SURVEY ON AUTOMORPHIC PERIOD FOR GENERAL BESSEL MODEL

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In this note, we introduce the refined conjecture on the automorphic period appeared in the study of Bessel model, which is a special sort of restriction problems of automorphic representations in the sense of (Gan–)Gross–Prasad [GP92, GP94, GGP12]. We also summarize some recent progress and approaches toward this problem.

1. Automorphic period

Let F be a number field with ring of adèles $\mathbb{A}$ and the set of places Σ. Fix a nontrivial character $\psi\colon F\backslash\mathbb{A}\to\mathbb{C}^\times$. Throughout the note, for a linear algebraic group G over F, we always use the Tamagawa measure $\mathrm{d}g$ on $G(\mathbb{A})$. In particular, the volume of $G(F)\backslash G(\mathbb{A})$ is 1 if G is unipotent.

Consider a reductive group G over F and a closed F-subgroup H. Denote $\mathcal{A}(G)$ the space of automorphic forms on $G(\mathbb{A})$ and $\mathcal{A}_{\mathrm{cusp}}(G)$ the subspace of cusp forms, both being representations of $G(\mathbb{A})$ via the right translation. Let

$$\nu\colon H(F)\backslash H(\mathbb{A})\to\mathbb{C}^\times$$

be an automorphic character.

Definition 1. For $\varphi\in\mathcal{A}(G)$, we define the following integral

$$\mathcal{P}_{H,\nu}(\varphi)=\int_{H(F)\backslash H(\mathbb{A})}\varphi(h)\nu(h)^{-1}\mathrm{d}h$$

to be the (H,ν)-*period* of φ, as long as it is absolutely convergent.

In what follows, we will mainly focus on the subspace $\mathcal{A}_{\mathrm{cusp}}(G)$ of cusp forms and the integral defining $\mathcal{P}_{H,\nu}(\varphi)$ is always absolutely convergent.

Definition 2. Let $\pi\subset\mathcal{A}_{\mathrm{cusp}}(G)$ be an irreducible (cuspidal automorphic) representation of $G(\mathbb{A})$. We say π is (globally) (H,ν)-*distinguished* if the linear functional $\mathcal{P}_{H,\nu}$ is nontrivial on π; in other words, $\mathcal{P}_{H,\nu}(\varphi)\neq 0$ for some $\varphi\in\pi$.

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It is clear that if π is (H,ν)-distinguished, then

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi,\nu)\neq\{0\},$$

where π is regarded as a representation of $H(\mathbb{A})$ by restriction.

Example 3 (Global genericity)**.** Let G be a quasi-split reductive group over F. Choose a Borel subgroup B of G (defined over F) with the unipotent radical U. Let $\nu\colon U(F)\backslash U(\mathbb{A})\to\mathbb{C}^\times$ be a generic character. Then $\mathcal{P}_{U,\nu}(\varphi)$ is absolutely convergent for $\varphi\in\mathcal{A}(G)$, which we usually call the (U,ν)-*Whittaker–Fourier coefficient* of φ. For an irreducible cuspidal representation $\pi\subset\mathcal{A}_{\mathrm{cusp}}(G)$, we say π is *globally generic* with respect to the pair (U,ν) if it is (U,ν)-distinguished.

Example 4 (Rankin–Selberg convolution)**.** Take $n,m,r\geq 0$ with $r\leq n-m-1$. Put $G=\mathrm{GL}(n)_F\times\mathrm{GL}(m)_F$. Consider the subgroup $H_r\subset\mathrm{GL}(n)_F$ consisting of matrices of the form

$$h=\begin{pmatrix} g_m & & & & & \star & \cdots & \star \\ \star & 1 & u_{1,2} & \cdots & \star & \star & \cdots & \star \\ \vdots & & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \star & & & 1 & u_{r,r+1} & \star & \cdots & \star \\ & & & & 1 & u_{r+1,r+2} & \cdots & \star \\ & & & & & 1 & \ddots & \vdots \\ & & & & & & \ddots & u_{n-m-1,n-m} \\ & & & & & & & 1 \end{pmatrix}$$

with $g_m\in\mathrm{GL}(m)_F$. We may view H_r as a subgroup of G by sending h to (h,g_m). Define

$$\nu(h)=\psi(u_{1,2}+\cdots+u_{n-m-1,n-m})$$

to be an automorphic character of $H_r(\mathbb{A})$. Then $\mathcal{P}_{H_r,\nu}(\varphi)$ is absolutely convergent for $\varphi\in\mathcal{A}_{\mathrm{cusp}}(G)$. When $r=0$, it is simply the Rankin–Selberg integral studied in [JPSS83]. In general, we prove in [Liua] that for an irreducible cuspidal representation $\pi\subset\mathcal{A}_{\mathrm{cusp}}(G)$, π is (H_r,ν)-distinguished if and only if $L(1/2,\pi,\mathrm{RS})\neq 0$, where $\mathrm{RS}\colon {}^LG\to{}^L\mathrm{GL}(nm)_F$ is the tensor product homomorphism.

