A $p$-ADIC WALDSPURGER FORMULA

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ABSTRACT. In this article, we study $p$-adic torus periods for certain $p$-adic valued functions on Shimura curves coming from classical origin. We prove a $p$-adic Waldspurger formula for these periods, as a generalization of the recent work of Bertolini, Darmon, and Prasanna. In pursuing such a formula, we construct a new anti-cyclotomic $p$-adic $L$-function of Rankin–Selberg type. At a character of positive weight, the $p$-adic $L$-function interpolates the central critical value of the complex Rankin–Selberg $L$-function. Its value at a finite order character, which is outside the range of interpolation, essentially computes the corresponding $p$-adic torus period.

Contents

1. Introduction 2
1.1. Complex Waldspurger formula 2
1.2. $p$-adic Maass functions 4
1.3. $p$-adic torus periods 4
1.4. $p$-adic characters 5
1.5. $p$-adic Waldspurger formula 5
1.6. Idea of proofs 6
1.7. A glance at the general case 7
1.8. Notation and conventions 8
2. Arithmetic of quaternionic Shimura curves 10
2.1. Fourier theory on Lubin–Tate groups 10
2.2. Shimura curves and Kodaira–Spencer isomorphism 14
2.3. Universal convergent modular forms 18
2.4. Comparison with archimedean differential operators 22
2.5. Proofs of claims via unitary Shimura curves 27
3. Statements of main theorems 31
3.1. Representations for incoherent quaternion algebras 31
3.2. $p$-adic Rankin–Selberg $L$-functions for abelian varieties of GL(2)-type 33
3.3. $p$-adic Waldspurger formula 37
3.4. $p$-adic Maass functions and alternative formulation 37
4. Proofs of main theorems 40
4.1. Distribution of matrix coefficient integrals 41
4.2. Universal torus periods 46
4.3. Interpolation of universal torus periods 48
4.4. Proof of main theorems 51
Appendix A. Compatibility of logarithm and Coleman integral 53

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Appendix B. Serre–Tate local moduli for $\mathcal{O}$-divisible groups (d’après N. Katz) 54
B.1. $\mathcal{O}$-divisible groups and Serre–Tate coordinates 55
B.2. Main theorem 57
B.3. Frobenius 59
B.4. Infinitesimal computation 62
References 64

1. Introduction

The aim of this article is to generalize a recent formula of Bertolini, Darmon, and Prasanna in [BDP13] which relates $p$-adic logarithm of Heegner points in abelian varieties parameterized by the modular curve $X_0(N)$ and certain $p$-adic $L$-values at a point outside its range of interpolation, for a prime $p$ split in the imaginary quadratic field. The paper [BDP13] works in the same setting as in the Gross–Zagier formula [GZ86] under the Heegner hypothesis. Prior to Bertolini–Darmon–Prasanna, Rubin [Rub92] has obtained a similar formula for elliptic curves with complex multiplication, and after Bertolini–Darmon–Prasanna, Brooks [Bro13] also obtained a similar formula allowing the modular curve to be a rational Shimura curve.

Our formula is for the general case concerning Heegner points on abelian varieties parameterized by Shimura curves over a totally real number field $F$, for a prime $p$ of $F$ split in the CM fields $E$. Even in the case $F = \mathbb{Q}$, our result is new since we remove all ramification restrictions in [BDP13] and [Bro13]. Moreover, we will place our formula in the setting of the Waldspurger formula [Wal85, YZZ13] which compares the global torus periods of automorphic forms with products of global central $L$-values and local torus periods. More precisely, we will define the relevant $p$-adic $L$-function, introduce the notion of $p$-adic Maass functions and their torus periods, and compare them with products of $p$-adic $L$-values and local torus periods. For practical applications of our formula, one may need a formula for local torus periods of Gross–Prasad test vectors. Fortunately, this formula has been worked out recently by Cai, Shu, and Tian in [CST14].

To construct the $p$-adic $L$-function and prove our $p$-adic Waldspurger formula, we study the congruence relation for both global (torus) periods and local (torus) periods appearing in the complex Waldspurger formula. A key ingredient of our construction is the existence of action of the Lubin–Tate formal group on Shimura curves at infinite level; this allows us to use $p$-adic Fourier analysis in [ST01].

In the rest of Introduction, we will sketch our construction and the proof for the formula in the case of elliptic curves over $\mathbb{Q}$. To be consistent with the notation in the main body of the article, we fix: (1) an elliptic curve $A$ over $\mathbb{Q}$; (2) an indefinite quaternion algebra $B$ over $\mathbb{Q}$; and (3) an imaginary quadratic field $E$ embedded into $B$.

As usual, put $\mathbb{A} = \mathbb{R} \times \hat{\mathbb{Q}}$ as the ring of adeles of $\mathbb{Q}$, and $\mathbb{A}_E := \mathbb{A} \otimes \mathbb{Q} E$. By the modularity theorem, the elliptic curve $A$ determines an irreducible cuspidal automorphic representation $\Pi$ of $\text{GL}_2(\mathbb{A})$. We assume that this representation has a nontrivial Jacquet–Langlands correspondence $\pi_\mathbb{C}$ to $B^\times$, uniquely realized on a subspace of $\mathcal{A}_\mathbb{C}(B^\times)$ – the space of automorphic forms on $B^\times \setminus (B \otimes \mathbb{Q} \mathbb{A})^\times$.

1.1. Complex Waldspurger formula. First let us review the (complex) Waldspurger formula [Wal85, YZZ13] for the cuspidal automorphic representation $\pi_\mathbb{C}$ of $(B \otimes \mathbb{Q} \mathbb{A})^\times$. 

Let $\chi : E^1 \mathbb{A}^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ be an automorphic character. Then we can form the (torus) period integrals

$$\mathcal{P}_C(\phi, \chi^{\pm 1}) := \int_{E^1 \mathbb{A}^\times \backslash \mathbb{A}_E^\times} f(t) \chi^{\pm 1}(t) \, dt, \quad \phi \in \pi_C.$$  

Here we adopt the Haar measure such that the totally volume of $E^1 \mathbb{A}^\times \backslash \mathbb{A}_E^\times$ is 2. We consider these integrals as elements in the linear dual of representation spaces as follows:

$$\mathcal{P}_C(\cdot, \chi) \in \text{Hom}_{\mathbb{A}_E^\times}(\pi_C \otimes \chi, \mathbb{C}); \quad \mathcal{P}_C(\cdot, \chi^{-1}) \in \text{Hom}_{\mathbb{A}_E^\times}(\pi_C \otimes \chi^{-1}, \mathbb{C}).$$

By the theorem of Saito–Tunnell [Tun83, Sai93], either both spaces have dimension 1 or they have dimension 0. Suppose that we are in the first case. Although we do not know how to construct a canonical basis in either space, we do know how to construct a canonical one in their tensor product. Namely, we have the element

$$\alpha = \prod_{v \leq \infty} \alpha_v \in \text{Hom}_{\mathbb{A}_E^\times}(\pi_C \otimes \chi, \mathbb{C}) \otimes \text{Hom}_{\mathbb{A}_E^\times}(\pi_C \otimes \chi^{-1}, \mathbb{C})$$

defined via the integration of local matrix coefficients:

$$\alpha_v(\phi_1, \phi_2; \chi) := \frac{L(1, \eta_v) L(1, \Pi_v, \text{Ad})}{\zeta_v(2) L(1/2, \Pi_v, \chi_v)} \int_{Q_E^\times \backslash E_E^\times} (\pi_C(t) \phi_1, \phi_2)_v \chi_v(t) \, dt$$

where $\eta = \prod_v \eta_v$ is the quadratic character corresponding to the quadratic field extension $E/\mathbb{Q}$, and $(\cdot, \cdot) = \prod_v (\cdot, \cdot)_v$ is the bilinear Petersson inner product pairing on $\pi_C$ defined by the Haar measure on $(B \otimes \mathbb{Q} \mathbb{A})^\times$ such that totally volume of $B^1 \mathbb{A}^\times \backslash (B \otimes \mathbb{Q} \mathbb{A})^\times$ is 2. It is proved by Waldspurger [Wal85, §3] that $\alpha$ is in fact a finite product for every pair of test vectors $(\phi_1, \phi_2)$.

Thus there is a unique constant $\Lambda(\pi_C, \chi) \in \mathbb{C}$, depending only on $\pi_C$ and $\chi$, such that

$$\mathcal{P}_C(\cdot, \chi) \cdot \mathcal{P}_C(\cdot, \chi^{-1}) = \Lambda(\pi_C, \chi) \cdot \alpha(\cdot, \cdot).$$

The Waldspurger formula gives an expression for $\Lambda(\pi_C, \chi)$ in terms of the Rankin–Selberg central value $\Lambda(1/2, \Pi, \chi)$.

**Theorem 1.1.1** (Waldspurger). We have

$$\Lambda(\pi_C, \chi) = \frac{\Lambda_Q(2)}{2 \Lambda(1, \eta) \Lambda(1, \Pi, \text{Ad})} \Lambda(1/2, \Pi, \chi).$$

In other words, for every pair of vectors $\phi_1, \phi_2 \in \pi_C$, we have

$$\mathcal{P}_C(\phi_1, \chi) \mathcal{P}_C(\phi_2, \chi^{-1}) = \frac{\Lambda_Q(2) \Lambda(1/2, \Pi, \chi)}{2 \Lambda(1, \eta) \Lambda(1, \Pi, \text{Ad})} \cdot \alpha(\phi_1, \phi_2; \chi).$$

**Remark 1.1.2.** In the above theorem, $\Lambda$ stands for complete global $L$-functions, that is, those as products of local $L$-functions over all places. However, in the main body of the article, we use global $L$-functions that are products of local $L$-functions over nonarchimedean places, which will be denoted by $L$ (except for $\zeta_F(s)$ with $F$ a number field). For example, if $\chi_{\infty}(z) = (z/\bar{z})^{\pm k}$ with $k \geq 1$, then we have

$$\Lambda(\pi_C, \chi) = \frac{k!(k-1)!}{(2\pi)^{2k-1}} \cdot \frac{\zeta_Q(2) L(1/2, \Pi, \chi)}{2 L(1, \eta) L(1, \Pi, \text{Ad})}.$$

It is a simple computation using the formulae in, for instance, [MN16, Lemma 2.3].
Remark 1.1.3. Note that, unlike our unified choice of the Tamagawa measure on $\mathbb{A}^\times \backslash \mathbb{A}_E^\times$, in [Wal85, YZZ13] the Haar measure in defining (1.1) has volume 1 on $E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times$, and the product Haar measure in defining (1.2) has volume $2\Lambda(1, \eta)$ on $E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times$. Therefore, the constant $\Lambda(\pi_\mathcal{C}, \chi)$ in their formulae differs from ours by $4\Lambda(1, \eta)$.

1.2. $p$-adic Maass functions. From now on, we fix a prime $p$, and equip $B$ with an isomorphism $B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$. For each (sufficiently small) open compact subgroup $U$ of $(B \otimes \mathbb{Q})^\times$, the double quotient

\[ B^\times \backslash (\mathbb{C} \backslash \mathbb{R}) \times (B \otimes \mathbb{Q})^\times / U \]

is the set of complex points of a Shimura curves $X_U$ defined over $\mathbb{Q}$. The curve $X_U$ is smooth over $\mathbb{Q}$, and proper if and only if $B$ is division. We put $X = \varprojlim U X_U$ as a scheme over $\mathbb{Q}$ with a right action of $(B \otimes \mathbb{Q})^\times$ under which $X_U = X/U$.

We say that a function $f: (\mathbb{C}_p) \rightarrow \mathbb{C}_p$ is a $p$-adic Maass function on $X$ if it is the pullback of some locally analytic function $X_U/(\mathbb{C}_p) \rightarrow \mathbb{C}_p$ on $X_U$. Denote by $\mathcal{A}_{\mathbb{C}_p}(B^\times)$ the $\mathbb{C}_p$-vector space of all $p$-adic Maass functions on $X$. It is a representation of $(B \otimes \mathbb{Q})^\times$.

Denote by $\pi_{\mathbb{C}_p}$ the subspace of $\mathcal{A}_{\mathbb{C}_p}(B^\times)$ spanned by functions of the form

\[ f^* \log_\omega \circ f: (\mathbb{C}_p) \rightarrow A(\mathbb{C}_p) \rightarrow \mathbb{C}_p, \]

where $f: X \rightarrow A$ is a nonconstant map; $\omega$ is a differential form on $A \otimes \mathbb{Q} \mathbb{C}_p$; and $\log_\omega$ is the $p$-adic logarithm map (see, for example, [Bou89]). The subspace $\pi_{\mathbb{C}_p} \subset \mathcal{A}_{\mathbb{C}_p}(B^\times)$ is a subrepresentation of $(B \otimes \mathbb{Q})^\times$. Thus on one hand we have a complex realization $\pi_\mathbb{C}$ and on the other hand we have a $p$-adic realization $\pi_{\mathbb{C}_p}$. They are related as follows: For every isomorphism $\iota: \mathbb{C}_p \sim \mathbb{C}$, we have a canonical isomorphism

\[ \pi_{\mathbb{C}_p} \otimes_{\mathbb{C}_p} \mathbb{C} \sim \pi_\mathbb{C}^{(2)}, \]

where $\pi_\mathbb{C}^{(2)} \subset \pi_\mathbb{C}$ is the subspace of weight 2 forms. It sends $f^* \log_\omega$ to $f^* \iota \omega$, which is well-defined. The latter is a differential form on $X \otimes_\mathbb{Q} \mathbb{C}$ hence induces an element in $\mathcal{A}_\mathbb{C}(B^\times)$.

1.3. $p$-adic torus periods. From now on, we also fix an embedding $E \subset \mathbb{C}_p$ and an isomorphism $\iota: \mathbb{C}_p \sim \mathbb{C}$. Then we have an induced isomorphism $E \otimes_\mathbb{Q} \mathbb{R} \simeq \mathbb{C}$. We assume that the isomorphism $B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$ is chosen such that the induced embedding $\mathbb{C} \simeq E \otimes_\mathbb{Q} \mathbb{R} \rightarrow B \otimes \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$ is the standard one sending $x + iy$ to $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$.

We proceed exactly as in the complex Waldspurger formula. Let $\chi: E^\times \mathbb{Q}^\times \backslash \bar{E}^\times \rightarrow \mathbb{C}_p^\times$ be a finite order character. Then parallel to (1.1), we can define the $p$-adic (torus) period integral as

\[ \mathcal{P}_{\mathbb{C}_p}(\phi, \chi^\pm) := \int_{E^\times \mathbb{Q}^\times \backslash \bar{E}^\times} \phi(\iota^{-1}[\pm i, t]) \chi^\pm(t) \operatorname{d}t, \quad \phi \in \pi_{\mathbb{C}_p}. \]

Here we have used the double coset presentation (1.3) of $X(\mathbb{C})$, and adopt the Haar measure on $E^\times \mathbb{Q}^\times \backslash \bar{E}^\times$ of total volume 2. Note that the above integrals are actually finite sums, and respectively induce elements

\[ \mathcal{P}_{\mathbb{C}_p}(\phi, \chi^\pm) \in \operatorname{Hom}_{\bar{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi^\pm, \mathbb{C}_p). \]
Similar to the complex case, both spaces $\text{Hom}_{\hat{E}_x}(\pi_{\mathbb{C}_p} \otimes \chi^\pm, \mathbb{C}_p)$ have the same dimension — either 1 or 0. Suppose that they have dimension 1. Now we construct a basis of their tensor product. For $\phi_1, \phi_2 \in \pi_{\mathbb{C}_p}$, we define

$$\alpha' (\phi_1, \phi_2; \chi) = \prod_{\nu < \infty} \iota^{-1} \alpha_{\nu}(\iota \phi_1, \iota \phi_2; \iota \chi),$$

where $\alpha_{\nu}$ is same as (1.2). Here, by abuse of notation, $\iota \phi$ denotes the image of $\phi$ under the map (1.4). Then $\alpha'$ is a basis of $\text{Hom}_{\hat{E}_x}(\pi_{\mathbb{C}_p} \otimes \chi, \mathbb{C}_p) \otimes \text{Hom}_{\hat{E}_x}(\pi_{\mathbb{C}_p} \otimes \chi^{-1}, \mathbb{C}_p)$. The invariant pairing $\prod_{\nu < \infty} (\ , \ )_{\nu}$ we use in the definition of $\alpha'$ is the one such that $\prod_{\nu < \infty} (\iota \phi_1, \iota \phi_2)_{\nu}$ is equal to the (bilinear) Petersson product of $\iota \phi_1$ and $\pi_{\mathbb{C}}(\begin{pmatrix} 1 & \ 0 \\ -1 & \ 1 \end{pmatrix}) \iota \phi_2$.

Thus there is a unique constant $L(\pi_{\mathbb{C}_p}, \chi) \in \mathbb{C}_p$, depending only on $\pi_{\mathbb{C}_p}$ and $\chi$, such that

$$\mathcal{P}_{\mathbb{C}_p}(\ , \chi) : \mathcal{P}_{\mathbb{C}_p}(\ , \chi^{-1}) = L(\pi_{\mathbb{C}_p}, \chi) : \alpha' (\ , \chi).$$

Our main objective is to give a formula for $L(\pi_{\mathbb{C}_p}, \chi)$, which we call the $p$-adic Waldspurger formula, under the only assumption that $p$ splits in $E$.

Thus from now on we assume that $p$ splits in $E$. Denote by $\mathfrak{P}$ the place of $E$ induced by the default embedding $E \subset \mathbb{C}_p$ and $\mathfrak{P}^c$ the other one above $p$.

1.4. $p$-adic characters. Put $G = E^\times \mathbb{Q}^\times \backslash \hat{E}^\times$, which is a profinite group. Denote by $\hat{G}$ the continuous dual over $\mathbb{Q}_p$. In other words, for every complete (commutative) $\mathbb{Q}_p$-algebra $R$, $\hat{G}(R)$ is the set of all continuous characters from $G$ to $R^\times$. Then $\hat{G}$ is represented by a (complete) $\mathbb{Q}_p$-algebra $\mathcal{D}(G)$. Thus there is a universal character $\delta: G \to \mathcal{D}(G)^\times$ such that composing with $\delta$ induces a bijection

$$(1.6) \quad \text{Hom}(\mathcal{D}(G), R) \simeq \hat{G}(R)$$

for every complete $\mathbb{Q}_p$-algebra $R$, where $\text{Hom}$ is taken in the category of topological $\mathbb{Q}_p$-algebras.

The place $\mathfrak{P}$ induces an injective homomorphism $\mathbb{Z}_p^\times \hookrightarrow G$. We say that a character $\chi \in \hat{G}(\mathbb{C}_p)$ has weight $w \in \mathbb{Z}$ if $\chi|_V$ is the $w$-th power homomorphism for some subgroup $V \subset \mathbb{Z}_p^\times$ of finite index. For a character $\chi: G \to \mathbb{C}_p^\times$ of weight $w$, there is a standard way to attach it an automorphic character $\chi^{(\iota)}: E^\times \mathbb{A}_E^\times \to \mathbb{C}_p^\times$ under $\iota$; in fact, $\chi^{(\iota)}$ is the unique automorphic character satisfying: (1) $\chi^{(\iota)}|_{\hat{E}^\times} = \iota \circ \chi|_{\hat{E}^\times}$; and (2) $\chi^{(\iota)}(z) = (z/\bar{z})^w$ for $z \in \mathbb{C} \simeq E \otimes_\mathbb{Q} \mathbb{R}$.

A character $\chi \in \hat{G}(\mathbb{C}_p)$ induces a homomorphism $\mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \to \mathbb{C}_p$ via (1.6), and we denote its kernel by $I_\chi$, which is a closed ideal of $\mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Put

$$\mathcal{D}(G; \pi_{\mathbb{C}_p}) = \mathcal{D}(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p / \cap_{\chi \in \Xi(\pi_{\mathbb{C}_p})} I_\chi,$$

where $\Xi(\pi_{\mathbb{C}_p})$ is the set of all $\chi$ such that $\text{dim} \text{Hom}_{\hat{E}^\times}(\pi_{\mathbb{C}_p} \otimes \chi, \mathbb{C}_p) = 1$. In particular, elements in $\mathcal{D}(G; \pi_{\mathbb{C}_p})$ can be viewed as functions on $\Xi(\pi_{\mathbb{C}_p})$ valued in $\mathbb{C}_p$.

1.5. $p$-adic Waldspurger formula. Our first theorem is about the existence of a $p$-adic $L$-function interpolating values $\Lambda(\pi \chi^{(\iota)})$, appeared in Theorem 1.1.1, for $\chi$ of positive weights.
Theorem 1.5.1. There is a unique element $L(\pi_{C_p}) \in \mathcal{D}(G; \pi_{C_p})$ such that for every $\chi \in \Xi(\pi_{C_p})$ of weight $k \geq 1$, we have

$$L(\pi_{C_p})(\chi) = \Lambda(\pi_{C_p}, \chi^{(i)}) \cdot 2\pi^{2k-1} \cdot \frac{\epsilon(1/2, \psi, \Pi_p \otimes \chi_{Q_p}^{(i)})}{L(1/2, \Pi_p \otimes \chi_{Q_p}^{(i)})^2}.$$ 

Here, $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ is the standard additive character.

Remark 1.5.2. We have the following remarks concerning the above theorem.

(1) By the theorem of Saito–Tunnell, a character $\chi \in \hat{G}(C_p)$ of an integer weight belongs to $\Xi(\pi_{C_p})$ if and only if for every finite place $v$ of $\mathbb{Q}$ other than $p$, we have $\epsilon(1/2, \Pi_v, \chi_v^{(i)}) = \eta_v(-1)\epsilon(B_v)$ where $\epsilon(B_v)$ is the Hasse invariant.

(2) The uniqueness part is clear since the subset of characters in $\Xi(\pi_{C_p})$ of positive weights is “dense” in $\Xi(\pi_{C_p})$.

Using this $p$-adic $L$-function, we can answer the question at the end of §1.3 about the ratio $L(\pi_{C_p}, \chi)$.

Theorem 1.5.3. Let $\chi \in \Xi(\pi_{C_p})$ be a finite order character, that is, $\chi$ has weight 0. Then we have

$$L(\pi_{C_p}, \chi) = L(\pi_{C_p})(\chi) \cdot \epsilon^{-1} \left( \frac{L(1/2, \Pi_p \otimes \chi_{Q_p}^{(i)})^2}{\epsilon(1/2, \psi, \Pi_p \otimes \chi_{Q_p}^{(i)})} \right).$$

In other words, for every pair of vectors $\phi_1, \phi_2 \in \pi_{C_p}$, we have

$$\mathcal{P}_{C_p}(\phi_1; \chi) \mathcal{P}_{C_p}(\phi_2; \chi^{-1}) = L(\pi_{C_p})(\chi) \cdot \epsilon^{-1} \left( \frac{L(1/2, \Pi_p \otimes \chi_{Q_p}^{(i)})^2}{\epsilon(1/2, \psi, \Pi_p \otimes \chi_{Q_p}^{(i)})} \right) \cdot \alpha'(\phi_1, \phi_2; \chi).$$

Theorems 1.5.1 and 1.5.3 follow from Theorems 3.2.10 and 3.4.4 in more general context. See Remark 3.4.5 for the reduction process.

1.6. Idea of proofs. We now explain the main idea of our proofs. The same ideas work for the general case as well. There are three major steps in the proofs of our main theorems:

1. construct universal torus periods;
2. construct universal matrix coefficient integrals;
3. construct the $p$-adic $L$-function.

For (1), by a universal torus period, we mean an element in $\mathcal{D}(G; \pi_{C_p})$, such that it specializes to Waldspurger periods at characters of positive weights. A key ingredient in our construction is a Mellin transform for forms on the Shimura curve with the infinite Iwahori level structure at $p$. This seems to be new and matches the philosophy that things look more canonical at the infinite level, which appears in some other works recently. The Mellin transform of a form $f$ has two variables: the Shimura curve itself, and the weight space. If we restrict the Shimura curve to an arbitrary open disc which reduces to a point on the special fiber, then we recover the (local) Mellin transform on the Lubin–Tate group in [ST01]. If we restrict to a classical point (a nonnegative integer, actually) on the weight space, then this recovers an iteration of the Atkin–Serre operator on the Shimura curve.

For (2), by a universal matrix coefficient integral, we mean again an element in $\mathcal{D}(G; \pi_{C_p})$, such that it specializes to classical matrix coefficient integrals at characters of integral weights. In our construction, we need to choose suitable test vectors in the representation $\pi_{C_p}$ and
show that the classical matrix coefficient integrals form a rigid analytic family. Our key idea is to use the Kirillov model to deal with arbitrary ramification at $p$ of $\pi_{C_p}$ and characters in $\Xi(\pi_{C_p})$ for the matrix coefficient integrals.

For (3), by the $p$-adic $L$-function, we mean an element in $\mathcal{D}(G; \pi_{C_p})$, such that it specializes to complex special $L$-values appearing in the complex Waldspurger formula at characters $\chi$ of positive weights. The $p$-adic $L$-function is defined essentially as the ratio of a universal torus period by a universal matrix coefficient integral. The complex Waldspurger formula will imply that such ratio is independent of the choice of the test vectors. In order to show that we have enough universal matrix coefficient integrals whose nonvanishing loci cover the entire space, we use a classical result of Saito–Tunnel on the dichotomy of matrix coefficient integrals and some argument in rigid analytic geometry. In particular, we need our constructions in (1) and (2) applicable to sufficiently many test vectors.

Finally, to obtain the $p$-adic Waldspurger formula, that is, the special value formula for finite order characters in terms of $p$-adic logarithm of Heegner cycles, we use the Multiplicity One property, a property from the global Mellin transform, and slight generalization of Coleman’s work in Appendix A.

1.7. A glance at the general case. In the main body of the article, we will put ourselves in a more general context. Since it is a $p$-adic theory, we fix a CM number field $E$ inside $\mathbb{C}_p$, with the maximal totally real subfield $F$. Let $p$ be the distinguished place of $F$ induced by the inclusion $F \subset \mathbb{C}_p$. Recall that an abelian variety $A$ over $F$ is of $GL(2)$-type if $M_A := \text{End}(A) \otimes \mathbb{Q}$ is a field of the same degree as the dimension of $A$.

Given a modular abelian variety $A$ over $F$ of $GL(2)$-type up to isogeny equipped an embedding $M := M_A \rightarrow \mathbb{C}_p$, we will construct a $p$-adic $L$-function $\mathcal{L}(A)$ and prove a $p$-adic Waldspurger formula, or rather a family of $p$-adic Waldspurger formulae for all relevant realizations of $A$ via $p$-adic Maass functions. Note that $A$ has a central character $\omega_A: F^\times \backslash \mathbb{F}^\times \rightarrow M^\times$.

The space of all locally $F_p$-analytic and smooth away from $p$ characters $\chi: E^\times \backslash \mathbb{F}^\times \rightarrow K^\times$ with a complete field extension $K/MF_p$ such that $\omega_A \cdot \chi|_{F^\times \backslash \mathbb{F}^\times} = 1$ can be organized into an ind-rigid analytic variety $\mathcal{E}$ over $MF_p$. It has a disjoint union decomposition $\mathcal{E} = \mathcal{E}_+ \sqcup \mathcal{E}_-$ defined by certain Rankin–Selberg $\epsilon$-factor of $A$. We denote by $\mathcal{D}(A, K)$ the coordinate algebra of $\mathcal{E}_- \otimes_{MF_p} K$ for every complete field extension $K/MF_p$. In Theorem 3.2.10, we construct our $p$-adic $L$-function for $A$ as an element

$$\mathcal{L}(A) \in (\text{Lie } A \otimes_{FM} \text{Lie } A^\vee) \otimes_{FM} \mathcal{D}(A, MF_p^\text{rt}),$$

where $MF_p^\text{rt}$ is the complete subfield of $\mathbb{C}_p$ generated by $M$, the maximal unramified extension of $F_p$, and the Lubin–Tate period (see §1.8); and $FM := F \otimes_\mathbb{Q} M$ which maps to $MF_p^\text{rt}$ naturally. Our $p$-adic $L$-function $\mathcal{L}(A)$ interpolates classical Rankin–Selberg central critical values for algebraic characters $\chi$ of positive weights, with respect to an arbitrary comparison isomorphism $\iota: \mathbb{C}_p \overset{\sim}{\rightarrow} \mathbb{C}$. In other words, we will not choose an archimedean place of $F$ as the theory should be entirely $p$-adic.

