \Omega[n],$$

such that $\chi_1 \circ (\mathrm{id}_{\mathscr{F}_1^\bullet} \otimes \alpha_2) = \chi_2 \circ (\alpha_1 \otimes \mathrm{id}_{\mathscr{G}_2^\bullet})$. Then we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^i(X, \mathscr{F}_1^\bullet) & \times & \mathrm{H}^{2n-i}(X, \mathscr{G}_1^\bullet) & \xrightarrow{\cup_{\chi_1}} & \mathrm{H}^n(X, \Omega) \\ \downarrow{\scriptstyle \mathrm{H}^i(X,\alpha_1)} & & \uparrow{\scriptstyle \mathrm{H}^{2n-i}(X,\alpha_2)} & & \| \\ \mathrm{H}^i(X, \mathscr{F}_2^\bullet) & \times & \mathrm{H}^{2n-i}(X, \mathscr{G}_2^\bullet) & \xrightarrow{\cup_{\chi_2}} & \mathrm{H}^n(X, \Omega) \end{array} \tag{6.3}$$

for every $i \in \mathbf{Z}$.

Proof of Theorem 6.6. Put $X = \mathcal{X}^{\mathrm{an}}$. Note that we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^p(X, \mathscr{T}_X^p) \times \mathrm{H}^{n-p}(X, \mathscr{T}_X^{n-p}) & \xrightarrow{\cup} & \mathrm{H}^n(X, \mathscr{T}_X^n) \\ \downarrow & & \downarrow \\ \mathrm{H}^p(X, \Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1}) \times \mathrm{H}^{n-p}(X, \Omega_X^{n-p,\mathrm{cl}}/\mathrm{d}\Omega_X^{n-p-1}) & \xrightarrow{\cup} & \mathrm{H}^n(X, \Omega_X^n/\mathrm{d}\Omega_X^{n-1}). \end{array}$$

The goal is to show that $\mathrm{cl}_{\log}(\mathcal{Z}) \cup \omega$ equals zero in $\mathrm{H}^n(X, \Omega_X^n/\mathrm{d}\Omega_X^{n-1})$. Denote by ζ the image of $\mathrm{cl}_{\log}(\mathcal{Z})$ in $\mathrm{H}^p(X, \Omega_X^{p,\mathrm{cl}}/\mathrm{d}\Omega_X^{p-1})$, and regard ω as in $\mathrm{H}^{n-p}(X, \Omega_X^{n-p,\mathrm{cl}}/\mathrm{d}\Omega_X^{n-p-1})$. We show $\zeta \cup \omega = 0$.

To explain the idea, we first give a "fake" proof. We have the conjugate spectral sequence $\mathrm{E}_r^{p,q}$ associated to the de Rham complex $(\Omega_X^\bullet, \mathrm{d})$, abutting to $\mathrm{H}^\bullet_{\mathrm{dR}}(X) = \mathrm{H}^\bullet(X, \Omega_X^\bullet)$ with

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \Omega_X^{q,\mathrm{cl}}/\mathrm{d}\Omega_X^{q-1}) = \bigoplus_{w=q}^{2q} \mathrm{H}^p(X, (\Omega_X^{q,\mathrm{cl}}/\mathrm{d}\Omega_X^{q-1})_w).$$

We make the following assumption: the differential map $\mathrm{d}_r^{p,q}$ of the spectral sequence preserves the above direct sum decomposition with respect to the weight w for $r \geq 2$. Then we have an induced map $\mathrm{H}^p(X, (\Omega_X^{q,\mathrm{cl}}/\mathrm{d}\Omega_X^{q-1})_{2q}) \to \mathrm{H}^{p+q}_{\mathrm{dR}}(X)$. It is not hard to see that the pairing (6.2) factors through the pairing $\mathrm{H}^{2p}_{\mathrm{dR}}(X) \times \mathrm{H}^{2n-2p}_{\mathrm{dR}}(X) \to K$ for de Rham cohomology. Therefore, the conclusion follows.

Unfortunately, we do not know whether the above assumption is true or not. Therefore, we need to consider a different de Rham complex in order to have certain Frobenius action on the entire complex so that the weight decomposition will be carried to the entire conjugate spectral sequence. The ad hoc de Rham complex uses log crystalline sites, which will be constructed through Steps 1–3 below. Steps 4–5 complete the remaining argument for the proof.

Step 1. To construct the ad hoc de Rham complex, we need the adic topology of X. By [Sch12, Theorem 2.24], we may associate to a K-analytic space X an adic space X^{ad}, and we have a canonical continuous map $\gamma_X\colon X^{\mathrm{ad}} \to X$ of topological spaces which makes X a maximal Hausdorff quotient of X^{ad}. Let $(\Omega^\bullet_{X^{\mathrm{ad}}}, \mathrm{d})$ be the de Rham complex on X^{ad}. Then we have a canonical map

$$\gamma_X^{-1}(\Omega^\bullet_X, \mathrm{d}) \to (\Omega^\bullet_{X^{\mathrm{ad}}}, \mathrm{d})$$

of complexes of sheaves of k-vector spaces on X^{ad}. Denote by ζ_{ad} (resp. ω_{ad}) the image of ζ (resp. ω) under the canonical map

$$\mathrm{H}^i(X, \Omega^{i,\mathrm{cl}}_X/\mathrm{d}\Omega^{i-1}_X) \to \mathrm{H}^i(X^{\mathrm{ad}}, \Omega^{i,\mathrm{cl}}_{X^{\mathrm{ad}}}/\mathrm{d}\Omega^{i-1}_{X^{\mathrm{ad}}})$$

for $i = p$ (resp. $i = n-p$). Note that when $i = n$, the above map is an isomorphism, by the same argument for Lemma 6.5.

We claim that there exists an alteration $f\colon \mathcal{X}' \to \mathcal{X}$ possibly after replacing k by a finite extension, such that $f^*\omega_{\mathrm{ad}}$ is in the image of the canonical map

$$\mathrm{H}^{2n-2p}(X'^{\mathrm{ad}}, \tau_{\leq n-p}\Omega^\bullet_{X'^{\mathrm{ad}}}) \to \mathrm{H}^{n-p}(X'^{\mathrm{ad}}, \Omega^{n-p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-p-1}_{X'^{\mathrm{ad}}}), \tag{6.4}$$

where $X' = \mathcal{X}'^{\mathrm{an}}$.

Assuming the above claim, we deduce the theorem as follows. Applying (6.3) to X'^{ad} and the sheaf $\Omega \coloneqq \Omega^n_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-1}_{X'^{\mathrm{ad}}}$, we obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^p(X'^{\mathrm{ad}}, \Omega^{p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{p-1}_{X'^{\mathrm{ad}}}) & \times & \mathrm{H}^{n-p}(X'^{\mathrm{ad}}, \Omega^{n-p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-p-1}_{X'^{\mathrm{ad}}}) & \longrightarrow & \mathrm{H}^n(X'^{\mathrm{ad}}, \Omega) \\
{\scriptstyle\alpha_1}\downarrow & & \uparrow{\scriptstyle\alpha_2} & & \| \\
\mathrm{H}^{2p}(X'^{\mathrm{ad}}, \tau_{\geq p}\Omega^\bullet_{X'^{\mathrm{ad}}}) & \times & \mathrm{H}^{2n-2p}(X'^{\mathrm{ad}}, \tau_{\leq n-p}\Omega^\bullet_{X'^{\mathrm{ad}}}) & \longrightarrow & \mathrm{H}^n(X'^{\mathrm{ad}}, \Omega) \\
{\scriptstyle\beta_1}\uparrow & & \downarrow{\scriptstyle\beta_2} & & \| \\
\mathrm{H}^{2p}(X'^{\mathrm{ad}}, \Omega^\bullet_{X'^{\mathrm{ad}}}) & \times & \mathrm{H}^{2n-2p}(X'^{\mathrm{ad}}, \Omega^\bullet_{X'^{\mathrm{ad}}}) & \longrightarrow & \mathrm{H}^n(X'^{\mathrm{ad}}, \Omega)
\end{array}$$

in which the maps among various complexes of sheaves are defined in the obvious way. By the above claim, there exists $\omega' \in \mathrm{H}^{2n-2p}(X'^{\mathrm{ad}}, \tau_{\leq n-p}\Omega^\bullet_{X'^{\mathrm{ad}}})$ such that $\alpha_2(\omega') = f^*\omega_{\mathrm{ad}}$. Thus,

$$f^*\zeta_{\mathrm{ad}} \cup f^*\omega_{\mathrm{ad}} = f^*\zeta_{\mathrm{ad}} \cup \alpha_2(\omega') = \alpha_1(f^*\zeta_{\mathrm{ad}}) \cup \omega'. \tag{6.5}$$