Example 5 (Unitary base change)**.** Let E/F be an étale algebra of degree 2. Put $G=\mathrm{Res}_{E/F}\,\mathrm{GL}(n)_E/\mathbf{G}_{m,F}$ and $H=\mathrm{GL}(n)_F/\mathbf{G}_{m,F}\subset G$, where $\mathbf{G}_{m,F}$ is identified with the subgroup of F-rational scalar matrices. Denote $\eta_{E/F}\colon F^\times\backslash\mathbb{A}^\times\to\{\pm1\}$ the quadratic character associated to E/F via global class field theory. Put

$$\nu=\eta_{E/F}^{n+1}\circ\det\colon H(F)\backslash H(\mathbb{A})\to\mathbb{C}^\times$$

which is well-defined. Then $\mathcal{P}_{H,\nu}(\varphi)$ is absolutely convergent for $\varphi \in \mathcal{A}_{\mathrm{cusp}}(G)$. Moreover, Y. Z. Flicker [Fli88] proved that for an irreducible cuspidal representation $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$, π is (H,ν)-distinguished if and only if the partial L-function $L^S(s,\pi,\mathrm{As}^{(-1)^{n+1}})$ has a (simple) pole at $s=1$, where $\mathrm{As}^{\pm}\colon {}^LG \to {}^L\mathrm{GL}(n^2)_F$ is the Asai (resp. twisted Asai) representation. Moreover, one expects that π is (H,ν)-distinguished if and only if it is the standard base change of a (cuspidal) automorphic representation of the quasi-split unitary group of n-variables defined by E/F.

Example 6 (Shalika period)**.** Put $G = \mathrm{GL}(2n)_F$. Let $H \subset G$ be the subgroup consisting of matrices

$$h = \begin{pmatrix} g & \\ X & g \end{pmatrix}, \quad g \in \mathrm{GL}(n)_F, X \in \mathrm{Mat}(n)_F.$$

Define the character $\nu\colon H(F)\backslash H(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\nu(h) = \psi(\mathrm{tr}\, X).$$

Then $\mathcal{P}_{H,\nu}(\varphi)$ is absolutely convergent for $\varphi \in \mathcal{A}_{\mathrm{cusp}}(G)$. By a well-known theorem of Jacquet and Shalika [JS90], for an irreducible cuspidal representation $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$, π is (H,ν)-distinguished if and only if the partial L-function $L^S(s,\pi,\wedge^2)$ has a (simple) pole at $s=1$, where $\wedge^2\colon {}^L\mathrm{GL}(2n)_F \to {}^L\mathrm{GL}(n(2n-1))_F$ is the wedge square representation. Moreover, one expects that π is (H,ν)-distinguished if and only if it is the functorial lift of a (cuspidal) automorphic representation of $\mathrm{SO}(n,n+1)_F$.

2. Bessel period and multiplicity one

Let E/F be a field extension of degree at most 2. Denote c the unique automorphism of E such that $E^{c=1} = F$. A E-hermitian space is a E-vector space V equipped with a nondegenerate pairing $(\ ,\ )\colon V \times V \to E$ (the hermitian form) that is E-linear in the first variable and satisfies $(u,v) = (v,u)^c$ for all $u,v \in V$. Denote $\mathrm{U}(V)$ the neutral connected component of the group of isometries of $V,(\ ,\ )$, which is a connected reductive group over F. More explicitly, $\mathrm{U}(V)$ is the special orthogonal (resp. unitary) group attached to $V,(\ ,\ )$ when $E=F$ (resp. $[E:F]=2$). We will fix a E-hermitian space $L = \mathrm{Span}_E\{x_0\}$ of dimension 1.

Consider a E-hermitian space $V,(\ ,\ )$ of dimension $m \geq 0$. Let $r \geq 0$. Put

$$X = \mathrm{Span}_E\{x_1,\dots,x_r\}, \quad X^* = \mathrm{Span}_E\{x_1^*,\dots,x_r^*\}$$

and equip $X \oplus X^*$ with a hermitian form such that $(x_i, x_j^*) = \delta_{ij}$ for $1 \leq i,j \leq r$. Put

$$V^\sharp = L \oplus V \oplus X \oplus X^*$$

as a E-hermitian space whose hermitian form is denoted by $(\ ,\ )^\sharp$. In particular, the dimension of $V^\sharp$ is $n = m+2r+1$. The group $G_0 = \mathrm{U}(V)$ is naturally a closed subgroup of $G_1 = \mathrm{U}(V^\sharp)$. Put $G = G_1 \times G_0$. Now we are going to define the subgroup $H = H_r$ of G. Consider the following complete flag of E-subspaces of X:

$$\{0\} = X_0 \subset X_1 \subset \cdots \subset X_r = X, \tag{2.1}$$

where $X_i = \mathrm{Span}_E\{x_r, \dots, x_{r+1-i}\} \subset X$ for $i = 0, \dots, r$. Let $P_r \subset G_1$ be the parabolic subgroup stabilizing the flag (2.1) and U_r its unipotent radical. The group G_0 acts on U_r by conjugation in G_1. Define

$$H_r = U_r \rtimes G_0$$

to be the *r-th Bessel subgroup* of G_1. In what follows, we will view H_r as a subgroup of $G = G_1 \times G_0$ via the quotient homomorphism to the second component. We have the following homomorphism

$$\begin{aligned} \rho\colon H_r &\to \mathrm{Res}_{E/F}\,\mathbf{G}^r_{a,E} \\ h &\mapsto \left((hx_{r-1}, x_r^*)^\sharp, \cdots, (hx_0, x_1^*)^\sharp\right). \end{aligned} \tag{2.2}$$