In Theorem 3.4.4, we prove a $p$-adic Waldspurger formula computing $p$-adic torus periods of $p$-adic Maass functions coming from $A$, in terms of special values of $\mathcal{L}(A)$ at finite order characters.
1.8. Notation and conventions. The article is self-contained from now on in the sense that if readers would like to study the general case directly, they can start from here, and nothing before will be used.

Throughout the article, we fix a prime \( p \), a CM number field \( E \subset \mathbb{C}_p \) with \( F \) the maximal totally real subfield contained in \( E \). Thus we obtain a distinguished place \( p \) (resp. \( \mathfrak{P} \)) of \( F \) (resp. \( E \)) above \( p \). We introduce the following key (and only) assumption of the article:

**Assumption:** \( p \) splits in \( E \); in other words, \( F_p = E_p \).

We introduce the following notation and conventions.

- Let \( g \) be the degree of \( F \).
- Let \( c \in \text{Gal}(E/F) \) be the (nontrivial) Galois involution.
- Denote by \( \mathbb{A} \) (resp. \( \mathbb{A}^\infty \)) the ring of adèles (resp. finite adèles) of \( F \). Put \( \mathbb{A}_E = \mathbb{A} \otimes_F E \) and \( \mathbb{A}^\infty_E = \mathbb{A}^\infty \otimes_F E \).
- Let \( \eta = \prod \eta_v: F^\times \backslash \mathbb{A}^\times \to \{\pm 1\} \) be the quadratic character associated to \( E/F \). In particular, we have the \( L \)-function \( L(s, \eta) = \prod_{v \in \mathbb{C}_p} L(s, \eta_v) \).
- Let \( d_E \in \mathbb{Z}_{>0} \) be the absolute value of the discriminant of \( E \).
- We denote by \( \mathcal{O}_p \) the ring of integers of \( F_p \). We denote by \( F_p^{nr} \) (resp. \( F_p^{ab} \)) the completion of the maximal unramified (resp. abelian) extension of \( F_p \) in \( \mathbb{C}_p \) and \( \mathcal{O}_p^{nr} \) (resp. \( \mathcal{O}_p^{ab} \)) its ring of integers. Denote by \( \kappa \) the residue field of \( \mathcal{O}_p^{nr} \), which is an algebraic closure of \( \mathbb{F}_p \).
- Denote by \( \overline{F^\times} \) (resp. \( \overline{E^\times} \)) the closure of \( F^\times \) (resp. \( E^\times \)) in \( \mathbb{A}^\times \) (resp. \( \mathbb{A}_E^\times \)). Put \( \mathcal{O}_p^{anti} = \mathcal{O}_{E_p}^\times / \mathcal{O}_p^\times \).
- We write elements \( t \in E_p = E \otimes_F F_p \) in the form \( (t_\bullet, t_\circ) \) where \( t_\bullet \in F_p \) (resp. \( t_\circ \in F_p \)) is the component of \( t \) at \( \mathfrak{P}^c \) (resp. \( \mathfrak{P} \)). We fix the following embedding \( E_p \to \text{Mat}_2(F_p) \) of \( F_p \)-algebras:

\[
(1.7) \quad t \mapsto \begin{pmatrix} t_\bullet & \cdot \\ \cdot & t_\circ \end{pmatrix}.
\]

- We adopt \( \mathbb{N} = \{m \in \mathbb{Z} \mid m \geq 0\} \) and write elements in \( A^{\oplus m} \) in columns for an object \( A \) (with a well-defined underlying set) in an abelian category.
- Denote by \( J \) the two-by-two matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).
- For \( m \in \mathbb{Z} \), define the \( p \)-Iwahori subgroup of level \( m \) of \( \text{GL}_2(\mathcal{O}_p) \) to be

\[
U_{p,m} = \left\{ g \in \text{GL}_2(\mathcal{O}_p) \mid g \equiv \begin{pmatrix} 1 & \* \\ 0 & 1 \end{pmatrix} \mod \mathcal{O}_p^{m} \right\} \quad \text{if } m \geq 0; \\
U_{p,m} = \left\{ g \in \text{GL}_2(\mathcal{O}_p) \mid g \equiv \begin{pmatrix} 1 & 0 \\ \* & 1 \end{pmatrix} \mod \mathcal{O}_p^{-m} \right\} \quad \text{if } m < 0.
\]

- We adopt the convention that the local or global Artin reciprocity maps send uniformizers to geometric Frobenii.
- If \( G \) is a reductive group over \( F \), we always take the Tamagawa measure when we integrate on the adèlic group \( G(\mathbb{A}) \). In particular, the total volume of \( E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times \) is 2.
- For a relative (formal) scheme \( X/S \), we will simply write \( \Omega^1_X \) instead of \( \Omega^1_{X/S} \) for the sheaf of relative differentials if the base is clear in the context. Tensor product of
quasi-coherent sheaves on $X$ will simply be denoted as $\otimes$, instead of $\otimes_{\mathcal{O}_X}$ where $\mathcal{O}_X$ is the structure sheaf of $X$.

- Denote by $\hat{G}_m$ (resp. $\hat{G}_a$) the multiplicative (resp. additive) formal group. They have the coordinate $T$. We denote by $\hat{G}_m[p^\infty]$ the induced (formal) $p$-divisible group of $\hat{G}_m$.

- Denote by $\mathcal{L}T$ the Lubin–Tate $O_p$-formal group over $O_p^{nr}$, which is unique up to isomorphism. We denote by $\mathcal{L}[p^\infty]$ the induced (formal) $O_p$-divisible group of $\mathcal{L}T$. (See §B.1 if not familiar with the terminology.)

- Denote by $F_p^{\text{ht}} \subset \mathbb{C}_p$ the complete field extension of $F_p^{nr}$ generated by the “period” of the Lubin–Tate group $\mathcal{L}T$ (see [ST01, page 460]). Its valuation is discrete only when $F_p = \mathbb{Q}_p$ by [ST01, Lemma 3.9], in which case $F_p^{\text{ht}} = \mathbb{Q}_p^{nr}$.

- In this article, we will only use basic knowledge about rigid analytic varieties over complete $p$-adic fields in the sense of Tate. Readers may use the book [BGR84] for a reference. If $\mathcal{X}$ is an $L$-rigid analytic variety for some complete nonarchimedean field $L$, we denote by $\mathcal{O}(\mathcal{X}, K)$ the complete $K$-algebra of $K$-valued rigid analytic functions on $\mathcal{X}$ for every complete field extension $K/L$.

**Definition 1.8.1** (Abelian Haar measure). We fix the Haar measure $dt_v$ on $F_v^\times \setminus E_v^\times$ for every place $v$ of $F$ determined by the following conditions:

- When $v$ is archimedean, the total volume of $F_v^\times \setminus E_v^\times \simeq \mathbb{R}^\times \setminus \mathbb{C}^\times$ is $1$;
- When $v$ is split, the volume of the maximal compact subgroup of $F_v^\times \setminus E_v^\times \simeq F_v^\times$ is $1$;
- When $v$ is nonsplit and unramified, the total volume is $1$;
- When $v$ is ramified, the total volume is $2$.

Then the product measure $\prod_v dt_v$ equals the product of $2^{-g}d_E^{-1/2}L(1, \eta)$ and the Tamagawa measure (compare with [YZZ13, 1.6]).

**Notation 1.8.2.** In the main part of the article, we will fix the choices of an additive character $\psi: F_p \to \mathbb{C}_p^\times$ of level $0$ and a generator $\nu: \mathcal{L}T \to \hat{G}_m$ in the free $O_p$-module $\text{Hom}(\mathcal{L}T, \hat{G}_m)$ of rank $1$. Then there are unique isomorphisms

\begin{align}
\nu_\pm: F_p/O_p \xrightarrow{\sim} \mathcal{L}[p^\infty](\mathbb{C}_p)
\end{align}

such that the induced composite maps

\[ \nu(\mathbb{C}_p) \circ \nu_+: F_p/O_p \to \hat{G}_m[p^\infty](\mathbb{C}_p) \subset \mathbb{C}_p^\times \]

coincide with $\psi^\pm$ respectively, where $\psi^+ = \psi$ and $\psi^- = \psi^{-1}$.

**Definition 1.8.3.** Recall from [YZZ13, 1.2.1] that an coherent/inoherent totally definite quaternion algebra over $\mathbb{A}$ is a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ such that the ramification set of $\mathbb{B}$, which is a finite set, contains all archimedean places and has even/odd cardinality. For such $\mathbb{B}$, put $\mathbb{B}^{\infty} = \mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}^{\infty}$.

An $E$-embedding of a totally definite quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ is an embedding

\begin{align}
\mathbf{e} = \prod_v e_v: \mathbb{A}^{\infty}_E = \prod_{v < \infty} E_0 \otimes_F F_v \hookrightarrow \mathbb{B}^{\infty}
\end{align}

of $\mathbb{A}^{\infty}$-algebras. We say that $\mathbb{B}$ is $E$-embeddable if there exists an $E$-embedding of $\mathbb{B}$. 

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2. Arithmetic of quaternionic Shimura curves

In this chapter, we study some $p$-adic arithmetic properties of quaternionic Shimura curves over a totally real field. We start from the local theory of some $p$-adic Fourier analysis on Lubin–Tate groups, following the work of [ST01], in §2.1. In §2.2, we study the Gauss–Manin connection and the Kodaira–Spencer isomorphism for quaternionic Shimura curves, followed by the discussion of universal convergent modular forms in §2.3. In particular, we prove Theorem 2.3.17, which is one of the most crucial technical results of the article. In §2.4, we prove some results involving comparison with transcendental constructions under a given complex uniformization. The last one §2.5 contains the proof of six claims in the previous ones, which requires the auxiliary use of unitary Shimura curves.

2.1. Fourier theory on Lubin–Tate groups. We use the reference [Sch02] for some terminologies from nonarchimedean functional analysis.

Let $G$ be a topologically finitely generated abelian locally $F_p$-analytic group – for example, $G = O_p$ which will be studied later. For a complete field $K$ containing $F_p$, denote by $C(G, K)$ the locally convex $K$-vector space of locally ($F_p$-)analytic $K$-valued functions on $G$, and $D(G, K)$ its strong dual (see Remark 2.1.1) which is a topological $K$-algebra with the multiplication given by convolution [ST01, §1]. We have a natural continuous injective homomorphism

$$\delta: G \to D(G, K)^*$$

sending $g \in G$ to the Dirac distribution $\delta_g$. Moreover, we have $D(G, K) \otimes_K K' \simeq D(G, K')$ for a complete field extension $K'/K$.

Remark 2.1.1. We briefly recall the notion of strong dual in [Sch02]. Let $V$ be a locally convex $K$-vector space, like $C(G, K)$ above. Denote by $\mathcal{L}(V, K)$ the $K$-vector space of continuous $K$-linear maps from $V$ to $K$. For every bound subset $B$ of $V$ (that is, for every open neighborhood $U \subset V$ of $0$, there exists $a \in K$ such that $B \subset aU$) and an ideal $I$ of $O_K$, the subset $\mathcal{L}(B, I) := \{ f \in \mathcal{L}(V, K) \mid f(B) \subset I \}$ is a lattice in $\mathcal{L}(V, K)$. Then the strong dual of $V$ is the (topological) $K$-vector space $\mathcal{L}(V, K)$ equipped with the topology defined by family of lattices $\mathcal{L}(B, I)$ for all bounded subsets $B$ of $V$ and ideals $I$ of $O_K$.

When $G$ is compact, there is a more explicit description of $D(G, K)$ on [ST01, Page 451].

Notation 2.1.2. Let $\mathcal{B}$ be the generic fiber of (the underlying formal scheme of) $\mathcal{L} \mathcal{J}$, which is isomorphic to the open unit disc over $F_p^{nr}$. We have a map

$$\alpha: \mathcal{B} \times_{Spf F_p^{nr}} \mathcal{B} \to \mathcal{B}$$

induced by the formal group law, and a map $O_p \times \mathcal{B} \to \mathcal{B}$ denoted by $(a, z) \mapsto a \cdot z$ coming from the $O_p$-action on $\mathcal{L} \mathcal{J}$. Denote by $\mathcal{O}(\mathcal{B}, K)$ the set of all $K$-valued rigid analytic functions on $\mathcal{B}$, which is a topological $K$-algebra.
Definition 2.1.3 (Stable function). A function $\phi \in \mathcal{O}(\mathcal{B}, K)$ is stable if
$$\sum_{z \in \ker[p]} \phi(\alpha(z, )) = 0,$$
where $\ker[p] \subset \mathcal{B}(K^\times)$ is the subset of $z$ such that $\varpi \cdot z = 0$ for one hence all uniformizers $\varpi$ of $O_p$. We denoted by $\mathcal{O}(\mathcal{B}, K)^\circ$ the subspace of $\mathcal{O}(\mathcal{B}, K)$ of stable functions.

From now on, we will assume that $K$ contains $\mathcal{T}_p^\times$ (see §1.8). By [ST01, Theorems 2.3 & 3.6] (together with the remark after [ST01, Corollary 3.7]), we have a Fourier transform,
$$\mathcal{F}: D(O_p, K) \xrightarrow{\sim} \mathcal{O}(\mathcal{B}, K),$$
which is an isomorphism of topological $K$-algebras, with respect to the homomorphism $\nu: \mathcal{L} \mathcal{F} \to \mathbb{G}_m$ (Notation 1.8.2).

Remark 2.1.4. In fact, the pairing $(a, z) \mapsto \nu(a \cdot z)$ on $O_p \times \mathcal{B}$ identifies $\mathcal{B}$ as the rigid analytic space parameterizing locally analytic characters of $O_p^\times$; and the Fourier transform $\mathcal{F}$ is the unique isomorphism satisfying $\mathcal{F}(\delta_0)(z) = \nu(a \cdot z)$ for $z \in \mathcal{B}$. In particular, the topological $K$-vector space $\mathcal{O}(\mathcal{B}, K)$ is topologically generated by rigid analytic functions $\nu^a$ on $\mathcal{B}$ defined by $\nu^a(z) = \nu(a \cdot z)$, for $a \in O_p$. See [ST01] for more details.

Remark 2.1.5. We have an action of $O_p$ on $\mathcal{B}$ coming from the Lubin–Tate group, hence an action of $O_p$ on $D(O_p, K)$ via $\mathcal{F}$. More precisely, the action of $t \in O_p$ on $D(O_p, K)$ is given by the multiplication of the Dirac distribution $\delta_t$.

We identify $D(O_p^\times, K)$ with the closed subspace of $D(O_p, K)$ consisting of distributions supported on $O_p^\times$.

Lemma 2.1.6. We have

1. The isomorphism $\mathcal{F}$ restricts to an isomorphism $\mathcal{F}: D(O_p^\times, K) \xrightarrow{\sim} \mathcal{O}(\mathcal{B}, K)^\circ$ of topological $K$-vector spaces;
2. The image of $\alpha^*|\mathcal{O}(\mathcal{B}, K)^\circ$ is contained in $\mathcal{O}(\mathcal{B}, K)^\circ \hat{\otimes}_K \mathcal{O}(\mathcal{B}, K)^\circ$.

Proof. By [ST01, §3], for $z \in \mathcal{B}(K)$, we have a locally analytic character $\kappa_z$ of $O_p$ such that $\kappa_z(a) = \nu(a \cdot z)$ for every $a \in O_p$, and $\lambda(\kappa_z) = \mathcal{F}(\lambda)(z)$ for $\lambda \in D(O_p, K)$. Moreover, the set of $\kappa_z$ is dense in $C(O_p, K)$.

Let $e_1$ (resp. $e_0$) be the characteristic function of $O_p^\times$ (resp. $O_p \setminus O_p^\times$), viewed as elements in $C(O_p, K)$.

For (1), we have the identity
$$\sum_{z \in \ker[p]} \kappa_z f = (\#O_p/p) e_0 f.$$
This means that $\mathcal{F}(O_p^\times, K) \subset \mathcal{O}(\mathcal{B}, K)^\circ$.

For (2), we consider the map $\alpha^\prime$, defined as the following composite map

\[ D(O_p, K) \overset{i}{\to} \mathcal{O}(\mathcal{B}, K) \overset{\alpha^\prime}{\to} \mathcal{O}(\mathcal{B} \times_{\text{Spf } F_p^\text{ur}} \mathcal{B}, K) \cong \mathcal{O}(\mathcal{B}, K) \otimes_K \mathcal{O}(\mathcal{B}, K) \]

\[ \overset{\mathcal{F}_1^{-1} \otimes \mathcal{F}_1^{-1}}{\to} D(O_p, K) \otimes_K D(O_p, K) \to (C(O_p, K) \otimes_K C(O_p, K))^\circ. \]

In view of (1), it suffices to show that for every $\alpha \in D(O_p, K)$ and $f_1, f_2 \in C(O_p, K)$, we have the formula

\[ \alpha^\prime(\lambda(f_1 \otimes f_2)) = \lambda(f_1 f_2). \]

For this, we may assume that $f_i = \kappa_{z_i}$ for some $z_i \in \mathcal{B}(K)$ ($i = 1, 2$) as the image of $D(O_p, K) \otimes_K D(O_p, K)$ consists of continuous linear forms. Then we have for $\alpha \in D(O_p, K)$,

\[ \alpha^\prime(\kappa_{z_1} \otimes \kappa_{z_2}) = ((\mathcal{F}_1^{-1} \otimes \mathcal{F}_1^{-1})(\alpha^\prime(\lambda)))((\kappa_{z_1} \otimes \kappa_{z_2})) = (\alpha^\prime(\lambda))(z_1, z_2) \]

by [ST01, Lemma 4.6.3]. But

\[ (\alpha^\prime(\lambda))(z_1, z_2) = \mathcal{F}(\lambda)(\alpha(z_1, z_2)) = \lambda(\kappa_{a(z_1, z_2)}) = \lambda(\kappa_{z_1, z_2}) \]

as $\kappa_{z_1, z_2} = \kappa_{a(z_1, z_2)}$. Thus (2.1) holds, and (2) follows. \hfill \Box

**Remark 2.1.7.** Lemma 2.1.6 implies that the function $\nu^a$ in Remark 2.1.4 is stable if and only if $a$ belongs to $O_p^\times$. Moreover, the topological $K$-vector space $\mathcal{O}(\mathcal{B}, K)^\circ$ is topologically generated by $\nu^a$ for $a \in O_p^\times$. The notion of stable functions is a local avatar of the notation of stabilization in the theory of $p$-adic modular/automorphic forms.

In the later argument, we will work on the compact abelian locally $F_p$-analytic group $O_p^{\text{anti}}$. Note that we have identified $O_p^{\text{anti}}$ with $O_p^\times$ via $t \mapsto t/t'$. Thus we have the following definition.

**Definition 2.1.8 (Local Mellin transform).** We call the following composite map

\[ M_{\text{loc}} : \mathcal{O}(\mathcal{B}, K)^\circ \to \mathcal{O}(\mathcal{B}, K)^\circ \otimes_K \mathcal{O}(\mathcal{B}, K)^\circ \overset{\text{id} \otimes \mathcal{F}_1^{-1}}{\to} \mathcal{O}(\mathcal{B}, K)^\circ \otimes_{F_p} D(O_p^{\text{anti}}, F_p), \]

fulfilled by Lemma 2.1.6, the **local Mellin transform**.

**Remark 2.1.9.** In fact, the composite map

\[ M' : \mathcal{O}(\mathcal{B}, K) \to \mathcal{O}(\mathcal{B}, K) \otimes_K \mathcal{O}(\mathcal{B}, K) \overset{\text{id} \otimes \mathcal{F}_1^{-1}}{\to} \mathcal{O}(\mathcal{B}, K) \otimes_{F_p} D(O_p, F_p) \]

is more like the analogy of the classical Mellin transform, as we may regard $M'$ as a map sending a function on $\mathcal{B}$ valued in $K$ to a function on $\mathcal{B}$ valued in $(K$-valued) distributions on the Lie group $O_p$. Recall that the classical Mellin transform $M$ sends a function $\phi$ on $\mathbb{R}_+^\times$ to a function $M(\phi)$ on $\mathbb{C}$. In fact, we may regard $M$ as a map sending a function $\phi$ on $\mathbb{R}_+^\times$ valued in $\mathbb{C}$ to a function $x \mapsto M(f(x \cdot ))$ on $\mathbb{R}_+^\times$ valued in $(C$-valued) distributions on the Lie group $G_a(\mathbb{C})$. We have analogies between $\mathbb{R}_+^\times$ and $\mathcal{B}$ – both are “spaces with abelian group structure”, and between $G_a(\mathbb{C})$ and $O_p$ – both are commutative Lie groups. Moreover, properties in [ST01, Lemma 4.6] are the analogies of those for the classical Mellin transform.

The continuous map $M'$ is uniquely determined by the formula $M'(\nu^a) = \nu^a \otimes \delta_a$ for $a \in O_p$, where $\nu^a$ is the function in Remark 2.1.4.

**Notation 2.1.10.** For every integer $k$, we have the character $\langle k \rangle : O_p^{\text{anti}} \simeq O_p^\times \to O_p^\times \subset K$ sending $t$ to $(t/t')^k$. It is an element in $C(O_p^{\text{anti}}, K)$. 
Lemma 2.1.11. For every integer $N$, the topological $K$-vector space $C(O_p^{\text{anti}}, K)$ is topologically generated by $\langle k \rangle$ for all $k \geq N$.

Proof. We may assume that $N = 0$ since for every $k \in \mathbb{Z}$, the function $\langle k \rangle$ is the limit of functions $\langle k' \rangle$ for $k' > 0$. By [ST01, Theorem 4.7], every function in $C(O_p, K)$ hence $C(O_p^{\text{anti}}, K)$ is the limit of finite linear combinations of polynomials on $O_p$. Thus the lemma holds for $N = 0$ and then every $N$. □

Definition 2.1.12 (Lubin–Tate differential operator). We define the Lubin–Tate differential operator $\Theta$ on $\mathcal{O}(\mathcal{B}, K)$ by the formula

$$\Theta \phi = \frac{d\phi}{\nu^* \frac{dt}{T}},$$

where we recall that $\nu$ is Notation 1.8.2 and $T$ is the standard coordinate of $\hat{G}_m$.

Example 2.1.13. For $a \in O_p$, we have $\Theta \nu^a = a \nu^a$ where $\nu^a \in \mathcal{O}(\mathcal{B}, K)$ is the function in Remark 2.1.4.

The following lemma reveals the relation between the local Mellin transform and the Lubin–Tate differential operator.

Lemma 2.1.14. Let $\phi \in \mathcal{O}(\mathcal{B}, K)^\circ$ be a stable function. Then $M_{\text{loc}}(\phi)$ is the unique element in $\mathcal{O}(\mathcal{B}, K)^\circ \otimes_{F_p} D(O_p^{\text{anti}}, F_p)$ satisfying

1. $M_{\text{loc}}(\phi)(\langle k \rangle) = \Theta^k \phi$ for every $k \geq 0$; and
2. $\Theta M_{\text{loc}}(\phi)(\langle -1 \rangle) = \phi$.

Here $\langle k \rangle$ is introduced in Notation 2.1.10.

Proof. This follows from [ST01, Lemma 4.6.8] and lemma 2.1.11. □

Definition 2.1.15 (Admissible function). We say that a stable function $\phi \in \mathcal{O}(\mathcal{B}, K)^\circ$ is $n$-admissible for some $n \in \mathbb{N}$ if

$$\phi(\alpha(-, z)) = \nu(z)\phi$$

for every $z \in \text{Ker}[p^n] \subset \mathcal{B}(K^{ac})$.

Lemma 2.1.16. Let $\phi \in \mathcal{O}(\mathcal{B}, K)^\circ$ be an $n$-admissible stable function for some $n \geq 1$. Then $\mathcal{F}^{-1}(\phi)$ is supported on $1 + p^n$. In particular, we have

$$M_{\text{loc}}(\phi)(\langle k \rangle) = M_{\text{loc}}(\phi)(\chi\langle k \rangle)$$

for every $k \in \mathbb{Z}$ and every (locally constant) character $\chi: O_p^{\text{anti}} \to K^\times$ that is trivial on $1 + p^n$.

Proof. This again follows from [ST01, Lemma 4.6.5]. In fact, by the similar strategy in the proof of Lemma 2.1.6 (1), it suffices to show that

$$\sum_{z \in \text{Ker}[p^n]} \kappa_z(a)\nu(z)^{-1} = 0$$

for $a \in O_p^{\circ}\setminus(1 + p^n)$, where $\text{Ker}[p^n] \subset \mathcal{B}(K^{ac})$ is the subset of $z$ such that $\varpi^n \cdot z = 0$ for one hence all uniformizers $\varpi$ of $O_p$. This holds as $\kappa_z(a) = \nu(a \cdot z)$. □

Remark 2.1.17. Let $n \geq 1$ be an integer. Lemma 2.1.16 implies that the function $\nu^a$ in Remark 2.1.4 is $n$-admissible stable if and only if $a$ belongs to $1 + p^n$. Moreover, the topological $K$-vector space of $n$-admissible stable functions is topologically generated by $\nu^a$ for $a \in 1 + p^n$. 
2.2. Shimura curves and Kodaira–Spencer isomorphism. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$ equipped with an isomorphism $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$. Then we have the system of (non-compactified) Shimura curves $\{X(\mathbb{B})_U\}$ indexed by (sufficiently small) open compact subgroups $U$ of $\mathbb{B}^\infty$ associated to $\mathbb{B}$ over Spec $F$ (see [YZZ13, §1.2.1] for example). More precisely, $X(\mathbb{B})_U$ is the scheme over Spec $F$, unique up to isomorphism, such that for every embedding $\iota: F \hookrightarrow \mathbb{C}$, $X(\mathbb{B})_U \otimes F \iota(F)$ is the canonical model of the complex Shimura curve

$$B(\iota)^\times \setminus \mathbb{H} \times \mathbb{B}^\infty /U$$

over the reflex field $\iota(F) \subset \mathbb{C}$, where $B(\iota)$ is nearby quaternion algebra over $F$ with respect to $\iota$ (see Definition 2.4.10 for more details).

As projective limits with affine transition morphisms exist in the category of schemes, we may put $X(\mathbb{B}) = \varprojlim_U X(\mathbb{B})_U$. We will simply write $X_U$ and $X_U$ if $\mathbb{B}$ is clear.

Notation 2.2.1. For an element $g \in \mathbb{B}^\infty$, we denote by $T_g: X \to X$ the morphism induced by the right translation of $g$, known as the Hecke morphism.

Denote by $\mathfrak{U}$ the set of all open compact subgroups of $\mathbb{B}^\infty = (\mathbb{B} \otimes \mathbb{A}^\infty)^\times$, which is a filtered partially ordered set under inclusion. For $U^p \in \mathfrak{U}$ and $m \in \mathbb{Z}$, put

$$X(m, U^p) = X_{U^p U_{p,m}} \otimes_F F^\text{nr}_p$$

where we recall that $U_{p,m}$ is the $p$-Iwahori subgroup of level $m$ as introduced in §1.8. Put

$$X(\pm \infty, U^p) := \varprojlim_{m \to \infty} X(\pm m, U^p).$$

For $m \in \mathbb{N} \cup \{\infty\}$, if we take the inverse limit over the partially ordered set $\mathfrak{U}$, then we obtain $F^\text{nr}_p$-schemes

$$X(\pm m) = \varprojlim_{U^p \in \mathfrak{U}} X(\pm m, U^p).$$

We have successive surjective morphisms

$$X(\pm \infty) \to \cdots \to X(\pm 1) \to X(0),$$

which are equivariant under the Hecke actions of $\mathbb{B}^\infty \times$. By the work of Carayol [Car86, §6], the $F^\text{nr}_p$-scheme $X(0)$ admits a canonical smooth model (see [Mil92, Definition 2.2] for its meaning) $\mathcal{X}$ over Spec $O^\text{nr}_p$.

Remark 2.2.2. Strictly speaking, Carayol assumed that $F \neq \mathbb{Q}$. But when $F = \mathbb{Q}$, one may take $\mathcal{X}$ to be the model defined by modular interpretation using elliptic curves (resp. abelian surfaces with quaternionic actions) when $\mathbb{B}^\infty$ is (resp. is not) the matrix algebra – this is well-known.

We recall the construction in [Car86, 1.4] of an $O_p$-divisible group $\mathfrak{G}$ on $\mathcal{X}$. We first introduce some notation.

