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Journal of Functional Analysis 265 (2013) 3325–3344

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Uniqueness of Fourier–Jacobi models: The Archimedean case [☆]

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Received 26 December 2012; accepted 31 August 2013

Available online 20 September 2013

Communicated by P. Delorme

Abstract

We prove uniqueness of Fourier–Jacobi models for general linear groups, unitary groups, symplectic groups and metaplectic groups, over an Archimedean local field.

Keywords: Classical group; Irreducible representation; Fourier–Jacobi model

1. Introduction and the main result

Uniqueness of Bessel models and Fourier–Jacobi models is the basic starting point to study L-functions for classical groups by Rankin–Selberg method [7,8]. Breakthroughs have been made towards the proof of the uniqueness in the recent years. Over a non-Archimedean local field of characteristic zero, uniqueness of Bessel models and Fourier–Jacobi models is now completely proved, by the works of Aizenbud–Gourevitch–Rallis–Schiffmann [1], Sun [19], Waldspurger [23], Ginzburg–Jiang–Rallis–Soudry [8], and Gan–Gross–Prasad [6]. Over an Archimedean local field, uniqueness of Bessel models is proved by Jiang–Sun–Zhu [11]. Only uniqueness of Fourier–Jacobi models in the Archimedean case remains open. This article is aimed to prove this remaining case.

☆ Y. Liu is supported by NSF grant DMS 1302000. B. Sun is supported by NSFC Grant 11222101.
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http://dx.doi.org/10.1016/j.jfa.2013.08.034

Theorem A. *Let G be a classical group $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, $\mathrm{Sp}_{2m}(\mathbb{C})$, or a metaplectic group $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{R})$, with an r-th Fourier–Jacobi subgroup S_r of it ($r \geqslant 1$, $n \geqslant 2r$, $p, q \geqslant r$, $m \geqslant r$). Denote by J_r and N_r the Jacobi quotient and the Whittaker quotient of S_r, respectively. Then for every irreducible Casselman–Wallach representation π of G, every nondegenerate irreducible Casselman–Wallach representation σ of J_r, and every nondegenerate unitary character ψ of N_r, one has that*

$$\dim \mathrm{Hom}_{S_r}(\pi \widehat{\otimes} \sigma, \psi) \leqslant 1.$$

Here σ and ψ are viewed as representations of S_r via inflations, and "$\widehat{\otimes}$" stands for the completed projective tensor product. By abuse of notation, we do not distinguish representations with their underlying vector spaces. Fourier–Jacobi subgroups as well as their Jacobi quotients and Whittaker quotients are defined in Section 2. The notion of "nondegenerate unitary character on N_r" is also explained in Section 2. The notions concerning Casselman–Wallach representations are explained in Section 3.3. Note that Theorem A for $\widetilde{\mathrm{Sp}}_{2m}(\mathbb{R})$ implies the analogous result for the symplectic group $\mathrm{Sp}_{2m}(\mathbb{R})$.

When $n = 2r$, or $p = q = r$, or $m = r$, Theorem A asserts uniqueness of Whittaker models for G. See [15,4] for uniqueness of Whittaker models for quasi-split linear groups over $\mathbb{R}$ (or [11] for a quick proof). When $G = \mathrm{U}(n, 1)$ and π is unitary, Theorem A is proved in [2].

As in the proof of uniqueness of Bessel models, our idea is to reduce Theorem A to the following basic case, which is called the multiplicity one theorem for Fourier–Jacobi models.

Theorem B. *Let J be one of the following Jacobi groups*

$$\begin{gathered} \mathrm{H}_{2n+1}(\mathbb{R}) \rtimes \mathrm{GL}_n(\mathbb{R}), \quad \mathrm{H}_{2n+1}(\mathbb{C}) \rtimes \mathrm{GL}_n(\mathbb{C}), \quad \mathrm{H}_{2p+2q+1}(\mathbb{R}) \rtimes \mathrm{U}(p,q), \\ \mathrm{H}_{2n+1}(\mathbb{C}) \rtimes \mathrm{Sp}_{2n}(\mathbb{C}), \quad \mathrm{H}_{2n+1}(\mathbb{R}) \rtimes \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}), \quad p, q, n \geqslant 0, \end{gathered} \tag{1}$$

where "H_{2k+1}" indicates the appropriate Heisenberg group of dimension $2k + 1$. Denote by G its respective subgroup

$$\mathrm{GL}_n(\mathbb{R}), \quad \mathrm{GL}_n(\mathbb{C}), \quad \mathrm{U}(p,q), \quad \mathrm{Sp}_{2n}(\mathbb{C}), \quad \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}).$$

Then for every nondegenerate irreducible Casselman–Wallach representation ρ of J, and every irreducible Casselman–Wallach representation of π of G, one has that

$$\dim \mathrm{Hom}_G(\rho \widehat{\otimes} \pi, \mathbb{C}) \leqslant 1.$$

In the above inequality, $\mathbb{C}$ stands for the trivial representation of G. Theorem B is proved in [21], except for the case of $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. But in this case, by the classification of nondegenerate irreducible Casselman–Wallach representations of Jacobi groups (see Section 3.3), Theorem B is obviously equivalent to the analogous result for $\mathrm{Sp}_{2n}(\mathbb{R})$, which is also proved in [21].

2. Fourier–Jacobi subgroups

In order to prove Theorem A uniformly, we introduce the following notation. Let $(\mathbb{K}, \iota)$ be one of the followings five $\mathbb{R}$-algebras with involutions:

$$(\mathbb{R}\times\mathbb{R}, \iota_{\mathbb{R}}), \quad (\mathbb{C}\times\mathbb{C}, \iota_{\mathbb{C}}), \quad (\mathbb{C}, {}^{-}), \quad (\mathbb{R}, 1_{\mathbb{R}}), \quad (\mathbb{C}, 1_{\mathbb{C}}), \tag{2}$$

where $\iota_{\mathbb{R}}$ and $\iota_{\mathbb{C}}$ are the maps interchanging the coordinates, $1_{\mathbb{R}}$ and $1_{\mathbb{C}}$ are the identity maps, and "$^{-}$" is the complex conjugation. Let E be a skew-Hermitian $\mathbb{K}$-module; namely, it is a free $\mathbb{K}$-module of finite rank, equipped with a nondegenerate $\mathbb{R}$-bilinear map

$$\langle\, ,\, \rangle_E : E\times E \to \mathbb{K}$$

satisfying

$$\langle u, v\rangle_E = -\langle v, u\rangle_E^{\iota}, \qquad \langle au, v\rangle_E = a\langle u, v\rangle_E, \quad a\in\mathbb{K},\ u, v\in E.$$

Denote by $\mathrm{U}(E)$ the group of all $\mathbb{K}$-module automorphisms of E which preserve the form $\langle\, ,\, \rangle_E$. According to the five cases of $(\mathbb{K}, \iota)$ in (2), it is respectively a real general linear group, a complex general linear group, a real unitary group, a real symplectic group, or a complex symplectic group. Put

$$\mathrm{U}'(E) := \begin{cases} \widetilde{\mathrm{Sp}}(E) & \text{if } \mathbb{K}=\mathbb{R};\\ \mathrm{U}(E) & \text{otherwise,}\end{cases}$$

where $\widetilde{\mathrm{Sp}}(E)$ denotes the metaplectic double cover of the symplectic group $\mathrm{Sp}(E)$. Then we have a short exact sequence

$$1 \to \mu_{\mathbb{K}} \to \mathrm{U}'(E) \to \mathrm{U}(E) \to 1, \tag{3}$$

where

$$\mu_{\mathbb{K}} := \begin{cases} \{\pm 1\} & \text{if } \mathbb{K}=\mathbb{R};\\ \{1\} & \text{otherwise.}\end{cases}$$

Let $r \geqslant 1$ and assume that there is a sequence

$$\mathcal{F}: \quad 0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

of totally isotropic free $\mathbb{K}$-submodules of E so that $\mathrm{rank}_{\mathbb{K}}(X_i) = i$, $i = 0, 1, \ldots, r$. Put

$$\mathrm{J}_{\mathcal{F}}(E) := \bigl\{ g\in \mathrm{U}(E) \bigm| (g-1)X_i \subset X_{i-1},\ i = 1, 2\ldots, r\bigr\}.$$

Denote by $\mathrm{J}'_{\mathcal{F}}(E)$ the inverse image of $\mathrm{J}_{\mathcal{F}}(E)$ under the covering map $\mathrm{U}'(E)\to\mathrm{U}(E)$. It is called an r-th Fourier–Jacobi subgroup of $\mathrm{U}'(E)$.