Define a character $\nu_r\colon H_r(F)\backslash H_r(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\nu_r(h) = \psi(\mathrm{Tr}_{E/F}\,\rho(h)).$$

Example 7. Consider the case where $G_0 \simeq \mathrm{SO}(2,2)_F$ and $G_1 \simeq \mathrm{SO}(4,3)_F$ (in particular, $E = F$ and $r = 1$). We identify G_i with the group

$$\{g \in \mathrm{GL}(3i+4)_F \mid {}^t g \mathrm{J}_{3i+4} g = \mathrm{J}_{3i+4}\}$$

for $i = 0, 1$, where J_n is the n-by-n matrix with 1 on the non-principal diagonal and 0 elsewhere. We regard G_0 as a subgroup of G_1 via the embedding

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \mapsto \begin{pmatrix} 1 &&&&&& \\ & g_{11} & g_{12} && g_{13} & g_{14} & \\ & g_{21} & g_{22} && g_{23} & g_{24} & \\ &&& 1 &&& \\ & g_{31} & g_{32} && g_{33} & g_{34} & \\ & g_{41} & g_{42} && g_{43} & g_{44} & \\ &&&&&& 1 \end{pmatrix}.$$

Then H_1 consists of matrices in G_1 of the form

$$h = \begin{pmatrix} 1 & u_1 & u_2 & u_0 & u_3 & u_4 & 0 \\ & g_{11} & g_{12} & & g_{13} & g_{14} & u_4^* \\ & g_{21} & g_{22} & & g_{23} & g_{24} & u_3^* \\ & & & 1 & & & -u_0 \\ & g_{31} & g_{32} & & g_{33} & g_{34} & u_2^* \\ & g_{41} & g_{42} & & g_{43} & g_{44} & u_1^* \\ & & & & & & 1 \end{pmatrix},$$

where $u_i \in \mathbf{G}_{a,F}$ for $i = 0, 1, 2, 3, 4$,

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \in G_0$$

and $u_{5-i}^* = -(g_{i1}u_4 + g_{i2}u_3 + g_{i3}u_2 + g_{i4}u_1)$ for $i = 1, 2, 3, 4$. The homomorphism $\rho\colon H_1 \to \mathbf{G}_{a,F}$ is given by $h \mapsto u_0$.

Note that the only split group that G_0 could be is $\mathbf{G}_{m,F}$, in which case we must have $E = F$ and $m = 2$. In what follows, we will assume that G_0 is *not* split, and $n + [E : F] \geq 4$ (equivalently, G_1 is not abelian). Then $\mathcal{P}_{H_r,\nu_r}(\varphi)$ is absolutely convergent for $\varphi \in \mathcal{A}_{\text{cusp}}(G)$.

Example 8. When $r = 0$, that is, $n = m + 1$. The unipotent group U_0 is trivial and we have $H_0 = G_0$, diagonally embedded into G. Moreover, the character ν_0 is trivial, and

$$\mathcal{P}_{H_0,\nu_0}(\varphi) = \int_{G_0(F)\backslash G_0(\mathbb{A})} \varphi(g)\mathrm{d}g.$$

When $m + [E : F] \leq 2$, the group G_0 is trivial and G_1 is quasi-split. The group U_r is a maximal unipotent subgroup of G_1 and ν_r is generic. Therefore,

$$\mathcal{P}_{H_r,\nu_r}(\varphi) = \int_{U_r(F)\backslash U_r(\mathbb{A})} \varphi(u)\nu(u)^{-1}\mathrm{d}u$$

is simply the Whittaker–Fourier coefficient of φ. This is a special case of Example 3.

Let $\pi \simeq \otimes'_{v\in\Sigma}\pi_v \subset \mathcal{A}_{\text{cusp}}(G)$ be an irreducible representation of $G(\mathbb{A})$. The assignment $\varphi \in \pi \mapsto \mathcal{P}_{H_r,\nu_r}(\varphi)$ defines an element in

$$\mathrm{Hom}_{H_r(\mathbb{A})}(\pi, \nu) = \bigotimes_{v\in\Sigma} \mathrm{Hom}_{H_r(F_v)}(\pi_v, \nu_v).$$

The following is the *Multiplicity One Theorem* for the Bessel subgroup H_r.

Theorem 9 ([AGRS10, GGP12, JSZ10, SZ12, Wal12b])**.** *Let v be a place of F. For every irreducible smooth representation π_v of $G(F_v)$,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_r(F_v)}(\pi_v, \nu_v) \leq 1.$$

In particular, the dimension of the space $\mathrm{Hom}_{H_r(\mathbb{A})}(\pi, \nu)$ is at most 1. For a place v of F, it is known that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_r(F_v)}(\pi_v, \nu_v) = \dim_{\mathbb{C}} \mathrm{Hom}_{H_r(F_v)}(\tilde{\pi}_v, \nu_v^{-1}),$$

where $\tilde{\pi}_v$ is the contragredient of π_v. In the next section, we will construct a candidate for the generator of the space

$$\mathrm{Hom}_{H_r(F_v)}(\pi_v, \nu_v) \otimes_{\mathbb{C}} \mathrm{Hom}_{H_r(F_v)}(\tilde{\pi}_v, \nu_v^{-1}),$$

whose dimension is at most 1, under the assumption that π_v is tempered.