Notation 2.2.3. For an integer $m \geq 1$, denote

1. $U^\text{pr}_{p,m} := \{g \in U_{p,0} \mid g \equiv 1 \mod p^m\}$ the principal congruence subgroup of level $p^m$;
2. $X(m)^\text{pr} \to X(0)^\text{pr}$ the corresponding covering with respect to the subgroup $U^\text{pr}_{p,m}$; and
3. $O^\times_{E, p,m} := O^\times_{E, p} \cap U^\text{pr}_{p,m}$, where $E_p$ is an $F_p$-subalgebra of $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$ via (1.7).
Consider the right action of $U_{p,0}/U_{p,m}^{\text{pr}}$ on $(\mathfrak{p}^{-m}/O_p)^{\otimes 2}$ such that $v.g = g^{-1}v$ for $g \in U_{p,0}/U_{p,m}^{\text{pr}} \simeq \text{GL}_2(O_p/\mathfrak{p}^m)$ and $v \in (\mathfrak{p}^{-m}/O_p)^{\otimes 2}$. Then the quotient scheme
\[
\left( (X(m))^{\text{pr}} \times (\mathfrak{p}^{-m}/O_p)^{\otimes 2} \right) / (U_{p,0}/U_{p,m}^{\text{pr}})
\]
defines a finite flat group scheme $G_m$ over $X(0)$, with the obvious $O_p$-action. The inductive system $\{G_m\}_{m \geq 1}$ defines an $O_p$-divisible group $G$ over $X(0)$ (which is however denoted by $E_\infty$ in [Car86, §5]). In particular, over $X(+\infty)$ (resp. $X(-\infty)$), we have an exact sequence
\[
(2.2) \quad 0 \longrightarrow F_p/O_p \longrightarrow G \longrightarrow F_p/O_p \longrightarrow 0
\]
such that the second arrow is the inclusion into the first (resp. second) factor and the third arrow is the projection onto the second (resp. first) factor.

By [Car86, 6.4], the $O_p$-divisible group $G$ extends uniquely to an $O_p$-divisible group $\mathcal{G}$ of dimension 1 and height 2 over $\mathcal{X}$, together with an action by $B^{\otimes p \times}$ that is compatible with the Hecke action on the base.

Put $h = [F_p : \mathbb{Q}_p]$. For $m \geq 1$, put $\mathcal{X}(m) = \mathcal{X} \otimes O_p/O_p$ and $\mathcal{G}(m) = \mathcal{G}|_{\mathcal{X}(m)}$. We have the following exact sequence
\[
(2.3) \quad 0 \longrightarrow \mathcal{L}^{(m)}_p \longrightarrow (\omega_p^{(m)})^\vee \longrightarrow 0,
\]
where
- $\mathcal{L}^{(m)}_p$ is the Dieudonné crystal of $\mathcal{G}^{(m)}$ evaluated at $\mathcal{X}(m)$, which is a locally free sheaf of rank $2h$;
- $\omega_p^{(m)}$ is the sheaf of invariant differentials of $\mathcal{G}^{(m)}/\mathcal{X}(m)$, which is a locally free sheaf of rank 1;
- $\omega_p^{(m)}$ is the sheaf of invariant differentials of $(\mathcal{G}(m)^\vee)/\mathcal{X}(m)$, which is a locally free sheaf of rank $2h - 1$.

They are equipped with actions of $O_p$ under which (2.3) is equivariant. The projective system of (2.3) for all $m \geq 1$ induces the following $O_p$-equivariant exact sequence
\[
(2.4) \quad 0 \longrightarrow \omega_* \longrightarrow \mathcal{L}_p \longrightarrow (\omega_p^{\otimes})^\vee \longrightarrow 0,
\]
of locally free sheaves over $\mathcal{X}$, the formal completion of $\mathcal{X}$ along its special fiber. Let $\mathcal{L}$ (resp. $\omega^{\otimes v}$) be the maximal subsheaf of $\mathcal{L}_p$ (resp. $(\omega_p^{\otimes})^\vee$) where $O_p$ acts via the structure map. Then we have the following $B^{\otimes p \times}$-equivariant exact sequence
\[
(2.5) \quad 0 \longrightarrow \omega_* \longrightarrow \mathcal{L} \longrightarrow \omega^{\otimes v} \longrightarrow 0,
\]
where $\omega_* = \omega_p^*$. We call (2.5) the formal Hodge exact sequence.

We have the Gauss–Manin connection
\[
(2.6) \quad \nabla_p : \mathcal{L}_p \rightarrow \mathcal{L}_p \otimes \Omega^1_{\mathcal{X}},
\]
for the Dieudonné crystal, which is equivariant under the Hecke action of $B^{\otimes p \times}$ and the action of $O_p$. Thus, it induces the Gauss–Manin connection
\[
(2.7) \quad \nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega^1_{\mathcal{X}},
\]
which is equivariant under the Hecke action of $B^{\otimes p \times}$.

We have the following Lemma 2.2.4 and Proposition 2.2.6 whose proof will be given in §2.5.
Lemma 2.2.4. The formal Hodge exact sequence (2.5) is algebraizable, that is, it is the formal completion of an exact sequence of locally free sheaves

\[(2.8)\quad 0 \longrightarrow \omega^* \longrightarrow \mathcal{L} \longrightarrow \omega^{\vee} \longrightarrow 0,\]
on $\mathcal{X}$. Here, by abuse of notation we adopt the same symbols for these quasi-coherent sheaves. Moreover, the Gauss–Manin connection (2.7) is algebraizable.

We simply call (2.8) the Hodge exact sequence.

Remark 2.2.5. For $m \geq 1$, one may consider the right action of $U_{p,0}/U_{p,m}^\text{pr}$ on $(O_p/p^m)^{\oplus 2}$ such that $v.g = g^{-1}v$ for $g \in U_{p,0}/U_{p,m}^\text{pr} \simeq \text{GL}_2(O_p/p^m)$ and $v \in (O_p/p^m)^{\oplus 2}$. Then the quotient scheme

\[
\left( X(m)^\text{pr} \times (O_p/p^m)^{\oplus 2} \right)/\left( U_{p,0}/U_{p,m}^\text{pr} \right)
\]
defines an $O_p/p^m$-local system $L_m$ on $X(0)$ of rank 2. Denote by $L$ the $O_p$-local system over $X(0)$ defined by $(L_m)_{m \geq 1}$. Then $\mathcal{E}_{X(0)} \otimes_{O_p} L$ is canonically isomorphic to the restriction of $\mathcal{L}$ on the generic fiber $X(0)$. Moreover, the induced connection on $\mathcal{E}_{X(0)} \otimes_{O_p} L$ coincides with the restriction of $\nabla$ on $X(0)$, by the proof of Lemma 2.2.4.

Proposition 2.2.6. The composite map

\[(2.9)\quad \omega^* \rightarrow \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega^1_{\mathcal{X}} \rightarrow \omega^{\vee} \otimes \Omega^1_{\mathcal{X}}\]
is an isomorphism of locally free sheaves on $\mathcal{X}$, where $\omega^\circ$ is the dual sheaf of $\omega^{\vee}$.

Definition 2.2.7 (Kodaira–Spencer isomorphism). We call the ($\mathbb{B}^{\infty,p\times}$-equivariant) isomorphism

\[(2.10)\quad \mathcal{KS} : \omega^* \otimes \omega^\circ \xrightarrow{\sim} \Omega^1_{\mathcal{X}},\]
induced by the isomorphism (2.9), the Kodaira–Spencer isomorphism.

For $w \in \mathbb{N}$, put $\mathcal{L}^{[w]} = \text{Sym}^w \mathcal{L} \otimes \text{Sym}^w \mathcal{L}^\vee$. The Gauss–Manin connection $\nabla^\vee$ on the dual sheaf $\mathcal{L}^\vee$ and the original one $\nabla$ induce a connection

\[
\nabla^{[w]} : \mathcal{L}^{[w]} \rightarrow \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}}.
\]

Define $\Theta^{[w]}$ to be the composite map

\[(2.11)\quad (\Omega^1_{\mathcal{X}})^{\otimes w} \xrightarrow{\mathcal{KS}^{-1}} (\omega^*)^{\otimes w} \otimes (\omega^\circ)^{\otimes w} \rightarrow \mathcal{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}},\]
where $\mathcal{KS}$ is the isomorphism (2.10).

Notation 2.2.8. Let $\mathcal{X}(0)$ be the (dense) open subscheme of $\mathcal{X}$ by removing all points on the special fiber where $\mathcal{G}$ is supersingular. For every integer $m \geq 1$, denote by $\mathcal{X}(m)$ the functor classifying $O_p/p^m$-equivariant frames over $\mathcal{X}(0)$, that is, homomorphisms $\mathcal{L}[p^m] \rightarrow \mathcal{G}[p^m]$ and $\mathcal{G}[p^m] \rightarrow p^{-m}/O_p$ such that the following sequence

\[
0 \longrightarrow \mathcal{L}[p^m] \longrightarrow \mathcal{G}[p^m] \longrightarrow p^{-m}/O_p \longrightarrow 0
\]
is exact.

Remark 2.2.9. The scheme $\mathcal{X}(m)$ is usually denoted by $\mathcal{X}(m)^{\text{ord}}$ in other literatures. But since we will only work with the ordinary locus, to reduce the burden of notation, we will omit the superscript.
For $m \in \mathbb{N}$, the functor $X(m)$ is representable by a scheme finite étale over $X(0)$, which we again denote by $X(m)$. Note that the generic fiber of $X(m)$ is canonically isomorphic to $X(m)$. Again as projective limits with affine transition morphisms exist in the category of schemes, we may put

$$X(\infty) := \lim_{m \to \infty} X(m).$$

We define $\mathcal{G}$, $\nabla^w$, $\Theta^w$, and the sequence (2.8) for $X(m)$ ($m \in \mathbb{N} \cup \{\infty\}$) via restriction and denote them by the same notation. Over $X(\infty)$, we have the universal frame

\begin{equation}
0 \longrightarrow \mathcal{L}[p^\infty] \xrightarrow{\rho^{\text{univ}}} \mathcal{G} \xrightarrow{\rho^{\text{univ}}_0} F_p/O_p \longrightarrow 0.
\end{equation}

By definition, there is an action of $O_{E_p}^\times$ on the morphism $X(\infty) \to X(0)$ such that the pullback of (2.12) along the action of $(t_\bullet, t_o) \in O_{E_p}^\times$ is the frame

\begin{equation}
0 \longrightarrow \mathcal{L}[p^\infty] \xrightarrow{t^{-1}_o \rho^{\text{univ}}_t} \mathcal{G} \xrightarrow{\rho^{\text{univ}}_0 t^{-1}_o} F_p/O_p \longrightarrow 0.
\end{equation}

This action is $\mathbb{B}^\infty_{E_p} \times$-equivariant. In what follows, we denote by

\begin{equation}
\Gamma_{t} : X(\infty) \to X(\infty)
\end{equation}

the morphism induced by the action of $t \in O_{E_p}^\times$.

**Definition 2.2.10.** We define the transition isomorphisms to be

$$\Upsilon_{\pm} : X(\pm \infty) \otimes_{F_p^{\text{nr}}} F_p^{\text{ab}} \xrightarrow{\sim} X(\infty) \otimes_{O_p^{\text{nr}}} F_p^{\text{ab}}$$

such that the pullbacks of (2.12) under $\Upsilon_{\pm}$ coincide with (2.2) in terms of the isomorphisms (1.8), respectively.

**Lemma 2.2.11.** The Hecke morphism $T_J$ (Notation 2.2.1) descends to an (iso)morphism $T_J : X(\infty) \to X(\infty)$, and the following diagram

$$\begin{array}{ccc}
X(\infty) \otimes_{F_p^{\text{nr}}} F_p^{\text{ab}} & \xrightarrow{T_J} & X(\infty) \otimes_{F_p^{\text{nr}}} F_p^{\text{ab}} \\
\Upsilon_+ & & \Upsilon_- \\
\downarrow & & \downarrow \\
\mathcal{X}(\infty) \otimes_{O_p^{\text{nr}}} F_p^{\text{ab}} & & \mathcal{X}(\infty) \otimes_{O_p^{\text{nr}}} F_p^{\text{ab}}
\end{array}$$

commutes. Here, we regard $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as an element in $\mathbb{B}_p$ via the fixed isomorphism $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$.

Moreover, the isomorphism $\Upsilon_+$ (resp. $\Upsilon_-$) is $\mathbb{B}^\infty_{E_p} \times$-equivariant and $O_{E_p}^\times$-equivariant (resp. $O_{E_p}^\times$-conjugate-equivariant).

**Proof.** It is clear that the Hecke morphism $T_J$ descends as the conjugation of $J$ turns $U_{p,m}$ to $U_{p,-m}$. The commutativity of the diagram follows from the fact that $T_JG|_{X(\infty)}$ is isomorphic to $G|_{X(\infty)}$.

The $\mathbb{B}^\infty_{E_p} \times$-equivariant property follows from the construction. By definition, $\Upsilon_+$ is $O_{E_p}^\times$-equivariant. The conjugate-equivariant property for $\Upsilon_-$ follows from the identity

$$J \begin{pmatrix} t_\bullet \\ t_o \end{pmatrix} J^{-1} = \begin{pmatrix} t_o \\ t_\bullet \end{pmatrix}.$$
for every $t = (t_*, t_0) \in O_{E_p}^\times$.

2.3. Universal convergent modular forms. For $m \in \mathbb{N} \cup \{\infty\}$, denote by $\mathcal{X}(m)$ the formal completion of $\mathcal{X}(m)$ along its special fiber. It is an affine formal scheme over $O_p^{ur}$, equipped with an $O_p$-divisible group $\mathcal{G}$ induced from $G$. In particular, $\mathcal{X}(\infty)$ is indeed the projective limit $\varprojlim_{m \to \infty} \mathcal{X}(m)$ in the category of formal schemes over $O_p^{ur}$.

Remark 2.3.2. There is a unique morphism $\Phi : \mathcal{X}(0) \to \mathcal{X}(0)$ lifting the Frobenius morphism on the special fiber of degree $\#O_p/\mathfrak{p}$ such that $\Phi^* \mathcal{G} \simeq \mathcal{G}/\mathcal{G}_0[\mathfrak{p}]$ where $\mathcal{G}_0$ is the formal part of $\mathcal{G}$. In particular, $\Phi$ induces an endomorphism $\Phi^*$ on $\mathcal{L}$.

Moreover, we have a unique $\Phi^*$-stable splitting

\begin{equation}
\mathcal{L} = \omega^\bullet \oplus \mathcal{L}^\circ
\end{equation}

with $\mathcal{L}^\circ$ an invertible quasi-coherent sheaf on $\mathcal{X}(0)$. In addition, $\mathcal{L}^\circ$ is horizontal with respect to the Gauss–Manin connection, that is, $\nabla \mathcal{L}^\circ \subset \mathcal{L}^\circ \otimes \Omega^1_{\mathcal{X}(0)}$.

Remark 2.3.2. The splitting (2.15) is called unit-root splitting. It induces an isomorphism $\mathcal{L}^\circ \sim \simeq \omega^{\circ, \circ}$. Dually, it induces a splitting $\mathcal{L}^{\circ, \circ} = \omega^\circ \oplus \mathcal{L}^\bullet$ possessing similar properties as in Lemma 2.3.1, with an isomorphism $\mathcal{L}^\bullet \sim \simeq \omega^{\circ, \circ}$.

If we restrict the unit-root splitting in both Lemma 2.3.1 and Remark 2.3.2 to $\mathcal{X}(m)$ for $m \in \mathbb{N} \cup \{\infty\}$, then we obtain a map

$\theta_{w,\text{ord}}^{[w]} : \mathcal{L}^{[w]} \to (\omega^\bullet)^{\otimes w} \otimes (\omega^\circ)^{\otimes w} \xrightarrow{\text{KS}} (\Omega^1_{\mathcal{X}(m)})^{\otimes w}$

for all $w \in \mathbb{N}$, where KS is the (formal completion of the restriction of the) map (2.10).

Definition 2.3.3 (Atkin–Serre operator). For $m \in \mathbb{N} \cup \{\infty\}$ and $w \in \mathbb{N}$, define the Atkin–Serre operator to be

$\Theta_{\text{ord}}^{[w]} : (\Omega^1_{\mathcal{X}(m)})^{\otimes w} \xrightarrow{\Theta_{\text{ord}}^{[w]}(\text{ord})} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}(m)} \xrightarrow{\Theta_{\text{ord}}^{[w]}(\text{ord})} (\Omega^1_{\mathcal{X}(m)})^{\otimes w+1}$,

where $\Theta_{\text{ord}}^{[w]}$ is defined in (2.11). For $k \in \mathbb{N}$, define the Atkin–Serre operator of degree $k$ to be

$\Theta_{\text{ord}}^{[w,k]} = \Theta_{\text{ord}}^{[w-k+1]} \circ \cdots \circ \Theta_{\text{ord}}^{[w]} : (\Omega^1_{\mathcal{X}(m)})^{\otimes w} \to (\Omega^1_{\mathcal{X}(m)})^{\otimes w+k}$.

In what follows, $w$ will always be clear from the text, hence we will suppress $w$ from notation; in other words, we simply write $\Theta_{\text{ord}}$ (resp. $\Theta_{\text{ord}}^k$) instead of $\Theta_{\text{ord}}^{[w]}$ (resp. $\Theta_{\text{ord}}^{[w,k]}$) for all $w \in \mathbb{N}$.

Using Serre–Tate coordinates (Theorem B.1.1), the formal deformation space of the $O_p$-divisible group $\mathcal{L}[p^\infty] \oplus F_p/O_p$ (over $\kappa$) is canonically isomorphic to $\mathcal{T}$. Thus, we have the classifying morphism

$c : \mathcal{X}(\infty) \to \mathcal{T}$

of $O_p^{ur}$-formal schemes. It induces a morphism

\begin{equation}
c_{/x} : \mathcal{X}(\infty)/x \to \mathcal{T}
\end{equation}
for every closed point \( x \in \mathfrak{X}(\infty)(\kappa) \), where \( \mathfrak{X}(\infty)/x \) denotes the formal completion of \( \mathfrak{X}(\infty) \) at \( x \). The following Lemma 2.3.4 and Proposition 2.3.5 will be proved in §2.5.

**Lemma 2.3.4.** The morphism \( c/x \) is an isomorphism for every \( x \).

By the above lemma, we have for every closed point \( x \in \mathfrak{X}(\infty)(\kappa) \), a restriction map

\[
\text{res}_x : \mathcal{M}^0(\infty, K) \to \mathcal{O}(\mathcal{B}, K)
\]

induced from \( c/x \) (2.16).

**Proposition 2.3.5.** There is a morphism \( \beta : \mathcal{LJ} \times_{\text{Spf} O_p^\text{nr}} \mathfrak{X}(\infty) \to \mathfrak{X}(\infty) \) such that

1. for every \( x \in \mathfrak{X}(\infty)(\kappa) \), \( \beta \) preserves \( \mathfrak{X}(\infty)/x \), and the induced morphism

\[
\beta_x : \mathcal{LJ} \times_{\text{Spf} O_p^\text{nr}} \mathfrak{X}(\infty)/x \to \mathfrak{X}(\infty)/x
\]

is simply the formal group law after identifying \( \mathfrak{X}(\infty)/x \) with \( \mathcal{LJ} \) via \( c/x \);

2. if we equip \( \mathcal{LJ} \) with the action of \( O_E^\times \times \mathbb{B}^{\times \mathbb{P}^1} \) via the inflation \( O_E^\times \to O_p^\times \) by \( t \mapsto t/t^c \)

and trivially on the second factor, then \( \beta \) is \( O_E^\times \times \mathbb{B}^{\times \mathbb{P}^1} \)-equivariant;

3. for every \( x \in F_p/O_p \), the following diagrams

\[
\begin{array}{ccc}
X(\pm \infty) \otimes_{F_p^\text{nr}} F_p^{ab} & \xrightarrow{T_{n^\pm(x)}} & X(\pm \infty) \otimes_{F_p^\text{nr}} F_p^{ab} \\
\Upsilon_{x^\pm}^{-1} \downarrow & & \downarrow \Upsilon_{x^\pm}^{-1} \\
\mathfrak{X}(\infty) \otimes_{O_p^\text{nr}} F_p^{ab} & \xrightarrow{\beta_{x^\pm}} & \mathfrak{X}(\infty) \otimes_{O_p^\text{nr}} F_p^{ab}
\end{array}
\]

commute, where

\[
n^+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad n^-(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},
\]

respectively; and \( \beta_x \) is the restriction of \( \beta \) to a point \( z \) of \( \mathcal{B} \).

In particular, \( \mathcal{LJ} \) acts trivially on the special fiber of \( \mathfrak{X}(\infty) \) and such \( \beta \) is unique.

**Definition 2.3.6.** We call

\[
\omega_\nu := c^* \nu^* \frac{dT}{T}
\]

the global Lubin–Tate differential, where \( \nu \) is the homomorphism in Notation 1.8.2. It is a nowhere vanishing global differential form on \( \mathfrak{X}(\infty) \otimes_{O_p^\text{nr}} O_{C_p} \), and in fact belongs to \( H^0(\mathfrak{X}(\infty), \Omega_{\mathfrak{X}(\infty)}^1) \otimes_{O_p^\text{nr}} F_p^{\text{nr}} \) by the definition of \( F_p^{\text{nr}} \).

**Remark 2.3.7.** The pullbacks \( \Upsilon_{x^\pm}^* \omega_\nu \) depend only \( \psi \) (or rather \( \psi^\pm \)), not on the choice of \( \nu \).

In the following definition, we generalize the notion of convergent modular forms first introduced by Katz [Kat75] to Shimura curves.

**Definition 2.3.8.** Suppose that we have \( m \in \mathbb{N} \cup \{\infty\} \), \( w \in \mathbb{Z} \) and a complete field extension \( K/F_p^{\text{nr}} \).

1. Define the space of \((K\text{-valued})\) convergent modular forms of weight \( w \) and \( p \text{-Iwahori level} \( m \) to be

\[
\mathcal{M}^w(m, K) = H^0(\mathfrak{X}(m), (\Omega_{\mathfrak{X}(m)}^1)^\otimes w) \otimes_{O_p^\text{nr}} K,
\]

which is naturally a complete \( K \)-vector space.
(2) A convergent modular form of weight 0 is simply called a convergent modular function.
(3) Put \( \mathcal{M}_o^w(m, K) = \bigcup_{U^p \in \mathcal{U}} \mathcal{M}_o^w(m, K)^{U^p} \subset \mathcal{M}_o^w(m, K) \).

Remark 2.3.9. For \( m \in \mathbb{N} \cup \{ \infty \} \) and \( w \in \mathbb{Z} \), the space \( \mathcal{M}_o^w(m, K) \) has a natural action by \( O_{E_p}^\times \times \mathbb{B}^{\infty \times \mathcal{X}} \) under which \( \mathcal{M}_o^w(m, K) \) is stable. Moreover, \( \mathcal{M}_0^0(m, K) \) is the complete tensor product of the coordinate ring of \( \mathcal{X}(m) \) and the field \( K \), and thus \( \mathcal{M}_o^w(m, K) \) is naturally a topological \( \mathcal{M}_0^0(m, K) \)-module.

In particular for \( w \in \mathbb{N} \), we have the Atkin–Serre operator
\[
\Theta_{\text{ord}} : \mathcal{M}_o^w(m, K) \to \mathcal{M}_o^{w+1}(m, K)
\]
induced from the corresponding operator of sheaves (Definition 2.3.3). The operator is \( O_{E_p}^\times \times \mathbb{B}^{\infty \times \mathcal{X}} \)-equivariant.

From now on, we suppose that \( K \) is a complete field extension of \( F_p^{\text{ab}} \). Then for every \( w \in \mathbb{Z} \), the multiplication by \( \omega_p^w \) (Definition 2.3.6) induces a canonical \( \mathbb{B}^{\infty \times \mathcal{X}} \)-equivariant isomorphism \( \mathcal{M}_0^0(\infty, K) \cong \mathcal{M}_o^w(\infty, K) \).

Definition 2.3.10 (Stable convergent modular forms). A convergent modular function \( f \in \mathcal{M}_o^0(\infty, K) \) is stable if
\[
\sum_{z \in \text{Ker}[p] \subset \mathcal{B}(K^{ac})} \beta_z^* f = 0,
\]
where \( \beta_z^* : \mathcal{M}_o^0(\infty, K) \to \mathcal{M}_o^0(\infty, K) \) is the map induced by \( \beta \) from Proposition 2.3.5. Denote by \( \mathcal{M}_0^0(\infty, K)^\circ \) the subspace of \( \mathcal{M}_o^0(\infty, K) \) of stable convergent modular functions.

For \( m \in \mathbb{N} \cup \{ \infty \} \) and \( w \in \mathbb{Z} \), put
\[
\mathcal{M}_o^w(m, K)^{\circ} = \mathcal{M}_o^w(m, K) \cap \mathcal{M}_o^0(\infty, K)^\circ \cdot \omega_p^w \subset \mathcal{M}_o^w(m, K).
\]

Remark 2.3.11. By Proposition 2.3.5, a convergent modular function \( f \) is stable if and only if \( \text{res}_x f \) (2.17) is stable (Definition 2.1.3) for every \( x \in \mathcal{X}(\infty)(\kappa) \).

Remark 2.3.12. The space \( \mathcal{M}_o^w(m, K)^{\circ} \) does not depend on the choices of \( \psi \) or \( \nu \).

Definition 2.3.13 (Admissible convergent modular forms). Let \( n \geq 0 \) be an integer. We say that a stable convergent modular function \( f \in \mathcal{M}_o^0(\infty, K)^\circ \) is \( n \)-admissible if \( \beta_z^* f = \nu(z) f \) holds for all \( z \in \text{Ker}[p^n] \subset \mathcal{B}(K^{ac}) \). We say that \( f \in \mathcal{M}_o^w(m, K) \) is an \( n \)-admissible stable convergent modular form if \( f \omega_p^{-w} \) is is an \( n \)-admissible stable convergent modular function.

Remark 2.3.14. By Proposition 2.3.5 (1), a stable convergent modular function \( f \) is \( n \)-admissible if and only if \( \text{res}_x f \) (2.17) is \( n \)-admissible (in the sense of Definition 2.1.15) for every \( x \in \mathcal{X}(\infty)(\kappa) \).

The following lemma is a comparison between the Atkin–Serre operator \( \Theta_{\text{ord}} \) (2.18) and the Lubin–Tate differential operator \( \Theta \) (Definition 2.1.12).

Lemma 2.3.15. For an element \( f \in \mathcal{M}_o^w(m, K) \) for some \( w, m \in \mathbb{N} \), we have
\[
\text{res}_x((\Theta_{\text{ord}} f)\omega_p^{-w-1}) = \Theta(\text{res}_x(f\omega_p^{-w}))
\]
for every \( x \in \mathcal{X}(\infty)(\kappa) \).

Proof. It follows from Lemma 2.3.4, Theorem B.2.3, and the definition of \( \Theta \). \( \square \)
**Definition 2.3.16** (Universal convergent modular form). A universal convergent modular form of depth \( m \in \mathbb{N} \) and tame level \( U^p \in \mathfrak{U} \) is an element
\[
M \in \mathcal{M}^0(\infty, K) \hat{\otimes}_{F_p} D(\mathcal{O}_p^{\text{anti}}, F_p)
\]
such that \( M \) is \( U^p \)-invariant and
\[
(2.19) \quad \Gamma_{t}^* M = \delta_{t}^{-1} \cdot M
\]
for \( t \in O_{E_p, m}^{\infty} \). Here, \( \Gamma_{t}: \mathfrak{X}(\infty) \to \mathfrak{X}(\infty) \) is the formal completion of the morphism (2.14); and we regard \( \delta_{t} \) as the Dirac distribution of the image of \( t \) under the quotient homomorphism \( O_{E_p}^{\infty} \to O_p^{\text{anti}} \).