When $r = 1$, we put

$$\mathrm{J}_X(E) := \mathrm{J}_{\mathcal{F}}(E) = \bigl\{ g\in \mathrm{U}(E) \bigm| (g-1)X \subset X\bigr\} \quad \text{and} \quad \mathrm{J}'_X(E) := \mathrm{J}'_{\mathcal{F}}(E).$$

Then $\mathrm{J}'_X(E)$ is isomorphic to a Jacobi group of Theorem B, and conversely, all Jacobi groups of Theorem B are isomorphic to some $\mathrm{J}'_X(E)$ (*cf.* [19, Section 1]).

For every subset S of E, set

$$S^\perp := \{v \in E \mid \langle v, u\rangle_E = 0 \text{ for all } u \in S\}.$$

Then $E' := X_{r-1}^\perp / X_{r-1}$ is obviously a skew-Hermitian $\mathbb{K}$-module. Put

$$X' := X_r / X_{r-1} \subset E',$$

which is a totally isotropic free $\mathbb{K}$-submodule of E' of rank 1. Restrictions yield a surjective homomorphism

$$j_{\mathcal{F}} : \mathrm{J}_{\mathcal{F}}(E) \to \mathrm{J}_{X'}(E').$$

There is a unique surjective homomorphism

$$j'_{\mathcal{F}} : \mathrm{J}'_{\mathcal{F}}(E) \to \mathrm{J}'_{X'}(E')$$

so that the squares in

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & \mathrm{J}'_{\mathcal{F}}(E) & \longrightarrow & \mathrm{J}_{\mathcal{F}}(E) & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle j'_{\mathcal{F}}} & & \downarrow{\scriptstyle j_{\mathcal{F}}} & & \\ 1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & \mathrm{J}'_{X'}(E') & \longrightarrow & \mathrm{J}_{X'}(E') & \longrightarrow & 1 \end{array}$$

are commutative. In view of the homomorphism $j'_{\mathcal{F}}$, we call $\mathrm{J}'_{X'}(E')$ the *Jacobi quotient* of $\mathrm{J}'_{\mathcal{F}}(E)$.

Put

$$\mathrm{N}_{\mathcal{F}}(X) := \{g \in \mathrm{GL}(X) \mid (g-1)X_i \subset X_{i-1},\ i = 1, 2, \ldots, r\}.$$

It is a maximal unipotent subgroup of the group $\mathrm{GL}(X)$ of $\mathbb{K}$-linear automorphisms of X. Restrictions yield a surjective homomorphism $w_{\mathcal{F}} : \mathrm{J}_{\mathcal{F}}(E) \to \mathrm{N}_{\mathcal{F}}(X)$. Composing it with the covering map $\mathrm{J}'_{\mathcal{F}}(E) \to \mathrm{J}_{\mathcal{F}}(E)$, we get a homomorphism

$$w'_{\mathcal{F}} : \mathrm{J}'_{\mathcal{F}}(E) \to \mathrm{N}_{\mathcal{F}}(X).$$

In view of this homomorphism, we call $\mathrm{N}_{\mathcal{F}}(X)$ the *Whittaker quotient* of the Fourier–Jacobi subgroup $\mathrm{J}'_{\mathcal{F}}(E)$.

We review the notion of nondegenerate characters on $\mathrm{N}_{\mathcal{F}}(X)$. Define a surjective homomorphism

$$\mathrm{N}_{\mathcal{F}}(X) \to \mathrm{A}_{\mathcal{F}}(X) := \prod_{i=1}^{r-1} \mathrm{Hom}_{\mathbb{K}}(X_{i+1}/X_i, X_i/X_{i-1}), \quad g \mapsto (a_1, a_2, \ldots, a_{r-1}), \tag{4}$$

where a_i is the map $v + X_i \mapsto (g-1)v + X_{i-1}$. Then every unitary character $\psi_{\mathrm{N}_{\mathcal{F}}(X)}$ on $\mathrm{N}_{\mathcal{F}}(X)$ descends to a character $\psi_{\mathrm{A}_{\mathcal{F}}(X)}$ on $\mathrm{A}_{\mathcal{F}}(X)$ though (4). Note that $\mathrm{A}_{\mathcal{F}}(X)$ is a free $\mathbb{K}^{r-1}$-module of rank 1. We say that $\psi_{\mathrm{N}_{\mathcal{F}}(X)}$ is nondegenerate if the restriction of $\psi_{\mathrm{A}_{\mathcal{F}}(X)}$ to every nonzero $\mathbb{K}^{r-1}$-submodule of $\mathrm{A}_{\mathcal{F}}(X)$ is nontrivial.

3. Preliminaries

3.1. Almost linear Nash groups

We work in the setting of Nash groups. The reader is referred to [16,17] for details. By a *Nash group*, we mean a group which is simultaneously a Nash manifold so that all group operations (the multiplication and the inversion) are Nash maps. Every semialgebraic subgroup of a Nash group is automatically closed and is called a *Nash subgroup*. It is canonically a Nash group.

A finite dimensional real representation $V_{\mathbb{R}}$ of a Nash group G is said to be a *Nash representation* if the action map $G \times V_{\mathbb{R}} \to V_{\mathbb{R}}$ is Nash. A Nash group is said to be *almost linear* if it admits a Nash representation with finite kernel. For every linear algebraic group G defined over $\mathbb{R}$, every finite fold topological group cover of an open subgroup of $\mathsf{G}(\mathbb{R})$ is naturally an almost linear Nash group. Conversely, every almost linear Nash group is of this form. In particular, all groups which occur in last section are almost linear Nash groups.

A Nash group is said to be *unipotent* if it admits a faithful Nash representation so that all group elements act as unipotent linear operators. It follows from the corresponding result for linear algebraic groups that every almost linear Nash group has a unipotent radical, namely, the largest unipotent normal Nash subgroup. A *reductive Nash group* is defined to be an almost linear Nash group with trivial unipotent radical.

Recall that a Nash manifold is said to be *affine* if it is Nash diffeomorphic to a closed Nash submanifold of some $\mathbb{R}^n$. Since every finite fold topological cover of an affine Nash manifold is an affine Nash manifold, all almost linear Nash groups are affine as Nash manifolds.

3.2. Schwartz inductions

If M is an affine Nash manifold and V_0 is a (complex) Fréchet space, then a V_0-valued smooth function $f \in \mathrm{C}^\infty(M; V_0)$ is said to be Schwartz if

$$|f|_{\mathrm{D},\nu} := \sup_{x \in M} \left|(\mathrm{D}f)(x)\right|_\nu < \infty$$

for all Nash differential operators D on M, and all continuous seminorms $|\cdot|_\nu$ on V_0. Recall that a differential operator D on M is said to be Nash if $\mathrm{D}\varphi$ is a Nash function whenever φ is a (complex-valued) Nash function on M. Denote by $\mathrm{C}^\varsigma(M; V_0) \subset \mathrm{C}^\infty(M; V_0)$ the space of Schwartz functions. Then both $\mathrm{C}^\varsigma(M; V_0)$ and $\mathrm{C}^\infty(M; V_0)$ are naturally Fréchet spaces, and the inclusion map $\mathrm{C}^\varsigma(M; V_0) \hookrightarrow \mathrm{C}^\infty(M; V_0)$ is continuous. Furthermore, we have that (*cf.* [22, p. 533])

$$\mathrm{C}^\varsigma(M; V_0) = \mathrm{C}^\varsigma(M) \widehat{\otimes} V_0 \quad \text{and} \quad \mathrm{C}^\infty(M; V_0) = \mathrm{C}^\infty(M) \widehat{\otimes} V_0,$$

where $\mathrm{C}^\varsigma(M) := \mathrm{C}^\varsigma(M; \mathbb{C})$ and $\mathrm{C}^\infty(M) := \mathrm{C}^\infty(M; \mathbb{C})$.