Note that we have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^p(\mathcal{X}')_{\mathbf{Q}} & \longrightarrow & \mathrm{H}^p(X', \Omega^{p,\mathrm{cl}}_{X'}/\mathrm{d}\Omega^{p-1}_{X'}) & \longrightarrow & \mathrm{H}^p(X'^{\mathrm{ad}}, \Omega^{p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{p-1}_{X'^{\mathrm{ad}}}) \\
{\scriptstyle\mathrm{cl}_{\mathrm{dR}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha_1} \\
\mathrm{H}^{2p}(\mathcal{X}', \Omega^\bullet_{\mathcal{X}'}) & \longrightarrow & \mathrm{H}^{2p}(X'^{\mathrm{ad}}, \Omega^\bullet_{X'^{\mathrm{ad}}}) & \xrightarrow{\beta_1} & \mathrm{H}^{2p}(X'^{\mathrm{ad}}, \tau_{\geq p}\Omega^\bullet_{X'^{\mathrm{ad}}})
\end{array}$$

by Remark 2.5. As $\mathrm{cl}_{\mathrm{dR}}(\mathcal{Z}) = 0$, we have $\mathrm{cl}_{\mathrm{dR}}(f^*\mathcal{Z}) = 0$. Thus $\alpha_1(f^*\zeta_{\mathrm{ad}}) = 0$ and $f^*\zeta \cup f^*\omega = 0$ by (6.5). However, as $f^*\colon \mathrm{H}^n(X^{\mathrm{ad}}, \Omega^n_{X^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-1}_{X^{\mathrm{ad}}}) \to \mathrm{H}^n(X'^{\mathrm{ad}}, \Omega^n_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-1}_{X'^{\mathrm{ad}}})$ is injective, we have $\zeta \cup \omega = 0$. The theorem is proved.

Step 2. Now we focus on the claim in Step 1. We first introduce some new sheaves on X. For a (proper flat) integral model $\mathcal{Y}$ of $\mathcal{X}$, define $\mathscr{K}^p_{X,\mathcal{Y}}$ to be the sheaf on $\mathcal{Y}_s$ associated to the presheaf

$$\mathcal{U} \mapsto \varinjlim_{\pi^{-1}\mathcal{U} \subset U} K^M_p(\mathcal{O}_X(U)) \otimes \mathbf{Q}, \quad \mathcal{U} \subset \mathcal{Y}_s$$

where the colimit is taken over all open neighborhoods U of $\pi^{-1}\mathcal{U}$ in X. Denote $\gamma_{\mathcal{Y}}\colon X^{\mathrm{ad}} \to \mathcal{Y}_s$ the induced continuous map. We have a canonical map $\mathscr{K}^p_{X,\mathcal{Y}} \to \gamma_{\mathcal{Y}*}\gamma_X^{-1}\mathscr{K}^p_X$ induced from the canonical map $\varinjlim_{\pi^{-1}\mathcal{U}\subset U} K^M_p(\mathcal{O}_X(U))\otimes\mathbf{Q} \to \varinjlim_{\pi^{-1}\mathcal{U}\subset U}\mathscr{K}^p_X(U)$. We claim that the induced map

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}\mathscr{K}^p_{X,\mathcal{Y}} \to \gamma_X^{-1}\mathscr{K}^p_X, \tag{6.6}$$

where the filtered colimit is taken over all integral models $\mathcal{Y}$ of X, is an isomorphism of sheaves on X^{ad}. In fact, denote by $\mathscr{K}^{\mathrm{pre},p}_X$ the presheaf $U\mapsto K^M_p(\mathcal{O}_X(U))\otimes\mathbf{Q}$ on X. Then $\mathscr{K}^p_{X,\mathcal{Y}}$ is simply $(\gamma_{\mathcal{Y}*}\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X)^+$, where $+$ denotes sheafification. Since sheafification commutes with pullback and taking colimit, we have

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}\mathscr{K}^p_{X,\mathcal{Y}} \xrightarrow{\sim} \varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}(\gamma_{\mathcal{Y}*}\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X)^+ \xrightarrow{\sim} \left(\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}\gamma_{\mathcal{Y}*}\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X\right)^+.$$

By [Sch12, Theorem 2.22], we have a cofiltered limit $X^{\mathrm{ad}}\simeq\varprojlim_{\mathcal{Y}}\mathcal{Y}_s$ induced by $\gamma_{\mathcal{Y}}$. Applying Lemma 6.7 below to the presheaf $\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X$ on X^{ad}, we have

$$\left(\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}\gamma_{\mathcal{Y}*}\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X\right)^+ \xrightarrow{\sim} (\gamma_X^{-1}\mathscr{K}^{\mathrm{pre},p}_X)^+ \xrightarrow{\sim} \gamma_X^{-1}(\mathscr{K}^{\mathrm{pre},p}_X)^+ = \gamma_X^{-1}\mathscr{K}^p_X.$$

Thus (6.6) is an isomorphism.

Put $\Omega^{\dagger,\bullet}_{X,\mathcal{Y}} = \gamma_{\mathcal{Y}*}\gamma_X^{-1}\Omega^\bullet_X$. Then we have a complex of sheaves of k-vector spaces $(\Omega^{\dagger,\bullet}_{X,\mathcal{Y}},\mathrm{d})$ on $\mathcal{Y}_s$. Again by Lemma 6.7, the canonical map

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}(\Omega^{\dagger,\bullet}_{X,\mathcal{Y}},\mathrm{d}) \to \gamma_X^{-1}(\Omega^\bullet_X,\mathrm{d}) \tag{6.7}$$

is an isomorphism.

For every $p\geq 0$, we have a canonical map

$$\lambda^p_{X,\mathcal{Y}}\colon \mathscr{K}^p_{X,\mathcal{Y}} \to \Omega^{\dagger,p,\mathrm{cl}}_{X,\mathcal{Y}}/\mathrm{d}\Omega^{\dagger,p-1}_{X,\mathcal{Y}}$$

similar to Definition 3.2. Denote by $\mathscr{L}^p_{X,\mathcal{Y}}$ the image sheaf of $\lambda^p_{X,\mathcal{Y}}$ in the above map. Passing to the quotient of isomorphisms (6.6) and (6.7), we obtain a canonical isomorphism

$$\varinjlim_{\mathcal{Y}} \gamma_{\mathcal{Y}}^{-1}\mathscr{L}^p_{X,\mathcal{Y}} \simeq \gamma_X^{-1}\mathscr{L}^p_X. \tag{6.8}$$

Step 3. By [Ber93, Proposition 1.3.6 (i)] and the fact that the topos on X^{ad} is equivalent to the G-topos on X under γ_X, we know that the canonical map

$$\mathrm{H}^{n-p}(X,\mathscr{L}^{n-p}_X) \to \mathrm{H}^{n-p}(X^{\mathrm{ad}},\gamma^{-1}\mathscr{L}^{n-p}_X)$$

is an isomorphism. On the other hand, by the isomorphism (6.8) and [SP, 0A37], the canonical map

$$\varinjlim_{\mathcal{Y}} \mathrm{H}^{n-p}(\mathcal{Y}_s,\mathscr{L}^{n-p}_{X,\mathcal{Y}}) \to \mathrm{H}^{n-p}(X^{\mathrm{ad}},\gamma^{-1}\mathscr{L}^{n-p}_X)$$

is an isomorphism. Together, we may assume that ω is in the image of the map

$$\mathrm{H}^{n-p}(\mathcal{Y}_s,\mathscr{L}^{n-p}_{X,\mathcal{Y}}) \to \mathrm{H}^{n-p}(X,\mathscr{L}^{n-p}_X).$$

for some integral model $\mathcal{Y}$. By [dJ96, Theorem 8.2], possibly after replacing k by a finite extension, we have a projective strictly semistable scheme $\mathcal{Y}'$ over k° with an alteration

$f\colon \mathcal{Y}' \to \mathcal{Y}$. Put $X' = \mathcal{Y}'^{\mathrm{an}}$. If we put $\Omega^\bullet_{X',\mathcal{Y}'} = \gamma_{\mathcal{Y}'*}\Omega^\bullet_{X'^{\mathrm{ad}}}$, then $(\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d})$ is a complex of sheaves of k-vector spaces on $\mathcal{Y}'_s$ and we have a canonical map

$$(\Omega^{\dagger,\bullet}_{X',\mathcal{Y}'}, \mathrm{d}) \to (\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d}). \tag{6.9}$$

Note that we have the exact sequence

$$0 \to \tau_{\leq n-p-1}\Omega^\bullet_{X'^{\mathrm{ad}}} \to \tau_{\leq n-p}\Omega^\bullet_{X'^{\mathrm{ad}}} \to \Omega^{n-p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-p-1}_{X'^{\mathrm{ad}}}[p-n] \to 0$$

of complexes of sheaves on X'^{ad}, which induces an short exact sequence

$$\mathrm{H}^{2n-2p}(X'^{\mathrm{ad}}, \tau_{\leq n-p}\Omega^\bullet_{X'^{\mathrm{ad}}}) \to \mathrm{H}^{n-p}(X'^{\mathrm{ad}}, \Omega^{n-p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-p-1}_{X'^{\mathrm{ad}}}) \to \mathrm{H}^{2n-2p+1}(X'^{\mathrm{ad}}, \tau_{\leq n-p-1}\Omega^\bullet_{X'^{\mathrm{ad}}}).$$