Theorem 2.3.17. Let \( f \in \mathcal{M}_w^w(m, K) \odot \) be a stable convergent modular form for some \( w, m \in \mathbb{N} \). Then there is a unique element
\[
M(f) \in \mathcal{M}^0(\infty, K) \hat{\otimes}_{F_p} D(\mathcal{O}_p^{\text{anti}}, F_p)
\]
such that for every \( k \in \mathbb{N} \),
\[
(2.20) \quad M(f)(\langle w + k \rangle) = (\Theta_{\text{ord}} f) \omega_{\nu}^{-w-k},
\]
where \( \Theta_{\text{ord}} \) is the map (2.18). Moreover, we have

1. if \( f \) is fixed by \( U^p \in \mathfrak{U} \), then so is \( M(f) \);
2. \( M(f) \) is a universal convergent modular form of depth \( m \) (Definition 2.3.16);
3. if \( w \geq 1 \), then we have
\[
\Theta_{\text{ord}}(M(f)(\langle w - 1 \rangle)) \omega_{\nu}^{w-1} = f;
\]
4. Suppose that \( f \) is \( n \)-admissible (Definition 2.3.13). Then we have
\[
M(f)(\langle k \rangle) = M(f)(\chi(\langle k \rangle))
\]
for every \( k \in \mathbb{Z} \) and every (locally constant) character \( \chi: O_p^{\text{anti}} \to K^\times \) that is trivial on \( (1 + p^n)^\infty \).

Proof. The uniqueness follows from Lemma 2.1.11.

The morphism \( \beta \) in Proposition 2.3.5 induces a map
\[
\beta^*: \mathcal{M}^0(\infty, K) \to \mathcal{M}^0(\infty, K) \otimes_K \mathcal{O}(\mathcal{B}, K).
\]
By Lemma 2.1.6 (2) and Remark 2.3.11, it sends \( \mathcal{M}^0(\infty, K) \odot \) into \( \mathcal{M}^0(\infty, K) \odot \mathcal{O}(\mathcal{B}, K) \odot \). Thus we may regard \( \beta^*(f \omega^{-w}) \) as an element in \( \mathcal{M}^0(\infty, K) \odot \mathcal{O}(\mathcal{B}, K) \odot \). We define \( \tau_w: D(\mathcal{O}_p^{\text{anti}}, F_p) \to D(\mathcal{O}_p^{\text{anti}}, F_p) \) via the Fourier transform \( \mathcal{F} \) since \( K \) contains \( F_p^{\text{st}} \). Define a (continuous \( F_p \)-linear) translation map
\[
\tau_w: D(\mathcal{O}_p^{\text{anti}}, F_p) \to D(\mathcal{O}_p^{\text{anti}}, F_p)
\]
such that \( (\tau_w \phi)(g) = \phi(g \cdot (-w)) \) for every \( g \in C(\mathcal{O}_p^{\text{anti}}, F_p) \). We take
\[
M(f) = \tau_w(\beta^*(f \omega^{-w})).
\]

For the formula (2.20), it suffices to check it after applying \( \text{res}_x \) for every \( x \in \mathfrak{X}(\infty)(\kappa) \). In fact, we have
\[
(2.21) \quad \text{res}_x M(f)(\langle w + k \rangle) = \text{res}_x(\beta^*(f \omega^{-w}))(\langle k \rangle) = M_{\text{loc}}(\text{res}_x(f \omega^{-w}))(\langle k \rangle)
\]
by Definition 2.1.8 and the definition of $\beta$. By Lemma 2.1.14, we have (2.21) = $\Theta^k(\text{res}_x(f\omega^{-w}))$. Finally by Lemma 2.3.15, we have $\Theta^k(\text{res}_x(f\omega^{-w})) = \text{res}_x((\Theta^k_{\text{ord}}f)\omega^{-w-k})$.

Property (1) follows from Proposition 2.3.5 (2). Properties (3) and (4) follow from Lemma 2.1.14 and Lemma 2.1.16, respectively. For property (2), we only need to show that (2.19) holds for $M = M(f)$ and $t \in O^*_F_{p,m}$. In fact, since $f$ is fixed by $O^*_F_{p,m}$, we have $M(f) = M(\Gamma^*_f)$ which equals $\delta_t \cdot \Gamma^*_f M(f)$ by Proposition 2.3.5 (2) and Remark 2.1.5.

The following definition is suggested by the formula (2.21) in the proof of the above theorem.

**Definition 2.3.18.** We call $M(f)$ in Theorem 2.3.17 the *global Mellin transform* of $f$.

### 2.4. Comparison with archimedean differential operators.

Now suppose that $\mathbb{B}$ is equipped with an $E$-embedding as in Definition 1.8.3 such that $e_p$ coincides with (1.7) under the fixed isomorphism $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$.

**Definition 2.4.1** (CM-subscheme). We define the *CM-subscheme* $Y$ to be $X^{E^x}$, the subscheme of $X$ fixed by the action of $e(E^x)$ for $e$ as in Definition 1.8.3. Define $Y^\pm$ to be the subschemes of $Y$ such that $E^x$ acts on the tangent space of points in $Y^\pm$ via the characters $t \mapsto (t/t^e)^{\pm 1}$ respectively. See also [YZZ13, §3.1.2].

In what follows, we will regard $Y^\pm$ as their base change to $F^\text{nr}_p$; in particular, they are closed subschemes of $X \otimes_{F_p} F^\text{nr}_p$.

**Lemma 2.4.2.** We have $Y = Y^+ \amalg Y^-$. Moreover, we have

1. both $Y^+(\mathbb{C}_p)$ and $Y^-(\mathbb{C}_p)$ are equipped with the natural profinite topology, isomorphic to $\overline{E^x} \setminus A^{\infty \times}_E$, and admit a transitive action of $A^{\infty \times}_E$ via Hecke morphisms (see §1.8 for the notation $E^x$);
2. the projection maps $X \otimes_{F_p} F^\text{nr}_p \to X(\pm \infty)$ restrict to isomorphisms from $Y^\pm$ to their images, respectively. In particular, we may regard $Y^\pm$ as closed subschemes of $X(\pm \infty)$.
3. the closed subschemes $Y^\pm(\infty) := \mathcal{Y}_\pm Y^\pm$ of $X(\infty) \otimes_{F^\text{nr}_p} F^\text{ab}_p$ descend to closed subschemes of $X(\infty) \otimes_{O^\text{nr}_p} F^\text{nr}_p$, where $\mathcal{Y}_\pm$ are the transition isomorphisms in Definition 2.2.10.

**Proof.** The decomposition follows directly from the definition. For the rest, we consider $Y^+(\mathbb{C}_p)$ without lost of generality.

Part (1) can be seen from the complex uniformization by choosing an arbitrary isomorphism $\mathbb{C}_p \simeq \mathbb{C}$.

Part (2) follows from the fact that $A^{\infty \times}_E$ does not contain any nontrivial unipotent element.

Part (3) follows from the fact that $\text{Gal}(F^\text{ab}_p/F^\text{nr}_p)$ acts via local class field theory as the right multiplication of $O^\times_p$ on the double coset presentation $X(\infty)(\mathbb{C}_p)$ hence preserves the subset $Y^+(\mathbb{C}_p)$. 

**Notation 2.4.3.** For $m \in \mathbb{N} \cap \{\infty\}$, denote by

1. $Y^\pm(m)$ the image of $Y^\pm(\infty)$ in $X(m) \otimes_{O^\text{nr}_p} F^\text{nr}_p$,
2. $\overline{Y}^\pm(m)$ the Zariski closure of $Y^\pm(m)$ in $X(m)$,
3. $\mathcal{Y}^\pm(m)$ the formal completion of $\overline{Y}^\pm(m)$ along the special fiber.
Lemma 2.4.4. For \( m \in \mathbb{N} \cup \{\infty\} \), we have \( \mathcal{Y}^\pm(m)(\mathbb{C}_p) = \mathcal{Y}^\pm(m)(F_p^{nr}) = \mathcal{Y}^\pm(m)(O_p^{nr}) \).
Here, for an \( O_p^{nr} \)-algebra \( R \), \( \mathcal{Y}^\pm(m)(R) \) are the sets of morphisms from \( \text{Spec} \; R \) to \( \mathcal{Y}^\pm(m) \) over \( \text{Spec} \; O_p^{nr} \), respectively.

Proof. Without lost of generality, we only prove the case for \( \mathcal{Y}^+(m) \).
We first consider the case where \( m = 0 \). The first identity \( \mathcal{Y}^\pm(0)(\mathbb{C}_p) = \mathcal{Y}^\pm(0)(F_p^{nr}) \) is well-known from the class field theory. Take an element \( x \in \mathcal{Y}^+(0)(F_p^{nr}) \). It induces a unique morphism \( y : \text{Spec} \; O_p^{nr} \to \mathcal{X} \). Since \( y \) is fixed by \( E^\times \), there are strict actions of \( E^\times \cap O_E^X \), hence \( O_{E_p} \) on the \( O_p \)-divisible group \( G_y \). Therefore, the reduction of \( G_y \) is ordinary. In other words, \( y \) factors through \( \mathcal{X}(0) \). As \( \mathcal{Y}^+(0) \) is defined as the Zariski closure of \( Y^+(0) \) in \( \mathcal{X}(0) \), we obtain an element \( x' \in \mathcal{Y}^+(0)(O_p^{nr}) \) uniquely determined by \( x \). Thus, we have \( \mathcal{Y}^+(0)(F_p^{nr}) = \mathcal{Y}^+(0)(O_p^{nr}) \).

The case for general \( m \) follows from the case for \( m = 0 \) and the following two facts: (1) \( \mathcal{Y}^+(m) \) is a closed subscheme of \( \mathcal{Y}^+(0) \times_{\mathcal{X}(0)} \mathcal{X}(m) \); (2) \( \mathcal{X}(m) \to \mathcal{X}(0) \) is a finite étale morphism (resp. a projective limit of finite étale morphisms) for \( m \in \mathbb{N} \) (resp. \( m = \infty \)). □

Notation 2.4.5. Let \( S \) be a scheme locally of finite type over \( \text{Spec} \; \mathbb{C} \). We denote by \( \tilde{S} \) the underlying real analytic space with the complex conjugation automorphism \( \zeta_S : \tilde{S} \to \tilde{S} \).
In what follows, we will sometimes deal with a complex scheme \( S \) that is of the form \( \lim_i S_i \) where \( S \) is a filtered partially ordered set and each \( S_i \) is a smooth complex scheme, with a sheaf \( \mathcal{F} \) that is the restriction of a quasi-coherent sheaf \( \mathcal{F}_0 \) on some \( S_0 \). Then we will write \( \tilde{S} = \{ \tilde{S}_i \}_{i \in I} \) for the projective system of the underlying real analytic spaces together with the complex conjugation \( \zeta_S \), and \( \tilde{\mathcal{F}} = \{ \tilde{\mathcal{F}}_i \}_{i \geq 0} \) the projective system of real analytification of the restricted sheaf \( \mathcal{F}_i \) for \( i \geq 0 \). Moreover, we denote
\[
H^0(\tilde{S}, \tilde{\mathcal{F}}) := \lim_{i \geq 0} H^0(\tilde{S}_i, \tilde{\mathcal{F}}_i).
\]

For an isomorphism \( \iota : \mathbb{C}_p \to \mathcal{C} \), put \( X_\iota = X \otimes_{F_\iota} \mathbb{C} \) and denote
\[
(2.22) \quad \zeta_\iota : \tilde{X}_\iota \to \tilde{X}_\iota
\]
the complex conjugation. Denote by \( (\mathcal{L}_i, \nabla_i) \) the restriction of the pair \( (\mathcal{L}, \nabla) \otimes_{O_p^{nr}} \mathcal{C} \) along \( \pi_i : X_\iota \to X \otimes_{O_p^{nr}} \mathbb{C} \), where \( (\mathcal{L}, \nabla) \) appears in Lemma 2.2.4. Applying the same procedure to the sequence (2.8), we obtain the following sequence
\[
0 \longrightarrow \omega_{\tilde{X}_\iota}^* \longrightarrow \mathcal{L}_\iota \longrightarrow \omega_{\tilde{X}_\iota}^{\nabla} \longrightarrow 0
\]
of locally free sheaves on \( X_\iota \). Similarly, we have the Kodaira–Spencer isomorphism
\[
(2.23) \quad \text{KS}_\iota : \omega_{\tilde{X}_\iota}^* \otimes \omega_{\tilde{X}_\iota}^0 \simeq \Omega^1_{\tilde{X}_\iota}
\]
induced by (2.10).

Lemma 2.4.6. The natural map \( \tilde{\omega}_{\tilde{X}_\iota}^* \oplus \zeta_\iota(\tilde{\omega}_{\tilde{X}_\iota}^*) \to \tilde{\mathcal{L}}_\iota \) is an isomorphism of sheaves on the real analytic space \( \tilde{X}_\iota \). Moreover, we have \( \nabla_i (\zeta_\iota(\tilde{\omega}_{\tilde{X}_\iota}^*)) \subset (\zeta_\iota(\tilde{\omega}_{\tilde{X}_\iota}^*)) \otimes \tilde{\Omega}^1_{X_\iota} \).

Proof. It follows from Lemma 2.4.12 later in this section. □

We have a remark similar to Remark 2.3.2, which together with Lemma 2.4.6 induce a map
\[
(2.24) \quad \theta_{1w}^i : \tilde{\mathcal{L}}_i^{[w]} \to \tilde{\Omega}^{1w}_{X_\iota} \otimes (\tilde{\omega}_{\tilde{X}_\iota}^*) \otimes w \xrightarrow{\text{KS}_\iota(2.23)} (\tilde{\Omega}^{1w}_{X_\iota})^{\otimes w}
\]
for all $w \in \mathbb{N}$.

**Definition 2.4.7.** Similar to Definition 2.3.3, define the **Shimura–Maass operator** to be

$$
\Theta_i^{[w]}: (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w} \xrightarrow{(2.11)} \mathcal{L}^{[w]} \otimes \tilde{\Omega}_{\mathcal{X}}^1 \xrightarrow{\Theta_i^{[w]}(2.24)} (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w+1}.
$$

For $k \in \mathbb{N}$, define the **Shimura–Maass operator of degree $k$** to be

$$
\Theta_i^{[w,k]} = \Theta_i^{[w+k-1]} \circ \cdots \circ \Theta_i^{[w]}: (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w} \rightarrow (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w+k}.
$$

As for $\Theta_{\text{ord}}$, we will suppress $w$ from notation and write $\Theta_i$ (resp. $\Theta_i^k$) for $\Theta_i^{[w]}$ (resp. $\Theta_i^{[w,k]}$).

In particular, we have the map

$$
(2.25) \quad \Theta_i: H^0(X_i, (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w}) \rightarrow H^0(X_i, (\tilde{\Omega}_{\mathcal{X}}^1)^{\otimes w+1}).
$$

**Notation 2.4.8.** Put

$$
X(m)_i = X(m) \otimes_{F^{\text{nr},i}} \mathbb{C}, \quad m \in \mathbb{Z} \cup \{\pm \infty\},
$$

$$
Y^{\pm}(m)_i = Y^{\pm}(m) \otimes_{F^{\text{nr},i}} \mathbb{C}, \quad m \in \mathbb{N} \cup \{\infty\}.
$$

Let $F^{\text{ab}}_p \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Take an element

$$
f \in H^0(X(m)_i, (\Omega_{X(m)_i}^1)^{\otimes w}) \otimes_F K
$$

with $m \in \mathbb{Z} \cup \{\pm \infty\}$ and $w \in \mathbb{N}$. Then by the transition isomorphism and by the restriction to ordinary locus, we have an element

$$
(2.26) \quad f_{\text{ord}} := \begin{cases} 
\Upsilon_+ f & \text{for } m \geq 0, \\
\Upsilon_- f & \text{for } m \leq 0.
\end{cases}
$$

By base change, $f$ induces another element

$$
f_i \in H^0(X(m)_i, (\Omega_{X(m)_i}^1)^{\otimes w}).
$$

The following lemma shows that the Atkin–Serre operator (2.18) and the Shimura–Maass operator (2.25) coincide on CM points. Note that the operator $\Theta_i$ descends along the projection map $X_i \rightarrow X(m)_i$.

**Lemma 2.4.9.** Let notation be as above. We have for $k \in \mathbb{N}$,

$$
G_\pm \left( (\Theta_i^k f_{\text{ord}})|_{Y^{\pm}(m)} \right) = (\Theta_i^k f_i)|_{Y^{\pm}(m)},
$$

as functions on $Y^{\pm}(m)_i$, regarded as closed subschemes of $X(\pm m)_i$ via the transition isomorphisms $\Upsilon_\pm$ in Definition 2.2.10 respectively.

**Proof.** Generally, once we restrict to stalks, we can not apply differential operators anymore. Therefore, we need alternative descriptions of $\Theta_{\text{ord}}^{[w]}$ and $\Theta_k^{[w,k]}$ (Here, we retrieve the original notation for clarity).

We denote by $\delta^{[w]}$ the following composite map

$$
(\omega^\ast)^{\otimes w} \times (\omega^\circ)^{\otimes w} \rightarrow \mathcal{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathcal{L}^{[w]} \otimes \Omega_{\mathcal{X}(n)}^1
$$

$$
\xrightarrow{\delta^{[w]}} (\omega^\ast)^{\otimes w} \otimes (\omega^\circ)^{\otimes w} \otimes \Omega_{\mathcal{X}(n)}^1 \xrightarrow{\text{KS}^{-1}} (\omega^\ast)^{\otimes w+1} \times (\omega^\circ)^{\otimes w+1},
$$

and by $\delta^{[w]}$ the following composite map

$$
\mathcal{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathcal{L}^{[w]} \otimes \Omega_{\mathcal{X}(n)}^1 \xrightarrow{\text{KS}^{-1}} \mathcal{L}^{[w]} \otimes (\omega^\ast \otimes \omega^\circ) \rightarrow \mathcal{L}^{[w+1]}.
$$
Since we have $\nabla\mathcal{L}^\circ \subseteq \mathcal{L}^\circ \otimes \Omega_1^1$ by Lemma 2.3.1 (2) and its dual version from Remark 2.3.2, the composition $\partial^{w+k-1} \circ \cdots \circ \partial^w$ coincides with the map

$$(\omega^\circ)^{\otimes w} \otimes (\omega^\circ)^{\otimes w} \to \mathcal{L}^w \xrightarrow{\delta^{w+k-1} \circ \cdots \circ \delta^w} \mathcal{L}^{[w+k]} \xrightarrow{\theta_{\text{ord}}^{[w+k]}} (\omega^\circ)^{\otimes w+k}.$$ 

Therefore, the map $\Theta_{\text{ord}}^{[w,k]}$ coincides with the composite map

$$(\Omega_1^1)^{\otimes w} \xrightarrow{\text{KS}^{-1}} (\omega^\circ)^{\otimes w} \otimes (\omega^\circ)^{\otimes w} \to \mathcal{L}^w \xrightarrow{\delta^{w+k-1} \circ \cdots \circ \delta^w} \mathcal{L}^{[w+k]} \xrightarrow{\text{KS} \theta_{\text{ord}}^{[w+k]}} (\Omega_1^1)^{\otimes w+k}.$$ 

The advantage of the above description is that $\theta_{\text{ord}}$ appears only at the end of the sequence of maps. Since we have $\nabla_l(c_i^*\hat{\omega}_i^\circ) \subset (c_i^*\hat{\omega}_i^\circ) \otimes \hat{\Omega}_1^1$, by Lemma 2.4.6, there is a similar description of $\Theta_{\text{ord}}^{[w,k]}$ as above. Therefore, to prove the lemma, we only need to show that the splitting in Lemma 2.4.6 coincides with the restriction of the splitting

$$\mathcal{L} \otimes \mathcal{O}_{p^r,i} \subset C = (\omega^\circ \otimes \mathcal{O}_{p^r,i} \subset C) \oplus (\mathcal{L}^\circ \otimes \mathcal{O}_{p^r,i} \subset C)$$

on $Y^+_i$ and $Y^-_i$.

Pick up an arbitrary point $y \in Y^+(m)_i(\mathbb{C}) \cup Y^-(m)_i(\mathbb{C})$. We have an action of $E^\times$ on both the splitting $\omega^\circ |_y \oplus \omega^\circ |_y$ and $(\omega^\circ \otimes \mathcal{O}_{p^r,i} \subset C)|_y \oplus (\mathcal{L}^\circ \otimes \mathcal{O}_{p^r,i} \subset C)|_y$. By definition, $\omega^\circ |_y$ and $\omega^\circ \otimes \mathcal{O}_{p^r,i} \subset C)|_y$ coincide, which is one of the two complex eigen-lines with respect to the $E^\times$-action. It follows that $c_i^*\omega^\circ |_y$ and $\mathcal{L}^\circ \otimes \mathcal{O}_{p^r,i} \subset C)|_y$ have to coincide as well, which contributes to the other complex eigen-line. \hfill $\Box$

We now study the behavior of the Shimura–Maass operator under complex uniformization.

**Definition 2.4.10 (i-nearby data).** Let $\iota: \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. An $i$-nearby data for $\mathbb{B}$ consists of

- a quaternion algebra $B(i)$ over $F$ such that $B(i)_v$ is definite for archimedean places $v$ other than $i|_F$,
- an isomorphism $B(i)_v \simeq \mathbb{B}_v$ for every finite place $v$ other than $p$,
- an isomorphism $B(i)_v \simeq B(i) \otimes_{F,v} \mathbb{R} \simeq \text{Mat}_2(\mathbb{R})$, and
- an embedding $e(i): E \to B(i)$ of $F$-algebras such that $e(i)_v$ coincides with $e_v$ under the isomorphism $B(i)_v \simeq \mathbb{B}_v$ for every finite place $v$ other than $p$, and $\mathcal{H}E^\times = \{ \pm i \}$.

We now choose an $i$-nearby data for $\mathbb{B}$. It induces a complex uniformization

$$X_i(\mathbb{C}) \simeq B(i)^\times \backslash \mathcal{H} \times \mathbb{B}^{\infty \times} / F^\times,$$

where $\mathcal{H} = \mathbb{C} \setminus \mathbb{R}$ denotes the union of Poincaré upper and lower half-planes. Let $z$ be the standard coordinate on $\mathcal{H}$.

**Lemma 2.4.11.** Denote by $L_i$ the $\mathbb{C}$-local system on $X_i$ defined by the quotient map

$$B(i)^\times \mathbb{C}^{\otimes 2} \times \mathcal{H} \times \mathbb{B}^{\infty \times} / F^\times \to B(i)^\times \mathcal{H} \times \mathbb{B}^{\infty \times} / F^\times \simeq X_i(\mathbb{C})$$

where the action of $\gamma \in B(i)^\times$ is given by the formula

$$\gamma[(a_1, a_2)^t, z, g] = [((a_1, a_2)\iota(\gamma)^{-1})^t, \iota(\gamma)(z), \gamma^\infty g].$$

Then we have a canonical isomorphism $L_i \simeq \mathcal{O}_{X_i} \otimes_{\mathbb{C}} L_i$ under which $\nabla_l$ coincides with the induced connection on $\mathcal{O}_{X_i} \otimes_{\mathbb{C}} L_i$.

**Proof.** It follows from the fact that $L_i$ is canonically isomorphic to the restriction of $L \otimes \mathcal{O}_{p^r,i} \subset C$ along the natural morphism $\pi_i$, where $L$ is the $\mathcal{O}_p$-local system on $\mathcal{X}$ defined in Remark 2.2.5. \hfill $\Box$
The following lemma will be proved in §2.5.

**Lemma 2.4.12.** Under the isomorphism \( L_s \simeq \mathcal{O}_{X_s} \otimes_{\mathbb{C}} L_s \) in Lemma 2.4.11, the subsheaf \( \omega_i^\bullet \) is generated by the section \( \omega_i^\bullet \) whose value at \( z \) is \((z,1)^t\).

The following lemma shows that our definition of Shimura–Maass operators coincides with the classical one.

**Lemma 2.4.13.** For every \( f \in H^0(\tilde{X}_s,(\tilde{\Omega}^1_{X_s})^\otimes w) \) with some \( w \in \mathbb{N} \), we have
\[
\Theta_i f \otimes dz^{\otimes-w-1} = \left( \frac{\partial}{\partial z} + \frac{2w}{z-\bar{z}} \right) f \otimes dz^{\otimes-w}.
\]

**Proof.** We may pass to the universal cover \( \mathcal{H} \times \mathbb{B}^\infty / \mathbb{F}^\infty \) and suppress the part \( \mathbb{B}^\infty / \mathbb{F}^\infty \) in what follows. Over \( \mathcal{H} \), the sheaf \( L_s \) is trivialized as \( \mathbb{C}^{\otimes 2} \) and the subsheaf \( \omega_i^\bullet \) is generated by the section \( \omega_i^\bullet \) whose value at \( z \) is \((z,1)^t\) by Lemma 2.4.12. Dually, the sheaf \( \mathcal{L}_s^\vee \) is trivialized as two-dimensional complex row vectors and the subsheaf \( \omega_i^\circ \) is generated by the section \( \omega_i^\circ \) whose value at \( z \) is \((1,-z)\). Then we have \( KS(\omega_i^\bullet \otimes \omega_i^\circ) = dz \).

It is easy to see that
\[
\Theta_i \left((\omega_i^\bullet)^{\otimes w} \otimes (\omega_i^\circ)^{\otimes w}\right) = \frac{2w}{z-\bar{z}} \left((\omega_i^\bullet)^{\otimes w} \otimes (\omega_i^\circ)^{\otimes w}\right) \otimes dz
\]
since \( c_i^w \omega_i^\bullet \) (resp. \( c_i^w \omega_i^\circ \)) is generated by the section \((z,1)^t\) (resp. \((1,-z)\)). The lemma follows. \( \square \)

We now introduce the notion of automorphic forms.

**Notation 2.4.14.** For every \( w \in \mathbb{Z} \), denote by \( \mathcal{A}^{(2w)}(B(i)^\times) \) (resp. \( \mathcal{A}^{(2w)}_{\text{cusp}}(B(i)^\times) \)) the space of real analytic (resp. and cuspidal) automorphic forms on \( B(i)^\times(\mathbb{A}) \) of weight \( 2w \) at \( i|F \) and invariant under the action of \( B(i)^\times \) at archimedean places \( v \) other than \( i|F \).

The spaces \( \mathcal{A}^{(2w)}(B(i)^\times) \) and \( \mathcal{A}^{(2w)}_{\text{cusp}}(B(i)^\times) \) are representations of \( B(i)^\times(\mathbb{A}) \) by the right translation \( R \).

**Lemma 2.4.15.** There is a natural \( \mathbb{B}^\infty / \mathbb{F}^\infty \)-equivariant map
\[
\phi_i : H^0(\tilde{X}_s,(\tilde{\Omega}^1_{X_s})^{\otimes w}) \to \mathcal{A}^{(2w)}(B(i)^\times)
\]
such that for \( g_i \in B(i)^\times = \text{GL}_2(\mathbb{R}) \),
\[
\phi_i(f)([g_i,1])j(g_i,i)^w = f(g_i(i)) \otimes dz^{\otimes-w},
\]
where \( j(g_i,i) = (\det g_i)^{-1} \cdot (ci+d)^2 \) is the square of the usual \( j \)-factor.

**Proof.** This is the well-known dictionary between modular forms and automorphic forms. \( \square \)

We denote by \( H^0_{\text{cusp}}(\tilde{X}_s,(\tilde{\Omega}^1_{X_s})^{\otimes w}) \subset H^0(\tilde{X}_s,(\tilde{\Omega}^1_{X_s})^{\otimes w}) \) the inverse image of \( \mathcal{A}^{(2w)}_{\text{cusp}}(B(i)^\times) \) under \( \phi_i \).

**Definition 2.4.16.** Define \( \Delta_\pm \) to be the matrices
\[
\frac{1}{4i} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}
\]
in \( \mathfrak{gl}_{2,\mathbb{C}} = \text{Mat}_2(\mathbb{C}) \), respectively. For an isomorphism \( i : \mathbb{C}_p \to \mathbb{C} \), define \( \Delta_{\pm,t} \) to be the matrices \( \Delta_\pm \) when regarded as elements in \( \text{Lie}_c(B(i) \otimes_{F,i} \mathbb{C}) = \mathfrak{gl}_{2,\mathbb{C}} \), respectively. Finally, define \( \Delta_{\pm,t}^k = \Delta_{\pm,t} \circ \cdots \circ \Delta_{\pm,t} \) to be the \( k \)-fold composition.
Lemma 2.4.17. For every \( f \in \mathcal{H}^0_{\text{cusp}}(\tilde{X}_\iota, (\tilde{\Omega}^1_{\tilde{X}_\iota})^{\otimes \nu}) \) and \( k \in \mathbb{N} \), we have
\[
\phi_\iota(\Omega^k f) = \Delta^k_{+, \iota} \phi_\iota(f),
\]
where \( \phi_\iota \) is defined in Lemma 2.4.15.