Now we recall Schwartz inductions from [5, Section 2]. Let G be an almost linear Nash group. Then it is affine as a Nash manifold. Let S be a Nash subgroup of G, and let V_0 be a smooth Fréchet representation of S of moderate growth (*cf.* [5, Definition 1.4.1] or [20, Section 2]). Define a continuous linear map

$$\begin{aligned} I_{S,V_0} : \mathrm{C}^{\varsigma}(G; V_0) &\to \mathrm{C}^{\infty}(G; V_0), \\ f &\mapsto \left(g \mapsto \int_S s.f(s^{-1}g)\,\mathrm{d}s\right), \end{aligned} \tag{5}$$

where $\mathrm{d}s$ is a left invariant Haar measure on S. We define the unnormalized Schwartz induction $\mathrm{ind}_S^G V_0$ to be the image of the map (5). Under the quotient topology of $\mathrm{C}^{\varsigma}(G; V_0)$ and under right translations, it is a smooth Fréchet representation of G of moderate growth.

A partition of unity argument shows the following

Lemma 3.1. *Let $f \in \mathrm{C}^{\infty}(G; V_0)$. If*

$$f(sg) = s.f(g), \quad s \in S,\ g \in G,$$

and f is compactly supported modulo S (that is, the support of f has compact image under the map $G \to S\backslash G$), then $f \in \mathrm{ind}_S^G V_0$.

If S' is a Nash subgroup of G containing S, then we have a canonical isomorphism of representations of G [5, Lemma 2.1.6]:

$$\mathrm{ind}_{S'}^G\big(\mathrm{ind}_S^{S'} V_0\big) \cong \mathrm{ind}_S^G V_0. \tag{6}$$

We will use the following result.

Lemma 3.2. *Let V_0 and V be smooth moderate growth Fréchet representations of S and G, respectively. If V is nuclear, then there is an isomorphism of representations of G:*

$$\mathrm{ind}_S^G\big(V_0 \widehat{\otimes} (V|_S)\big) \cong \big(\mathrm{ind}_S^G V_0\big) \widehat{\otimes} V. \tag{7}$$

Proof. Note that the diagram

$$\begin{array}{ccc} \mathrm{C}^{\varsigma}(G; V_0) \widehat{\otimes} V & \xrightarrow{I_{S,V_0} \widehat{\otimes} \mathrm{Id}_V} & \mathrm{C}^{\infty}(G; V_0) \widehat{\otimes} V \\ \downarrow & & \downarrow \\ \mathrm{C}^{\varsigma}(G; V_0 \widehat{\otimes} V) & \xrightarrow{I_{S,V_0 \widehat{\otimes} (V|_S)}} & \mathrm{C}^{\infty}(G; V_0 \widehat{\otimes} V) \end{array} \tag{8}$$

commutes, where the vertical arrows are G-representation isomorphisms given by

$$f \otimes v \mapsto \big(g \mapsto f(g) \otimes g.v\big).$$

The image of the bottom horizontal arrow of (8), to be viewed as a representation of G with the quotient topology, equals to the left-hand side of (7). The top horizontal arrow of (8) is the composition of the map

$$\mathrm{C}^{\varsigma}(G;V_0)\,\widehat{\otimes}\,V \to \left(\mathrm{ind}_S^G V_0\right)\widehat{\otimes}\,V \tag{9}$$

and the map

$$\left(\mathrm{ind}_S^G V_0\right)\widehat{\otimes}\,V \to \mathrm{C}^{\infty}(G;V_0)\,\widehat{\otimes}\,V. \tag{10}$$

The continuous linear map (9) is surjective by [22, Proposition 43.9], and is then open by the open mapping theorem for Fréchet spaces. The map (10) is injective since V is nuclear (*cf.* [22, Proposition 50.4]). Therefore, the image of the top horizontal arrow of (8) equals to the right-hand side of (7). This proves the lemma. □

3.3. *Casselman–Wallach representations*

Let G be an almost linear Nash group as before. Denote by $\mathrm{D}^{\varsigma}(G)$ the space of (complex-valued) Schwartz densities on G. Recall that $\mathrm{D}^{\varsigma}(G) = \mathrm{C}^{\varsigma}(G)\,\mathrm{d}g$, where $\mathrm{d}g$ is a left invariant Haar measure on G. It is an associative algebra under convolutions.

Let V be a smooth Fréchet representation of G of moderate growth. Then V is a $\mathrm{D}^{\varsigma}(G)$-module:

$$\big(f(g)\,\mathrm{d}g\big).v := \int_G f(g)g.v\,\mathrm{d}g, \quad f \in \mathrm{C}^{\varsigma}(G),\ v \in V.$$

We say that V is a Casselman–Wallach representation of G if

- every $\mathrm{D}^{\varsigma}(G)$-submodule of V is closed in V, and
- V is of finite length as an abstract $\mathrm{D}^{\varsigma}(G)$-module.

It is clear that a Casselman–Wallach representation of G is irreducible if and only if it is irreducible as an abstract $\mathrm{D}^{\varsigma}(G)$-module. Furthermore, by the open mapping theorem for Fréchet spaces, it is easy to see that if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is a topologically exact sequence of smooth Fréchet representation of G of moderate growth, then V_2 is a Casselman–Wallach representation if and only if both V_1 and V_3 are so.

When G is reductive, du Cloux proves that V is a Casselman–Wallach representation if and only if its underlying Harish–Chandra module is of finite length (*cf.* [5, Section 3]), and Casselman and Wallach prove that every finite length Harish–Chandra module has a unique Casselman–Wallach representation as its globalization (*cf.* [3] and [24, Chapter 11]).

We return to the setup in Section 2. Put

$$\Bbbk := \left\{a \in \mathbb{K} \,\middle|\, a^{\iota} = a\right\},$$

which is either $\mathbb{R}$ or $\mathbb{C}$. Let E be a skew-Hermitian $\mathbb{K}$-module as before. Put $E_{\Bbbk} := E$, to be viewed as a symplectic space over $\Bbbk$ under the form

$$\langle u, v\rangle_{E_{\Bbbk}} := \frac{\langle u, v\rangle_E - \langle v, u\rangle_E}{2}.$$

The associated Heisenberg group is defined to be

$$\mathrm{H}(E) := \Bbbk \times E,$$

with group multiplication

$$(a, v) \cdot \left(a', v'\right) := \left(a + a' + \left\langle v', v\right\rangle_{E_{\Bbbk}}, v + v'\right).$$

The group $\mathrm{U}(E)$ acts (from left) on $\mathrm{H}(E)$ as group automorphisms through its natural action on E. This defines a semidirect product (the Jacobi group)

$$\mathbb{J}(E) := \mathrm{H}(E) \rtimes \mathrm{U}(E),$$

and its covering

$$\mathbb{J}'(E) := \mathrm{H}(E) \rtimes \mathrm{U}'(E).$$

They are almost linear Nash groups, both having $\mathrm{H}(E)$ as their unipotent radicals. Recall that $\Bbbk$ is the center of $\mathrm{H}(E)$.

Let ρ be an irreducible Casselman–Wallach representation of $\mathbb{J}'(E)$. By a version of Schur Lemma [5, Proposition 5.1.4], $\Bbbk$ acts through a character ψ_ρ in ρ. We say that ρ is nondegenerate if ψ_ρ is nontrivial. Note that the moderate growth condition implies that ψ_ρ is unitary. Since all the Jacobi groups which occur in Theorem A and Theorem B are isomorphic (as Nash groups) to some $\mathbb{J}'(E)$, we also get the notion of "nondegenerate irreducible Casselman–Wallach representations" for these groups.