Thus ω is in the image of (6.4) if and only if ω maps to zero in $\mathrm{H}^{2n-2p+1}(X'^{\mathrm{ad}}, \tau_{\leq n-p-1}\Omega^\bullet_{X'^{\mathrm{ad}}})$. Since we have assumed that ω comes from $\mathrm{H}^{n-p}(\mathcal{Y}_s, \mathscr{L}^{n-p}_{X,\mathcal{Y}})$, it suffices to show that the composite map

$$\begin{aligned}\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) &\to \mathrm{H}^{n-p}(X', \mathscr{L}^{n-p}_{X'}) \\ &\to \mathrm{H}^{n-p}(X'^{\mathrm{ad}}, \Omega^{n-p,\mathrm{cl}}_{X'^{\mathrm{ad}}}/\mathrm{d}\Omega^{n-p-1}_{X'^{\mathrm{ad}}}) \to \mathrm{H}^{2n-2p+1}(X'^{\mathrm{ad}}, \tau_{\leq n-p-1}\Omega^\bullet_{X'^{\mathrm{ad}}})\end{aligned}$$

vanishes. However, the above composite map coincides with the following one

$$\begin{aligned}\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) &\to \mathrm{H}^{2n-2p+1}(\mathcal{Y}'_s, \tau_{\leq n-p-1}\Omega^{\dagger,\bullet}_{X',\mathcal{Y}'}) \\ &\to \mathrm{H}^{2n-2p+1}(\mathcal{Y}'_s, \tau_{\leq n-p-1}\Omega^\bullet_{X',\mathcal{Y}'}) \to \mathrm{H}^{2n-2p+1}(X'^{\mathrm{ad}}, \tau_{\leq n-p-1}\Omega^\bullet_{X'^{\mathrm{ad}}}).\end{aligned}$$

Therefore, the claim in Step 1 is reduced to the vanishing of the map

$$\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) \to \mathrm{H}^{2n-2p+1}(\mathcal{Y}'_s, \tau_{\leq n-p-1}\Omega^\bullet_{X',\mathcal{Y}'}). \tag{6.10}$$

Step 4. The advantage of $(\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d})$ is that the *entire* complex admits a canonical Frobenius action as we explain as follows. Fix a uniformizer of k; let $\mathcal{Y}'^\times$ and $\mathrm{Spec}(k^\circ)^\times$ be the log schemes equipped with the canonical log structure as in [HK94, (2.13.2)]. Then the induced morphism $\mathcal{Y}'^\times \to \mathrm{Spec}(k^\circ)^\times$ is log smooth as $\mathcal{Y}'$ is strictly semistable over k°. Let $\mathcal{Y}'^\times_s \to \mathrm{Spec}(\widetilde{k})^\times$ be the induced morphism of log schemes equipped with the pullback log structure as in [HK94, (2.13.2)]. Finally, let $\mathrm{Spf}\, W(\widetilde{k})^\times$ be the formal log scheme whose log structure is the canonical lifting of $\mathrm{Spec}(\widetilde{k})^\times$ as in [HK94, Definition (3.1)]. Here, we use the Zariski topology in the construction of log schemes and log crystal sites instead of the étale one in [HK94]. There is a canonical morphism $u\colon (\mathcal{Y}'^\times_s/\,\mathrm{Spf}\, W(\widetilde{k})^\times)_{\text{log-cris}} \to \mathcal{Y}'_s$ of sites. By [HK94, (5.4) & Proposition (2.20)], we have a canonical isomorphism

$$\mathrm{R}u_*\mathcal{O}^{\text{log-cris}}_{\mathcal{Y}'^\times_s/\,\mathrm{Spf}\, W(\widetilde{k})^\times} \otimes_{W(\widetilde{k})} k \simeq (\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d})$$

in the derived category of abelian sheaves on $\mathcal{Y}'_s$, where $\mathcal{O}^{\text{log-cris}}_{\mathcal{Y}'^\times_s/\,\mathrm{Spf}\, W(\widetilde{k})^\times}$ denotes the structure sheaf in the log crystal site. Note that when applying [HK94, Proposition (2.20)], we use the trivial covering as $\mathcal{Y}'^\times \to \mathrm{Spec}(k^\circ)^\times$ is log smooth. Since $\mathcal{O}_{\mathcal{Y}'^\times_s/\,\mathrm{Spf}\, W(\widetilde{k})^\times}$ admits a Frobenius action over $\mathrm{Spec}\,\widetilde{k}$, we obtain a Frobenius action on the entire complex $(\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d})$ in the derived category.

For $w \in \mathbf{Z}$, denote by $(\Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'})_w$ the maximal subsheaf of $\Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'}$ generated by sections of generalized weight w. We claim that

(a) the image of the canonical map $\mathscr{K}^p_{X',\mathcal{Y}'} \to \mathscr{L}^p_{X',\mathcal{Y}'} \to \Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'}$ is contained in the subsheaf $(\Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'})_{2p}$ for every p;

(b) the image of the canonical map $\Omega^{\dagger,p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,p-1}_{X',\mathcal{Y}'} \to \Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'}$ is contained in the subsheaf $\bigoplus_{w=0}^{2p}(\Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'})_w$ for every p.

Then the triviality of the map (6.10) follows easily from an argument of spectral sequences. In fact, we have a map of spectral sequences ${}^\dagger\mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p,q}$ abutting to the canonical map $\mathrm{H}^\bullet(\mathcal{Y}'_s, \Omega^{\dagger,\bullet}_{X',\mathcal{Y}'}) \to \mathrm{H}^\bullet(\mathcal{Y}'_s, \Omega^{\bullet}_{X',\mathcal{Y}'})$ induced from (6.9), whose second page consists of canonical maps $\mathrm{H}^p(\mathcal{Y}'_s, \Omega^{\dagger,q,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,q-1}_{X',\mathcal{Y}'}) \to \mathrm{H}^p(\mathcal{Y}'_s, \Omega^{q,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{q-1}_{X',\mathcal{Y}'})$. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) & \longrightarrow & {}^\dagger\mathrm{E}_2^{n-p,n-p} & \longrightarrow & \mathrm{E}_2^{n-p,n-p} \\
 & & \downarrow {\scriptstyle {}^\dagger\mathrm{d}_2^{n-p,n-p}} & & \downarrow {\scriptstyle \mathrm{d}_2^{n-p,n-p}} \\
 & & {}^\dagger\mathrm{E}_2^{n-p+2,n-p-1} & \longrightarrow & \mathrm{E}_2^{n-p+2,n-p-1}.
\end{array}$$

By (a), (b) and the fact that $(\Omega^\bullet_{X',\mathcal{Y}'}, \mathrm{d})$ has a Frobenius action, the composite map

$$\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) \to \mathrm{E}_2^{n-p+2,n-p-1}$$

obtained from the above diagram factors through

$$(\mathrm{E}_2^{n-p,n-p})_{2n-2p} \to \bigoplus_{w=0}^{2n-2p-2} (\mathrm{E}_2^{n-p+2,n-p-1})_w,$$

hence must be zero. By a similar argument, we know that for every $r \geq 2$, the (inductively defined) map

$$\mathrm{H}^{n-p}(\mathcal{Y}'_s, \mathscr{L}^{n-p}_{X',\mathcal{Y}'}) \to \mathrm{E}_r^{n-p,n-p} \xrightarrow{\mathrm{d}_r^{n-p,n-p}} \mathrm{E}_r^{n-p+r,n-p-r+1}$$

vanishes. Therefore, (6.10) vanishes as a consequence of spectral sequence.

Step 5. The last step is devoted to the proof of the two claims (a) and (b) in Step 4. We remark that they are not formal consequences of Theorem 1.4.

By definition, $\Omega^{p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{p-1}_{X',\mathcal{Y}'}$ is the sheaf on $\mathcal{Y}'_s$ associated to the presheaf $\mathcal{U} \mapsto \mathrm{H}^p_{\mathrm{dR}}(\pi^{-1}\mathcal{U})$, and $\Omega^{\dagger,p,\mathrm{cl}}_{X',\mathcal{Y}'}/\mathrm{d}\Omega^{\dagger,p-1}_{X',\mathcal{Y}'}$ is the sheaf on $\mathcal{Y}'_s$ associated to the presheaf $\mathcal{U} \mapsto \mathrm{H}^p_{\mathrm{dR}}(X', \pi^{-1}\mathcal{U})$ by [Ber07, Lemma 5.2.1]. We check (a) and (b) on stalks and thus fix a point $x \in \mathcal{Y}'_s$.