3. Local invariant functional

In this section, we will suppress the place v from notation. In particular, F is a local field; E/F is an étale algebra of degree at most 2; $G = G(F_v)$ and similarly for other groups. Let π be an irreducible smooth (resp. Casselman–Wallach) representation of G if F is non-archimedean (resp. archimedean), and $\tilde{\pi}$ its contragredient. Recall that we may associate to a pair of elements $\varphi \in \pi$ and $\tilde{\varphi} \in \tilde{\pi}$ a smooth function $\Phi_{\varphi,\tilde{\varphi}}(g) = \langle \pi(g)\varphi, \tilde{\varphi} \rangle$ on G, called a *matrix coefficient* of π, where $\langle\ ,\ \rangle \colon \pi \times \tilde{\pi} \to \mathbb{C}$ is the canonical pairing. We say π is *tempered* if for every K-finite φ and $\tilde{\varphi}$, $\Phi_{\varphi,\tilde{\varphi}}$ belongs to $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$. The function $\Phi_{\varphi,\tilde{\varphi}}$ may be viewed as a local analogue of automorphic form. Thus if one would like to mimic the definition of automorphic periods in the global case, the natural thinking would be the following integral

$$\int_{H_r} \Phi_{\varphi,\tilde{\varphi}}(h)\nu(h)^{-1}\mathrm{d}h. \tag{3.1}$$

Formally, it does define an element in $\mathrm{Hom}_{H_r}(\pi, \nu) \otimes_{\mathbb{C}} \mathrm{Hom}_{H_r}(\tilde{\pi}, \nu^{-1})$. Unfortunately, unlike the global case, the above integral is not absolutely convergent in general, except when $r = 0$ (which is studied originally by Ichino and Ikeda [II10] in the orthogonal case and later by N. Harris [Har] in the unitary case). Therefore, we need to regularize the integral (3.1).

We will provide two methods for such regularization. The first one works only for F non-archimedean which now describe. Thus assume F is non-archimedean. The following lemma is proved in [Liub].

Lemma 10. *Let π be an irreducible smooth representation of G. For every $\varphi \in \pi$ and $\tilde{\varphi} \in \tilde{\pi}$, there exists a compact open subgroup $\mathcal{U}$ of U_r such that for all other compact open subgroups $\mathcal{U}'$ of U_r containing $\mathcal{U}$,*

$$\int_{\mathcal{U}'} \Phi_{\varphi,\tilde{\varphi}}(u)\nu(u)^{-1}\mathrm{d}u = \int_{\mathcal{U}} \Phi_{\varphi,\tilde{\varphi}}(u)\nu(u)^{-1}\mathrm{d}u.$$

If we denote the integral on the right-hand side in the above proposition by

$$\int_{U_r}^{\mathrm{st}} \Phi_{\varphi,\tilde{\varphi}}(u)\nu(u)^{-1}\mathrm{d}u,$$

then one easily sees that it is independent of $\mathcal{U}$ that satisfies the property in the proposition. For $g \in G$, put

$$\mathcal{F}_\nu(\Phi_{\varphi,\tilde{\varphi}})(g) = \int_{U_r}^{\mathrm{st}} \Phi_{\pi(g)\varphi,\tilde{\varphi}}(u)\nu(u)^{-1}\mathrm{d}u$$

which is a smooth function on G. We have the following proposition.

Proposition 11 ([Liub, Theorem 2.1]). *Assume π is tempered. Then the integral*

$$\alpha(\varphi, \tilde{\varphi}) := \int_{G_0} \mathcal{F}_\nu(\Phi_{\varphi,\tilde{\varphi}})(g)\mathrm{d}g$$

is absolutely convergent, where G_0 is diagonally embedded into G and $\mathrm{d}h = \mathrm{d}u \cdot \mathrm{d}g$. In particular, the functional $(\varphi, \tilde{\varphi}) \mapsto \alpha(\varphi, \tilde{\varphi})$ defines an element in $\mathrm{Hom}_{H_r}(\pi, \nu) \otimes_{\mathbb{C}} \mathrm{Hom}_{H_r}(\tilde{\pi}, \nu^{-1})$.

The second way works for an arbitrary local field F of characteristic 0. Denote U_r° to be the kernel of $\rho|_{U_r}$ where ρ is defined in (2.2). Then ρ induces an isomorphism $U_r/U_r^\circ \xrightarrow{\sim} E^r$. One may view ν as an element in the dual space $\widehat{U_r/U_r^\circ} \simeq \widehat{E^r}$. In fact, it is in the regular locus of $\widehat{E^r}$, that is, those characters which are nontrivial on each component of E^r. We fix a decomposition $\mathrm{d}h = \mathrm{d}u \cdot \mathrm{d}u^\circ \cdot \mathrm{d}g$ of the Haar measure $\mathrm{d}h$ on H_r into Haar measures on U_r/U_r°, U_r° and G_0, respectively.