Fix a nontrivial unitary character $\psi_{\Bbbk}$ on $\Bbbk$. Let ω be a smooth oscillator representation of $\mathbb{J}'(E)$ associated to it, namely, it is a Casselman–Wallach representation of $\mathbb{J}'(E)$, and when viewed as a representation of $\mathrm{H}(E)$, it is irreducible with central character $\psi_{\Bbbk}$. Smooth oscillator representations of $\mathbb{J}'(E)$ exist by the well-known result of splitting metaplectic covers [14], see also [13, Proposition 4.1]. If π_0 is a Casselman–Wallach representations of $\mathrm{U}'(E)$, to be viewed as a representation of $\mathbb{J}'(E)$ via inflation, then $\omega \widehat{\otimes} \pi_0$ is a Casselman–Wallach representation of $\mathbb{J}'(E)$ so that $\Bbbk \subset \mathbb{J}'(E)$ acts by the character $\psi_{\Bbbk}$. Conversely, all such representations are of the form $\omega \widehat{\otimes} \pi_0$ for some π_0. Furthermore, $\omega \widehat{\otimes} \pi_0$ is irreducible if and only if π_0 is. See [20] for details.

4. Reduction to the basic case

We continue with the notation of Section 2. We reformulate Theorem A more precisely as follows:

Theorem 4.1. *For every irreducible Casselman–Wallach representation π of $\mathrm{U}'(E)$, every nondegenerate irreducible Casselman–Wallach representation σ of $\mathrm{J}'_{X'}(E')$, and every nondegenerate unitary character ψ on $\mathrm{N}_{\mathcal{F}}(X)$, one has that*

$$\dim \mathrm{Hom}_{\mathrm{J}'_{\mathcal{F}}(E)}(\pi \widehat{\otimes} \sigma, \psi) \leqslant 1.$$

In this section, we explain the strategy of the proof of Theorem 4.1.

Fix two totally isotropic free submodules $Y \supset Y_{r-1}$ of E such that the parings

$$\langle\ ,\ \rangle_E : X \times Y \to \mathbb{K} \quad \text{and} \quad \langle\ ,\ \rangle_E : X_{r-1} \times Y_{r-1} \to \mathbb{K}$$

are nondegenerate. Fix $x_r \in X$ and $y_r \in Y$ so that

$$\langle x_r, Y_{r-1}\rangle_E = 0, \qquad \langle X_{r-1}, y_r\rangle_E = 0, \quad \text{and} \quad \langle x_r, y_r\rangle_E = 1.$$

Identify $E' := X_{r-1}^{\perp}/X_{r-1}$ with $(X_{r-1} \oplus Y_{r-1})^{\perp}$, and $E_0 := X^{\perp}/X$ with $(X \oplus Y)^{\perp}$. Then we get decompositions

$$E = X_{r-1} \oplus E' \oplus Y_{r-1} = X \oplus E_0 \oplus Y \quad \text{and} \quad E' = \mathbb{K}x_r \oplus E_0 \oplus \mathbb{K}y_r.$$

Identify $X' := X/X_{r-1} \subset E'$ with $\mathbb{K}x_r$, and $\mathrm{J}_{X'}(E')$ with $\mathbb{J}(E_0)$ via the isomorphism

$$g \mapsto \big(\langle gy_r, y_r\rangle_{E'_{\mathbb{k}}}, [gy_r - y_r]_{E_0}; [g]_{E_0}\big). \tag{11}$$

Here for every $v \in \mathbb{K}x_r \oplus E_0$, denote by $[v]_{E_0} \in E_0$ the image of v under the projection $\mathbb{K}x_r \oplus E_0 \to E_0$, and for every $g \in \mathrm{J}_{X'}(E')$, denote by $[g]_{E_0}$ the element of $\mathrm{U}(E_0)$ so that the diagram

$$\begin{array}{ccc} \mathbb{K}x_r \oplus E_0 & \xrightarrow{g|_{\mathbb{K}x_r \oplus E_0}} & \mathbb{K}x_r \oplus E_0 \\ \downarrow & & \downarrow \\ E_0 & \xrightarrow{[g]_{E_0}} & E_0 \end{array}$$

commutes.

Fix the unique identification $\mathrm{J}'_{X'}(E') = \mathbb{J}'(E_0)$ so that the squares in

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & \mathrm{J}'_{X'}(E') & \longrightarrow & \mathrm{J}_{X'}(E') & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \mu_{\mathbb{K}} & \longrightarrow & \mathbb{J}'(E_0) & \longrightarrow & \mathbb{J}(E_0) & \longrightarrow & 1 \end{array}$$

are commutative.

Denote by P'_X the parabolic subgroup of $\mathrm{U}'(E)$ stabilizing X, and by M'_X the Levi subgroup of $\mathrm{U}'(E)$ stabilizing both X and Y. Then we have a Levi decomposition

$$\mathrm{P}'_X = \mathrm{N}_X \rtimes \mathrm{M}'_X,$$

where N_X is the unipotent radical of P'_X. Moreover, we have

$$\mathrm{M}'_X = \frac{\mathrm{U}'(E_0)\times \mathrm{GL}'(X)}{\Delta\mu_{\mathbb{K}}},$$

where $\mathrm{GL}'(X)$ is the subgroup of M'_X fixing E_0 pointwise, and $\Delta\mu_{\mathbb{K}}$ is the group $\mu_{\mathbb{K}}$ diagonally embedded in $\mathrm{U}'(E_0)\times \mathrm{GL}'(X)$.

Put $\mathrm{H}(X^\perp) := \mathbb{k}\times X^\perp$, which is a subgroup of $\mathrm{H}(E)$. Define a homomorphism

$$\begin{aligned} p_X : \mathrm{H}\bigl(X^\perp\bigr)\rtimes \mathrm{P}'_X &\to \mathrm{H}(E_0)\rtimes \mathrm{M}'_X,\\ (t, v+v_0; uh) &\mapsto (t, v_0; h), \end{aligned} \tag{12}$$

where $t\in\mathbb{k}$, $v\in X$, $v_0\in E_0$, $u\in \mathrm{N}_X$ and $h\in \mathrm{M}'_X$. The kernel of p_X is $X\rtimes \mathrm{N}_X$. We always view $\mathrm{H}(E_0)\rtimes \mathrm{M}'_X$ as a quotient of $\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X$ via the map p_X.

Since the covering map $\mathrm{GL}'(X)\to \mathrm{GL}(X)$ uniquely splits over $\mathrm{N}_{\mathcal{F}}(X)$, we also view $\mathrm{N}_{\mathcal{F}}(X)$ as a subgroup of $\mathrm{GL}'(X)$.

The following lemma is routine to check.

Lemma 4.2. *The diagram*

$$\begin{array}{ccc} \mathrm{J}'_{\mathcal{F}}(E) & \xrightarrow{\;g\mapsto y_r^{-1} g y_r\;} & \mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X \\ \Big\downarrow{\scriptstyle j'_{\mathcal{F}}\times w'_{\mathcal{F}}} & & \Big\downarrow{\scriptstyle p_X} \\ \mathrm{J}'_{X'}(E')\times \mathrm{N}_{\mathcal{F}}(X) & \longrightarrow & \mathrm{H}(E_0)\rtimes \mathrm{M}'_X \end{array} \tag{13}$$

commutes, where the bottom horizontal arrow is the map

$$\begin{aligned} \mathrm{J}'_{X'}\bigl(E'\bigr)\times \mathrm{N}_{\mathcal{F}}(X) &\subset \mathrm{J}'_{X'}\bigl(E'\bigr)\times \mathrm{GL}'(X)\\ &\to \frac{\mathrm{J}'_{X'}(E')\times \mathrm{GL}'(X)}{\Delta\mu_{\mathbb{K}}} = \frac{\mathbb{J}'(E_0)\times \mathrm{GL}'(X)}{\Delta\mu_{\mathbb{K}}}\\ &= \frac{\mathrm{H}(E_0)\rtimes (\mathrm{U}'(E_0)\times \mathrm{GL}'(X))}{\Delta\mu_{\mathbb{K}}} = \mathrm{H}(E_0)\rtimes \mathrm{M}'_X. \end{aligned}$$

For the top horizontal arrow of (13), note that $\mathrm{J}'_{\mathcal{F}}(E)\subset \mathrm{P}'_X\subset \mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X\subset \mathbb{J}'(E)$, and $y_r\in E\subset \mathrm{H}(E)\subset \mathbb{J}'(E)$.