To prove (a), it suffices to consider the case where $p = 1$. Let $\mathcal{U} \subset \mathcal{Y}'$ be an open affine neighborhood of x. Take $f \in \mathcal{O}^*(X', \pi^{-1}\mathcal{U}_s)$; we have to show that the image of $\frac{\mathrm{d}f}{f}$ in $\mathrm{H}^1_{\mathrm{dR}}(\pi^{-1}\mathcal{U}_s)$ is of generalized weight 2 for a possibly smaller open neighborhood $\mathcal{U}$ of x. First, by [Gub13, Proposition 7.2] and possibly shrinking $\mathcal{U}$, we may replace f by the restriction of an (algebraic) function $f \in \mathcal{O}(\mathcal{U}_k)$ without changing $|f|$, hence the image of $\frac{\mathrm{d}f}{f}$ in $\mathrm{H}^1_{\mathrm{dR}}(X', \pi^{-1}\mathcal{U}_s)$ by [Ber07, Theorem 4.2.1], and the image of $\frac{\mathrm{d}f}{f}$ in $\mathrm{H}^1_{\mathrm{dR}}(\pi^{-1}\mathcal{U}_s)$. Since $\mathcal{Y}'$ is projective, we may choose a closed embedding $\mathcal{Y}' \hookrightarrow \mathbf{P}^N_{k^\circ}$ into a projective space. Choose an open affine neighborhood $\mathcal{V}$ of x in $\mathbf{P}^N_{k^\circ}$ such that $\mathcal{V} \cap \mathcal{Y}' \subset \mathcal{U}$ and $f|_{\mathcal{V}\cap\mathcal{Y}'} = g|_{\mathcal{V}\cap\mathcal{Y}'}$ for some $g \in \mathcal{O}^*(\mathcal{V}_k)$. Since $\mathbf{P}^N_{k^\circ}$ is smooth, by Lemma 5.6, $\frac{\mathrm{d}g}{g}$ belongs to $\mathrm{H}^1_{\mathrm{rig}}(\mathcal{V}_s/k)_2$ and thus its image in $\mathrm{H}^1_{\mathrm{dR}}(\pi^{-1}\mathcal{V}_s)$ is of generalized weight 2. By functoriality of log crystal sites for the morphism $\mathcal{Y}' \to \mathbf{P}^N_{k^\circ}$, we conclude that the image of $\frac{\mathrm{d}f}{f}$ in $\mathrm{H}^1_{\mathrm{dR}}(\pi^{-1}(\mathcal{V}\cap\mathcal{Y}')_s)$ is of generalized weight 2. Here, $\pi^{-1}(\mathcal{V}\cap\mathcal{Y}')_s$ is the inverse image in X'.

Claim (b) is a consequence of [GK05, Theorem 0.1] and [Chi98, Theorem 2.3]. In fact, we have a functorial map of spectral sequences ${}'\mathrm{E}_r^{p,q} \to {}''\mathrm{E}_r^{p,q}$ abutting to $\mathrm{H}^\bullet_{\mathrm{dR}}(X', \pi^{-1}\mathcal{U}) \to$

$\mathrm{H}^\bullet_{\mathrm{dR}}(\pi^{-1}\mathcal{U})$ with the first page being

$$'\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\mathrm{rig}}(\mathcal{U}_s^{(p)}/\operatorname{Spf} W(\widetilde{k})^\times) \otimes_{W(\widetilde{k})} k \to {''\mathrm{E}}_1^{p,q} = \mathrm{H}^q_{\text{log-cris}}(\mathcal{U}_s^{(p)}/\operatorname{Spf} W(\widetilde{k})^\times) \otimes_{W(\widetilde{k})} k,$$

where $\mathcal{U}_s^{(p)}$ is the disjoint union of irreducible components of $\mathcal{U}_s^{[p]}$, equipped with the induced log structure from $\mathcal{Y}_s'^\times$. By [GK05, Theorem 3.1, Lemma 4.6] and [Chi98, Theorem 2.3], we know that the weights of (the finite dimensional k-vector space) $'\mathrm{E}_1^{p,q}$ are in the range $[p, 2p]$, and thus the weights of $\mathrm{H}^p_{\mathrm{dR}}(X', \pi^{-1}\mathcal{U})$ are in the range $[0, 2p]$. □

Lemma 6.7. *Let $\mathcal{I}$ be a cofiltered category. Let $i \mapsto X_i$ be a diagram of topological spaces over $\mathcal{I}$. Let $X = \varprojlim_i X_i$ be the limit with projection maps $\gamma_i\colon X \to X_i$. Let $\mathscr{P}$ be a presheaf on X. Then the canonical map*

$$\left(\varinjlim_i \gamma_i^{-1}\gamma_{i*}\mathscr{P}\right)^+ \to \mathscr{P}^+$$

is an isomorphism. Here, we regard γ_i^{-1} and γ_{i*} as pullback and pushforward of presheaves respectively, and $+$ denotes sheafification.

Proof. Consider the collection $\mathcal{B}$ of open subsets of X of the form $\gamma_i^{-1}U_i$ for $i \in \mathcal{I}$ and an open subset U_i of X_i. Then $\mathcal{B}$ is a basis of the topology on X. It suffices to show that for every $U \in \mathcal{B}$, the canonical map

$$\left(\varinjlim_i \gamma_i^{-1}\gamma_{i*}\mathscr{P}\right)(U) \to \mathscr{P}(U)$$

is an isomorphism. This is obvious as $\mathcal{I}$ is cofiltered. □

7. Cohomological triviality for tropical cycle classes

In this section, we study tropical cycle class maps and prove Theorem 1.8, by first establishing the relation of maps τ_X^p and λ_X^p in the following result.

Theorem 7.1. *Let K be a non-Archimedean field embeddable into $\mathbf{C}_\mathbf{F}$, and X a smooth K-analytic space. Then $\ker \tau_X^p = \ker \lambda_X^p$. In other words, we have a canonical isomorphism*

$$\mathscr{T}_X^p \simeq \mathscr{L}_X^p$$

of sheaves on X for every $p \geq 0$.

Proof. It suffices to check the equality on stalks. Thus we fix a point $x \in X$ with $s = s(x)$ and $t = t(x)$.

Let U be an open neighborhood of x. Take an element

$$F = \sum_{\alpha=1}^N c_\alpha\{f_{\alpha 1}, \dots, f_{\alpha p}\} \in \mathscr{K}_X^p(U)$$

where $c_\alpha \in \mathbf{Q}$ and $f_{\alpha\beta} \in \mathcal{O}_X^*(U)$. We apply Lemma 6.3 to the finite collection $\{f_{\alpha\beta}\}$ and adopt the notation there. In particular, $|\mu^* f_{\alpha\beta} \cdot (\nu^* g_{\alpha\beta} \prod_{\gamma=1}^t T_\gamma^{d_{\alpha\beta\gamma}})^{-1}|$ is a positive constant on V, which we denote by $c_{\alpha\beta}$.

Define three tropical charts as follows.

- The first one uses $f_{\alpha\beta}$ $(1 \leq \alpha \leq N, 1 \leq \beta \leq p)$, which induce a moment morphism $U \to (\mathbf{G}_{m,K}^{\mathrm{an}})^{Np}$, and thus a tropicalization map
$$\mathrm{trop}_U \colon U \to (\mathbf{G}_{m,K}^{\mathrm{an}})^{Np} \xrightarrow{-\log|\,|} \mathbf{R}^{Np}.$$
- The second one uses functions $g_{\alpha\beta}$ $(1 \leq \alpha \leq N, 1 \leq \beta \leq p)$, which induce a moment morphism $W \to (\mathbf{G}_{m,L}^{\mathrm{an}})^{Np}$, and thus a tropicalization map
$$\mathrm{trop}_W \colon W \to (\mathbf{G}_{m,L}^{\mathrm{an}})^{Np} \xrightarrow{-\log|\,|} \mathbf{R}^{Np}.$$
- The third one uses functions $\nu^* g_{\alpha\beta}$ $(1 \leq \alpha \leq N, 1 \leq \beta \leq p)$ and T_γ $(1 \leq \gamma \leq t)$, which induce a moment morphism $V \to (\mathbf{G}_{m,L}^{\mathrm{an}})^{Np+t}$, and thus a tropicalization map
$$\mathrm{trop}_V \colon V \to (\mathbf{G}_{m,L}^{\mathrm{an}})^{Np+t} \xrightarrow{-\log|\,|} \mathbf{R}^{Np+t}.$$

We have a commutative diagram
$$\begin{array}{ccc} W & \xrightarrow{\mathrm{trop}_W} & \mathbf{R}^{Np} \\ {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\breve{\nu}} \\ V & \xrightarrow{\mathrm{trop}_V} & \mathbf{R}^{Np+t} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\breve{\mu}} \\ U & \xrightarrow{\mathrm{trop}_U} & \mathbf{R}^{Np} \end{array}$$
in which $\breve{\mu}$ sends a point $(x_{\alpha\beta}, x_\gamma) \in \mathbf{R}^{Np+t} = \mathbf{R}^{Np} \times \mathbf{R}^t$ to $(y_{\alpha\beta})$ where
$$y_{\alpha\beta} = -\log c_{\alpha\beta} + x_{\alpha\beta} + \sum_{\gamma=1}^{t} d_{\alpha\beta\gamma} x_\gamma,$$
and $\breve{\nu}$ is the projection onto the first Np factors. Note that
$$\tau_X^p(F) = \sum_{\alpha=1}^{N} c_\alpha \bigwedge_{\beta=1}^{p} \mathrm{d}' y_{\alpha\beta},$$
and thus
$$\breve{\mu}^* \tau_X^p(F) = \sum_{\alpha=1}^{N} c_\alpha \bigwedge_{\beta=1}^{p} \left(\mathrm{d}' x_{\alpha\beta} + \sum_{\gamma=1}^{t} d_{\alpha\beta\gamma} \mathrm{d}' x_\gamma \right)$$
as a $(q, 0)$-superform on $\mathbf{R}^{Np+t}$.