Proposition 12. *Assume π is tempered. For any K-finite vectors $\varphi \in \pi$ and $\tilde{\varphi} \in \tilde{\pi}$,*

(1) *the integral*

$$\mathfrak{a}_{\varphi,\tilde{\varphi}}(u) := \int_{U_r^\circ G_0} \Phi_{\varphi,\tilde{\varphi}}(uu^\circ g)\mathrm{d}u^\circ \mathrm{d}g$$

is absolutely convergent for every $u \in U_r/U_r^\circ$;

(2) *the measure $\mathfrak{a}_{\varphi,\tilde{\varphi}}(u)\mathrm{d}u$ is a tempered distribution on U_r/U_r°;*

(3) *the Fourier transform $\widehat{\mathfrak{a}_{\varphi,\tilde{\varphi}}}$, as a tempered distribution on $\widehat{U_r/U_r^\circ} \simeq \widehat{E^r}$, is smooth on the regular locus. In particular,*

$$\alpha(\varphi, \tilde{\varphi}) := \widehat{\mathfrak{a}_{\varphi,\tilde{\varphi}}}(\nu)$$

is well-defined.

The functional $(\varphi, \tilde{\varphi}) \mapsto \alpha(\varphi, \tilde{\varphi})$ defines an element in the space $\mathrm{Hom}_{H_r}(\pi, \nu) \otimes_{\mathbb{C}} \mathrm{Hom}_{H_r}(\tilde{\pi}, \nu^{-1})$.

In [Wal12a], Waldspurger also provided a regularization of the integral (3.1) when F is non-archimedean, using a way different from the previous two. It is not hard to see that in fact, three ways give rise to the same $\alpha(\varphi,\tilde{\varphi})$ when F is non-archimedean. We have the following properties about $\alpha(\varphi,\tilde{\varphi})$, in which (1) is proved by Waldspurger [Wal12a] and (2), (3) are proved in [Liub].

Theorem 13. *Let π be tempered and $\alpha(\varphi,\tilde{\varphi})$ as defined above.*

(1) *Assume $E=F$ that is non-archimedean. If $\dim_{\mathbb{C}}\mathrm{Hom}_{H_r}(\pi,\nu)=1$, then $\alpha(\varphi,\tilde{\varphi})\neq 0$ for some v and $\tilde{\varphi}$.*

(2) *Assume π is unitary, that is, $\tilde{\pi}\simeq\bar{\pi}$. Then for $\varphi\in\pi$,*

$$\alpha(\varphi,\bar{\varphi})\geq 0.$$

(3) *Assume F is non-archimedean and G is unramified. Choose a hyperspecial maximal subgroup K_G of G and assume that φ and $\tilde{\varphi}$ are both fixed by K_G with $\langle\varphi,\tilde{\varphi}\rangle=1$; ψ has the conductor $U_r\cap K_G$; and the volume of $K_G\cap H_r$ under $\mathrm{d}h$ is 1. Then*

$$\alpha(\varphi,\tilde{\varphi})=\frac{\Delta_{G_1}L(1/2,\pi,\mathrm{Std})}{L(1,\pi,\mathrm{Ad})},$$

where Δ_{G_1} is certain L-value of the Gross motive of G_1 [Gro97], whose explicit formula can be found in [Liub, Section 2].

We conjecture that part (1) in the above theorem holds in all other cases as well, especially when F is archimedean.

4. Distinguished packet

Let us go back to the global situation and thus F is a number field. Let $\mathfrak{V}$ be the set of all isomorphism classes of E-hermitian spaces of dimension m (and a fixed discriminant if $E=F$), which we call a *pure inner class*. For $\beta\in\mathfrak{V}$, denote V_β the corresponding E-hermitian space. We have groups $G_0^\beta=\mathrm{U}(V_\beta)$, $G_1^\beta=\mathrm{U}(V_\beta^\sharp)$, $G^\beta=G_1^\beta\times G_0^\beta$, $H_r^\beta\subset G^\beta$, and the character ν_r^β. Note that $G^\beta\simeq G^{\beta'}$ if and only if $\beta=\beta'$. For G^β, denote $\Pi(G^\beta)_{\mathrm{cusp}}^{\mathrm{temp}}$ the set of isomorphism classes of irreducible tempered cuspidal automorphic representations[1] of $G^\beta(\mathbb{A})$. By Arthur's conjecture, the multiplicity of any member of $\Pi(G^\beta)_{\mathrm{cusp}}^{\mathrm{temp}}$ in $\mathcal{A}_{\mathrm{cusp}}(G^\beta)$ is 1.

The Langlands dual group ${}^LG^\beta$ depends only on the pure inner class $\mathfrak{V}$, which we will denote by ${}^LG_{\mathfrak{V}}$ by abuse of notation. There is a conjectural Langlands group $\mathcal{L}_F$ associated to the field F. A (global) *tempered elliptic*

[1] An irreducible cuspidal automorphic representation $\pi\simeq\otimes'_{v\in\Sigma}\pi_v$ is tempered if π_v is tempered for all v.