As in Theorem 4.1, let σ be a nondegenerate irreducible Casselman–Wallach representation of $\mathbb{J}'(E_0) = \mathrm{J}'_{X'}(E')$. Fix a generic irreducible Casselman–Wallach representation τ of $\mathrm{GL}'(X)$ so that $\chi_\tau = \chi_\sigma$, where χ_σ is the character of $\mu_{\mathbb{K}}$ through which $\mu_{\mathbb{K}}\subset \mathbb{J}'(E_0)$ acts in σ, and likewise for χ_τ. Then $\sigma\,\widehat{\otimes}\,\tau$ descends to a representation of

$$\frac{\mathbb{J}'(E_0)\times \mathrm{GL}'(X)}{\Delta\mu_{\mathbb{K}}} = \mathrm{H}(E_0)\rtimes \mathrm{M}'_X.$$

We recall some notations in [11]. Put

$$\mathrm{d}_{\mathbb{K}} := \begin{cases} 1 & \text{if } \mathbb{K} \text{ is a field;} \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\mathbb{K}_+^\times := \left(\mathbb{R}_+^\times\right)^{\mathrm{d}_{\Bbbk}} \quad \left(\mathbb{R}_+^\times \text{ is the multiplicative group of positive real numbers}\right).$$

For all $a \in \mathbb{K}_+^\times$ and $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$, put

$$a^s := a_1^{s_1} a_2^{s_2} \in \mathbb{C}^\times, \quad \text{if } \mathrm{d}_{\mathbb{K}} = 2,\ a = (a_1, a_2),\ s = (s_1, s_2);$$

if $\mathrm{d}_{\mathbb{K}} = 1$, $a^s \in \mathbb{C}^\times$ retains the usual meaning.

For every $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$, denote by τ_s the representation of $\mathrm{GL}'(X)$ which has the same underlying space as that of τ, and has the action

$$\tau_s(g) = |g|^s \tau(g), \quad g \in \mathrm{GL}'(X),$$

where $|g|^s$ is the image of g under the composition map

$$\mathrm{GL}'(X) \to \mathrm{GL}(X) \xrightarrow{\text{determinant}} \mathbb{K}^\times \xrightarrow{|\cdot|} \mathbb{K}_+^\times \xrightarrow{(\cdot)^s} \mathbb{C}^\times,$$

and

$$|\cdot| : \mathbb{K}^\times \to \mathbb{K}_+^\times$$

is the map of taking componentwise absolute values. Then $\sigma \widehat{\otimes} \tau_s$ is an irreducible Casselman–Wallach representation of $\mathrm{H}(E_0) \rtimes \mathrm{M}'_X$. By inflation through p_X, we view it as an irreducible Casselman–Wallach representation of $\mathrm{H}(X^\perp) \rtimes \mathrm{P}'_X$. For simplicity in notation, put

$$\mathrm{I}_s := \mathrm{ind}_{\mathrm{H}(X^\perp) \rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \sigma \widehat{\otimes} \tau_s.$$

Proposition 4.3. *Except for a measure zero set of $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$, the unnormalized Schwartz induction I_s is a nondegenerate irreducible Casselman–Wallach representation of $\mathbb{J}'(E)$.*

Proof. Assume that $\Bbbk \subset \mathbb{J}'(E_0)$ acts through the nontrivial unitary character $\psi_\Bbbk$ in σ. Then $\Bbbk \subset \mathbb{J}'(E)$ also acts through $\psi_\Bbbk$ in I_s. As in Section 3.3, let ω be a smooth oscillator representation of $\mathbb{J}'(E)$ associated to $\psi_\Bbbk$.

Denote by ω_X the topological X-coinvariant space of ω, namely, it is the maximal Hausdorff quotient of ω on which $X \subset \mathrm{H}(E)$ acts trivially. This is a representation of $\mathrm{H}(X^\perp) \rtimes \mathrm{P}'_X$. By using the mixed Schrödinger model (*cf.* [9], [12, Section 5]), we know that $X \rtimes \mathrm{N}_X$ acts trivially on ω_X, and it descends to a smooth oscillator representation of $\mathrm{H}(E_0) \rtimes \mathrm{M}'_X$. By Frobenius reciprocity and using Schrödinger models, we know that the quotient map $\omega \to \omega_X$ induces an isomorphism of $\mathrm{H}(E) \rtimes \mathrm{P}'_X$-representations:

$$\omega|_{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \cong \mathrm{ind}_{\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X}^{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \omega_X. \tag{14}$$

Let $s \in \mathbb{C}^{\mathrm{d}_\mathbb{K}}$. Put

$$\varrho_s := \mathrm{Hom}_{\mathrm{H}(E_0)}(\omega_X, \sigma \widehat{\otimes} \tau_s),$$

equipped with the compact open topology. It is an irreducible Casselman–Wallach representation of M'_X, and we have (*cf.* [20])

$$\sigma \widehat{\otimes} \tau_s \cong \omega_X \widehat{\otimes} \varrho_s \tag{15}$$

as representations of $\mathrm{H}(E_0) \rtimes \mathrm{M}'_X$.

Then as a $\mathbb{J}'(E)$-representation,

$$\begin{aligned}
\mathrm{I}_s &= \mathrm{ind}_{\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \sigma \widehat{\otimes} \tau_s \\
&\cong \mathrm{ind}_{\mathrm{H}(E)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \big(\mathrm{ind}_{\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X}^{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \sigma \widehat{\otimes} \tau_s\big) \quad \text{by (6)} \\
&\cong \mathrm{ind}_{\mathrm{H}(E)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \big(\mathrm{ind}_{\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X}^{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \omega_X \widehat{\otimes} \varrho_s\big) \quad \text{by (15)} \\
&\cong \mathrm{ind}_{\mathrm{H}(E)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \big(\big(\mathrm{ind}_{\mathrm{H}(X^\perp)\rtimes \mathrm{P}'_X}^{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \omega_X\big) \widehat{\otimes} \varrho_s\big) \quad \text{by Lemma 3.2} \\
&\cong \mathrm{ind}_{\mathrm{H}(E)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} (\omega|_{\mathrm{H}(E)\rtimes \mathrm{P}'_X} \widehat{\otimes} \varrho_s) \quad \text{by (14)} \\
&\cong \omega \widehat{\otimes} \mathrm{ind}_{\mathrm{H}(E)\rtimes \mathrm{P}'_X}^{\mathbb{J}'(E)} \varrho_s \quad \text{by Lemma 3.2} \\
&= \omega \widehat{\otimes} \mathrm{ind}_{\mathrm{P}'_X}^{\mathrm{U}'(E)} \varrho_s.
\end{aligned}$$

By using Langlands classification and the result of Speh–Vogan [18, Theorem 1.1], we know that except for a measure zero set of $s \in \mathbb{C}^{\mathrm{d}_\mathbb{K}}$, the parabolic induction $\mathrm{ind}_{\mathrm{P}'_X}^{\mathrm{U}'(E)} \varrho_s$ is an irreducible Casselman–Wallach representation of $\mathrm{U}'(E)$. This finishes the proof by the argument in the last paragraph of Section 3.3. □

Fix a nonzero element λ of the one-dimensional space $\mathrm{Hom}_{\mathrm{N}_\mathcal{F}(X)}(\tau, \psi^{-1})$. It induces a continuous linear map

$$\Lambda : \sigma \widehat{\otimes} \tau \to \sigma, \qquad u \otimes v \mapsto \lambda(v)u.$$