It is elementary to see that
$$\mu^* \lambda_X^p(F) = \sum_{\Gamma \subset \{1,\dots,t\}, |\Gamma| \leq p} \nu^* \omega_\Gamma \wedge \left(\bigwedge_{\gamma \in \Gamma} \frac{\mathrm{d} T_\gamma}{T_\gamma} \right),$$
where
$$\omega_\Gamma = \sum_{\alpha=1}^{N} c_\alpha \sum_{\iota \colon \Gamma \hookrightarrow \{1,\dots,q\}} \epsilon(\iota) \prod_{\gamma \in \Gamma} d_{\alpha\iota(\gamma)\gamma} \left(\bigwedge_{\beta \notin \iota(\Gamma)} \frac{\mathrm{d} g_{\alpha\beta}}{g_{\alpha\beta}} \right).$$
Here, $\epsilon(\iota) \in \{\pm 1\}$ is certain sign determined by ι, and all multiple wedge products are taken in the increasing order of the relevant indices.

On the tropical side, we have

$$\breve{\mu}^*\tau_X^p(F) = \sum_{\Gamma\subset\{1,\dots,t\},|\Gamma|\leq p} \breve{\nu}^*\breve{\omega}_\Gamma \wedge \left(\bigwedge_{\gamma\in\Gamma} \mathrm{d}'x_\gamma\right),$$

where

$$\breve{\omega}_\Gamma = \sum_{\alpha=1}^{N} c_\alpha \sum_{\iota\colon \Gamma\hookrightarrow\{1,\dots,q\}} \epsilon(\iota)\prod_{\gamma\in\Gamma} d_{\alpha\iota(\gamma)\gamma}\left(\bigwedge_{\beta\notin\iota(\Gamma)} \mathrm{d}'x_{\alpha\beta}\right).$$

We show $(\ker\lambda_X^p)_x \subset (\ker\tau_X^p)_x$. We assume that $\lambda_X^p(F)$ is an exact form on U, and we need to show that $\tau_X^p(F) = 0$ on a possibly smaller open neighborhood of x. Since $\mu^*\lambda_X^p(F)$ is exact, by Example 4.3, we know that ω_Γ induces the zero class in $\mathrm{H}_{\mathrm{dR}}^{q-|\Gamma|}(W,(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k L)$ for every Γ. Therefore by Lemma 7.2 below, there exists a compact neighborhood W' of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k L$ contained in W such that $\breve{\omega}_\Gamma = 0$ on $\mathrm{trop}_W(W')$ for every Γ. Therefore, $\mu^*\tau_X^p(F) = 0$ on $\mu^{-1}W'$, and we are done since μ is an affine surjection.

We show $(\ker\tau_X^p)_x \subset (\ker\lambda_X^p)_x$. We assume that $\tau_X^p(F) = 0$ on U. Then by the similar argument as above, we may conclude that there is an open neighborhood V' of y contained in V such that $\mu^*\lambda_X^p(F)$ is exact. By shrinking V', we may assume that the restricted morphism $\mu\colon V' \to U'$ is finite étale, where $U' \coloneqq \mu(V')$ is an open neighborhood of x in U. Write $\mu^*\lambda_X^p(F)|_{V'} = \mathrm{d}\omega'$ for some $(p-1)$-form ω' on V'. Then $\lambda_X^p(F)|_{U'} = (\deg\mu)^{-1}\mathrm{d}\omega$, where ω is the trace of ω' along $\mu\colon V' \to U'$. So $\lambda_X^p(F)$ is zero in the stalk at x.

The theorem is proved. □

Lemma 7.2. *Let k be a finite extension of $\mathbf{Q}_\mathbf{F}$ and K a non-Archimedean field containing k. Suppose that we have*

- *a proper strictly semistable scheme $\mathcal{S}$ over k° of dimension s such that every irreducible component of $\mathcal{S}_s^{[r]}$ is geometrically irreducible for every $r \geq 0$,*
- *an irreducible component $\mathcal{E}$ of $\mathcal{S}_s$,*
- *an open neighborhood W of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K$ in $\mathcal{S}_K^{\mathrm{an}}$ where $\mathcal{E}^\heartsuit = \mathcal{E}\backslash\mathcal{S}_s^{[1]}$,*
- *invertible functions $g_1,\dots,g_h \in \mathcal{O}^*(W,(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K)$, which induce the tropicalization map $\mathrm{trop}_W\colon W \to \mathbf{R}^h$,*
- *$c_1,\dots,c_N \in \mathbf{Q}$ and $\beta_1,\dots,\beta_N\colon \{1,\dots,p\} \to \{1,\dots,h\}$.*

Put

$$\omega = \sum_{\alpha=1}^{N} c_\alpha \left(\frac{\mathrm{d}g_{\beta_\alpha(1)}}{g_{\beta_\alpha(1)}} \wedge\cdots\wedge \frac{\mathrm{d}g_{\beta_\alpha(p)}}{g_{\beta_\alpha(p)}}\right)$$

and

$$\breve{\omega} = \sum_{\alpha=1}^{N} c_\alpha \left(\mathrm{d}'x_{\beta_\alpha(1)} \wedge\cdots\wedge \mathrm{d}'x_{\beta_\alpha(p)}\right),$$

where $x_1,\dots,x_h$ are the natural coordinates on $\mathbf{R}^h$.

Then ω induces the zero class in $\mathrm{H}_{\mathrm{dR}}^p(W,(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K)$ if and only if $\breve{\omega} = 0$ on $\mathrm{trop}_W(W')$ for some compact neighborhood W' of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K$ contained in W.

Proof. We adopt the notation in §4. Let $e \in \mathcal{E}$ be a point. There is a unique subset $I(e)$ of $\{1,\dots,M\}$ such that $e \in \mathcal{E}_{I(e)}^\heartsuit$. We choose an affine open neighborhood $\mathcal{S}_e$ of e together with functions $f_{e,0}$ and $f_{e,i}$ in $\mathcal{O}(\mathcal{S}_e)$ for $i \in I(e)$ such that $\mathcal{S}_e \cap \mathcal{E}_I$ is defined by the ideal $(f_{e,0}, f_{e,i} \mid i \in I)$ for every subset $I \subset I(e)$. We may further assume that $\mathcal{S}_e \cap \mathcal{E}_I$ is irreducible for every I. Put $\mathcal{E}_e = \mathcal{S}_e \cap \mathcal{E}$ and $\mathcal{E}_{e,i} = \mathcal{S}_e \cap \mathcal{E}_i$. For $1 \leq \beta \leq h$, suppose

$\mathrm{div}(g_\beta)|_{\mathcal{E}_e} = \sum_{i\in I(e)} a_{\beta,e,i}\mathcal{E}_{e,i}$ for $a_{\beta,e,i}\in\mathbf{Z}$. Then $|g_\beta\cdot\prod_{i\in I(e)} f_{e,i}^{-a_{\beta,e,i}}|$ is equal to a constant $c_{\beta,e}\in\mathbf{R}_{>0}$ on some compact neighborhood W_e (for all β) of $\pi^{-1}(\mathcal{S}_e\cap\mathcal{E}^\heartsuit)_{\widetilde{K}}$ in $W\cap\pi^{-1}(\mathcal{E}_e)_{\widetilde{K}}$.

Put

$$Z=\sum_{\alpha=1}^{N} c_\alpha\left(\mathrm{div}(g_{\beta_\alpha(1)})\wedge\cdots\wedge\mathrm{div}(g_{\beta_\alpha(p)})\right).$$

Then the restriction of D to $\mathcal{E}_e$ is equal to

$$Z_e\coloneqq\sum_{\{i_1<\cdots<i_p\}\subset I(e)} b_{e,i_1<\cdots<i_p}\left(\mathrm{div}(f_{e,i_1})\wedge\cdots\wedge\mathrm{div}(f_{e,i_p})\right),$$

where $\beta_{e,i_1<\cdots<i_p}\in\mathbf{Q}$ is uniquely determined by c_α and $a_{\beta,e,i}$ via an explicit formula which we omit.