Proof. This follows from Lemma 2.4.13, together with [Bum97] p.130, p.143, and Proposition 2.2.5 on p.155. \( \square \)

2.5. Proofs of claims via unitary Shimura curves. In this section, we prove the six claims (2.2.4, 2.2.6, 2.3.1, 2.3.4, 2.3.5, and 2.4.12) left in previous sections. Suppose that we are in the case of modular curves, that is, \( F = \mathbb{Q} \) and \( \mathbb{B} \) is unramified at every prime, then these statements except Proposition 2.3.5 are clear. In fact, in this case,

- Lemma 2.2.4 follows from the fact that (2.5) is the formal completion of the Hodge sequence coming from the universal elliptic curve;
- Proposition 2.2.6 is well-known;
- Lemma 2.3.1 is proved by Katz as [Kat78, Theorem 1.11.27];
- Lemma 2.3.4 follows from Serre–Tate coordinates in [Kat81];
- Proposition 2.3.5 again can be proved via Serre–Tate coordinates (one can adjust our proof below to the case of modular curves); and
- Lemma 2.4.12 is again well-known.

The main idea is to use the existence of universal family of elliptic curves with deformation theory. However, in the general case, \( \mathcal{X} \) is not a moduli space; therefore we have to use some auxiliary moduli space to deduce these statements. The reader may skip the rest part of this section for the first reading.

Our strategy is to use the unitary Shimura curves considered by Carayol in [Car86]. Thus we will fix an isomorphism \( \iota: \mathbb{C}_p \xrightarrow{\sim} \mathbb{C} \). In particular, \( F^\nr_p \) is a subfield of \( \mathbb{C} \). We also fix an \( \nu \)-nearby data for \( \mathbb{B} \) (Definition 2.4.10) and put \( B = B(\iota) \) for short.

Note that when \( F = \mathbb{Q} \) there is no need to change the Shimura data as \( \mathcal{X} \) is already a moduli space. In order to unify the argument, we will choose to do so in this case as well. We will also assume that we are not in the case of classical modular curves (that is, \( F = \mathbb{Q} \) and \( \mathbb{B} \) is unramified at every prime) where all these statements are known, as explained above.

Fix an element \( \lambda \in \mathbb{C} \) such that \( \text{Im} \lambda > 0; -\lambda^2 \in \mathbb{N} \); \( p \) splits in \( \mathbb{Q}(\lambda) \subset \mathbb{C} \); and \( \mathbb{Q}(\lambda) \) is not contained in \( E \). We have subfields \( F(\lambda) \) and \( E(\lambda) \) of \( \mathbb{C} \), and identify their completion inside \( \mathbb{C} \simeq \mathbb{C}_p \) with \( F_p \). In [Car86, §2] (see also [Kas04, §2]), a reductive group \( G' \) over \( \mathbb{Q} \) is defined as a subgroup of \( \text{Res}_{F/\mathbb{Q}}(B^\times \times F^\times \mathcal{F}(\lambda)^\times) \) (which itself is a subgroup of \( \text{Res}_{F(\lambda)/\mathbb{Q}}(B^\times F(\lambda)^\times) \)) with “rational norms”. In particular, we have
\[
G'(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \text{GL}_2(F_p) \times (\mathbb{B}^\times_{p_2} \times \cdots \times \mathbb{B}^\times_{p_m}),
\]
where \( p_2, \ldots, p_m \) are primes of \( F \) over \( p \) other than \( p \). Put
\[
G^p = G'((\prod_{q \neq p} \mathbb{Z}_q) \otimes \mathbb{Q}) \times (\mathbb{B}_p^\times \times \cdots \times \mathbb{B}^\times_{p_m}),
\]
and let \( \mathbf{U}' \) be the set of all (sufficiently small) open compact subgroups \( U^p \) of \( G^p \). Then for each \( U^p \in \mathbf{U}' \), there is a unitary Shimura curve \( \mathcal{X}'_{U^p} \), smooth and projective over \( \text{Spec} F_p \), of the level structure \( \mathbb{Z}_p^\times \times \text{GL}_2(O_p) \times U^p \). It has a canonical smooth model \( \mathcal{X}'_{U^p} \) over \( \text{Spec} O^\nr_p \) defined via a moduli problem [Car86, §6]. In particular, there is a universal abelian variety \( \pi: \mathcal{A}_{U^p} \to \mathcal{X}'_{U^p} \) with a specific \( p \)-divisible subgroup \( \mathcal{G}'_{U^p} := (\mathcal{A}_{U^p}[p^\infty])^{1,1} \subset \mathcal{A}_{U^p}[p^\infty] \) (it is denoted as \( E'_\infty \) in loc. cit.), which is an \( O_p \)-divisible group of dimension 1 and height
2. Here, for an object $M$ with $O_{F(\lambda)} \otimes \mathbb{Z}_p$-action, we denote by $M^{2,1}$ the direct summand corresponding to the $p$-adic place $F(\lambda) \xrightarrow{\lambda \rightarrow \lambda} F(\lambda) \subset \mathbb{C}_p$. Then $(A_{U^p}[p^\infty])^{2,1}$ admits an action by $O_B \otimes O_p$, $O_p \cong \text{Mat}_2(O_p)$. Put $e_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ and $(A_{U^p}[p^\infty])^{2,1}_1 = e_1(A_{U^p}[p^\infty])^{2,1}$. See [Car86, §2.6] for more details.

We let $\mathcal{Y}'(0)_{U^p}$ be the (dense) open subscheme of $\mathcal{X}'_{U^p}$ by removing all points on the special fiber where $\mathcal{G}'$ is supersingular. For $n \in \mathbb{N}$, define $\mathcal{X}'(n)_{U^p}$ to be the functor classifying $O_p$-equivariant extensions

$$0 \longrightarrow \mathcal{L}^p[p^n] \longrightarrow \mathcal{G}'[p^n] \longrightarrow p^{-n}/O_p \longrightarrow 0$$

of $\mathcal{G}'$ over $\mathcal{X}'(0)_{U^p}$. The obvious map $\mathcal{X}'(n)_{U^p} \to \mathcal{X}'(0)_{U^p}$ is étale. Finally, put $\mathcal{X}'(\infty)_{U^p} = \varprojlim_n \mathcal{X}'(n)_{U^p}$.

The construction of Carayol amounts to saying that for every sufficiently small $U^p \in \mathcal{U}$ and a connected component $\mathcal{X}'_{U^p}$ of $\mathcal{X}_U$, there exists a member $U'^p \in \mathcal{U}'$ such that

- $\mathcal{X}(n)_{U^p}^\dagger := \mathcal{X}'_{U^p} \times_{\mathcal{X}'_{U^p}} \mathcal{X}(n)_{U^p}$ is isomorphic to the neutral connected component of $\mathcal{X}'(n)_{U^p}$ for $n \in \mathbb{N} \cup \{\infty\}$;

- under the above isomorphism, $\mathcal{G}_{U^p}|_{\mathcal{X}(n)_{U^p}^\dagger}$ is isomorphic to the restriction of $\mathcal{G}'_{U^p}$ to (the neutral connected component of) $\mathcal{X}'(n)_{U^p}$.

In what follows we may and will fix a sufficiently small subgroup $U^p \in \mathcal{U}$, a connected component $\mathcal{X}'_{U^p}$ of $\mathcal{X}_U$, and a corresponding subgroup $U'^p \in \mathcal{U}'$. To simplify notation, we will suppress $U^p$ and $U'^p$, and will regard $\mathcal{X}'$ as a connected component of $\mathcal{X}$ as well.

Consider the Hodge exact sequence

$$0 \longrightarrow \pi_*\Omega^1_{\mathcal{A}/\mathcal{X}'} \longrightarrow \mathcal{H}^1_{\text{dR}}(\mathcal{A}/\mathcal{X'}) \longrightarrow R^1\pi_*\mathcal{O}_\mathcal{A} \longrightarrow 0.$$ 

It has a direct summand

$$0 \longrightarrow (\pi_*\Omega^1_{\mathcal{A}/\mathcal{X}'})^{2,1}_1 \longrightarrow \mathcal{H}^1_{\text{dR}}(\mathcal{A}/\mathcal{X'})^{2,1}_1 \longrightarrow (R^1\pi_*\mathcal{O}_\mathcal{A})^{2,1}_1 \longrightarrow 0,$$

which is $O_p$-equivariant, where $(-)^{2,1}_1$ is defined similarly as above. Here in (2.27), the three sheaves are locally constant of rank $1$, $2h$, and $2h - 1$, respectively, where $h = [F_p : \mathbb{Q}_p]$.

We introduce the following notation.

**Notation 2.5.1.** If $M$ is a locally free sheaf on a scheme over $\text{Spec} \, O_p$ equipped with an $O_p$-action $O_p \to \text{End} \, M$, then we denote by $M^{O_p}$ the maximal subsheaf on which $O_p$ acts via the structure homomorphism.

In what follows, we denote the sequence (2.27) after applying $(-)^{O_p}$ by

$$0 \longrightarrow \Omega^\bullet \longrightarrow \mathcal{L}' \longrightarrow \Omega^\bullet \overset{\text{c}}{\longrightarrow} 0.$$ 

**Proof of Lemma 2.2.4.** It suffices to consider the problem after the restriction to an arbitrarily chosen connected component $\mathcal{X}^\dagger$ of $\mathcal{X}$. By the definition of $\mathcal{G}'$ in [Car86, 5.4], we know that (2.27) is the Hodge exact sequence for $\mathcal{G}'$. Since $\mathcal{G}$ is isomorphic to $\mathcal{G}'$ on $\mathcal{X}^\dagger$, (2.27) is also the Hodge exact sequence for $\mathcal{G}$. Therefore, if we restrict (2.27) to $\mathcal{X}^\dagger$ and take formal completion, we recover the exact sequence (2.4) (restricted to $\mathcal{X}^\dagger$); and if we further apply the functor $(-)^{O_p}$, then we recover the exact sequence (2.5). In other words, the formal completion of (2.28) coincides with (2.5), both restricted to $\mathcal{X}^\dagger$. This shows that (2.5) is algebraizable.
For the next assertion, we have the Gauss–Manin connection
\[ \nabla'_A : \mathcal{H}^1_{dR}(A/X') \to \mathcal{H}^1_{dR}(A/X') \otimes \Omega^1_{\mathcal{X}'}, \]
and the induced connection
\[ (2.29) \quad \nabla'_p : \mathcal{H}^1_{dR}(A/X')^{2,1} \to \mathcal{H}^1_{dR}(A/X')^{2,1} \otimes \Omega^1_{\mathcal{X}'}. \]
Since applying \((-1)^{2,1} \right)_{1} \to \mathcal{X}'\text{ coincides with } \nabla'(2.7), \text{ when restricted to } \mathcal{X}', \text{ under which } \nabla'(2.30) \text{ coincides with } \nabla(2.7). \text{ Therefore, under the previous identification, } \text{KS}^{r} \text{ coincides with } KW(2.10). \text{ Then Proposition 2.2.6 follows from the following analogous statement for } \mathcal{X}': (2.31) \text{ is an isomorphism.} \]

The proof is similar to [DT94, Lemma 7], which essentially follows from the Grothendieck–Messing theory. Denote by \(A^\vee\) the dual abelian variety of \(A\). Then \(\omega^{\vee\text{op}}\) is canonically isomorphic to \((\text{Lie}(A^\vee/X')^{2,1})^{O_p}\). We only need to show that for every closed point \(t : \text{Spec } k(t) \to \mathcal{X}'\), the induced map
\[ (2.32) \quad \omega^{\vee\text{op}} \otimes k(t) \to (\text{Lie}(A^\vee/X')^{2,1})^{O_p} \otimes \Omega^1_{\mathcal{X}'} \otimes k(t) \]
is surjective, where \(\text{Lie}\) denotes the sheaf of tangent vectors.

Let \(A/\text{Spec } k(t)\) be the abelian variety classified by \(t\). Put \(T = \text{Spec } k(t)[\varepsilon]/(\varepsilon^2)\). The lifts \(A_\phi\) of \(A\) (with other PEL structures) to \(T\) correspond to homomorphisms
\[ \phi : t^*\omega^{\vee\text{op}} \to (\text{Lie}(A^\vee/X')^{2,1})^{O_p} \otimes k(t). \]
Since both sides are \(k(t)\)-vector spaces of dimension 1, we may choose a homomorphism \(\phi\) that is surjective. Let \(t_\phi : T \to \mathcal{X}'\) be the morphism that classifies \(A_\phi/T\). Compose the isomorphism \(t_\phi^*\omega^{\vee\text{op}} \otimes k(t) \to t^*\omega^{\vee\text{op}}\) and the surjective map \(\phi\). By the isomorphism
\[ (\text{Lie}(A^\vee/X')^{2,1})^{O_p} \otimes k(t) \simeq t_\phi^*(\text{Lie}(A^\vee/X')^{2,1})^{O_p} \otimes \Omega^1_{T/k(t)} \otimes k(t), \]
we obtain a surjective map
\[ t_\phi^*\omega^{\vee\text{op}} \otimes k(t) \to t_\phi^*(\text{Lie}(A^\vee/X')^{2,1})^{O_p} \otimes \Omega^1_{T/k(t)} \otimes k(t), \]
which is the pullback of (2.32) under \(t_\phi\). Therefore, (2.32) is surjective.

For \(n \in \mathbb{N} \cup \{\infty\}\), denote by \(\mathcal{X}'(n)\) the formal completion of \(\mathcal{X}'(n)\) along its special fiber, which is equipped with an \(O_{p^r}\)-divisible group \(G'\) induced from \(G\).
Proof of Lemma 2.3.1. By the proof of Lemma 2.2.4, it suffices to prove the same statement for $\mathcal{X}'(0)$. The desired morphism $\Phi': \mathcal{X}'(0) \to \mathcal{X}(0)$ is constructed through the moduli interpretation of $\mathcal{X}'(0)$ by “dividing $\mathcal{G}'[p]$”, which lifts the Frobenius on the special fiber of degree $\#O_p/p$. The uniqueness of such $\Phi'$ is ensured by (the proof of) Lemma 2.3.4 and Theorem B.1.1.

The proof of the remaining part is similar to [Kat78, Theorem 1.11.27]. The only modification we need is to show that the subsheaf $\mathcal{L}'_s$ of $\mathcal{L}'$ glues to a formal quasi-coherent sheaf. For this, we adopt the proof of [Kat73, Theorem 4.1] in the case where $\mathbb{Z}_p$ is replaced by $O_p$ and $p$ is replaced by a uniformizer $\varpi$ of $F$. The assumptions are satisfied because the Newton polygon of the underlying $p$-divisible group of $\mathcal{G}|_x$ for every $x \in \mathcal{X}'(0)(\kappa)$ is the one starting with $(0,0)$, ending with $(2h,1)$ and having the unique breaking point at $(h,0)$. □

Remark 2.5.2. In fact, the induced map of $\Phi'$ constructed in the above proof on the coordinate ring is simply the operator Frobenius defined in [Kas04, Definition 11.1].

Proof of Lemma 2.3.4. We only need to prove the similar statement for $\mathcal{X}'(\infty)$. By the moduli interpretation of $\mathcal{X}'(\infty)$ and the Serre–Tate Theorem on deformation of abelian varieties, we have an isomorphism $\mathcal{X}'(\infty)/_x \simeq \mathcal{M}_x$ where $\mathcal{M}_x$ is the formal scheme representing deformations of $\mathcal{G}|_x$. By Theorem B.1.1, we know that $\mathcal{M}_x$ is canonically isomorphic to $\mathcal{L}T$, and the induced isomorphism $\mathcal{X}'(\infty)/_x \simeq \mathcal{L}T$ is just $c/x$ by definition. □

Proof of Proposition 2.3.5. Recall that we have a similarly defined formal scheme $\mathcal{X}'(\infty)$ over $\text{Spf} O^\text{nr}_p$. The uniqueness of $\beta$ is clear. Thus by comparison, it suffices to construct the morphism $\beta': \mathcal{L}T \times_{\text{Spf} O^\text{nr}_p} \mathcal{X}'(\infty) \to \mathcal{X}'(\infty)$ with similar properties in Proposition 2.3.5, since the action of $\mathcal{L}T$ is supposed to preserve the special fiber.

We use the moduli interpretation of $\mathcal{X}'(\infty)$. For a scheme $S$ over $\text{Spec} O^\text{nr}_p$ where $p$ is locally nilpotent, $\mathcal{X}'(\infty)(S)$ is the set of isomorphism classes of quintuples $(A, \iota, \theta, k^p, \kappa_p)$, where $(A, \iota, \theta, k^p)$ is the same data in [Car86, §5.2] but $k^p$ is an isomorphism instead of a class, and $\kappa_p$ is an exact sequence

$$0 \longrightarrow \mathcal{L}T[p^\infty] \longrightarrow (A_p^\infty)_{1,1} \longrightarrow F_p/O_p \longrightarrow 0.$$

On the other hand, $\mathcal{L}T(S)$ is the set of isomorphism classes of $(G, \kappa_G)$ where $\kappa_G$ is an exact sequence

$$0 \longrightarrow \mathcal{L}T[p^\infty] \longrightarrow G \longrightarrow F_p/O_p \longrightarrow 0.$$

Using the group structure on $\mathcal{L}T$, we may add the above two exact sequences to a new one, denoted by $\alpha(\kappa_p, \kappa_G)$, which can be written as

$$0 \longrightarrow \mathcal{L}T[p^\infty] \longrightarrow \alpha((A_p^\infty)_{1,1}, G) \longrightarrow F_p/O_p \longrightarrow 0.$$

By the Serre–Tate Theorem on deformation of abelian varieties and the fact that étale level structures are determined on the special fiber, we associate canonically a quintuple $(A', \iota', \theta', k'^p, \kappa_p')$ with $\kappa_p' = \alpha(\kappa_p, \kappa_G)$. This defines the morphism $\beta'$. The properties of Proposition 2.3.5 for $\beta'$ follow directly from the construction. □

Proof of Lemma 2.4.12. We define $X'_l$ similarly as the projective limit over all level structures over $\mathbb{C}$. Then we have the complex uniformization

$$X'_l(\mathbb{C}) \simeq G'(\mathbb{Q}) \backslash \mathcal{H} \times G'(A^\infty),$$
where $G'(\mathbb{Q})$ acts on $\mathcal{H}$ via the $\iota$-component of $G'(\mathbb{R})$. We similarly define a $\mathbb{C}$-local system $L'_t$ on $X'_t$ via the quotient map

$$G'(\mathbb{Q})\backslash \mathbb{C}^{\oplus 2} \times \mathcal{H} \times G'(\mathbb{A}^\infty) \to G'(\mathbb{Q})\backslash \mathcal{H} \times G'(\mathbb{A}^\infty)$$

where the action of $\gamma \in G'(\mathbb{Q})$ is given by the formula

$$\gamma[(a_1, a_2)^t, z, g] = \left[((a_1, a_2)\iota(\gamma)^{-1})^t, \iota(\gamma)(z), \gamma^{-1}g\right],$$

where we regard $\iota(\gamma)$ as an element in $GL_2(\mathbb{C})$ in the formula $(a_1, a_2)\iota(\gamma)^{-1}$. By the same reason as in Lemma 2.4.11, we have a canonical isomorphism $\mathcal{L}'_t \simeq \mathcal{O}_{X'_t} \otimes_{\mathcal{C}} L'_t$. Here, we regard $\mathcal{L}'_t$ as the restriction of $\mathcal{L}'$ in $(2.28)$ to $X'_t$. By the comparison between $(2.28)$ and $(2.8)$ established in the proof of Lemma 2.2.4, it suffices to show that the subsheaf $\omega^\bullet$ is generated by the section $\omega^\bullet_{z'}$ whose value at $z$ is $(z, 1)^t$.

However, the coherent sheaf $\mathcal{H}_{dR}(A/X')$ is obtained from the local system

$$G'(\mathbb{Q})\backslash \mathbb{C}^{\oplus 2g} \times \mathcal{H} \times G'(\mathbb{A}^\infty) \to G'(\mathbb{Q})\backslash \mathcal{H} \times G'(\mathbb{A}^\infty),$$

where $G'(\mathbb{Q})$ acts on $\mathbb{C}^{\oplus 2g} = \mathbb{C}^{\oplus 2} \oplus \cdots \oplus \mathbb{C}^{\oplus 2}$ diagonally via all archimedean places of $F$. From the Hodge homomorphism in the Shimura data of $G'$, we see that the restriction of $\omega^\bullet \simeq (\pi_* \Omega^1_{A/X'})^{2,1}$ to $X'_t$ is generated by the section $\omega^\bullet_{z'}$ whose value at $z$ is $(z, 1)^t$. This follows from the same computation for the case of modular curves. Therefore, Lemma 2.4.12 is proved.

\section{Statements of main theorems}

In this chapter, we state our main theorems about $p$-adic $L$-functions and $p$-adic Waldspurger formula for the general case. We start from recalling some background about representations of incoherent algebras and abelian varieties of $GL(2)$-type in §3.1. In §3.2, we state the main theorem about $p$-adic $L$-functions in terms of Heegner cycles on abelian varieties. In §3.3, we state the main theorem about $p$-adic Waldspurger formula in terms of Heegner cycles on abelian varieties. In §3.4, we provide an alternative formulation of our main theorems in terms of periods of $p$-adic Maass functions, in the same spirit as Introduction, and deduce them from the previous formulation via Heegner cycles on abelian varieties.

\subsection{Representations for incoherent quaternion algebras}

We recall some materials from [YZZ13, §3.2]. Let $\iota_1, \ldots, \iota_g$ be all archimedean places of $F$. Let $\mathbb{B}$ be a totally definite quaternion algebra over $\mathbb{A}$. As in §2.2, there is an associated projective system of Shimura curves $\{X_U = X(\mathbb{B})_U\}$ over $F$, and $X = \varprojlim_U X_U$. We recall the following definition in [YZZ13, §3.2.2].

\textbf{Notation 3.1.1.} Let $L$ be a field embeddable into $\mathbb{C}$. Denote by $A(\mathbb{B}^\times, L)$ the set of isomorphism classes of irreducible (admissible) representations $\Pi$ of $\mathbb{B}^{\infty\times}$ over $L$ such that for some and hence all embeddings $L \hookrightarrow \mathbb{C}$, the Jacquet–Langlands transfer of $\Pi \otimes_L \mathbb{C}$ to $GL_2(\mathbb{A}^\infty)$ is a finite direct sum of (finite components of) irreducible cuspidal automorphic representations of $GL_2(\mathbb{A})$ of parallel weight 2.

Let $A$ be an abelian variety over $F$.

\textbf{Notation 3.1.2.} Recall from [YZZ13, §3.2.3] the following notation

$$\Pi(\mathbb{B})_A := \lim_U \text{Hom}_{U^+}(X_U^*, A),$$

where
• the colimit is taken over all open compact subgroups \( U \) of \( \mathbb{B}^{\infty \times} \);
• \( X_U^* \) is the smooth compactification of \( X_U \) (which is simply \( X_U \) unless in the case of classical modular curves);
• \( \xi_U \) is the normalized Hodge class on \( X_U^* \) [YZZ13, §3.1.3]; and
• \( \text{Hom}_{\xi_U}(X_U^*, A) \) denotes the \( \mathbb{Q} \)-vector space of modular parameterizations, that is, the abelian group of morphisms from \( X_U^* \) to \( A \) that send \( \xi_U \) to a torsion point, tensoring with \( \mathbb{Q} \).

We simply write \( \Pi_A \) for \( \Pi(\mathbb{B})_A \) if \( \mathbb{B} \) is clear from the context.

If we denote by \( J_U \) the Jacobian of \( X_U^* \), then \( \xi_U \) induces a morphism \( X_U^* \rightarrow J_U \). Thus, \( \text{Hom}_{\xi_U}(X_U^*, A) \) is canonically identified with \( \text{Hom}^0(J_U, A) := \text{Hom}(J_U, A) \otimes \mathbb{Q} \).

Put \( M_A := \text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q} \). It is clear that both \( \Pi(\mathbb{B})_A \) and \( M_A \) depend only on \( A \) up to isogeny.

**Definition 3.1.3.** We say that \( A \) can be parameterized by \( \mathbb{B} \) if there is a non-constant morphism from \( X = X(\mathbb{B}) \) to \( A \). Denote by \( \text{AV}^0(\mathbb{B}) \) the set of simple abelian varieties over \( F \) that can be parameterized by \( \mathbb{B} \) up to isogeny.

The set \( \text{AV}^0(\mathbb{B}) \) is stable under duality. Take an element \( A \in \text{AV}^0(\mathbb{B}) \). Then \( \Pi_A \) is a nonzero rational irreducible representation of \( \mathbb{B}^{\infty \times} \), which is an element in \( \mathcal{A}(\mathbb{B}^{\times}, \mathbb{Q}) \) (Notation 3.1.1). The assignment \( A \mapsto \Pi_A \) induces a bijection between \( \text{AV}^0(\mathbb{B}) \) and \( \mathcal{A}(\mathbb{B}^{\times}, \mathbb{Q}) \).

Moreover, \( M_A \) is a field of degree equal to the dimension of \( A \) and it acts on the representation \( \Pi_A \). Denote by \( A^\vee \) the dual abelian variety (up to isogeny) of \( A \) and we have \( \Pi_{A^\vee} \) similarly. There is a canonical isomorphism \( M_{A^\vee} \cong M_A \) as in [YZZ13, §3.2.4].

**Definition 3.1.4** (Canonical pairing, [YZZ13, §3.2.4]). We have a canonical pairing

\[
(\cdot, \cdot)_A : \Pi_A \times \Pi_{A^\vee} \rightarrow M_A
\]

induced by maps

\[
(\cdot, \cdot)_U : \text{Hom}^0(J_U, A) \times \text{Hom}^0(J_U, A^\vee) \rightarrow M_A
\]

defined by the assignment \((f_+, f_-) \mapsto \text{vol}(X_U)^{-1} \circ f_+ \circ f_-^\vee \in \text{End}^0(A) = M_A \) for all open compact subgroups \( U \) of \( \mathbb{B}^{\infty \times} \).

Recall that an abelian variety \( A \) (up to isogeny) over \( F \) is of GL(2)-type if \( M_A \) is a field of degree equal to the dimension of \( A \). Let \( A \) be such an abelian variety (up to isogeny) and denote by

\[
\omega_A : F^x \backslash \mathbb{A}^{\infty \times} \rightarrow M_A^x
\]

the central character associated to \( A \). For a finite place \( v \) of \( F \), choose a rational prime \( \ell \) that does not divide \( v \). We have a Galois representation \( \rho_{A,v} \) of \( D_v \), the decomposition group at \( v \), on the \( \ell \)-adic Tate module \( V_\ell(A) \) of \( A \), which is a free module over \( M_{A,\ell} := M_A \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \) of rank 2. It is well-known that the characteristic polynomial

\[
P_v(T) = \det_{M_{A,\ell}}(1 - \text{Frob}_v | V_\ell(A)^1)
\]

belongs to \( M_A[T] \) and is independent of \( \ell \), where \( I_v \subset D_v \) is the inertia subgroup and \( \text{Frob}_v \in D_v/I_v \) is the geometric Frobenius.

**Remark 3.1.5.** We use \( \omega \) to denote both differential forms and central characters since both ways are standard. We hope this does not cause any confusion to readers.

**Definition 3.1.6** (L-functions and \( \epsilon \)-factors). Let \( K \) be a field containing \( M_A \).
(1) Define the local $L$-function of $A$ as $L(s, \rho_{A,v}) := P_v(N_v)^{-s-1/2} \in M_A \otimes \mathbb{Q} \mathbb{C}$. In a similar manner, we define the local adjoint $L$-function of $A$ denote as $L(s, \rho_{A,v}, \text{Ad})$; in particular, $L(1, \rho_{A,v}, \text{Ad}) \in M_A$.

(2) For a locally constant character $\chi_v : F_v^\times \to K^\times$, we have the twisted local $L$-function $L(s, \rho_{A,v} \otimes \chi_v) \in K \otimes \mathbb{Q} \mathbb{C}$. If $\psi : F_v \to \mathbb{C}^\times$ is a nontrivial additive character, then we have the $\epsilon$-factor $\epsilon(1/2, \psi, \rho_{A,v} \otimes \chi_v)$.