Let π be an irreducible Casselman–Wallach representation of $\mathrm{U}'(E)$ as in Theorem 4.1, and let

$$\langle\ ,\ \rangle_\mu : \pi \times \sigma \to \mathbb{C}$$

be a continuous bilinear map which represents an element $\mu \in \mathrm{Hom}_{\mathrm{J}'_{\mathcal{F}}(E)}(\pi \widehat{\otimes} \sigma, \psi)$. As before, let $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$. For every $f \in \mathrm{I}_s$ and $u \in \pi$, consider the following function on $\mathrm{U}'(E)$:

$$g \mapsto \langle g.u, (\Lambda \circ f)(y_r^{-1} g)\rangle_\mu. \tag{16}$$

Here, both $g \in \mathrm{U}'(E)$ and $y_r \in E \subset \mathrm{H}(E)$ are viewed as elements in $\mathbb{J}'(E) = \mathrm{H}(E) \rtimes \mathrm{U}'(E)$, and f is viewed as a $\sigma \widehat{\otimes} \tau$-valued function ($\tau_s = \tau$ as vector spaces). It follows from Lemma 4.2 that the function (16) is left $\mathrm{J}'_{\mathcal{F}}(E)$-invariant.

The following zeta integrals, which appear as local factors of certain global Rankin–Selberg integrals (*cf.* [8, Eq. (38)]), play a key role in the proof of Theorem A:

$$\mathrm{Z}_\mu(f, u) := \int_{\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)} \langle g.u, (\Lambda \circ f)(y_r^{-1} g)\rangle_\mu \, \mathrm{d}g, \quad f \in \mathrm{I}_s,\ u \in \pi,$$

where $\mathrm{d}g$ is a fixed right $\mathrm{U}'(E)$-invariant positive Borel measure on $\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)$.

We postpone the proof of the following result to Section 5.

Proposition 4.4. *Assume that $\mu \neq 0$. Then for every $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$, there are elements $f \in \mathrm{I}_s$ and $u \in \pi$ such that the integral $\mathrm{Z}_\mu(f, u)$ is absolutely convergent and nonzero.*

For every $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$, denote by $\mathrm{Re}\, s \in \mathbb{R}^{\mathrm{d}_{\mathbb{K}}}$ its componentwise real part. We write $\mathrm{Re}\, s > c$ for a real number c if every component of $\mathrm{Re}\, s$ is $> c$. In Section 6, we prove the following

Proposition 4.5. *There is a real constant c_μ, depending on π, σ, τ and μ, such that for every $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$ with $\mathrm{Re}\, s > c_\mu$, the integral $\mathrm{Z}_\mu(f, u)$ is absolutely convergent for every $f \in \mathrm{I}_s$ and $u \in \pi$, and defines a $\mathrm{U}'(E)$-invariant continuous linear functional on $\mathrm{I}_s \widehat{\otimes} \pi$.*

Now we are ready to prove Theorem 4.1, as in the discussion of [11, Section 3.4]. Let F be a finite dimensional subspace of $\mathrm{Hom}_{\mathrm{J}'_{\mathcal{F}}(E)}(\pi \widehat{\otimes} \sigma, \psi)$. By Proposition 4.5, we have a linear map

$$F \to \mathrm{Hom}_{\mathrm{U}'(E)}(\mathrm{I}_s \widehat{\otimes} \pi, \mathbb{C}), \quad \mu \mapsto \mathrm{Z}_\mu$$

for $\mathrm{Re}\, s > c_F$, where c_F is a real constant depending on π, σ, τ and F. Moreover, by Proposition 4.4, the above map is an injection. In view of Proposition 4.3, choose s such that $\mathrm{Re}\, s > c_F$ and I_s is irreducible. Then $\dim \mathrm{Hom}_{\mathrm{U}'(E)}(\mathrm{I}_s \widehat{\otimes} \pi, \mathbb{C}) \leqslant 1$ by Theorem B. Therefore $\dim_{\mathbb{C}} F \leqslant 1$ and Theorem 4.1 is proved.

5. Proof of Proposition 4.4

We continue with the notation of the last section. Denote by P'_Y the parabolic subgroup of $\mathrm{U}'(E)$ stabilizing Y. It has a Levi decomposition

$$\mathrm{P}'_Y = \frac{\mathrm{U}'(E_0) \times \mathrm{GL}'(Y)}{\Delta \mu_{\mathbb{K}}} \ltimes \mathrm{N}_Y,$$

where N_Y is the unipotent radical, and $\mathrm{GL}'(Y)=\mathrm{GL}'(X)$ is the subgroup of $\mathrm{U}'(E)$ stabilizing X and Y and fixing E_0 pointwise. Denote by $\mathrm{P}'_{y_r}(Y)$ the subgroup of $\mathrm{GL}'(Y)$ fixing y_r, and by $\mathrm{P}'_{Y_{r-1}}(Y)$ the subgroup of $\mathrm{GL}'(Y)$ fixing Y_{r-1} pointwise. Then the multiplication map

$$\frac{\mathrm{P}'_{y_r}(Y)\times \mathrm{P}'_{Y_{r-1}}(Y)}{\Delta\mu_{\mathbb{K}}}\to \mathrm{GL}'(Y)$$

is an open embedding, and its image has full measure in $\mathrm{GL}'(Y)$. It is also routine to check that the map

$$\frac{(\mathrm{H}(X^{\perp})\rtimes \mathrm{P}'_X)\times (\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)}{\Delta\mu_{\mathbb{K}}}\to \mathbb{J}'(E),\qquad (g,h)\mapsto g y_r^{-1}h \tag{17}$$

is an open embedding.

Take two vectors $u_\sigma\in\sigma$ and $u_s\in\tau_s$. Take a compactly supported smooth function ϕ on $\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y$ so that

$$\phi(zh)=\chi_\tau(z)\phi(h),\quad z\in\mu_{\mathbb{K}},\ h\in \mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y.$$

Recall that $\sigma\widehat{\otimes}\tau_s$ is a representation of $\mathrm{H}(E_0)\rtimes \mathrm{M}'_X$, and is viewed as a representation of $\mathrm{H}(X^{\perp})\rtimes \mathrm{P}'_X$ by inflation. Put

$$\phi'(g,h):=\phi(h)\big(g.(u_\sigma\otimes u_s)\big),\quad g\in \mathrm{H}\big(X^{\perp}\big)\rtimes \mathrm{P}'_X,\ h\in \mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y.$$

Extension by zero of ϕ' through (17) yields a $\sigma\widehat{\otimes}\tau_s$-valued smooth function f on $\mathbb{J}'(E)$. By Lemma 3.1, $f\in \mathrm{I}_s$.

It is elementary to see that there is a positive smooth function γ_r on $(\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y)$ so that

$$\int_{\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)}\varphi(g)\,\mathrm{d}g=\int_{((\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y))\times(\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)}\gamma_r(h)\varphi(hk)\,\mathrm{d}h\,\mathrm{d}k,$$

for all nonnegative continuous functions φ on $\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)$, where $\mathrm{d}h$ is a right $\mathrm{P}'_{y_r}(Y)$-invariant positive Borel measure on $(\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y)$, and $\mathrm{d}k$ is a right invariant Haar measure on $\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y$.