On the other hand, on W_e, the tropicalization map trop_W factors through another one

$$\mathrm{trop}_{W_e}\colon W_e\to\mathbf{R}^{I(e)}$$

induced by functions f_i for $i\in I(e)$. In other words, we have an affine map $\nu_e\colon\mathbf{R}^{I(e)}\to\mathbf{R}^h$ determined by $a_{\beta,e,i}$ and $c_{\beta,e}$, whose explicit formula is left to the reader, such that $\mathrm{trop}_W|_{W_e}=\nu_e\circ\mathrm{trop}_{W_e}$. Moreover, it is straightforward to check that we have

$$\nu_e^*\breve{\omega}=\sum_{\{i_1<\cdots<i_p\}\subset I(e)} b_{e,i_1<\cdots<i_p}\left(\mathrm{d}'x_{e,i_1}\wedge\cdots\wedge\mathrm{d}'x_{e,i_p}\right)$$

where $x_{e,i}$ for $i\in I(e)$ are the natural coordinates on $\mathbf{R}^{I(e)}$.

We first consider the "only if" direction: Assume that ω induces the zero class in $\mathrm{H}^p_{\mathrm{dR}}(W,(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K)$. By Proposition 4.6, we have $Z=0$. Thus $Z_e=0$ for every $e\in\mathcal{E}$. As

$$\mathrm{div}(f_{e,i_1})\wedge\cdots\wedge\mathrm{div}(f_{e,i_p})=\mathcal{S}_e\cap\mathcal{E}_{\{i_1,\dots,i_p\}}\neq 0,$$

we have $b_{e,i_1<\cdots<i_p}=0$ for all subsets $\{i_1<\cdots<i_p\}\subset I(e)$. Therefore, $\nu_e^*\breve{\omega}=0$. Now we take a finite subset $E\subset\mathcal{E}$ such that $\{\mathcal{E}_e|e\in E\}$ form a covering of $\mathcal{E}$. Then $W'\coloneqq\bigcup_{e\in E}W_e$ is a compact neighborhood of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K$ contained in W, and moreover $\breve{\omega}=0$ on $\mathrm{trop}_W(W')$.

We then consider the "if" direction: Assume $\breve{\omega}=0$ on $\mathrm{trop}_W(W')$ for some compact neighborhood W' of $(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K$ contained in W. Then for every $e\in\mathcal{E}$, the image $\mathrm{trop}_{W_e}(W_e\cap W')$ always contains a p-dimensional cell that is parallel to the plane spanned by $\{x_{e,i_1},\dots,x_{e,i_p}\}$ for every subset $\{i_1<\cdots<i_p\}\subset I(e)$. In particular, we must have $b_{e,i_1<\cdots<i_p}=0$ for all subsets $\{i_1<\cdots<i_p\}\subset I(e)$. This implies that $Z=0$. By Proposition 4.6, Lemma 4.5, and Theorem 1.4 (ii), we know that ω induces the zero class in $\mathrm{H}^p_{\mathrm{dR}}(W,(\pi^{-1}\mathcal{E}^\heartsuit)\widehat{\otimes}_k K)$.

The lemma is proved. □

The following theorem shows the finiteness of $\mathrm{H}^{1,1}_{\mathrm{trop}}$ and studies the tropical cycle class of line bundles.

Theorem 7.3. *Let $\mathcal{X}$ be a proper smooth scheme over $\mathbf{C}_\mathbf{F}$. Then*

(1) $\mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X})$ is of finite dimension.
(2) $\dim\mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X})=1$ if $\mathcal{X}$ is an irreducible curve.
(3) For a line bundle $\mathcal{L}$ on $\mathcal{X}$ whose (algebraic) de Rham Chern class $\mathrm{cl}_{\mathrm{dR}}(\mathcal{L})\in\mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/\mathbf{C}_\mathbf{F})$ is trivial, we have $\mathrm{cl}_{\mathrm{trop}}(\mathcal{L})=0$.

Proof. We put $X = \mathcal{X}^{\mathrm{an}}$. By Theorem 7.1, we have a canonical isomorphism $\mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X}) \simeq \mathrm{H}^1(X, \mathscr{L}^1_X)$. By Proposition 6.2 and Theorem 1.4, we have

$$\mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X) = \mathrm{H}^1(X, \Psi_X) \oplus \mathrm{H}^1(X, \mathscr{L}^1_X) \otimes_{\mathbf{Q}} \mathbf{C_F}.$$

For (1), it suffices to show that $\dim \mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X) < \infty$. In fact, we have a spectral sequence $\mathrm{E}^{p,q}_r$ abutting to $\mathrm{H}^\bullet_{\mathrm{dR}}(X) = \mathrm{H}^\bullet(X, \Omega^\bullet_X)$ with the second page terms $\mathrm{E}^{p,q}_2 = \mathrm{H}^p(X, \Omega^{q,\mathrm{cl}}_X/\mathrm{d}\Omega^{q-1}_X)$. Thus, it suffices to show that both $\mathrm{H}^3(X, \mathbf{C_F})$ and $\mathrm{H}^2(X, \Omega^\bullet_X)$ are finite dimensional. Since the homotopy type of X is a finite CW complex by [HL16], $\dim \mathrm{H}^i(X, \mathbf{C_F}) < \infty$ for every $i \in \mathbf{Z}$. By GAGA, $\mathrm{H}^i(X, \Omega^\bullet_X)$ is canonically isomorphic to the algebraic de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathbf{C_F})$ for every i, and thus finite dimensional.

For (2), we know that $\dim \mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X) = 1$ from the discussion for (1). Thus $\dim \mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X}) \leq 1$. However, from Corollary 3.11, we have $\dim \mathrm{H}^{1,1}_{\mathrm{trop}}(\mathcal{X}) \geq 1$. Thus (2) follows.

For (3), by Theorem 7.1, it suffices to show that $\mathrm{cl}_{\log}(\mathcal{L})$ is zero in $\mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)$. Note that the map $\mathrm{CH}^1(\mathcal{X})_{\mathbf{Q}} \to \mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)$ factors as

$$\mathrm{CH}^1(\mathcal{X})_{\mathbf{Q}} \to \mathrm{H}^1(\mathcal{X}, \Omega^{1,\mathrm{cl}}_{\mathcal{X}}/\mathrm{d}\mathcal{O}_{\mathcal{X}}) \to \mathrm{H}^1(X, \Omega^{1,\mathrm{cl}}_X/\mathrm{d}\mathcal{O}_X)$$

in which the first map factors through the algebraic equivalence (up to torsion) by [BO74, (0.5)]. Thus, $\mathrm{cl}_{\log}(\mathcal{L}) = 0$ since $\mathcal{L}$ has the trivial de Rham Chern class. □

Remark 7.4. Suppose that $\mathcal{X}$ is an irreducible proper smooth curve over $\mathbf{C_F}$. Then we also have $\dim \mathrm{H}^{1,0}_{\mathrm{trop}}(\mathcal{X}) = \dim \mathrm{H}^{0,1}_{\mathrm{trop}}(\mathcal{X}) = b^1(\mathcal{X}^{\mathrm{an}})$. In fact, by the argument for Theorem 7.3, we have $\dim \mathrm{H}^{1,0}_{\mathrm{trop}}(\mathcal{X}) = \dim \mathrm{H}^0(\mathcal{X}^{\mathrm{an}}, \Upsilon^1_{\mathcal{X}^{\mathrm{an}}}) = b^1(\mathcal{X}^{\mathrm{an}})$ by [Ber07, Corollary 4.3.5 (iii)].

From now on, we fix an embedding $\mathbf{R} \hookrightarrow \mathbf{C_F}$.

Definition 7.5 (Tropical trace map)**.** Let X be a proper smooth $\mathbf{C_F}$-analytic space of dimension n. We have a total integration map

$$\int_X \colon \mathrm{H}^n(X, \mathscr{T}^n_X) \to \mathrm{H}^{n,n}(X) \to \mathbf{R}.$$

By the isomorphism $\mathscr{T}^n_X \simeq \mathscr{L}^n_X$ in Theorem 7.1, Lemma 6.2, and by extending the above map linearly over $\mathbf{C_F}$, we obtain a $\mathbf{C_F}$-linear map

$$\mathrm{Tr}^{\mathrm{trop}}_X \colon \mathrm{H}^n(X, \Upsilon^n_X) \simeq \mathrm{H}^n(X, \mathscr{L}^n_X) \otimes_{\mathbf{Q}} \mathbf{C_F} \to \mathbf{C_F},$$

called the *tropical trace map* for X.