Arthur parameter[2] for the pure inner class $\mathfrak{V}$ is an equivalence class of homomorphisms

$$\phi\colon \mathcal{L}_F \to {}^L G_{\mathfrak{V}}$$

such that the image is bounded, and

$$\mathcal{S}_\phi := \mathrm{Cent}_{{}^L G^0_{\mathfrak{V}}}(\mathrm{Im}\,\phi), \tag{4.1}$$

the centralizer of $\mathrm{Im}\,\phi$ in the neutral connected component ${}^L G^0_{\mathfrak{V}}$ of ${}^L G_{\mathfrak{V}}$, is finite. In fact, $\mathcal{S}_\phi$ will then be a finite 2-group in our case. Denote $\Phi(\mathfrak{V})^{\mathrm{temp}}_{\mathrm{ell}}$ the set of equivalent classes of tempered elliptic Arthur parameters for $\mathfrak{V}$. A part of Arthur's conjecture predicts a natural finite-to-one surjective map

$$\mathrm{rec}\colon \coprod_{\mathfrak{V}} \Pi(G^\beta)^{\mathrm{temp}}_{\mathrm{cusp}} \to \Phi(\mathfrak{V})^{\mathrm{temp}}_{\mathrm{ell}}$$

such that $\pi \in \mathrm{rec}^{-1}(\phi)$ if and only if they have the same partial (unramified) L-function

$$L^S(s,\pi,r) = L^S(s,\phi,r)$$

for any L-homomorphism $r\colon {}^L G_{\mathfrak{V}} \to {}^L\mathrm{GL}(N)_F$. Moreover, for any $v \in \Sigma$, $L(s,\phi_v,\mathrm{Std})L(2s,\phi_v,\mathrm{Ad})^{-1}$ is expected to be holomorphic and nonzero at $s = 1/2$. In what follows, we are going to assume all these conjectures. The inverse image $\Pi_\phi := \mathrm{rec}^{-1}(\phi)$ for an element $\phi \in \Phi(\mathfrak{V})^{\mathrm{temp}}_{\mathrm{ell}}$ is called a *tempered elliptic Vogan packet* for (the pure inner class) $\mathfrak{V}$.

Definition 14. Let Π_ϕ be a tempered elliptic Vogan packet for $\mathfrak{V}$. We say Π_ϕ is (globally) *Bessel distinguished* if there is $\beta \in \mathfrak{V}$ and $\varphi_\beta \in \pi_\beta \in \Pi_\phi \cap \Pi(G^\beta)^{\mathrm{temp}}_{\mathrm{cusp}}$ such that $\mathcal{P}_{H^\beta_r,\nu^\beta_r}(\varphi_\beta) \neq 0$.

The following conjecture is known as the global Gan–Gross–Prasad conjecture for Bessel periods.

Conjecture 15 ([GGP12])**.** *Let Π_ϕ be a tempered elliptic Vogan packet for $\mathfrak{V}$. Then Π_ϕ is Bessel distinguished if and only if $L(1/2,\phi,\mathrm{Std}) \neq 0$.*

The following result is a consequence of the recent work of Waldspurger [Wal10, Wal12a] (resp. Beuzart-Plessis [BP]) on the local Gan–Gross–Prasad conjecture for special orthogonal (resp. unitary) groups in the non-archimedean case.

Proposition 16. *Let Π_ϕ be a tempered elliptic Vogan packet for $\mathfrak{V}$. If there are two members π_β and $\pi'_{\beta'}$ that are (H^β_r,ν^β_r)- and $(H^{\beta'}_r,\nu^{\beta'}_r)$-distinguished, respectively, then their finite components are isomorphic; in other words, $(\pi_\beta)^\infty \simeq (\pi'_{\beta'})^\infty$ (in particular, $G^\beta(\mathbb{A}^\infty) \simeq G^{\beta'}(\mathbb{A}^\infty)$).*

[2] See [GGP12] for a formulation without using the Langlands group $\mathcal{L}_F$.

We expect that $\pi_\beta \simeq \pi'_{\beta'}$ (in particular, $\beta = \beta'$), which will follow by a similar result of Waldspurger and Beuzart-Plessis in the archimedean local case.

5. Refined Gan–Gross–Prasad conjecture and examples

Let $\pi \simeq \otimes'_{v\in\Sigma}\pi_v \subset \mathcal{A}_{\mathrm{cusp}}(G)$ be an irreducible tempered cuspidal automorphic representation of $G(\mathbb{A})$. Then π is unitary and $\bar{\pi} = \{\bar{\varphi} \mid \varphi \in \pi\}$. For each $v \in \Sigma$, choose a nontrivial $G(F_v)$-invariant hermitian paring $\langle\ ,\ \rangle_v \colon \pi_v \times \bar{\pi}_v \to \mathbb{C}$. Motivated by Theorem 13 (3), we put

$$\alpha_v^\natural = \frac{L(1,\pi_v,\mathrm{Ad})}{\Delta_{G_1,v}L(1/2,\pi_v,\mathrm{Std})}\alpha_v,$$

where $\alpha_v \in \mathrm{Hom}_{H_r(F_v)}(\pi_v,\nu_v)\otimes_{\mathbb{C}}\mathrm{Hom}_{H_r(F_v)}(\bar{\pi}_v,\nu_v^{-1})$ is the invariant functional defined in Proposition 12 using the paring $\langle\ ,\ \rangle_v$ and the local Haar measure $\mathrm{d}h_v$ on $H_r(F_v)$. We assume that the product measure $\prod_{v\in\Sigma}\mathrm{d}h_v$ on $H_r(\mathbb{A})$ coincides with the Tamagawa measure, and $\otimes_{v\in\Sigma}\langle\ ,\ \rangle_v$ gives the Petersson inner product on $\pi \times \bar{\pi}$ (defined by the Tamagawa measure on $G(\mathbb{A})$). The product

$$\alpha_{\mathbb{A}}^\natural = \prod_{v\in\Sigma}\alpha_v^\natural$$

is a well-defined element in $\mathrm{Hom}_{H_r(\mathbb{A})}(\pi,\nu)\otimes_{\mathbb{C}}\mathrm{Hom}_{H_r(\mathbb{A})}(\bar{\pi},\nu^{-1})$.