For every $u\in\pi$, we have that

$$\begin{aligned}\mathrm{Z}_\mu(f,u)&=\int_{\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)}\big\langle g.u,(\Lambda\circ f)\big(y_r^{-1}g\big)\big\rangle_\mu\,\mathrm{d}g\\&=\int_{((\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y))\times(\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)}\gamma_r(h)\big\langle (hk).u,(\Lambda\circ f)\big(y_r^{-1}hk\big)\big\rangle_\mu\,\mathrm{d}h\,\mathrm{d}k\end{aligned}$$

$$= \int_{((\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y))\times(\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)} \gamma_r(h)\big\langle (hk).u, \Lambda\big(f\big(hy_r^{-1}k\big)\big)\big\rangle_\mu \,\mathrm{d}h\,\mathrm{d}k$$

$$= \int_{((\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y))\times(\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)} \gamma_r(h)\,\phi(k)\big\langle (hk).u, \Lambda\big(u_\sigma\otimes\tau_s(h)u_s\big)\big\rangle_\mu \,\mathrm{d}h\,\mathrm{d}k$$

$$= \int_{((\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y))\times(\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y)} \lambda\big(\tau_s(h)u_s\big)\Phi(h,k)\,\mathrm{d}h\,\mathrm{d}k,$$

where

$$\Phi(h,k) := \gamma_r(h)\phi(k)\big\langle (hk).u, u_\sigma\big\rangle_\mu, \quad h\in \mathrm{P}'_{y_r}(Y),\ k\in \mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y.$$

Choose ϕ, u and u_σ appropriately so that the function

$$\Psi(h) := \int_{\mathrm{P}'_{Y_{r-1}}(Y)\ltimes \mathrm{N}_Y} \Phi(h,k)\,\mathrm{d}k$$

does not vanish at 1.

Note that the smooth function Ψ on $\mathrm{P}'_{y_r}(Y)$ satisfies

$$\Psi(bzh) = \psi(b)\chi_\tau(z)\Psi(h), \quad b\in \mathrm{N}_{\mathcal{F}}(X),\ z\in\mu_{\mathbb{K}},\ h\in \mathrm{P}'_{y_r}(Y).$$

Recall from [10, Section 3] that for every smooth function W on $\mathrm{P}'_{y_r}(Y)$ such that

$$W(bzh) = \psi(b)^{-1}\chi_\tau(z)W(h), \quad b\in \mathrm{N}_{\mathcal{F}}(X),\ z\in\mu_{\mathbb{K}},\ h\in \mathrm{P}'_{y_r}(Y),$$

if W has compact support modulo $\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}}$, then there is a vector $u'_s\in\tau_s$ such that

$$W(h) = \lambda\big(\tau_s(h)u'_s\big), \quad h\in \mathrm{P}'_{y_r}(Y).$$

Therefore we may choose u_s appropriately so that the function $h\mapsto \lambda(\tau_s(h)u_s)$ on $\mathrm{P}'_{y_r}(Y)$ has compact support modulo $\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}}$, and that

$$\int_{(\mathrm{N}_{\mathcal{F}}(X)\times\mu_{\mathbb{K}})\backslash \mathrm{P}'_{y_r}(Y)} \lambda\big(\tau_s(h)u_s\big)\Psi(h)\,\mathrm{d}h \neq 0. \tag{18}$$

Note that the integral $\mathrm{Z}_\mu(f,u)$ equals to the left-hand side of (18), and its integrant is smooth and compactly supported. This finishes the proof of Proposition 4.4.

6. Proof of Proposition 4.5

Extend $x_r \in X$ to a $\mathbb{K}$-basis $\{x_1, x_2, \dots, x_r\}$ of X so that $x_i \in X_i$ for $i = 1, \dots, r$. Under this basis, $(\mathbb{K}_+^\times)^r$ embeds in $\mathrm{GL}(X)$:

$$(\mathbb{K}_+^\times)^r \subset (\mathbb{K}^\times)^r = \prod_{i=1}^{r} \mathrm{GL}(\mathbb{K}x_i) \hookrightarrow \mathrm{GL}(X).$$

Since the covering $\mathrm{GL}'(X) \to \mathrm{GL}(X)$ uniquely splits over $(\mathbb{K}_+^\times)^r$, it also embeds in $\mathrm{GL}'(X)$. For every $\mathbf{t} = (t_1, t_2, \dots, t_r) \in (\mathbb{K}_+^\times)^r$, denote by $a_{\mathbf{t}}$ the corresponding element in $\mathrm{GL}'(X)$, and put

$$\|\mathbf{t}\| := \prod_{i=1}^{r} \varpi\left(t_i + t_i^{-1}\right) \quad \text{and} \quad \xi(\mathbf{t}) := \prod_{i=1}^{r-1} \varpi\left(1 + \frac{t_i}{t_{i+1}}\right),$$

where for every $t \in \mathbb{K}_+^\times$,

$$\varpi(t) := \begin{cases} t & \text{if } \mathrm{d}_{\mathbb{K}} = 1; \\ t't'' & \text{if } \mathrm{d}_{\mathbb{K}} = 2 \text{ and } t = (t', t''). \end{cases}$$

Fix a maximal compact subgroup K of $\mathrm{U}'(E)$. It is elementary to see that there is a positive character δ_r on $(\mathbb{K}_+^\times)^r$ such that

$$\int_{\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)} \varphi(g)\,\mathrm{d}g = \int_{(\mathbb{K}_+^\times)^r \times K} \delta_r(\mathbf{t})\varphi(a_{\mathbf{t}}k)\,\mathrm{d}^\times\mathbf{t}\,\mathrm{d}k \tag{19}$$

for all nonnegative continuous functions φ on $\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)$, where $\mathrm{d}k$ is the normalized Haar measure on K, and $\mathrm{d}^\times\mathbf{t}$ is an appropriate Haar measure on $(\mathbb{K}_+^\times)^r$. Pick a positive constant c_0 so that

$$\delta_r(\mathbf{t}) \leqslant \|\mathbf{t}\|^{c_0}, \quad \mathbf{t} \in (\mathbb{K}_+^\times)^r. \tag{20}$$

Recall that

$$\langle\,,\,\rangle_\mu : \pi \times \sigma \to \mathbb{C}$$

is a continuous bilinear map which represents an element $\mu \in \mathrm{Hom}_{\mathrm{J}'_{\mathcal{F}}(E)}(\pi \widehat{\otimes} \sigma, \psi)$. Pick a continuous seminorm $|\cdot|_{\pi,1}$ on π and a continuous seminorm $|\cdot|_\sigma$ on σ so that

$$\left|\langle u, v\rangle_\mu\right| \leqslant |u|_{\pi,1}|v|_\sigma, \quad u \in \pi,\ v \in \sigma. \tag{21}$$

By the moderate growth condition on π, there is a constant $c_1 > 0$ and a continuous seminorm $|\cdot|_{\pi,2}$ on π such that

$$\left|(a_{\mathbf{t}}k).u\right|_{\pi,1} \leqslant \|\mathbf{t}\|^{c_1}|u|_{\pi,2}, \quad \mathbf{t} \in (\mathbb{K}_+^\times)^r,\ k \in K,\ u \in \pi. \tag{22}$$

Recall that $\lambda \in \mathrm{Hom}_{\mathrm{N}_{\mathcal{F}(X)}}(\tau, \psi^{-1})$ induces a continuous linear map

$$\Lambda : \sigma \widehat{\otimes} \tau \to \sigma, \quad u \otimes v \mapsto \lambda(v)u.$$

Still denote by τ the following continuous linear action of $\mathrm{GL}'(X)$ on $\sigma \widehat{\otimes} \tau$:

$$\tau(h)(u \otimes v) := u \otimes \bigl(\tau(h)v\bigr), \quad u \in \sigma,\ v \in \tau.$$

The moderate growth condition on τ implies that there are a constant $c_2 > 0$ and a continuous seminorm $|\cdot|_{\sigma \widehat{\otimes} \tau}$ on $\sigma \widehat{\otimes} \tau$ such that

$$\bigl|\Lambda\bigl(\tau(a_{\mathbf{t}})w\bigr)\bigr|_\sigma \leqslant \|\mathbf{t}\|^{c_2} |w|_{\sigma \widehat{\otimes} \tau}, \quad \mathbf{t} \in \bigl(\mathbb{K}_+^\times\bigr)^r,\ w \in \sigma \widehat{\otimes} \tau.$$

Lemma 6.1. *For every positive integer N, there is a continuous seminorm $|\cdot|_{\sigma \widehat{\otimes} \tau, N}$ on $\sigma \widehat{\otimes} \tau$ such that*

$$\bigl|\Lambda\bigl(\tau(a_{\mathbf{t}})w\bigr)\bigr|_\sigma \leqslant \xi(\mathbf{t})^{-N} \|\mathbf{t}\|^{c_2} |w|_{\sigma \widehat{\otimes} \tau, N}, \quad \mathbf{t} \in \bigl(\mathbb{K}_+^\times\bigr)^r,\ w \in \sigma \widehat{\otimes} \tau. \tag{23}$$

Proof. This is similar to the proof of [11, Lemma 6.2]. We omit the details. □

Put $c_\mu := c_0 + c_1 + c_2$. Recall that $s \in \mathbb{C}^{\mathrm{d}_{\mathbb{K}}}$.