Remark 7.6. We expect that $\dim \mathrm{H}^n(X, \mathscr{T}^n_X) = 1$ in the setup of Definition 7.5. Then it is an interesting question to ask the value of $\int_X$ on $\mathrm{H}^n(X, \mathscr{T}^n_X)$. Suppose that $X = \mathcal{X}^{\mathrm{an}}_{\mathrm{a}}$ for $\mathcal{X}$ in Proposition 7.7, then we know (assuming $\dim \mathrm{H}^n(X, \mathscr{T}^n_X) = 1$) that the restriction of $\int_X$ to $\mathrm{H}^n(X, \mathscr{T}^n_X)$ takes values in $\mathbf{Q}$ by Corollary 3.10. Therefore, the tropical trace map $\mathrm{Tr}^{\mathrm{trop}}_X$ does not depend on the choice of the embedding $\mathbf{R} \hookrightarrow \mathbf{C_F}$, and factors through the canonical isomorphism $\mathrm{H}^n(X, (\Omega^n_X/\mathrm{d}\Omega^{n-1}_X)_{2n}) \cong \mathbf{C_F}$ in Lemma 6.5.

The following proposition can be regarded as certain algebraicity property of the tropical trace map, which is defined via transcendental way.

Proposition 7.7. *Let $k \subset \mathbf{C_F}$ be a discrete non-Archimedean subfield, and $\mathcal{X}$ a geometrically connected proper smooth scheme over k of dimension n. If we put $\mathcal{X}_{\mathrm{a}} = \mathcal{X} \otimes_k \mathbf{C_F}$, then the* *tropical trace map* $\mathrm{Tr}^{\mathrm{trop}}_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}$ *factors through the canonical map*

$$\mathrm{H}^n(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}, \Upsilon^n_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}) \to \mathrm{H}^n(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}, (\Omega^n_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}})_{2n}). \tag{7.1}$$

In particular, $\mathrm{H}^n(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}, (\Omega^n_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}})_w) \simeq \mathbf{C}_{\mathbf{F}}$ *if* $w = 2n$*, and is trivial otherwise.*

Proof. Put $X = \mathcal{X}^{\mathrm{an}}_{\mathrm{a}}$. The last assertion follows from the previous one and the following facts:

- $\mathrm{Tr}^{\mathrm{trop}}_X$ is nonzero as one can write down an (n,n)-form on X with nonzero total integral;
- $\mathrm{H}^n(X, \Omega^n_X/\mathrm{d}\Omega^{n-1}_X) = \bigoplus_{w\in\mathbf{Z}} \mathrm{H}^n(X, (\Omega^n_X/\mathrm{d}\Omega^{n-1}_X)_w)$ by Theorem 1.4;
- $\mathrm{H}^n(X, \Omega^n_X/\mathrm{d}\Omega^{n-1}_X) \simeq \mathbf{C}_{\mathbf{F}}$ by Lemma 6.5; and
- the image of $\mathrm{H}^n(X, \Upsilon^n_X) \to \mathrm{H}^n(X, \Omega^n_X/\mathrm{d}\Omega^{n-1}_X)$ is contained in $\mathrm{H}^n(X, (\Omega^n_X/\mathrm{d}\Omega^{n-1}_X)_{2n})$ by Theorem 1.4.

To prove the factorization of $\mathrm{Tr}^{\mathrm{trop}}_X$, we may replace $\mathcal{X}$ by $\mathcal{Y}$ through an alteration $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ smooth and geometrically connected, due to the functoriality of $\mathrm{Tr}^{\mathrm{trop}}_\bullet$ and the fact that the pullback map $\mathrm{H}^n(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}, \Omega^n_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}) \to \mathrm{H}^n(\mathcal{Y}^{\mathrm{an}}_{\mathrm{a}}, \Omega^n_{\mathcal{Y}^{\mathrm{an}}_{\mathrm{a}}}/\mathrm{d}\Omega^{n-1}_{\mathcal{Y}^{\mathrm{an}}_{\mathrm{a}}})$ is an isomorphism. Thus, we may assume that $\mathcal{X}$ is projective.

Define Ξ^n_X to be the quotient sheaf in the following exact sequence

$$0 \to \Upsilon^n_X \to (\Omega^n_X/\mathrm{d}\Omega^{n-1}_X)_{2n} \to \Xi^n_X \to 0. \tag{7.2}$$

It is functorial in X. One can show that the sheaf Ξ^n_X is supported on $\{x \in X \mid s(x) \geq 2\}$. This suggests that one should expect $\mathrm{H}^i(X, \Xi^n_X) = 0$ for $i \geq n-1$, which suffices for the proposition. In fact, such vanishing result can be proved if we have semistable resolution instead of alteration. In the absence of semistable resolution, we need an *ad hoc* argument as follows.

Take an arbitrary cohomology class $\alpha \in \mathrm{H}^{n-1}(X, \Xi^n_X)$. Since X is (Hausdorff and) compact, by [SP, 09V2, 01FM], there is a finite open covering $\underline{U} = \{U_i \mid i = 1, \dots, M\}$ of X such that α is represented by an (alternative) Čech cocycle $\underline{\alpha} = \{\alpha_I \in \Xi^n_X(U_I) \mid I \subset \{1, \dots, M\}, |I| = n\}$ on $\underline{U}$, where $U_I = \bigcap_{i\in I} U_i$ as always. By refining $\underline{U}$, we may assume that α_I is in the image of the map $(\Omega^n_X/\mathrm{d}\Omega^{n-1}_X)(U_I)^{\mathrm{pre}}_{2n} \to \Xi^n_X(U_I)$ for every I (See Definition 5.9 for the notation). By Lemma 7.8 below and possibly replacing k by a finite extension inside $\mathbf{C}_{\mathbf{F}}$, we have a proper flat integral model $\mathcal{Y}$ of $\mathcal{X}$ such that if $\mathcal{Z}_1, \dots, \mathcal{Z}_M$ are all reduced irreducible components of $\mathcal{Y}_s$, then the covering $\{(\pi^{-1}\mathcal{Z}_i)\widehat{\otimes}_k\mathbf{C}_{\mathbf{F}} \mid i = 1, \dots, M\}$ refines $\underline{U}$ (here, M might be different from the previous one). By [dJ96, Theorem 8.2], possibly after further replacing k by a finite extension inside $\mathbf{C}_{\mathbf{F}}$, we have a (proper) strictly semistable scheme $\mathcal{Y}'$ over k° with an alteration $\mathcal{Y}' \to \mathcal{Y}$. For simplicity, we may also assume that every irreducible component of $\mathcal{Y}'^{[r]}_s$ is geometrically irreducible for every $r \geq 0$. In particular, if we denote by $\mathcal{Z}'_1, \dots, \mathcal{Z}'_{M'}$ all irreducible components of $\mathcal{Y}'_s$, then the covering $\{(\pi^{-1}\mathcal{Z}'_i)\widehat{\otimes}_k\mathbf{C}_{\mathbf{F}} \mid i = 1, \dots, M'\}$ refines $f^{-1}\underline{U} \coloneqq \{f^{-1}U_i \mid i = 1, \dots, M\}$.

Put $X' = (\mathcal{Y}' \otimes_k \mathbf{C}_{\mathbf{F}})^{\mathrm{an}}$. We claim that $f^*\alpha = 0$, where $f^*\alpha$ is the canonical image of $f^{-1}\alpha$ in $\mathrm{H}^{n-1}(X', \Xi^n_{X'})$. Assume the claim. We prove the factorization of $\mathrm{Tr}^{\mathrm{trop}}_X$. Take an element ω in the kernel of (7.1). We have to show that $\mathrm{Tr}^{\mathrm{trop}}_X(\omega) = 0$. By (7.2), we know that $\omega = \delta(\alpha)$ for some $\alpha \in \mathrm{H}^{n-1}(X, \Xi^n_X)$ where $\delta\colon \mathrm{H}^{n-1}(X, \Xi^n_X) \to \mathrm{H}^n(X, \Upsilon^n_X)$ is the induced coboundary map. By the claim, we know that $f^*\omega = 0$. In other words, $\mathrm{Tr}^{\mathrm{trop}}_{X'}(f^*\omega) = 0$. However, since $\mathrm{Tr}^{\mathrm{trop}}_{X'}(f^*\omega) = \deg(f)\,\mathrm{Tr}^{\mathrm{trop}}_X(\omega)$, we have $\mathrm{Tr}^{\mathrm{trop}}_X(\omega) = 0$. Thus the proposition is proved modulo the previous claim.