Assume that Theorem 13 (1) holds in all cases. Then

$$\mathrm{Hom}_{H_r(\mathbb{A})}(\pi,\nu)\otimes_{\mathbb{C}}\mathrm{Hom}_{H_r(\mathbb{A})}(\bar{\pi},\nu^{-1}) = \mathrm{Span}_{\mathbb{C}}\{\alpha_{\mathbb{A}}^\natural\}.$$

On the other hand, for $\varphi_1 \in \pi, \bar{\varphi}_2 \in \bar{\pi}$,

$$\beta(\varphi_1,\bar{\varphi}_2) = \mathcal{P}_{H_r,\nu_r}(\varphi_1)\mathcal{P}_{H_r,\nu_r^{-1}}(\bar{\varphi}_2) \tag{5.1}$$

also defines an element in $\mathrm{Hom}_{H_r(\mathbb{A})}(\pi,\nu)\otimes_{\mathbb{C}}\mathrm{Hom}_{H_r(\mathbb{A})}(\bar{\pi},\nu^{-1})$. Thus by Multiplicity One Theorem 9, there exists a constant c_π depending only π, such that

$$\beta = c_\pi \cdot \alpha_{\mathbb{A}}^\natural. \tag{5.2}$$

The following conjecture, proposed originally by Ichino and Ikeda [II10] in the case $E = F$ and $r = 0$ and then extended by N. Harris [Har] in the case $[E : F] = 2$ and $r = 0$, is formulated in [Liub] for all Bessel periods. This is often called the *refined global Gan–Gross–Prasad conjecture.*

Conjecture 17 ([Liub])**.** *Let $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$ be an irreducible tempered cuspidal automorphic representation of $G(\mathbb{A})$. We have*

$$c_\pi = \frac{1}{|\mathcal{S}_{\mathrm{rec}(\pi)}|}\frac{\Delta_{G_1}L(1/2,\pi,\mathrm{Std})}{L(1,\pi,\mathrm{Ad})},$$

where $\mathcal{S}_{\mathrm{rec}(\pi)}$ is the finite 2-group defined in (4.1).

In the Whittaker–Fourier case, that is, $m+[E:F]\leq 2$, the above conjecture is also formulated by Lapid and Mao in [LM], as special cases of their conjecture on Whittaker–Fourier coefficients for more general reductive groups.

The above conjecture is fully known when

- $E=F$, $m=2$ and $r=0$, which is essentially the Waldspurger formula [Wal85];
- $E=F$, $m=3$ and $r=0$, which is known as the triple product formula, finally settled by A. Ichino [Ich08];
- $[E:F]=2$, $m=1$ and $r=0$, which can be deduced from the Waldspurger formula.

Moreover, if $E=F$ and $r=0$, we have some other known special cases when m is small. See [II10] for a summary of all other examples in this case. Toward G of higher ranks, we have the following celebrated result recently obtained by W. Zhang.

Theorem 18 ([Zhab])**.** *We assume certain standard hypotheses from Arthur's conjecture. Suppose $[E:F]=2$, $m\geq 2$ and $r=0$. Denote Σ_s (resp. Σ_i) the subset of Σ consisting of places of F that are split (resp. non-split) in E. Let $\pi\subset\mathcal{A}_{\mathrm{cusp}}(G)$ be an irreducible (tempered) cuspidal automorphic representation of $G(\mathbb{A})$. If*

1. *Σ_s contains all archimedean places;*
2. *there exists $v_0\in\Sigma_s$ such that π_{v_0} is supercuspidal;*
3. *at any place $v\in\Sigma_i$ (which is non-archimedean by (1)), either $G_0(F_v)$ is compact, or π_v is supercuspidal, or π_v is unramified and the Jacquet–Rallis fundamental lemma for G holds at F_v,*

then Conjecture 17 holds for such π.

Remark 19. In the above theorem, condition (2) will force π to be tempered and moreover $|\mathcal{S}_{\mathrm{rec}(\pi)}|=4$, under certain hypothesis from Arthur's conjecture. The Jacquet–Rallis fundamental lemma is known by the work of Z. Yun [Yun11] providing the residue characteristic of F_v greater that $c(m)$, a constant depending only on m.

We have two more examples in the case $r=1$, which are new.

Theorem 20. [Liub] *Assume $E=F$, $m=2$ and $r=1$. Let $\pi=\pi_1\boxtimes\pi_0$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})=G_1(\mathbb{A})\times G_0(\mathbb{A})$. Suppose that $G_1\simeq\mathrm{SO}(2,3)_F$ and π_1 is an endoscopic Yoshida lift. Then Conjecture 17 holds for such π.*

In the above theorem, G_0 is a non-split torus and hence π_0 is simply an automorphic character of $G_0(\mathbb{A})$. The representation π_1 is expected to be tempered, but we do *not* assume this. In particular, we have a way to define $\alpha^\natural_{\mathbb{A}}$ without using the assumption that π_1 is tempered. Finally, $|\mathcal{S}_{\mathrm{rec}(\pi)}|=8$.