Lemma 6.2. *If* $\mathrm{Re}\, s > c_\mu$*, then there is a positive integer N such that*

$$\int_{(\mathbb{K}_+^\times)^r} \|\mathbf{t}\|^{c_\mu} \Pi(\mathbf{t})^{\mathrm{Re}\, s} \xi(\mathbf{t})^{-N} \varpi(1 + t_r)^{-N}\, \mathrm{d}^\times \mathbf{t} < \infty, \tag{24}$$

where

$$\mathbf{t} = (t_1, t_2, \ldots, t_r) \quad \textit{and} \quad \Pi(\mathbf{t}) := \prod_{i=1}^r t_i \in \mathbb{K}_+^\times. \tag{25}$$

Proof. We assume that $\mathrm{d}_{\mathbb{K}} = 1$. The other case obviously follows from this one. Write

$$\alpha_i := \frac{t_i}{t_{i+1}}, \quad i = 1, \ldots, r-1; \qquad \alpha_r := t_r.$$

Then

$$\begin{cases} \|\mathbf{t}\| \leqslant \prod_{i=1}^r \bigl(\alpha_i + \alpha_i^{-1}\bigr)^i, \\ \Pi(\mathbf{t}) = \prod_{i=1}^r \alpha_i^i, \\ \xi(\mathbf{t})\varpi(1 + t_r) = \prod_{i=1}^r (1 + \alpha_i). \end{cases}$$

Therefore the left-hand side of (24) is at most

$$\prod_{i=1}^{r} \int_{\mathbb{R}_+^\times} \left(\alpha_i + \alpha_i^{-1}\right)^{ic_\mu} \alpha_i^{i \operatorname{Re} s} (1+\alpha_i)^{-N} \, \mathrm{d}^\times \alpha_i,$$

where $\mathrm{d}^\times \alpha_i$ is an appropriate Haar measure on $\mathbb{R}_+^\times$, $i = 1, 2, \ldots, r$. It is elementary to see that if $N > r(c_\mu + \operatorname{Re} s)$, then

$$\int_{\mathbb{R}_+^\times} \left(\alpha_i + \alpha_i^{-1}\right)^{ic_\mu} \alpha_i^{i \operatorname{Re} s} (1+\alpha_i)^{-N} \, \mathrm{d}^\times \alpha_i < \infty, \quad i = 1, 2, \ldots, r.$$

This finishes the proof. □

Now assume that $\operatorname{Re} s > c_\mu$. Let $f \in \mathrm{I}_s$, to be viewed as a $\sigma \widehat{\otimes} \tau$-valued function on $\mathbb{J}'(E)$, and let $u \in \pi$. We want to show that the integral $\mathrm{Z}_\mu(f, u)$ is absolutely convergent and defines a $\mathrm{U}'(E)$-invariant continuous linear functional of $\mathrm{I}_s \widehat{\otimes} \pi$. The $\mathrm{U}'(E)$-invariance is obvious as soon as the absolute convergence is proved.

We have

$$\begin{aligned}
\left|\mathrm{Z}_\mu(f, u)\right| &\leqslant \int_{\mathrm{J}'_{\mathcal{F}}(E)\backslash \mathrm{U}'(E)} \left|\left\langle g.u, \, (\Lambda \circ f)\left(y_r^{-1} g\right)\right\rangle_\mu\right| \mathrm{d}g \\
&= \int_{(\mathbb{K}_+^\times)^r \times K} \delta_r(\mathbf{t}) \left|\left\langle (a_{\mathbf{t}} k).u, \, (\Lambda \circ f)\left(y_r^{-1} a_{\mathbf{t}} k\right)\right\rangle_\mu\right| \mathrm{d}^\times \mathbf{t} \, \mathrm{d}k \quad \text{by (19)} \\
&\leqslant \int_{(\mathbb{K}_+^\times)^r \times K} \|\mathbf{t}\|^{c_0} \left|(a_{\mathbf{t}} k).u\right|_{\pi,1} \left|(\Lambda \circ f)\left(y_r^{-1} a_{\mathbf{t}} k\right)\right|_\sigma \mathrm{d}^\times \mathbf{t} \, \mathrm{d}k \quad \text{by (20) and (21)} \\
&\leqslant \int_{(\mathbb{K}_+^\times)^r \times K} \|\mathbf{t}\|^{c_0 + c_1} |u|_{\pi,2} \left|(\Lambda \circ f)\left(y_r^{-1} a_{\mathbf{t}} k\right)\right|_\sigma \mathrm{d}^\times \mathbf{t} \, \mathrm{d}k \quad \text{by (22).}
\end{aligned}$$

Let N be a positive integer as in Lemma 6.2, and let $|\cdot|_{\sigma \widehat{\otimes} \tau, N}$ be a continuous seminorm on $\sigma \widehat{\otimes} \tau$ as in Lemma 6.1. Then for every $\mathbf{t} = (t_1, t_2, \ldots, t_r) \in (\mathbb{K}_+^\times)^r$ and $k \in K$, we have

$$\begin{aligned}
\left|(\Lambda \circ f)\left(y_r^{-1} a_{\mathbf{t}} k\right)\right|_\sigma &= \left|\Pi(\mathbf{t})^s\right| \left|\Lambda\left(\tau(a_{\mathbf{t}}) f\left(a_{\mathbf{t}}^{-1} y_r^{-1} a_{\mathbf{t}} k\right)\right)\right|_\sigma \quad \Pi(\mathbf{t}) \text{ is as in (25)} \\
&\leqslant \Pi(\mathbf{t})^{\operatorname{Re} s} \xi(\mathbf{t})^{-N} \|\mathbf{t}\|^{c_2} \left|f\left(a_{\mathbf{t}}^{-1} y_r^{-1} a_{\mathbf{t}} k\right)\right|_{\sigma \widehat{\otimes} \tau, N} \quad \text{by (23)} \\
&= \Pi(\mathbf{t})^{\operatorname{Re} s} \xi(\mathbf{t})^{-N} \|\mathbf{t}\|^{c_2} \left|f\left((-t_r y_r) k\right)\right|_{\sigma \widehat{\otimes} \tau, N} \\
&\leqslant \Pi(\mathbf{t})^{\operatorname{Re} s} \xi(\mathbf{t})^{-N} \|\mathbf{t}\|^{c_2} \varpi (1 + t_r)^{-N} |f|_{\mathrm{I}_s, N}.
\end{aligned}$$

Here

$$|f|_{\mathrm{I}_s,N} := \sup\left\{\varpi(1+t)^N \left|f\left((-ty_r)k\right)\right|_{\sigma\,\widehat{\otimes}\,\tau,N} \;\middle|\; t \in \mathbb{K}_+^\times,\ k \in K\right\}.$$

It is easy to see that $|\cdot|_{\mathrm{I}_s,N}$ is a continuous seminorm on I_s.

Therefore

$$\left|\mathrm{Z}_\mu(f,u)\right| \leqslant |f|_{\mathrm{I}_s,N}|u|_{\pi,2} \int\limits_{(\mathbb{K}_+^\times)^r} \|\mathbf{t}\|^{c_\mu} \Pi(\mathbf{t})^{\operatorname{Re} s} \xi(\mathbf{t})^{-N} \varpi(1+t_r)^{-N}\, \mathrm{d}^\times \mathbf{t},$$

and Proposition 4.5 follows by (24).

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