(3) For a locally constant character $\chi_v : E_v^\times \to K^\times$ such that $\omega_{A,v} \cdot \chi_v|_{F_v^\times} = 1$, we have the local Rankin–Selberg $L$-function $L(s, \rho_{A,v}, \chi_v) \in K \otimes \mathbb{Q} \mathbb{C}$ and the $\epsilon$-factor $\epsilon(1/2, \rho_{A,v}, \chi_v)$. See Remark 3.1.7 for more details.

(4) Let $\iota : K \hookrightarrow \mathbb{C}$ be an embedding, which induces a homomorphism $\iota : K \otimes \mathbb{Q} \mathbb{C} \to \mathbb{C}$ by abuse of notation. We define the global $L$-function of $A$ (with respect to $\iota$) to be

$$L(s, \rho_{A}^{(i)}):= \prod_{v<\infty} \iota L(s, \rho_{A,v}).$$

Similarly, we have the global version $L(s, \rho_{A}^{(i)}, \text{Ad})$ and $L(s, \rho_{A}^{(i)}, \chi^{(i)})$ of other $L$-functions as well.

(5) We say that $A$ is automorphic if $L(s, \rho_{A}^{(i)}), \text{ for some and hence all } \iota$, is (the finite component of) the $L$-function of an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$.

Remark 3.1.7. If $v$ splits into two places $v_1$ and $v_2$ of $E$, then $L(s, \rho_{A,v}, \chi_v)$ is defined to be the product $L(s, \rho_{A,v_1} \otimes \chi_{v_1})L(s, \rho_{A,v_2} \otimes \chi_{v_2})$. If $v$ induces a single place $w$ of $E$, then we define $L(s, \rho_{A,v}, \chi_v) := L(s, (\rho_{A,v}|_{D_w}) \otimes \chi_v)$. By choosing a nontrivial additive character $\psi : F_v \to \mathbb{C}^\times$, we have the local Rankin–Selberg $\epsilon$-factor $\epsilon(s, \psi, \rho_{A,v}, \chi_v)$. It is well-known that $\epsilon(1/2, \psi, \rho_{A,v}, \chi_v)$ belongs to $\{\pm 1\}$ and does not depend on the choice of $\psi$. We denote its value by $\epsilon(1/2, \rho_{A,v}, \chi_v)$.

The global $L$-functions $L(s, \rho_{A}^{(i)}), L(s, \rho_{A}^{(i)}, \text{Ad}), \text{ and } L(s, \rho_{A}^{(i)}, \chi^{(i)})$ are always absolutely convergent for $\text{Re } s > 1$.

Remark 3.1.8. It is conjectured that every abelian variety of $\text{GL}(2)$-type is automorphic. In particular, when $F = \mathbb{Q}$, every abelian variety of $\text{GL}(2)$-type is parameterized by modular curves. This follows from Serre’s modularity conjecture (for $\mathbb{Q}$) [Rib92, Theorem 4.4], where the latter has been proved by Khare and Wintenberger [KW09].

3.2. $p$-adic Rankin–Selberg $L$-functions for abelian varieties of $\text{GL}(2)$-type. From now on, we fix an abelian variety $A$ of $\text{GL}(2)$-type over $F$ up to isogeny that is automorphic (Definition 3.1.6 (5)), equipped with an embedding $M_A \subset \mathbb{C}_p$. For simplicity, in what follows, we put $M := M_A = M_{A^v}$ regarded as a subfield of $\mathbb{C}_p$, and put $F^M = F \otimes \mathbb{Q} M$ which is naturally equipped with a homomorphism to $\mathbb{C}_p$.

Notation 3.2.1. Denote by $\mathcal{B}(A)$ the (finite) set of isomorphism classes of totally definite incoherent quaternion algebra $\mathcal{B}$ over $\mathbb{A}$ that is $E$-embeddable (Definition 1.8.3) and such that $A$ can be parameterized by $\mathcal{B}$ (Definition 3.1.3).

For each (representative) $\mathcal{B} \in \mathcal{B}(A)$, we fix an isomorphism $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$ and an $E$-embedding (Definition 1.8.3) under which $e_p$ coincides with (1.7). Then we have the $F$-scheme $X = X(\mathcal{B})$ and its closed subscheme $Y = Y^+ \amalg Y^-$ (Definition 2.4.1). We also fix an $\mathbb{A}_E^\infty$-equivariant isomorphism

(3.1) \[ c : Y^+(\mathbb{C}_p) \cong Y^-(\mathbb{C}_p), \]
which we call an abstract conjugation for \( \mathbb{B} \).

**Definition 3.2.2.** Denote by \( \mathcal{U} \) the set of open compact subgroups of \( \mathbb{A}_E^{\infty \times} \), which is a filtered partially ordered set under inclusion. Let \( K \) be a complete field extension of \( F_p \).

1. A \((K\text{-valued})\) character
   \[
   \chi : E^\times \backslash \mathbb{A}_E^{\infty \times} \to K^\times
   \]
   is a character of weight \( w \in \mathbb{Z} \) if
   - \( \chi \) is invariant under some \( V^p \in \mathcal{U} \);
   - there is an open compact subgroup \( V_p \) of \( E_p^\times \) such that \( \chi(t) = (t_q/t_q\cdot)^w \) for \( t \in V_p \).
   We call \( V^p \) the tame level of \( \chi \).
2. For a character \( \chi \) of weight \( w \) as above, we define two characters \( \check{\chi}_p \) and \( \check{\chi}_q \) of \( F^\times_p \) by the formula \( \check{\chi}_p(t) = t^{-w}\chi_p(t) \) and \( \check{\chi}_q(t) = t^w\chi_q(t) \).
3. Suppose that \( K \) is contained in \( \mathbb{C}_p \). Let \( \chi \) be a \( K \text{-valued} \) character of weight \( w \). Given an isomorphism \( \iota : \mathbb{C}_p \cong \mathbb{C} \), we define the following local characters
   - \( \chi^{(i)}_v = 1 \) if \( v \infty \) but not equal to \( \iota|_F \);
   - \( \chi^{(i)}_v(z) = (z/\bar{z})^w \) for \( v = \iota|_F \), where \( z \in E \otimes_{F_v} \mathbb{R} \xrightarrow{\iota|_{E_v}} \mathbb{C} \);
   - \( \chi^{(i)}_v = i\chi_v \) for \( v < \infty \) but \( v \neq p \);
   - \( \chi^{(i)}_v(t) = \iota((\check{\chi} \iota(t_0))(\check{\chi} \iota(t_a))) \) for \( t \in E^\times_p \).
   The product \( \chi^{(i)} := \otimes_v \chi^{(i)}_v : \mathbb{A}_E \to \mathbb{C}^\times \) is called the \( \iota \)-avatar of \( \chi \).
4. Suppose that \( K \) contains \( M \). Denote by \( \Xi(A,K)_w \) the set of all \( K \text{-valued} \) characters of weight \( w \) such that
   - \( \o A : \chi|_{\mathbb{A}_E^{\infty \times}} = 1 \);
   - \( \# \{ v < \infty, v \neq p \mid \epsilon(1/2, \rho_{A,v}, \chi_v) = -1 \} \equiv g - 1 \mod 2 \).
   Put \( \Xi(A,K) = \bigcup_w \Xi(A,K)_w \).

**Remark 3.2.3.** The character \( \chi^{(i)} \) is automorphic, that is, it factors through \( E^\times \backslash \mathbb{A}_E^{\infty \times} \).

**Lemma 3.2.4.** For a character \( \chi \in \Xi(A,K) \), there is a unique element \( \mathbb{B}_\chi \in \mathcal{D}(A) \) such that \( \epsilon(1/2, \rho_{A,v}, \chi_v) = \chi_v(-1)\eta_v(-1)\epsilon(\mathbb{B}_\chi,v) \) for every finite place \( v \neq p \) of \( F \).

**Proof.** The existence of such \( \mathbb{B}_\chi \) follows from Condition (b) in Definition 3.2.2 (4). The uniqueness is clear since \( \mathbb{B}_\chi \) is unramified at \( p \) and \( \epsilon(\mathbb{B}_\chi,v) \) are prescribed at all other places \( v \).

The following definition generalizes the discussion in [ST01, §1].

**Definition 3.2.5 (Distribution algebra).** Let \( K/F_p \) be a complete field extension that contains \( M \).

1. For a locally constant character \( \omega : F^\times \backslash \mathbb{A}_E^{\infty \times} \to M^\times \), denote by \( \mathcal{C}(\omega,K) \) the locally convex \( K \text{-vector space of} \ K \text{-valued} \) locally analytic functions \( f \) on the locally \( F_p \text{-analytic group} \ E^\times \backslash \mathbb{A}_E^{\infty \times} \) satisfying
   - \( f \) is invariant under translation by some \( V^p \in \mathcal{U} \);
   - \( f(xt) = \omega(t)^{-1}f(x) \) for all \( x \in E^\times \backslash \mathbb{A}_E^{\infty \times} \) and \( t \in F^\times \backslash \mathbb{A}_E^{\infty \times} \).
   Let \( \mathcal{D}(\omega,K) \) be the strong dual of \( \mathcal{C}(\omega,K) \) as a topological \( K \text{-algebra} \) (see Remarks 2.1.1 & 3.2.6).
2. Define \( \mathcal{D}(A,K) \) to be the quotient \( K \text{-algebra of} \mathcal{D}(\omega_A,K) \) divided by the closed ideal generated by elements that vanish on \( \Xi(A,K) \subset \mathcal{C}(\omega_A,K) \).
(3) For \( B \in \mathcal{B}(A) \), define \( \mathcal{D}(A, B, K) \) to be the quotient \( K \)-algebra of \( \mathcal{D}(\omega_A, K) \) divided by the closed ideal generated by elements that vanish on \( \chi \in \Xi(A, K) \) with \( B_\chi \simeq B \) for \( B \) as in Lemma 3.2.4.

(4) Define
\[
\delta: E^X \backslash A_E^{\infty } \to \mathcal{D}(\omega_A, K)^X \to \mathcal{D}(A, K)^X \to \mathcal{D}(A, B, K)^X
\]
to be various continuous homomorphisms given by Dirac distributions.

Remark 3.2.6. The topological \( K \)-vector space \( \mathcal{D}(\omega, K) \) is a commutative topological \( K \)-algebra with the multiplication given by convolution \( [ST01, \S 1] \). For a complete field extension \( K'/K \), we have \( \mathcal{D}(\omega, K) \otimes_K K' \simeq \mathcal{D}(\omega, K') \). Moreover, if \( K \) is discretely valued, then \( \mathcal{D}(\omega, K) \) may be written as a projective limit, indexed by tame levels \( V^p \in \mathfrak{V} \), of nuclear Fréchet–Stein \( K \)-algebras with finite étale transition homomorphisms (see Remark 4.4.5), and thus complete. We have similar remarks for \( \mathcal{D}(A, K) \) and \( \mathcal{D}(A, B, K) \).

Remark 3.2.7. Suppose \( F = M = \mathbb{Q} \) and that \( \omega = 1 \) is the trivial character. Fix a (sufficiently small) tame level \( V^p \in \mathfrak{V} \). Define \( \mathcal{C}(1, Q_p, V^p) \) similarly as in Definition 3.2.5 by requiring that \( f \) is invariant under translation by \( A_\omega \), and \( \mathcal{D}(1, Q_p, V^p) \) the strong dual of \( \mathcal{C}(1, Q_p, V^p) \) as a topological \( Q_p \)-algebra. Then for every complete field extension \( K/Q_p \), there is a natural bijection between continuous characters \( \mathcal{D}(1, Q_p, V^p) \to K^\times \) and continuous characters \( E^X A_\omega \to A_\omega /V^p \to K^\times \). In particular, \( \mathcal{D}(1, Q_p, V^p) \) is isomorphic to the coordinate ring of a finite disjoint union of open unit discs over \( Q_p \) (compare with \( \S 2.1 \)). See Remark 4.4.5 for an interpretation in more general case.

For a representative \( B \in \mathcal{B}(A) \), put \( \Omega_{X,Y^\pm} = \Omega_{1, X^\pm} |_{Y^\pm} \). For \( t \in E^X \backslash A_E^{\infty } \), there are canonical isomorphisms \( T^*_1 \Omega_{X,Y^\pm} \simeq \Omega_{X,Y^\pm} \). Put
\[
(3.2) \quad \omega_{\psi^\pm} = (T^*_1 \omega_{\nu}) |_{Y^\pm} \otimes_{F^{|p|}} F^{|p|} F^{|p|}_{ab},
\]
where \( T^*_1 \) are in Definition 2.2.10 and \( \omega_{\nu} \) is the global Lubin–Tate differential in Definition 2.3.6. Then \( \omega_{\psi^\pm} \) are sections of \( \Omega_{X,Y^\pm} \otimes_{F^{|p|}} F^{|p|} F^{|p|}_{ab} \) respectively, depending only on the additive character \( \psi \) (Remark 2.3.7).

Let \( M F^{|p|}_{ab} F^{|p|} \subset K \subset \mathbb{C}_p \) be a complete intermediate field. Take a character \( \chi \in \Xi(A, K)_k \) with \( k \geq 0 \) and take \( B = B_\chi \). Define \( \sigma^\pm_\chi \) to be the \( K \)-subspaces of \( \mathbb{H}^0(Y^\pm, \Omega_{X,Y^\pm}^{\infty-k}) \otimes_F K \) consisting of \( \varphi \) such that \( T^*_1 \varphi = \chi(t)^{\pm 1} \varphi \), respectively. By Lemma 2.4.2 (1), both \( \sigma^+_\chi \) and \( \sigma^-_\chi \) have dimension 1. The abstract conjugation \( c(3.1) \) induces an \( A_E^{\infty } \)-invariant bilinear pairing
\[
(\cdot, \cdot)_\chi: \sigma^+_\chi \times \sigma^-_\chi \to K
\]
by the formula \( (\varphi^+, \varphi^-)_\chi = (\varphi^+ \otimes \omega^k_{\psi^+}) \cdot c^*(\varphi^- \otimes \omega^k_{\psi^-}) \), where the right-hand side is a \( K \)-valued constant function on \( Y^+ \), hence can be regarded as an element in \( K \).

Put \( A^+ = A \) and \( A^- = A^\vee \), and \( \Pi^\pm = \Pi_{A^\pm} \in A(B^X, \mathbb{Q}) \). We have the canonical pairing
\[
(\cdot, \cdot)_A: \Pi^+ \times \Pi^- \to M \subset \mathbb{C}_p \quad (\text{Definition 3.1.4}).
\]

Lemma 3.2.8. Assume that \( k \geq 1 \). For every \( \nu: \mathbb{C}_p \to \mathbb{C} \), we have a unique \( \mathbb{B}^\infty \times A_E^{\infty } \)-invariant bilinear pairing
\[
(\cdot, \cdot)^{\nu(l)}_{A, \chi}: (\Pi^+ \otimes_{F^{|p|}} \sigma^+_\chi) \times (\Pi^- \otimes_{F^{|p|}} \sigma^-_\chi) \to (\text{Lie } A^+ \otimes_{F^{|p|}} \text{Lie } A^-) \otimes_{F^{|p|}, \nu} \mathbb{C},
\]
such that for every \( f_{\pm} \in \Pi^\pm, \varphi_{\pm} \in \sigma_{\chi}^\pm \) and \( \omega_{\pm} \in H^0(A^\pm, \Omega_{A^\pm}^1) \), we have

\[
\langle \omega_+ \otimes \omega_-, (f_+ \otimes \varphi_+, f_- \otimes \varphi_-) \rangle_{\Lambda^A_X}^{(i)} = \left( \iota \varphi_+ \otimes c_i^* \varphi_- \otimes \mu^k \right) \int_{X_0(\C)} \frac{\Theta_{i}^{k-1} f_+ \omega_+ \otimes c_i^* \Theta_{i}^{k-1} f_- \omega_-}{\mu^k} \, dx,
\]

where

- \( \langle , \rangle \) is the canonical pairing between \( H^0(A^+, \Omega_{A^+}^1) \otimes_{FM} H^0(A^-, \Omega_{A^-}^1) \) and \( \text{Lie } A^+ \otimes_{FM} \text{Lie } A^- \);
- \( \mu \) is an arbitrary Hecke invariant hyperbolic metric on \( X_0(\C) \);
- \( c_i \) is the complex conjugation on \( X_0(\C) \);
- \( \iota \varphi_+ \otimes c_i^* \varphi_- \otimes \mu^k \) is a constant function on \( Y_i^+(\C) \), hence viewed as a complex number;
- \( \Theta_i \) is the Shimura–Maass operator \( (\text{Definition 2.4.7}) \); and
- \( dx \) is the Tamagawa measure on \( X_0(\C) \).

Moreover, there is a unique (nonzero) element \( P_i(A, \chi) \in (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM, i} \C \) such that

\[
\langle , \rangle_{\Lambda^A_X}^{(i)} = P_i(A, \chi) \cdot \iota( , )_A \otimes \iota( , )_\chi.
\]

**Proof.** For given \( \omega_{\pm} \in H^0(A^\pm, \Omega_{A^\pm}^1) \), the formula (3.3) defines a bilinear pairing

\[
(\Pi^+ \otimes_{FM} \sigma_{\chi}^+) \times (\Pi^- \otimes_{FM} \sigma_{\chi}^-) \to \C,
\]

which is \( \mathbb{B}_{\infty, \infty} \times \mathbb{A}_E^{\infty, \infty} \)-invariant. By duality, all these pairings for different \( \omega_{\pm} \) give rise to a nonzero pairing

\[
\langle , \rangle_{\Lambda^A_X}^{(i)} : (\Pi^+ \otimes_{FM} \sigma_{\chi}^+) \times (\Pi^- \otimes_{FM} \sigma_{\chi}^-) \to (\text{Lie } A^+ \otimes_{Q} \text{Lie } A^-) \otimes_{FM, i} \C,
\]

and it is easy to see that \( \langle , \rangle_{\Lambda^A_X}^{(i)} \) takes values in \( (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM, i} \C \). The existence of \( P_i(A, \chi) \) follows from the uniqueness of the Petersson inner product and the fact that \( (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM, i} \C \) is a \( \C \)-vector space of dimension 1. \(
\)

**Remark 3.2.9.** The element \( P_i(A, \chi) \) can be viewed as a function on the set \( \bigcup_{k \geq 1} \Xi(A, K) \) valued in the 1-dimensional \( \C \)-vector space \( (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM, i} \C \). It depends on the choices of \( c \) and \( \psi \).

**Theorem 3.2.10.** There is a unique element

\[
\mathcal{L}(A) \in (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{B}(A, MF_p^\mathfrak{M})
\]

such that for every character \( \chi \in \Xi(A, K)_k \) with \( k \geq 1 \) and \( MF_p^\mathfrak{M}F_{p^\mathfrak{M}} \subset K \subset \mathbb{C}_p \) a complete intermediate field, and every \( \iota : \mathbb{C}_p \simeq \C \), we have

\[
\langle \mathcal{L}(A)(\chi) \rangle = L(1/2, \rho_{\Lambda}^{(i)}, \chi^{(i)}) \cdot \frac{\zeta_F(2) \zeta_F(2) \zeta_F(2)}{L(1, \eta^2 L(1, \rho_{\Lambda}^{(i)}, \text{Ad})}} \cdot \left( \frac{L(1/2, \rho_{\Lambda \mathfrak{P}} \otimes \chi^{\mathfrak{P}})}{L(1/2, \rho_{\Lambda \mathfrak{P}} \otimes \chi^{\mathfrak{P}})^2} \right)
\]

as an equality in \( (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM, i} \C \).

**Remark 3.2.11.** The element \( \mathcal{L}(A) \) depends only on the choices of (1) an additive character \( \psi \) of \( F_p \) of level 0, and (2) the abstract conjugation \( c \) \( (3.1) \) for each (representative) \( \mathfrak{B} \in \mathcal{B}(A) \), in an elementary way. More precisely,

1. if we change \( \psi \) to \( \psi_a \) for some \( a \in O_p^\times \), where \( \psi_a(x) = \psi(ax) \) for \( x \in F_p \), then \( \mathcal{L}(A) \) is multiplied by \( \omega_\mathfrak{P}(a) \cdot \delta_\mathfrak{P}^\mathfrak{C} \), where \( a \) is regarded at the place \( \mathfrak{P}^\mathfrak{C} \) in the Dirac distribution \( \delta_\mathfrak{a} \);
(2) if we write $\mathcal{L}(A) = \{ \mathcal{L}(A, B) \}_{B \in \mathcal{D}(A)}$ under the canonical isomorphism $\mathcal{D}(A, K) \simeq \prod_{B \in \mathcal{D}(A)} \mathcal{D}(A, B, K)$ (Remark 4.4.4) and change $c$ (for $B$) to $c' = T_t \circ c$ for some $t \in \mathcal{A}_E^\infty$, then the component $\mathcal{L}(A, B)$ is multiplied by $\delta_t$ (Definition 3.2.5 (4)).

3.3. $p$-adic Waldspurger formula. Let $K$ be a complete field extension of $M$. Consider an element $\chi \in \Xi(A, K)_0$. We take $B = B_\chi \in \mathcal{O}(A)$. Choose a CM point $P^+ \in Y^+(E^\text{ab}) = Y^+(\mathbb{C}_p)$ and put $P^- = cP^+$.

**Definition 3.3.1.** For every $f_\pm \in \Pi^\pm$, we define the Heegner cycles $P^\pm_{\chi}(f_\pm)$ on $A^\pm$ to be

$$P^\pm_{\chi}(f_\pm) = \int_{E \setminus \mathcal{A}_E^\infty} f_\pm(T_tP^\pm) \otimes_M \chi(t)^{\pm 1} dt,$$

a finite sum in fact. Here, we recall that $T_t$ is the Hecke morphism (Notation 2.2.1) and we adopt the Haar measure $dt$ of total volume $2$.

Suppose now that $K$ contains $MF_p^{\text{ab}}$. We have $K$-linear maps

$$\log_{A^\pm}: A^\pm(K) \otimes_M K \to \text{Lie} A^\pm \otimes_{FM} K$$

given by $p$-adic logarithms on $A^\pm$ (see, for example, [Bou89]). As a functional on $\Pi^+ \times \Pi^-$, the product $\log_{A^+} P^+_{\chi}(f_+) \cdot \log_{A^-} P^-_{\chi}(f_-)$ defines an element in the following one dimensional $K$-vector space

$$\text{Hom}_{\mathcal{A}_E^\infty}(\Pi^+ \otimes \chi, K) \otimes_K \text{Hom}_{\mathcal{A}_E^\infty}(\Pi^- \otimes \chi^{-1}, K) \otimes_{FM} (\text{Lie} A^+ \otimes_{FM} \text{Lie} A^-).$$

It depends on the choice of $c$ but not on the choice of $P^+$.

**Theorem 3.3.2** ($p$-adic Waldspurger formula). There exists a unique element

$$\alpha_{\chi}(\cdot, \cdot) \in \text{Hom}_{\mathcal{A}_E^\infty}(\Pi^+ \otimes \chi, K) \otimes_K \text{Hom}_{\mathcal{A}_E^\infty}(\Pi^- \otimes \chi^{-1}, K)$$

such that for every $\iota: \mathbb{C}_p \sim \to \mathbb{C}$,

$$\iota \alpha_{\chi}(f_+, f_-) = \alpha^\chi(f_+, f_-; \chi^{(\iota)})$$

for every $f_\pm \in \Pi^\pm$, where the right-hand side is the (normalized) matrix coefficient integral appearing in the complex Waldspurger formula (which will be recalled in Definition 4.1.4). Moreover, for a character $\chi \in \Xi(A, K)_0$, we have

$$\log_{A^+} P^+_{\chi}(f_+) \cdot \log_{A^-} P^-_{\chi}(f_-) = \mathcal{L}(A)(\chi) \cdot \frac{L(1/2, \rho_{A^+} \otimes \chi_{\mathfrak{F}}^\text{ab})^2}{\epsilon(1/2, \psi, \rho_{A^+} \otimes \chi_{\mathfrak{F}}^\text{ab})} \cdot \alpha_{\chi}(f_+, f_-)$$

for every $f_\pm \in \Pi^\pm$.

3.4. $p$-adic Maass functions and alternative formulation. Let $B$ be an arbitrary totally definite incoherent quaternion algebra over $A$. As in §2.2, we have $X(B) = \varprojlim_U X(B)_U$ as the projective limit of Shimura curves associated to $B$ over $\text{Spec} F$. The following definition generalizes the one in §1.2.

**Definition 3.4.1** ($p$-adic Maass function). We say that a function $\phi: X(B)(\mathbb{C}_p) \to \mathbb{C}_p$ is a $p$-adic Maass function on $X(B)$ if it is the pullback of some locally analytic function $X(B)_U(\mathbb{C}_p) \to \mathbb{C}_p$. Denote by $\mathcal{O}_{\mathbb{C}_p}(B^\infty)$ the $\mathbb{C}_p$-vector space of all $p$-adic Maass functions on $X(B)$. It is a representation of $B^\infty$. 
We go back to the setting in §3.2 where we have fixed an abelian variety \( A \) of \( \text{GL}(2) \)-type over \( F \) up to isogeny that is automorphic, equipped with an embedding \( M = M_A \subset \mathbb{C}_p \). Denote by \( \pi(\mathcal{B})_{\text{rat}}^{A} \) the subspace of \( \mathcal{A}_{\mathcal{B}}(\mathbb{B}^{\times}) \) spanned by functions of the form

\[
 f^{*} \log \omega : X(\mathcal{B})(\mathbb{C}_p) \xrightarrow{f} A(\mathbb{C}_p) \xrightarrow{\log \omega} \mathbb{C}_p,
\]

where \( f : X(\mathcal{B}) \to A \) is a nonconstant map; \( \omega \) is a differential form on \( A \otimes_{\mathbb{Q}} \mathbb{C}_p; \) and \( \log \omega = \langle \log A, \omega \rangle \). The subspace \( \pi(\mathcal{B})_{\text{rat}}^{A} \) is a subrepresentation of \( \mathbb{B}^{\times\times} \), which also receives an action of \( M \) by acting on \( A \). Denote by \( \pi(\mathcal{B})_A \) the subspace of \( \pi(\mathcal{B})_{\text{rat}}^{A} \) on which \( M \) acts via the default embedding \( M \subset \mathbb{C}_p \), which is again a subrepresentation of \( \mathbb{B}^{\times\times} \).

**Lemma 3.4.2.** Suppose that \( \mathcal{B} \) belongs to \( \mathcal{B}(A) \) (Notation 3.2.1). For every nonzero differential form \( \omega \in H^0(\mathcal{A}, \Omega^1_A) \), the map

\[
 \varsigma_\omega : \Pi(\mathcal{B})_A \to \pi(\mathcal{B})_A
\]

sending \( f \) to \( f^{*} \log \omega |_{X(\mathcal{B})(\mathbb{C}_p)} \) is \( \mathbb{B}^{\times\times} \)-equivariant and \( M \)-linear, and the induced map \( \Pi(\mathcal{B})_A \otimes_M \mathbb{C}_p \to \pi(\mathcal{B})_A \) is an isomorphism.

**Proof.** It follows directly from the definition that \( \varsigma_\omega \) is \( \mathbb{B}^{\times\times} \)-equivariant and \( M \)-linear. To show the isomorphism, it suffices to show that \( \pi(\mathcal{B})_A \) is a nonzero irreducible representation of \( \mathbb{B}^{\times\times} \). Since \( \mathcal{B} \) belongs to \( \mathcal{B}(A) \), the space \( \pi(\mathcal{B})_{\text{rat}}^{A} \) is nonzero, hence so is \( \pi(\mathcal{B})_A \).

For the irreducibility, we choose an isomorphism \( \iota : \mathbb{C}_p \xrightarrow{\sim} \mathbb{C} \). Consider the map \( \pi(\mathcal{B})_{A \otimes_{\mathbb{C}_p}} \mathbb{C} \to \mathcal{A}_{\text{cusp}}(B(i)^{\times}) \) sending \( f^{*} \log \omega \) to \( if^{*} \omega \) regarded as a weight 2 holomorphic cusp form on \( B(i)^{\times}(\mathbb{A}) \). The map is well-defined, injective, and \( \mathbb{B}^{\times\times} \)-equivariant. Its image coincide with the weight 2 subspace of the cuspidal automorphic representation of \( B(i)^{\times}(\mathbb{A}) \) determined by \( A \) and the embedding \( \iota : M \subset \mathbb{C} \) (see [YZZ13, Theorem 3.3.2]). It follows that the image is irreducible as a representation of \( \mathbb{B}^{\times\times} \). Therefore, \( \pi(\mathcal{B})_A \) itself is an irreducible representation of \( \mathbb{B}^{\times\times} \).