Theorem 21. [Liub] *Assume $E = F$, $m = 3$ and $r = 1$. Let $\pi = \pi_1 \boxtimes \pi_0$ be an irreducible tempered cuspidal automorphic representation of $G(\mathbb{A}) = G_1(\mathbb{A}) \times G_0(\mathbb{A})$. Suppose that G_1 is quasi-split but not split and π_1 is a certain endoscopic tempered cuspidal automorphic representation. Assume that Conjecture 17 holds for $m = 2$, $r = 0$ and E being the discriminant field of the (6-dimensional) quadratic space $V^\sharp$. Then Conjecture 17 holds for such π.*

In fact, in the above theorem, G_1 is naturally isomorphic to a subgroup of $\mathrm{PGU}(2,2)_E$ whose quotient is abelian, where the quasi-split unitary group is associated to the quadratic extension E/F with E the discriminant field of the quadratic space $V^\sharp$. The representation π_1 we consider comes from the global theta lift of $\mathrm{U}(W)$ where W is a 3-dimensional E-hermitian space.

6. Relative trace formula

In the last section, we would like to introduce an approach toward Conjecture 17 in the unitary case, namely, the *relative trace formula*. Several years ago, Jacquet and Rallis [JR11] proposed such approach in the case $[E : F] = 2$ and $r = 0$. After that, significant progress has been achieved, including the work of Z. Yun on the proof of the corresponding (relative) fundamental lemma [Yun11], of W. Zhang on the proof of the corresponding smooth matching [Zhaa] and finally Theorem 18 in [Zhab]. On the other hand, the author extended such relative trace formula to all r (and even the case of Fourier–Jacobi periods, which are not discussed in this note) in [Liua]. In the rest of the section, we would like to explain its basic idea. Before doing so, let us remark that in certain orthogonal case – when $E = F$, $m = 2$ and $r = 1$, there are also some relative trace formulas and related results, mainly due to M. Furusawa, K. Martin and J. Shalika. See [FM] for a summary.

Let us now assume $[E : F] = 2$ and fix a pure inner class $\mathfrak{V}$. Put $\mathbf{G} = \mathrm{Res}_{E/F}(\mathrm{GL}(n)_E \times \mathrm{GL}(m)_E)$ and one may similarly define a subgroup $\mathbf{H}_r \subset \mathbf{G}$ like H_r. In addition, put $\mathbf{G}' = \mathrm{GL}(n)_F \times \mathrm{GL}(m)_F$ which is a subgroup of $\mathbf{G}$. We define two distributions: $\mathcal{I}$ on the space $\mathcal{C}_c^\infty(G(\mathbb{A}))$ and $\mathcal{J}$ on the space $\mathcal{C}_c^\infty(\mathbf{G}(\mathbb{A}))$. Very roughly speaking, $\mathcal{I}$ encodes the information of the linear functional β (5.1), that is, the left-hand side of (5.2), and $\mathcal{J}$ encodes the information on the special L-value, that is, the right-hand side of (5.2).

Formally, both distributions have two sorts of decompositions: the *spectral decomposition*

$$\mathcal{I} = \sum_{\pi} \mathcal{I}_\pi, \quad \mathcal{J} = \sum_{\Pi} \mathcal{J}_\Pi,$$

and the *orbital decomposition*

$$\mathcal{I} = \sum_{H_r(F)\backslash G(F)/H_r(F)} \mathcal{I}_\gamma, \quad \mathcal{J} = \sum_{\mathbf{H}_r(F)\backslash \mathbf{G}(F)/\mathbf{G}'(F)} \mathcal{J}_\delta.$$

Inspired by Proposition 16, we should consider all $\beta \in \mathfrak{V}$ at the same time. Our goal is to compare the spectral side, that is, $\mathcal{I}_\pi$ and $\mathcal{J}_\Pi$, which can be transferred to the comparison on the orbital side. The following lemma is the first step toward such comparison.

Lemma 22 ([Liua]). *Denote* $(\mathbf{H}_r(F)\backslash\mathbf{G}(F)/\mathbf{G}'(F))_{\mathrm{reg}}$ *the subset of* $\mathbf{H}_r(F)\backslash\mathbf{G}(F)/\mathbf{G}'(F)$ *consisting of orbits whose stabilizer is trivial, and similarly for* $(H_r^\beta(F)\backslash G^\beta(F)/H_r^\beta(F))_{\mathrm{reg}}$. *Then they are Zariski dense (resp. v-adically dense) if F is a number field (resp. F is viewed as its localization at v). Moreover, we have a natural bijection*

$$(\mathbf{H}_r(F)\backslash\mathbf{G}(F)/\mathbf{G}'(F))_{\mathrm{reg}} \simeq \coprod_{\beta\in\mathfrak{V}} (H_r^\beta(F)\backslash G^\beta(F)/H_r^\beta(F))_{\mathrm{reg}}.$$

We also propose the conjectures on the corresponding relative fundamental lemma and smooth matching. The fundamental lemma is open unless $r = 0$ or $m = 0$ (which follows from a slightly modified argument of B. C. Ngô [NBC99] in the proof of the Jacquet–Ye fundamental lemma). In general, it is hopeful that one may attack it by an algebro-geometric method that is a combination of [NBC99] and [Yun11]. The smooth matching is wide open unless $r = 0$, even in the case $m = 0$.

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