To prove the claim, we may replace $f^*\alpha$ by α and assume that $\mathcal{Y}' = \mathcal{Y}$ to ease notation. Moreover, we may assume that α is represented by a Čech cocycle with respect to the open covering $\underline{U} = \{U_i \mid i = 1, \dots, M\}$ in which $U_i = (\pi^{-1}\mathcal{Z}_i)\widehat{\otimes}_k \mathbf{C}_\mathbf{F}$. We study a typical n-fold intersection of $\underline{U}$. Without lost of generality, we consider $U_{\{1,\dots,n\}} \coloneqq \bigcap_{i=1}^n U_i$. If $\bigcap_{i=1}^n \mathcal{Z}_i = \emptyset$, then $U_{\{1,\dots,n\}} = \emptyset$. So we may assume that $\bigcap_{i=1}^n \mathcal{Z}_i = \coprod_{j=1}^L \mathcal{C}_j$, where each $\mathcal{C}_j$ is a geometrically irreducible proper smooth curve over $\widetilde{k}$. Then $U_{\{1,\dots,n\}} = \coprod_{j=1}^L U_{\mathcal{C}_j}$ where $U_{\mathcal{C}_j} \coloneqq (\pi^{-1}\mathcal{C}_j)\widehat{\otimes}_k \mathbf{C}_\mathbf{F}$.

Now we take a typical member $\mathcal{C}$ in $\{\mathcal{C}_1, \dots, \mathcal{C}_L\}$. Put $\mathcal{C}^\heartsuit = \mathcal{C} \backslash \mathcal{Y}_s^{[n]}$. Then we have

$$\mathcal{C} \backslash \mathcal{C}^\heartsuit = \mathcal{C} \cap \mathcal{Y}_s^{[n]} = \coprod_{c=1}^{C} \mathcal{P}_c$$

where $\mathcal{P}_c$ is isomorphic to $\operatorname{Spec} \widetilde{k}$. We may cover $\mathcal{C}^\heartsuit$ by affine opens $\{\mathcal{C}_b \mid 1 \le b \le B\}$ such that $\pi^{-1}\mathcal{C}_b \simeq \mathbf{E}^{n-1}_{k,\varpi} \times_k \mathcal{M}(C_b)$ for some integrally smooth affine k-affinoid algebra C_b such that $\mathcal{C}_b \cong \operatorname{Spec}(C_b)_s$, where $\mathbf{E}^t_{k,\varpi}$ is defined in Example 4.3. Applying Theorem 1.4 (ii) to $\mathcal{M}(C_b)$ and by Example 4.3, we may find an open neighborhood V_b of $\pi^{-1}\mathcal{C}_b$ in $U_\mathcal{C}$ and an element $\beta_b \in \Upsilon^n_X(V_b)$ such that $\alpha_{\{1,\dots,n\}}\mid_{V_b} - \beta_b$ induces the zero class in $\mathrm{H}^n_{\mathrm{dR}}(V_b)$. On the other hand, $\pi^{-1}\mathcal{P}_c$ is isomorphic to $\mathbf{E}^n_{k,\varpi}$ for every $1 \le c \le C$. Thus by Example 4.3, we may find an element $\gamma_c \in \Upsilon^n_X(W_c)$ where $W_c \coloneqq (\pi^{-1}\mathcal{P}_c)\widehat{\otimes}_k \mathbf{C}_\mathbf{F}$ such that $\alpha_{\{1,\dots,n\}}\mid_{W_c} - \gamma_c$ induces the zero class in $\mathrm{H}^n_{\mathrm{dR}}(W_c)$. Note that $\{V_b \mid 1 \le b \le B\} \cup \{W_c \mid 1 \le c \le C\}$ form an open covering of $U_\mathcal{C}$. The existence of β_b and γ_c (for an arbitrary subset of $\{1, \dots, M\}$ of cardinality n and an arbitrary $\mathcal{C}$) ensures that we can refine the covering $\underline{U}$ such that α_I becomes trivial everywhere in the refined one. In particular, $\alpha = 0$ in $\mathrm{H}^{n-1}(X, \Xi^n_X)$. □

Lemma 7.8. *Let K be a non-Archimedean field and $\mathcal{X}$ a projective scheme over K. For every finite open covering $\underline{U} = \{U_i | i = 1, \dots, M\}$ of $(\mathcal{X} \otimes_K \widehat{K^\mathrm{a}})^\mathrm{an}$, there are a finite extension K'/K in $\widehat{K^\mathrm{a}}$ and a proper flat integral model $\mathcal{Y}$ of $\mathcal{X} \otimes_K K'$ over K'° such that for every irreducible component $\mathcal{Z}$ of $\mathcal{Y}_s$, there is some $U_i \in \underline{U}$ containing $(\pi^{-1}\mathcal{Z})\widehat{\otimes}_{K'}\widehat{K^\mathrm{a}}$.*

Proof. The proof of [Pay09, Theorem 4.2] in fact implies the canonical isomorphism

$$(\mathcal{X} \otimes_K \widehat{K^\mathrm{a}})^\mathrm{an} \cong \varprojlim_{\iota} \mathbf{Trop}(\mathcal{X} \otimes_K \widehat{K^\mathrm{a}}, \iota)$$

of topological spaces, where the limit is taken over all projective embeddings $\iota \colon \mathcal{X} \otimes_K K_1 \to \mathbf{P}^{N(\iota)}_{K_1}$ defined over finite extensions K_1/K in $\widehat{K^\mathrm{a}}$. Here, we adopt the notation in [Pay09]. In particular, $\mathbf{Trop}(\mathcal{X} \otimes_K \widehat{K^\mathrm{a}}, \iota)$ is a closed subset of $\mathbf{Trop}(\mathbf{P}^{N(\iota)}_{\widehat{K^\mathrm{a}}})$. By an easy topological argument, we may find a single projective embedding $\iota \colon \mathcal{X} \otimes_K K_1 \to \mathbf{P}^N_{K_1}$ with a finite open covering $\underline{U'}$ of $\mathbf{Trop}(\mathbf{P}^N_{\widehat{K^\mathrm{a}}})$ such that the induced open covering under the above isomorphism refines $\underline{U}$. By successive blow-up of linear subspaces in the special fiber of $\mathbf{P}^N_{K_1^\circ}$, we can find a further finite extension K'/K_1 in $\widehat{K^\mathrm{a}}$ and a proper flat integral model $\mathcal{Y}'$ of $\mathbf{P}^N_{K'}$ over K'° such that for every irreducible component $\mathcal{Z}'$ of $\mathcal{Y}'_s$, there is some $U'_i \in \underline{U'}$ containing the image of $(\pi^{-1}\mathcal{Z}')\widehat{\otimes}_{K'}\widehat{K^\mathrm{a}}$ in $\mathbf{Trop}(\mathbf{P}^N_{\widehat{K^\mathrm{a}}})$. Finally, we take $\mathcal{Y}$ to be the closure of $\iota(X \otimes_K K')$ in $\mathcal{Y}'$. □

Proof of Theorem 1.8. By Lemma 7.9 below and possibly replacing k by a finite extension in $\mathbf{C}_\mathbf{F}$, we may assume $\omega \in \mathscr{A}^{n-p,n-p}_{\mathcal{X}^\mathrm{an}}(\mathcal{X}^\mathrm{an})$. Put $X = \mathcal{X}^\mathrm{an}$. We regard ω as the induced class in

$\mathrm{H}^{n-p,n-p}(X)$. By Corollary 3.5, we may further assume that ω is an element in $\mathrm{H}^p(X, \mathscr{T}_X^p)$. By Theorem 3.7, we need to show that

$$\int_X \mathrm{cl}_{\mathrm{trop}}(\mathcal{Z}) \wedge \omega = 0.$$

However, by Proposition 7.7, it suffices to show that $\langle \mathrm{cl}_{\log}(\mathcal{Z}), \omega \rangle_{\mathcal{X}} = 0$ where $\mathrm{cl}_{\log}$ is defined in Definition 6.4. However, this follows from Theorem 6.6. □

Lemma 7.9. *Let the assumption and notation be as in Theorem 6.6. Then for every $p, q \geq 0$, the canonical map*

$$\varinjlim_{k'} \mathscr{A}^{p,q}_{\mathcal{X}^{\mathrm{an}}_{k'}}(\mathcal{X}^{\mathrm{an}}_{k'}) \to \mathscr{A}^{p,q}_{\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}}(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}})$$

is an isomorphism, where the colimit is taken over all finite extensions k' of k in $\mathbf{C_F}$.

Proof. The lemma follows from the fact that for every $f \in \mathcal{O}^*(\mathcal{X}^{\mathrm{an}}_{\mathrm{a}}, V)$ where V is a rational affinoid domain, there is a function $g \in \mathcal{O}^*(\mathcal{X}^{\mathrm{an}}_{k'}, V')$ for some k' such that $\varsigma_{k'}^{-1}(V') = V$ and $f^{-1} \cdot \varsigma_{k'}^* g$ has norm 1 on some open neighborhood of V, where $\varsigma_{k'} \colon \mathcal{X}^{\mathrm{an}}_{\mathrm{a}} \to \mathcal{X}^{\mathrm{an}}_{k'}$ is the canonical projection. Here, we have used [Ber07, Lemma 2.1.3 (ii)]. □

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