From now on, we fix a representative \( \mathcal{B} \) in \( \mathcal{B}(A) \), and we will prove a \( p \)-adic Waldspurger formula for \( p \)-adic Maass functions on \( \mathcal{X} := X(\mathcal{B}) \) contained in \( \pi(\mathcal{B})_A \).

Take two nonzero differential forms \( \omega_\pm \in H^0(\mathcal{A}^\pm, \Omega^1_{A^\pm}) \). By Lemma 3.4.2, we have isomorphisms

\[
 \varsigma_{\omega_\pm} : \Pi(\mathcal{B})_{A^\pm} \otimes_M \mathbb{C}_p \xrightarrow{\sim} \pi(\mathcal{B})_{A^\pm}.
\]

Let \( \chi \in \Xi(A, \mathbb{C}_p)_0 \) be a character such that \( \mathcal{B}_\chi \simeq \mathcal{B} \). For \( \phi_\pm \in \pi(\mathcal{B})_{A^\pm} \), we put

\[
 \alpha_\chi^{\omega_\pm, \omega_\pm}(\phi_+, \phi_-) = \alpha_\chi(\varsigma_{\omega_+}^{-1}\phi_+, \varsigma_{\omega_-}^{-1}\phi_-) \in \mathbb{C}_p,
\]

where \( \alpha_\chi(\ , \ ) \) is the pairing in Theorem 3.3.2. Then \( \alpha_\chi^{\omega_\pm, \omega_\pm}(\ , \ ) \) is a basis of the 1-dimensional space

\[
 (3.5) \quad \text{Hom}_{\mathbb{B}^{\times\times}}(\pi(\mathcal{B})_{A^+} \otimes \chi, \mathbb{C}_p) \otimes_{\mathbb{C}_p} \text{Hom}_{\mathbb{B}^{\times\times}}(\pi(\mathcal{B})_{A^-} \otimes \chi^{-1}, \mathbb{C}_p).
\]

Globally, we have the following definition. Choose a CM point \( P^+ \in Y^+(E^{ab}) = \mathbb{Y}^+(\mathbb{C}_p) \) and put \( P^- = cP^+ \) as in §3.3.

**Definition 3.4.3.** For \( \phi_\pm \in \pi(\mathcal{B})_{A^\pm} \) and \( \chi \in \Xi(A, \mathbb{C}_p)_0 \), we define the \( p \)-adic torus period to be

\[
 \mathcal{P}_{\mathbb{C}_p}(\phi_\pm, \chi^{\pm 1}) := \int_{E\times \mathcal{A}_E^{\times}} \phi_\pm(T_t P^\pm) \cdot \chi(t)^{\pm 1} dt,
\]

where the Haar measure \( dt \) has total volume 2 as in Definition 3.3.1.
The above integrals are in fact finite sums valued in $\mathbb{C}_p$. The product $\mathcal{P}_{\mathbb{C}_p}(, \chi) \cdot \mathcal{P}_{\mathbb{C}_p}(, \chi^{-1})$ defines another element in (3.5), which depends on the choice of $c$ but not on the choice of $P^+$. In particular, it is proportional to $\alpha_{\chi^+}^{\omega_+}(, )$.

The following theorem is the $p$-adic Waldspurger formula for $p$-adic Maass functions. Recall that we have the $p$-adic $L$-function $\mathcal{L}(A)$ from Theorem 3.2.10. Put $\pi = \pi(\mathbb{B})_A$ as an irreducible subrepresentation of $\mathcal{H}_{\mathbb{C}_p}(\mathbb{B}^\times)$.

**Theorem 3.4.4** ($p$-adic Waldspurger formula for $p$-adic Maass functions). Put

$$\mathcal{L}_{\omega_+}(\pi) = \langle \omega_+ \otimes \omega_-, \mathcal{L}(A) \rangle,$$

regarded as an element in $\mathcal{D}(A, \mathbb{C}_p)$. Then for a character $\chi \in \Xi(A, \mathbb{C}_p)_0$, we have

$$\mathcal{P}_{\mathbb{C}_p}(\phi_+, \chi) \mathcal{P}_{\mathbb{C}_p}(\phi_-, \chi^{-1}) = \mathcal{L}_{\omega_+}(\pi) \cdot \frac{L(1/2, \pi \otimes \chi_p)}{\epsilon(1/2, \psi, \pi \otimes \chi_p)} \cdot \alpha_{\chi^+}^{\omega_+}(\phi_+, \phi_-)$$

for every $\phi_+ \in \pi(\mathbb{B})_A$.

**Proof.** It follows from Theorem 3.3.2, after pairing with $\omega_+ \otimes \omega_-$. □

**Remark 3.4.5.** In this remark, we explain how to deduce Theorems 1.5.1 and 1.5.3 in Introduction. So we take $A$ to be an elliptic curve over $\mathbb{Q}$. In particular, we have $F = M = \mathbb{Q}$, $A^+ = A^- = A$, and that $\omega_A = 1$ is the trivial character. We also fix an isomorphism $\iota : \mathbb{C}_p \cong \mathbb{C}$. We have the indefinite quaternion algebra $B$ over $\mathbb{Q}$. Take $\mathbb{B} \in \mathcal{B}(A)$ such that $\mathbb{B}^\infty \cong B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$. So we may identify $B$ with $B(\iota)$ in the $\iota$-nearby data for $\mathbb{B}$ (Definition 2.4.10). Moreover, we take $\psi : \mathbb{Q}_p \to \mathbb{C}^\times_p$ to be the additive character such that $\iota \circ \psi$ is the standard one. We choose the abstract conjugation $c$ (3.1) such that $c \otimes_{\mathbb{C}_p, \iota} \mathbb{C}$ coincides with the restriction of the complex conjugation on $\hat{X}_L$.

We also note that $\mathcal{D}(G)$ is simply $\mathcal{D}(1, \mathbb{C}_p)$; and $\mathcal{D}(G; \pi_{\mathbb{C}_p})$ is simply $\mathcal{D}(A, \mathbb{B}, \mathbb{C}_p)$ (Definition 3.2.5.)

1. We first deduce Theorem 1.5.1. Take the $p$-adic $L$-function $\mathcal{L}(A)$ as in Theorem 3.2.10, regarded as element in $\mathcal{D}(A, \mathbb{C}_p)$. Take a basis $\omega$ of $H^0(A, \Omega^1_A)$. Then there is a unique element $P_\omega \in \mathbb{C}_p^\times$ such that $\iota(P_\omega^{-1}(f_1, f_2)_A)$ is equal to the (bilinear) Petersson inner product of $\phi_i(f^*_i \omega)$ and $R\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\phi_i(f^*_i \omega)$ for every $f_1, f_2 \in \Pi_A$. Now we define $\mathcal{L}(\pi_{\mathbb{C}_p})$ to be the image of

$$P_\omega \cdot \iota^{-1}(d_{E, F}^{1/2}L(1, \eta)) \cdot \langle \omega \otimes \omega, \mathcal{L}(A) \rangle$$

under the canonical projection $\mathcal{D}(A, \mathbb{C}_p) \to \mathcal{D}(A, \mathbb{B}, \mathbb{C}_p) = \mathcal{D}(G; \pi_{\mathbb{C}_p})$. It is clear that $\mathcal{L}(\pi_{\mathbb{C}_p})$ does not depend on the choice of $\omega$, hence well-defined. Then Theorem 1.5.1 follows from Theorem 3.2.10, Remark 1.1.2, and Lemma 3.4.6 below (with $r = 2$).

2. Now we deduce Theorem 1.5.3. In Definition 3.4.3, we choose $P^+$ such that $\iota P^+ = [+i, 1]$, and thus $\iota P^- = [-i, 1]$. Then $\mathcal{P}_{\mathbb{C}_p}(, \chi^\pm 1)$ in Definition 3.4.3 coincide with those in (1.5). Therefore, Theorem 1.5.3 follows from Theorem 3.4.4.

**Lemma 3.4.6.** Let $\pi$ be the discrete series representation of weight $r \geq 2$ of $GL_2(\mathbb{R})$ with trivial central character. Fix a nonzero $GL_2(\mathbb{R})$-equivariant bilinear pairing $(, ) : \pi \times \pi \to \mathbb{C}$. Let $f_+ \in \pi$ be a generator of weight $r$, that is, the archimedean component of holomorphic modular forms of weight $r$. Put $f_- = \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)f_+$, which is a generator of weight $-r$. 
Then we have
\[
\frac{(\Delta^k f_+, \Delta^k f_-)}{(f_+, f_-)} = \frac{k!(k + r - 1)!}{4^k(r - 1)!},
\]
where $\Delta_{\pm}$ are in Definition 2.4.16.

Proof. It is well-known that $(f_+, f_-) \neq 0$. Put
\[
X_{\pm} = \frac{1}{2} \left( \frac{1}{\pm i} \right) = 2i\Delta_{\pm}, \quad H = -i\binom{1}{-1}.
\]
Recall from [Bum97, p157, (2.39)] the following Casimir element
\[
\Delta = -\frac{1}{4}(2X_+X_- + 2X_-X_+ + H^2).
\]
It acts on $\pi$ by the scalar
\[
\lambda_r := \frac{r}{2} \left( 1 - \frac{r}{2} \right).
\]
We say that a vector $g \in \pi$ has weight $\mu$ if $Hg = \mu g$. For such $g$, we have
\[
X_-X_+g = -\left( \frac{H^2 + 2H}{4} + \Delta \right)g = -\left( \frac{\mu^2 + 2\mu}{4} + \lambda_r \right)g.
\]
Now for each $k \geq 0$, the vector $X_+^k f_+$ is of weight $r + 2k$. Therefore, we have for $k \geq 1$ the formula
\[
X_-X_+^k f_+ = X_-X_+(X_+^{k-1} f_+) = -\left( \frac{(r + 2k - 2)(r + 2k)}{4} + \lambda_r \right)X_+^{k-1} f_+.
\]
We prove the identity
\[
\frac{(X_+^k f_+, X_-^k f_-)}{(f_+, f_-)} = \frac{k!(k + r - 1)!}{(r - 1)!}
\]
by induction on $k \geq 0$. The case $k = 0$ is trivial. Suppose that we know this for $k - 1$. Then we have
\[
(X_+^k f_+, X_-^k f_-) = -(X_-X_+^k f_+, X_-^{k-1} f_-)
= \left( \frac{(r + 2k - 2)(r + 2k)}{4} + \lambda_r \right) \cdot (X_+^{k-1} f_+, X_-^{k-1} f_-)
= k(k + r - 1) \cdot (X_+^{k-1} f_+, X_-^{k-1} f_-).
\]
The lemma follows as $X_{\pm} = 2i\Delta_{\pm}$. □

4. Proofs of main theorems

This chapter is dedicated to the proof of Theorems 3.2.10 and 3.3.2. In §4.1, we construct the distribution interpolating matrix coefficient integrals appearing in the complex Waldspurger formula. We construct the universal torus period in §4.2, which is a crucial construction toward the $p$-adic $L$-function. In §4.3, we study the relation between universal torus periods and classical torus periods, based on which we accomplish the proof of our main theorems in §4.4.
4.1. Distribution of matrix coefficient integrals. Recall that we have fixed an abelian variety $A$ of $GL(2)$-type over $F$ up to isogeny that is automorphic, equipped with an embedding $M = M_A \subset \mathbb{C}_p$, as in §3.2.

Let $K/M F_p$ be a complete field extension. Take a representative $\mathbb{B}$ in $\mathcal{B}(A)$. As in §3.2, we fix an isomorphism $\mathbb{B}_p \simeq \text{Mat}_2(F_p)$ and an $E$-embedding under which $e_p$ coincides with (1.7). Recall that we put $\Pi^\pm = \Pi_{A^+} = \Pi(\mathbb{B})_{A^+}$ (Notation 3.1.2).

**Definition 4.1.1** (Stable/admissible vector). We say that elements $f_\pm$ in $\Pi^\pm \otimes_M K$ or $\Pi_\pm^\otimes_M K$ are stable vectors if respectively,

1. $f_\pm$ are fixed by $N^\pm(O_p)$;
2. $f_\pm$ satisfy the relation

$$\sum_{g \in N^\pm(p^{-1})/N^\pm(O_p)} \Pi_p^\pm(g)f_\pm = 0.$$ 

We denote by $(\Pi^\pm)^{\otimes}_K$ the subsets of $\Pi^\pm \otimes_M K$ consisting of stable vectors, respectively. We denote by $(\Pi^\pm_{\otimes})^\otimes_K$ the subsets of $\Pi_{\otimes}^\pm \otimes_M K$ consisting of stable vectors, respectively.

For $n \in \mathbb{N}$, we say that stable vectors $f_\pm$ in $(\Pi^\pm)^{\otimes}_K$ or $(\Pi^\pm_{\otimes})^\otimes_K$ are $n$-admissible if respectively,

$$\Pi_p^\pm(n^\pm(x))f_\pm = \psi^\pm(x)f_\pm$$

for every $x \in p^{-n}/O_p$, where $n^\pm(x)$ are the same as in Proposition 2.3.5.

**Remark 4.1.2.** If we realize $\Pi_p^\pm$ in their Kirillov models with respect to the pair $(N^+, \psi^\pm)$, then $f_{\pm p}$ belong to $(\Pi_p^\pm)^{\otimes}_K$ if and only if $f_{\pm p}$ (resp. $\Pi_p^\pm(J)f_{\pm p}$) is supported on $O_p^\times$; and is $n$-admissible if and only if $f_{\pm p}$ (resp. $\Pi_p^\pm(J)f_{\pm p}$) is supported on $(1 + p^n)^\times$.

**Definition 4.1.3.** Let $w \in \mathbb{Z}$, $n \in \mathbb{N}$ be integers, and $\omega: F_\infty^{\times}\mathbb{A}_E^{\times} \to M^\times$ a locally constant character. We say that a $K$-valued character $\chi: E_\infty^{\times}\mathbb{A}_E^{\times} \to K^\times$ of weight $w$ is of central type $\omega$ and depth $n$ if

- $\omega \cdot \chi|_{A^{\infty}} = 1$; and
- $\chi_{\mathbb{F}_q}(t) = t^{-w}$ for all $t \in (1 + p^n)^\times$.

We denote by $\Xi(\omega, K)^n_w$ the set of all $K$-valued characters of weight $w$, central type $\omega$ and depth $n$. Moreover, put $\Xi(\omega, K)^n = \bigcup_w \Xi(\omega, K)^n_w$.

We recall the definition of the classical (normalized) matrix coefficient integral. Suppose that $K$ is contained in $\mathbb{C}_p$. We take a character $\chi \in \Xi(\omega, K)$. Let $\iota: \mathbb{C}_p \rightarrow \mathbb{C}$ be an isomorphism. Morally speaking, the integral should be defined as

$$\alpha^2(f_+, f_-; \chi^{(i)}) = \int_{\mathbb{A}_E^{\times}\mathbb{A}_E^{\times}} \iota(\Pi(t))f_+, f_-A \cdot \chi^{(i)}(t) dt.$$ 

However, it is not absolutely convergent, so we need regularization recalled as follows.

**Definition 4.1.4** (Regularized matrix coefficient integral). Take an arbitrary decomposition $\iota(\cdot) = \Pi_{v<\infty}(\cdot)_{|_{\mathbb{C}_p}}$ where $(\cdot, \cdot)_{|_{\mathbb{C}_p}}: \Pi^+_v \otimes \Pi^-_v \to \mathbb{C}$ is a $\mathbb{B}_v^\times$-invariant bilinear pairing. For $f_\pm = \otimes_{v<\infty} f_\pm$ such that $(f_+, f_-)_{|_{\mathbb{C}_p}} = 1$ for all but finitely many $v$, we put

$$\alpha(f_+, f_-; \chi^{(i)}_v) = \int_{\mathbb{F}_q^{\times}\mathbb{E}_v^\times}(\Pi_v(t)f_+, f_-)_{|_{\mathbb{C}_p}} \chi^{(i)}_v(t) dt;$$

$$\alpha^2(f_+, f_-; \chi^{(i)}_v) = \left(\frac{\zeta_F(2)L(1/2, \rho^{(i)}_{A^+, v}, \chi^{(i)}_v)}{L(1, \eta_v)L(1, \rho^{(i)}_{A^+, v}, \text{Ad})}\right)^{-1} \alpha(f_+, f_-; \chi^{(i)}_v).$$
Here, $dt$ is the measure on $F_v^\times \backslash E_v^\times$ given determined in §1.8, and $\rho_{A,v}^{(i)}$ is the corresponding admissible complex representation of $E_v^\times$ via $\iota$. Then by [Wal85, §3] we have

$$\alpha^\sharp(f_{+v}, f_{-v}; \chi_v^{(i)}) = 1$$

for all but finitely many $v$, and the product

$$\alpha^\sharp(f_+, f_-; \chi^{(i)}) := \prod_{v<\infty} \alpha^\sharp(f_{+v}, f_{-v}; \chi_v^{(i)})$$

is well-defined. We extend the functional $\alpha^\sharp(\cdot; \chi^{(i)})$ to all $f_+, f_-$ by linearity.

**Remark 4.1.5.** The functional $\alpha^\sharp(\cdot; \chi^{(i)})$ does not depend on the choice of the decomposition of $\iota(\cdot, \cdot)_A$.

The following proposition is our main result, whose proof will be given at the end of this section. Note that since $\Xi(\omega, K)^n$ is a subset of $\mathcal{G}(\omega, K)$, we have a natural pairing $\mathcal{D}(\omega, K) \times \Xi(\omega, K)^n \to K$.

**Proposition 4.1.6.** Let $M F_p \subset K \subset C_p$ be a complete intermediate field. Let $f_\pm \in (\Pi^\pm)_{K}^{\otimes}$ be two $n$-admissible stable vectors for some (common) $n \in \mathbb{N}$. Then there is a unique element $\mathcal{D}(f_+, f_-) \in \mathcal{D}(\omega_A, K)$ such that for all $K$-valued characters $\chi \in \Xi(\omega_A, K)^n$ of central type $\omega_A$ and depth $n$, and $\iota: C_p \to \mathbb{C}$, we have

$$\iota \mathcal{D}(f_+, f_-)(\chi) = \int \left( \frac{L(1/2, \rho_{A,p} \otimes \hat{\chi}^p)}{\epsilon(1/2, \rho_{A,p} \otimes \hat{\chi}^p)} \right)^2 \alpha^\sharp(f_+, f_-; \chi^{(i)}).$$

**Definition 4.1.7.** The element $\mathcal{D}(f_+, f_-)$ is called the $(K$-valued) local period distribution.

Before giving the proof, we make a convenient choice of a decomposition of $(\cdot, \cdot)_A$. Realize the representations $\Pi^\pm_p$ in their Kirillov models as in Remark 4.1.2. We may assume that $f_\pm = \otimes f_{\pm v}$, with $f_{\pm v} \in \Pi^\pm_v \otimes_M K$, are decomposable and are fixed by some (common) sufficiently small open compact subgroup $V_p \in \mathcal{Y}$. Choose a decomposition $(\cdot, \cdot)_A = \prod_{v<\infty}(\cdot, \cdot)_v$ such that

1. $(f_{+v}, f_{-v})_v = 1$ for all but finitely many $v$;
2. $(f'_{+v}, f'_{-v})_v \in K$ for all $f'_{\pm v} \in \Pi^\pm_v \otimes_M K$;
3. for $f'_{\pm p} \in \Pi^\pm_p \otimes_M K$ that are compactly supported on $F_p^\times$,

$$(f'_{+p}, f'_{-p})_p = \int_{F_p^\times} f'_{+p}(a) f'_{-p}(a) \, da,$$

where $da$ is the Haar measure on $F_p^\times$ such that the volume of $O_p^\times$ is 1.

We need two lemmas for the proof of Proposition 4.1.6. For simplicity, write $\omega = \omega_A$. For each finite place $v \neq p$ and an open compact subgroup $V_v$ of $E_v^\times$, let $\mathcal{D}(\omega_v, K, V_v)$ be the quotient $K$-algebra of $\text{D}(E_v^\times / V_v, K)$ divided by the closed ideal generated by $\{\omega_v(t)\delta_t - 1 | t \in F_v^\times\}$. Put

$$\mathcal{D}(\omega_v, K) = \lim_{V_v} \mathcal{D}(\omega_v, K, V_v),$$

where the limit runs over all $V_v$. Let $\mathcal{D}(\omega_p, K)$ be the quotient of $\text{D}(E_p^\times, K)$ by the closed ideal generated by $\{\omega_p(t)\delta_t - 1 | t \in F_p^\times\}$. For every finite place $v$, we have a natural homomorphism $\mathcal{D}(\omega_v, K) \to \mathcal{D}(\omega, K)$.

**Lemma 4.1.8.** Let $v \neq p$ be a finite place of $F$. 

(1) There exists a unique element
\[
\mathcal{L}^{-1}(\rho_{A,v}) \in \mathcal{D}(\omega_v, MF_p)
\]
such that for every locally constant character \(\chi_v : E_v^\times \to K^\times\) satisfying \(\omega_v \cdot \chi_v|_{E_v^\times} = 1\), we have
\[
\mathcal{L}^{-1}(\rho_{A,v})(\chi_v) = L(1/2, \rho_{A,v}, \chi_v)^{-1}.
\]
(2) For \(f_{\pm v} \in \Pi_v^\pm \otimes_M K\), there exists a unique element
\[
\mathcal{D}(f_{+v}, f_{-v}) \in \mathcal{D}(\omega_v, K)
\]
such that for every locally constant character \(\chi_v : E_v^\times \to K^\times\) satisfying \(\omega_v \cdot \chi_v|_{E_v^\times} = 1\), and \(i : \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}\), we have
\[
i \mathcal{D}(f_{+v}, f_{-v})(\chi_v) = \alpha^2(f_{+v}, f_{-v}; \chi_v^{(i)}).
\]

Proof. The uniqueness is clear. In the following proof, we suppress \(v\) from the notation and we will use the subscript \(i\) for all changing of coefficients of representations via \(i\).

To prove (1), we first consider the following situation. Let \(\tilde{F}\) be either \(F\) or \(E\), and \(\tilde{\Pi}\) be an irreducible admissible \(M\)-representation of \(GL_2(\tilde{F})\). We claim that there is a (unique) element \(\mathcal{L}^{-1}(\tilde{\Pi}) \in D_\nu(\tilde{F}^\times, MF_p)\), where
\[
D_\nu(\tilde{F}^\times, K) := \lim_{\nu} D(\tilde{F}^\times/V, K)
\]
with \(V\) running over all open compact subgroups of \(\tilde{F}^\times\), such that for every locally constant character \(\chi : \tilde{F}^\times \to K^\times\) and \(i : \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}\),
\[
i \mathcal{L}^{-1}(\tilde{\Pi})(\chi) = L(1/2, \tilde{\Pi}_i \otimes \chi)\).
\]
In fact, for a locally constant character \(\mu : \tilde{F}^\times \to M^\times\), define \(\mathcal{L}^{-1}(\tilde{\Pi}) \in D_\nu(\tilde{F}^\times, MF_p)\) by the formula
\[
\mathcal{L}^{-1}(\mu)(h) = 1 - \int_{O_{\tilde{F}}^\times} \mu(\tilde{w}a)h(\tilde{w}a) \, da
\]
for \(h \in \lim_{\nu} C(\tilde{F}^\times/V, MF_p)\). Here, \(\tilde{w}\) is an arbitrary uniformizer of \(\tilde{F}\), and \(da\) is the Haar measure on \(O_{\tilde{F}}^\times\) with total volume 1. Then we have three cases:

- If \(\tilde{\Pi}\) is supercuspidal, put \(\mathcal{L}^{-1}(\tilde{\Pi}) = 1\).
- If \(\tilde{\Pi}\) is the unique irreducible subrepresentation of the unnormalized parabolic induction of \((\mu, |\mu|^{-2})\) for a character \(\mu : \tilde{F}^\times \to M^\times\), then we put \(\mathcal{L}^{-1}(\tilde{\Pi}) = \mathcal{L}^{-1}(\mu)\).
- If \(\tilde{\Pi}\) is the irreducible unnormalized parabolic induction of \((\mu_1^2, |\mu_2|^{-1})\) for a pair of characters \(\mu_i : \tilde{F}^\times \to M^\times (i = 1, 2)\), then we put \(\mathcal{L}^{-1}(\tilde{\Pi}) = \mathcal{L}^{-1}(\mu_1) \odot \mathcal{L}^{-1}(\mu_2)\).

Here, we adopt the unnormalized induction in order to track the rationality properties.

Go back to (1). First, assume that \(E/F\) is non-split. Then we define \(\mathcal{L}^{-1}(\rho_A)\) to be the image of \(\mathcal{L}^{-1}(\Pi_E)\) in \(\mathcal{D}(\omega, MF_p)\) where \(\Pi_E\) is the base change of \(\Pi\) to \(GL_2(E)\), which depends only on \(\rho_A\). Second, assume that \(E = E_\bullet \times F_\circ\) is split where \(E_\bullet = F_\circ = F\). Then we define \(\mathcal{L}^{-1}(\rho_A)\) to be the image of \(\mathcal{L}^{-1}(\Pi_\bullet) \otimes \mathcal{L}^{-1}(\Pi_\circ)\) in \(\mathcal{D}(\omega, MF_p)\).

Now we consider (2). First, assume that \(E/F\) is non-split. Then the torus \(F^\times/E^\times\) is compact, hence the matrix coefficient \(\Phi_{f_+, f_-}(g) := (\Pi^+(g)f_+, f_-)\) is finite under \(E^\times\)-translation. We may assume that the restriction \(\Phi_{f_+, f_-}|_{E^\times} = \sum_i a_i \chi_i\) is a finite \(K\)-linear combination of \(K\)-valued (locally constant) characters \(\chi_i\) of \(E^\times\) such that \(\omega \cdot \chi_i|_{E^\times} = 1\).
Assigning to every locally constant function $h$ on $E^\times$ satisfying $\omega(t)h(at) = h(a)$ for all $a \in E^\times$ and $t \in F^\times$ the integral
\[
\sum_i a_i \int_{F^\times \setminus E^\times} \chi_i(t)h(t) \, dt,
\]
which is a finite sum, defines an element $\alpha(f_+, f_-)$ in $D_\omega(\omega, K)$. Put
\[
\mathcal{D}(f_+, f_-) = \left( \frac{\zeta_F(2)}{L(1, \rho_A, \text{Ad})L(1, \eta)} \right)^{-1} L^{-1}(\rho_A)\alpha(f_+, f_-).
\]

Second, assume that $E = F_\bullet \times F_\circ$ is split. We may suppose that the embedding $E \rightarrow \text{Mat}_2(F)$ is given by
\[
(t_\bullet, t_\circ) \mapsto \begin{pmatrix} t_\bullet & \ast \\ 0 & t_\circ \end{pmatrix}
\]
for $t_\bullet, t_\circ \in F$. Moreover, a character $\chi$ of $E^\times$ is given by a pair $(\chi_\bullet, \chi_\circ)$ of characters of $F^\times$ such that $\chi((t_\bullet, t_\circ)) = \chi_\bullet(t_\bullet)\chi_\circ(t_\circ)$.

Now we realize $\Pi^\pm$ in their Kirillov models with respect to (nontrivial) additive characters $\psi^\pm : F \rightarrow \mathbb{C}^\times$ of conductor 0 respectively, where $\psi^- = (\psi^+)^{-1}$. Moreover, we may assume for $f_\pm \in \Pi^\pm \otimes_M K$ that are compactly supported on $F^\times$, that
\[
(f_+, f_-) = \int_{F^\times} f_+(a)f_-(a) \, da,
\]
where $da$ is the Haar measure on $F^\times$ such that the volume of $O_F^\times$ is $c$ for some $c \in M$. We have the following formula
(4.1)
\[
\alpha^\pm(f_+, f_-; \chi_i) = \left( \frac{\zeta_F(2)L(1/2, \rho_A^{(i)}, \chi_i)}{L(1, \eta)L(1, \rho_A^{(i)}, \text{Ad})} \right)^{-1} \int_{F^\times} tf_+(a) \cdot \chi_\bullet(a) \, da \int_{F^\times} tf_-(b) \cdot \chi_\circ(b^{-1}) \, db
\]
\[
= \left( \frac{\zeta_F(2)}{L(1, \eta)L(1, \rho_A^{(i)}, \text{Ad})} \right)^{-1} Z(tf_+, \chi_i)Z(tf_-, \chi_i^{-1}),
\]
where
\[
Z(tf_\pm, \chi_i^{\pm 1}) = L(1/2, \Pi^\pm_i \otimes \chi_i^{\pm 1})^{-1} \int_{F^\times} tf_\pm(a) \cdot \chi_i^{\pm 1}(a) \, da.
\]
Note that the above integrals are simply local zeta integrals. To conclude, it suffices to show that there exist elements $Z(f_\pm) \in D_\omega(F^\times, K)$ such that for every locally constant character $\chi : F^\times \rightarrow K^\times$ and every isomorphism $i : \mathbb{C}_p \overset{\sim}{\rightarrow} \mathbb{C}$, we have $iZ(f_\pm)(\chi) = Z(tf_\pm, \chi_i^{\pm 1})$, respectively. Without loss of generality, we only construct $Z(f_+)$.

Enlarging $M$ if necessary to include $l^{1/2}$ where $l$ is the cardinality of the residue field of $F$, there is a subspace $\Pi^{+,c}$ of $\Pi^+$ such that $\Pi^{+,c} \otimes_M K$ is the subspace of $\Pi^+ \otimes_M K$ of functions that are compactly supported on $F^\times$. For $f_+ \in \Pi^{+c} \otimes_M K$, we may define $Z(f_+)$ such that for every locally constant function $h$ on $F^\times$
\[
Z(f_+)(h) = \mathcal{L}_F^{-1}(\Pi^+)(h) \times \int_{F^\times} f_+(a)h(a) \, da.
\]
Therefore, we may conclude the proof if $\dim \Pi^+ / \Pi^{+c} = 0$. There are two cases remaining.
First, $\Pi^+$ is a special representation, that is, $\dim \Pi^+ / \Pi^{+, c} = 1$. We may choose a representative $f_+ = \mu(a) \cdot \text{ch}_{\mathcal{O}_F \setminus \{0\}}(a)$ for some character $\mu: F^\times \to M^\times$. Then $Z(tf_+, \chi_t) = c$ (resp. $0$) if $\mu \cdot \chi$ is unramified (resp. otherwise). Therefore, we may define $Z(f_+)$ such that

$$Z(f_+(h)) = \int_{O_F^\times} \mu(a) h(a) \, da$$

for every locally constant function $h$ on $F^\times$.

Second, $\Pi^+$ is a principal series, that is, $\dim \Pi^+ / \Pi^{+, c} = 2$. There are two possibilities. In the first case, we may choose representatives $f_+^1 = \mu(a) \cdot \text{ch}_{\mathcal{O}_F \setminus \{0\}}(a)$ and $f_+^2 = (1 - \log |a|) \mu(a) \cdot \text{ch}_{\mathcal{O}_F \setminus \{0\}}(a)$ for some character $\mu: F^\times \to M^\times$. The function $f_+^1$ has been treated above. For $f_+^2$, we have $Z(tf_+^2, \chi_t) = \log a$ (resp. $0$) if $\mu \cdot \chi$ is unramified (resp. otherwise). Therefore, we may define $Z(f_+^2)$ such that

$$Z(f_+^2(h)) = \mathcal{L}_F^{-1}(\mu^1)(h) \times \int_{O_F^\times} \mu(a) h(a) \, da$$

for every locally constant function $h$ on $F^\times$. In the second case, we may choose representatives $f_+^1 = \mu(a) \cdot \text{ch}_{\mathcal{O}_F \setminus \{0\}}(a)$ and $f_+^2 = (1 - \log |a|) \mu(a) \cdot \text{ch}_{\mathcal{O}_F \setminus \{0\}}(a)$ for some character $\mu: F^\times \to M^\times$. The function $f_+^1$ has been treated above. For $f_+^2$, we have $Z(tf_+^2, \chi_t) = c$ (resp. $0$) if $\mu \cdot \chi$ is unramified (resp. otherwise). Therefore, we may define $Z(f_+)$ such that

$$Z(f_+(h)) = \int_{O_F^\times} \mu(a) h(a) \, da$$

for every locally constant function $h$ on $F^\times$. $\square$

**Lemma 4.1.9.** Let $f_{\pm} \in (\Pi_{\mathfrak{p}}^+)_{K}^\circ$ be two $n$-admissible stable vectors. There exists a unique element

$$\mathcal{D}(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}}) \in \mathcal{D}(\omega_{\mathfrak{p}}, K)$$

with the following property: for every character $\chi_{\mathfrak{p}}: E^\times_{\mathfrak{p}} \to K^\times$ satisfying $\omega_{\mathfrak{p}} \cdot \chi_{\mathfrak{p}}|_{F_{\mathfrak{p}}^\times} = 1$ and $\chi_{\mathfrak{p}}(t) = t^{-w}$ for $t \in (1 + \mathfrak{p}^n)^\times$ and some $w \in \mathbb{Z}$, and $\iota: \mathbb{C}_{\mathfrak{p}} \cong \mathbb{C}$, we have

$$\iota \mathcal{D}(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}})(\chi_{\mathfrak{p}}) = \iota \left( \frac{L(1/2, \rho_{A_{\mathfrak{p}}} \otimes \tilde{\chi}_{\mathfrak{p}})}{\epsilon(1/2, \psi, \rho_{A_{\mathfrak{p}}} \otimes \tilde{\chi}_{\mathfrak{p}})} \right)^{\alpha^2(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}}; \chi_{\mathfrak{p}}(\iota))}.$$

Here, $\tilde{\chi}$ is defined similarly as in Definition 3.2.2 (2).

Moreover, there are $n$-admissible stable vectors $f_{\pm \mathfrak{p}} \in (\Pi_{\mathfrak{p}}^+)_{K}^\circ$ such that $\mathcal{D}(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}})(\chi_{\mathfrak{p}}) \neq 0$ for every such $\chi_{\mathfrak{p}}$.

**Proof.** The uniqueness of $\mathcal{D}(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}})$ is clear as those characters $\chi_{\mathfrak{p}}$ in the statement span a dense subspace of $\mathcal{C}(\omega_{\mathfrak{p}}, K)$ by Lemma 2.1.11.

For the existence of $\mathcal{D}(f_{+ \mathfrak{p}}, f_{- \mathfrak{p}})$, first note that the formula (4.1) also works for $v = \mathfrak{p}$. Moreover, we have the functional equation

$$Z(\iota f_{- \mathfrak{p}}, \chi_{\mathfrak{p}}(\iota)) = \iota \epsilon(1/2, \psi, \rho_{A_{\mathfrak{p}}} \otimes \tilde{\chi}_{\mathfrak{p}}) \cdot Z(\iota(\Pi_{\mathfrak{p}}^{-}(J)f_{- \mathfrak{p}}), \chi_{\mathfrak{p}}(\iota)).$$

By Remark 4.1.2, we only need to show that for $f \in \Pi_{\mathfrak{p}}^+ \otimes_{M} K$ that is supported on $(1 + \mathfrak{p}^n)^\times$, there exists $\mathcal{D}'(f) \in \mathcal{D}(\omega_{\mathfrak{p}}, K)$ such that for $\chi_{\mathfrak{p}}$ as in the statement and $\iota$,

$$\iota \mathcal{D}'(f)(\chi_{\mathfrak{p}}) = \int_{O_F^\times} \iota f(a) \cdot \chi_{\mathfrak{p}}(\iota)(a) \, da.$$
Then we may set
\[ Q(f_+, f_-) = \left( \frac{\zeta_{f_p}(2)}{L(1, \rho_{A, p}: \text{Ad})L(1, \eta_p)} \right)^{-1} Q'(f_p) Q'(\Pi_p^{-1}(J)f_-) \]

For the existence \( Q'(f) \), since \( \chi_{\Psi}^{(i)} \) restricts to the trivial character on \((1 + p^n)^\times\), we have
\[ \int_{O_p^\times} i f(a) \cdot \chi_{\Psi}^{(i)}(a) \, da = i \int_{O_p^\times} f(a) \, da. \]
We may put
\[ Q'(f) = \int_{O_p^\times} f(a) \, da \in K \]
which is a constant (depending only on \( f \)).

The last part of the lemma follows from (4.2). □

Proof of Proposition 4.1.6. Let \( f_\pm \in (\Pi^\pm)_K^\times \) be two \( n \)-admissible stable vectors. It is clear that \( Q(f_+, f_-) \) constructed in Lemma 4.1.8 is equal to 1 for almost all \( v \). Therefore, we may simply define \( Q(f_+, f_-) \) to be the image of \( Q(f_+, f_-) \otimes \bigotimes_{v \neq p} Q(f_+, f_-) \)
in \( \mathcal{D}(\omega_A, K) \). □

4.2. Universal torus periods. Let \( \mathbb{B} \) be as in the previous section. As in §3.3, we choose a CM point \( P^+ \in Y^+(E_{ab}) \) and put \( P^- = cP^+ \). By Lemma 2.4.2, we regard \( P^\pm \) as points in \( X(\pm\infty)(F_{ab}^\times) \), respectively. By the same lemma, the morphism \( \Gamma_t (2.14) \) preserves \( Y^\pm(\infty) \) for \( t \in O_E^\times \), respectively.

Recall that for \( m \in \mathbb{N} \cup \{\infty\} \), we have the closed formal subscheme \( Y^\pm(m) \) of \( X(m) \) as in §2.4. For a complete field extension \( K/F_{ab}^{nr} \), put
\[ N^\pm(m, K) = H^0(Y^\pm(m), G_{Y^\pm(m)} \otimes_{O_{E_p}^\times} K). \]

Lemma 4.2.1. Suppose that \( K \) is a complete field extension of \( F_{ab}^\times \). Then the respective maps from \( N^\pm(\infty, K) \) to the \( K \)-algebra of continuous \( K \)-valued functions on \( E^\times \setminus A_E^\infty \) that send \( f \in N^\pm(\infty, K) \) to the functions \( x \mapsto f(\Upsilon_\pm T_x P^\pm) \) are isomorphisms. We recall that \( \Upsilon_\pm \) are in Definition 2.2.10,

Moreover, the induced actions of \( t \in O_{E_p}^\times \) on \( N^\pm(\infty, K) \) are respectively given by
\[ (\Gamma_t^* f)(x) = \begin{cases} f(x t), & \text{for } f \in N^+(\infty, K), \\ f(x t^c), & \text{for } f \in N^-(\infty, K), \end{cases} \]
for \( x \in E^\times \setminus A_E^\infty \).

Proof. The isomorphism follows from Lemmas 2.4.2 and 2.4.4. The action is a consequence of Lemma 2.2.11. □

Notation 4.2.2. Consider a locally constant character
\[ \omega: F^\times \setminus A^\infty \to M^\times. \]
Let $K$ be a complete field extension of $MF_p$. For every $V^p \in \mathfrak{M}$ on which $\omega$ is trivial, denote by $\mathcal{D}(\omega, K, V^p)$ the quotient $K$-algebra of $D(E^E A_{E}^{\infty} \setminus V^p, K)$ divided by the closed ideal generated by $\{ \omega(t) \delta_t - 1 \mid t \in A_{E}^{\infty} \}$.

Then by some standard facts from functional analysis (see [Tay, Propositions 2.11 & 2.12]) and Remark 2.1.1, we have a canonical isomorphism

$$\mathcal{D}(\omega, K) \simeq \lim_{\longleftarrow V^p \in \mathfrak{M}} \mathcal{D}(\omega, K, V^p)$$

of topological $K$-algebras, where the former one is in Definition 3.2.5 (1). The (unique) continuous homomorphism $D(O^\times_{E_p}, K) \to D(E^E A_{E}^{\infty} \setminus V^p, K)$ sending $\delta_t$ to $\omega_p(t_o) \delta_t$ for $t = (t_o, t_o) \in O^\times_{E_p}$ descends to a continuous homomorphism $\mathbf{w}: D(O^\text{anti}_p, K) \to \mathcal{D}(\omega, K, V^p)$ of $K$-algebras, which is compatible with respect to the change of $V^p$. In other words, we have a homomorphism

$$(4.3) \quad \mathbf{w}: D(O^\text{anti}_p, K) \to \mathcal{D}(\omega, K).$$

**Definition 4.2.3 (Universal character).** We respectively define the $\pm$-universal character to be

$$\chi_{\text{univ}}^\pm: E^E A_{E}^{\times} / A_{E}^{\times} \to \mathcal{D}(\omega, MF_p)^\times,$$

where $\delta$ is defined in Definition 3.2.5 (4).

The universal characters depend on $\omega$. Since we will always take $\omega = \omega_A$, we suppress it from notation.

**Lemma 4.2.4.** The universal characters $\chi_{\text{univ}}^\pm$ are elements in $N^{\pm}(\infty, F_p^\text{ab}) \otimes F_p^\text{ab} \mathcal{D}(\omega, MF_p^\text{ab})$ satisfying

$$\Gamma^\times_t \chi_{\text{univ}}^\pm = \delta_t \cdot \chi_{\text{univ}}^\pm,$$

respectively, for $t \in O^\times_{E_p,m}$ if $\omega$ is trivial on $F_p^\times \cap O^\times_{E_p,m}$.

**Proof.** It follows from the definition, Lemma 4.2.1, and the observation that conjugation and inversion coincide on $O^\text{anti}_p$. \hfill \square

Suppose that $K$ is a complete field extension of $MF_p^\text{lt} F_p^\text{ab}$. Given a stable convergent modular form $f \in \mathcal{M}^u(m, K)^\odot$ for some $w, m \in \mathbb{N}$ (Definition 2.3.10), we have the global Mellin transform $\mathbf{M}(f)$ by Theorem 2.3.17, and by (4.3),

$$\mathbf{w}\mathbf{M}(f) \in \mathcal{M}^u(m, K)^\odot \otimes K \mathcal{D}(\omega, K).$$

By restriction, we obtain elements

$$\mathbf{w}\mathbf{M}(f)|_{\mathfrak{M}^\pm(\infty)} \in N^{\pm}(\infty, K) \otimes K \mathcal{D}(\omega, K).$$

By Theorem 2.3.17 (2) and Lemma 4.2.4, the product $\mathbf{w}\mathbf{M}(f)|_{\mathfrak{M}^\pm(\infty)} \cdot \chi_{\text{univ}}^\pm$ descends to an element in $N^{\pm}(m, K) \otimes K \mathcal{D}(\omega, K)$ if $\omega$ is trivial on $F_p^\times \cap O^\times_{E_p,m}$.

For every $V^p \in \mathfrak{M}$ under which $f$ is invariant, we regard $\mathbf{w}\mathbf{M}(f)|_{\mathfrak{M}^\pm(\infty)} \cdot \chi_{\text{univ}}^\pm$ as elements in $N^{\pm}(m, K) \otimes K \mathcal{D}(\omega, K, V^p)$ respectively. They are invariant under the action of $V^p$ on $N^{\pm}(m, K)$. 


Definition 4.2.5 (Universal torus period). We define the universal torus periods of \( f \) to be the elements
\[
\mathcal{P}_\omega^\pm(f) := \frac{2}{|E^x \backslash \mathbb{A}_E^\times / \mathbb{A}^\times_p|} \sum_{E^x \backslash \mathbb{A}_E^\times / \mathbb{A}^\times_p} \left( (\omega \cdot M(f)|_{\mathcal{E}(\infty)} \cdot \chi_{\text{univ}}^\pm)(t) \right)
\]
in \( \mathcal{D}(\omega, K, V^p) \).

Remark 4.2.6. We add the factor 2 in the above definition in order to be consistent with the Tamagawa measure we choose in the complex Waldspurger formula recalled in §1.1.

By construction, the elements \( \mathcal{P}_\omega^\pm(f) \) are independent of \( m \), and are compatible with respect to the change of \( V^p \). Therefore, they are elements in \( \mathcal{D}(\omega, K) \). In fact, for a character \( \chi \in \Xi(\omega, K) \), Lemma 4.2.1 allows us to write
\[
\mathcal{P}_\omega^\pm(f)(\chi) = \int_{E^x \backslash \mathbb{A}_E^\times} \omega(t) M((\xi_{\pm}^* \cdot \omega_{\pm})(\chi)(\frac{\iota}{\mathcal{E}})(\frac{\chi}{t}) \cdot \chi(t)^\pm) \, dt.
\]

4.3. Interpolation of universal torus periods. We keep the setting in the previous section. Let \( M_{F_{p}}^{\text{ur}} K_{F_p} \subset K \subset \mathbb{C}_{p} \) be a complete intermediate field.

By Definition 4.1.1, elements \( f^\pm \in \Pi^\pm \otimes_F K \) can be realized as \( K \)-linear combinations of \( \mathcal{C} \)-valued modular forms \( \phi \). Using the notation in (2.26), we have convergent modular forms
\[
(\xi^\pm_{\omega})(\phi)|_{\mathcal{E}} \in \mathcal{M}_\phi^{\omega}(\mathbb{C}_{p}).
\]
Then \( (\xi^\pm_{\omega})(\phi)|_{\mathcal{E}} \) are stable (in the sense of Definition 2.3.10) if and only if \( f^\pm \) are stable (in the sense of Definition 4.1.1), respectively. By Proposition 2.3.5 (3), \( (\xi^\pm_{\omega})(\phi)|_{\mathcal{E}} \) are \( n \)-admissible (in the sense of Definition 2.3.13) if and only if \( f^\pm \) are \( n \)-admissible (in the sense of Definition 4.1.1), respectively.

Notation 4.3.1. For stable vectors \( f^\pm \in (\Pi)^0_K \), define the elements
\[
\mathcal{P}_{\text{univ}}^\pm(f^\pm) \in \text{Lie } A^\pm \otimes_{FM} \mathcal{D}(\omega^A, K)
\]
by the formulae
\[
\langle \omega_{\pm}, \mathcal{P}_{\text{univ}}^\pm(f^\pm) \rangle = \mathcal{P}_{\omega^A}^\pm((\xi^\pm_{\omega_{\text{ord}}}))(\phi).
\]

In this section, we study the relation between
\[
\iota \mathcal{P}_{\text{univ}}^+ f^+(\chi) \cdot \iota \mathcal{P}_{\text{univ}}^- (f^-)(\chi) \in \text{Lie } A^+ \otimes_{FM} \text{Lie } A^- \otimes_{FM, \chi} \mathbb{C}
\]
for a given isomorphism \( \iota : \mathbb{C} \rightarrow \mathbb{C} \), with classical torus periods, for \( f^\pm \) as above and a character \( \chi \in \Xi(\omega^A, K) \) of weight \( k \geq 1 \) and depth \( n \) (Definition 4.1.3). For this purpose, we choose an \( \iota \)-nearby data for \( \mathbb{B} \) (Definition 2.4.10). In particular, we have
\[
Y^\pm_{\iota}(\mathbb{C}) = E^x \setminus \{\pm i\} \times \mathbb{A}_{E}^\times K_{\mathbb{A}^\times} \subset X_{\iota}(\mathbb{C}).
\]
Choose elements \( t^\pm \in \mathbb{A}_{E}^\times \) such that \( \iota P^\pm \) are represented by \([\pm i, t^\pm]\), respectively. Define \( \zeta^\pm_{\iota} \in \mathbb{C}^\times \) such that
\[
(4.4) \quad d\zeta(\pm i, t^\pm) = \zeta^\pm_{\iota} \cdot s_{\omega, \psi | P^\pm},
\]
where \( \omega, \psi \) are defined in (3.2). We also introduce matrices
\[
(4.4') \quad j^\pm_{\iota} = \left( \begin{array}{rr} 1 & \pm 1 \\ \end{array} \right) \text{ in } \text{Mat}_2(\mathbb{R}) = B(\iota) \otimes_{F_{\mathbb{A}}} \mathbb{R}.
\]
Notation 4.3.2. For a cusp form $\Phi \in \mathcal{S}_{\text{cusp}}(B(1)^{\infty})$ with central character $\iota \circ \omega_{A}^{\pm}$, we respectively define

$$\mathcal{P}_{\iota}(\Phi, \chi^{(\pm)}) := \int_{E \times \mathbb{A}^{	imes} \setminus \mathbb{A}^{	imes}_{\mathbb{K}}} \Phi(t) \chi^{(\pm)}(t) \, dt$$

to be the complex torus periods appearing in the complex Waldspurger formula.

Lemma 4.3.3. Let the notation be as above. We have

$$\iota \langle \omega_{\pm}, \mathcal{P}_{\text{univ}}^{\pm}(f_{\pm})(\chi) \rangle \cdot \iota \langle \omega_{-}, \mathcal{P}_{\text{univ}}^{-}(f_{-})(\chi) \rangle = \left( \zeta_{\iota}^{\pm} \right)^{k} \cdot \chi^{(\pm)}(t_{\pm}^{-1} t_{-}) \times \mathcal{P}_{\iota}(\Delta_{+,\iota}^{k-1}(R(j_{+})\phi_{i}(f_{+}^{*}\omega_{+})), \chi^{(\pm)}(\iota)) \mathcal{P}_{\iota}(\Delta_{-,\iota}^{k-1}(R(j_{-})\phi_{i}(f_{-}^{*}\omega_{-})), \chi^{(\pm)}(\iota^{-1})),$$

where $\phi_{i}$ is defined in Lemma 2.4.15; $\chi^{(\pm)}$ is the $\iota$-avatar of $\chi$ as in Definition 3.2.2 (3).

Proof. Take $V^{\mp} \subset \mathcal{V}$ under which $f_{\pm}$ and $\chi$ are invariant. By Theorem 2.3.17 and Definition 4.2.5, we have

$$\iota \langle \omega_{\pm}, \mathcal{P}_{\text{univ}}^{\pm}(f_{\pm})(\chi) \rangle = \frac{2}{|E \times \mathbb{A}^{	imes} / V^{\mp}O_{E,m}^{\times}|} \sum_{E \times \mathbb{A}^{	imes} / V^{\mp}O_{E,m}^{\times}} \Theta^{\pm}_{\iota}(f_{\pm}^{\mp}\omega_{\pm})(\gamma_{\pm} T_{\iota} P^{\pm}) \cdot \omega_{\psi_{\pm}}^{-k}(\gamma_{\pm} T_{\iota} P^{\pm}) \cdot \chi^{\pm}(t)$$

for some sufficiently large $m \geq n$. By (4.4) and Lemma 2.4.9, we have

$$\iota \langle \omega_{\pm}, \mathcal{P}_{\text{univ}}^{\pm}(f_{\pm})(\chi) \rangle = \frac{2}{|E \times \mathbb{A}^{	imes} / V^{\mp}O_{E,m}^{\times}|} \sum_{E \times \mathbb{A}^{	imes} / V^{\mp}O_{E,m}^{\times}} \Theta^{\pm}_{\iota}(f_{\pm}^{\mp}\omega_{\pm})(T_{\iota} P^{\pm}) \cdot \omega_{\psi_{\pm}}^{-k}(T_{\iota} P^{\pm}) \cdot \chi^{\pm}(t)$$

which by Lemma 2.4.17 equals

$$\left( \zeta_{\iota}^{\pm} \right)^{k} \chi^{(\pm)}(t_{\pm}^{-1}) \int_{E \times \mathbb{A}^{	imes} \setminus \mathbb{A}^{	imes}_{\mathbb{K}}} R(j_{+}^{\pm}) \phi_{i}(\Theta^{\pm}_{\iota}(f_{\pm}^{\mp}\omega_{\pm}))(t) \cdot \chi^{(\pm)}(t) \, dt$$

This completes the proof. \qed

Proposition 4.3.4. Given $n$-admissible stable vectors $f_{\pm} \in (\Pi^{\pm})_{K}^{\circ}$, and a character $\chi \in \Xi(\omega_{A}, K)^{n}$ of weight $k \geq 1$ and depth $n$, we have

$$\iota \langle \mathcal{P}_{\text{univ}}^{\pm}(f_{+})(\chi) \rangle \cdot \iota \langle \mathcal{P}_{\text{univ}}^{-}(f_{-})(\chi) \rangle = \iota \mathcal{P}(f_{+}, f_{-})(\chi)$$

$$\times L(1/2, \rho_{A}^{(\chi)} \cdot \chi^{(\iota)}(\iota)) \cdot \frac{2^{\eta_{1}} d_{E}^{1/2} \zeta_{E}(2) P_{\iota}(A, \chi)}{L(1, \eta)^{2} L(1, \rho_{A}^{(\chi)}(\iota), \text{Ad})} \cdot \iota \left( \frac{e(1/2, \psi, \rho_{A,p} \otimes \tilde{\chi}_{p}^{\psi})}{L(1/2, \rho_{A,p} \otimes \tilde{\chi}_{p}^{\psi})} \right),$$

as an equality in $(\text{Lie} \, A^{\pm} \otimes_{F^{M}} \text{Lie} \, A^{-}) \otimes_{F^{M}, \iota} \mathbb{C}$. 


Proof. It suffices to show the equality after pairing with $\omega_+ \otimes \omega_-$ for an arbitrary pair of differential forms $\omega_\pm \in H^0(A^\pm, \Omega^1_{A^\pm})$.

By the complex Waldspurger formula [Wal85] (or see [YZZ13, Theorem 1.4.2]) and Proposition 4.1.6, we have

$$
\mathcal{P}_C(\Delta^{k-1}_{+,+}(R(j^+_i)\phi_i(f^*_+\omega_+)), \chi^{(i)+1}) \mathcal{P}_C(\Delta^{k-1}_{-,+}(R(j^-_i)\phi_i(f^*_+\omega_-)), \chi^{(i)-1})
$$

$$
= C_i \frac{\zeta_F(2)L(1/2, \rho_A^{(i)} \chi^{(i)})}{L(1, \rho_A^{(i)} \chi^{(i)} \Ad)L(1, \eta)} \frac{2}{2^{-g}d_1^{1/2}1/2(1, \eta)} \cdot \frac{t \left( \epsilon(1/2, \psi, \rho_{A,p} \otimes \chi_{\mathbb{Q}}) \right)}{\left( L(1/2, \rho_{A,p} \otimes \chi_{\mathbb{Q}})^2 \right)} \cdot \mathcal{P}(f_+, f_-)(\chi),
$$

where $C_i$ is the complex constant such that

$$
(\Delta^{k-1}_{+,+}(R(j^+_i)\phi_i(f^*_+\omega_+)), \Delta^{k-1}_{-,+}(R(j^-_i)\phi_i(f^*_+\omega_-)))_{\text{Pet}} = C_i \cdot t(f_+ f_-)A
$$

holds for all $f_+$ and $f_-$. Here $(\ , \ )_{\text{Pet}}$ is the bilinear Petersson inner product pairing. By Lemma 4.3.3, the proposition is reduced to the following formula

$$
(5.5) \quad \langle \omega_+ \otimes \omega_-, P_i(A, \chi) \rangle = C_i \cdot (\zeta_+^\xi \zeta_-^\eta)^k \cdot \chi^{(i)}(t_+^1 t_-^1).
$$

By Lemma 3.2.8, it suffices to show that

$$
\frac{\nu \varphi_+ \circ c^\ast \nu \varphi_- \circ \mu^k}{\nu(\varphi_+ \varphi_-)_\chi} \int_{X_i(C)} \frac{\Theta^{k-1}_i f^*_+ \omega_+ \otimes c^\ast \Theta^{k-1}_i f^*_- \omega_-}{\mu^k} \ dx
$$

$$
\quad = (\zeta_+^\xi \zeta_-^\eta)^k \cdot \chi^{(i)}(t_+^1 t_-^1) \cdot (\Delta^{k-1}_{+,+}(R(j^+_i)\phi_i(f^*_+\omega_+)), \Delta^{k-1}_{-,+}(R(j^-_i)\phi_i(f^*_+\omega_-)))_{\text{Pet}}
$$

for some choice of $\varphi_\pm \in \sigma^\xi_+ \mathfrak{C}_p$. We take elements $\varphi_\pm$ such that $\nu \varphi_\pm(P^\pm) = dz([\pm i, t_\pm]^k)$, respectively. Then $\nu(\varphi_+ \varphi_-)_\chi = (\zeta_+^\xi \zeta_-^\eta)^{-k}$ by (4.4). Now we take $\mu$ to be the standard invariant hyperbolic metric on $X_i = B(i)^\times \times \mathbb{H}^\infty / F^\times$. Then $\nu \varphi_+ \circ c^\ast \nu \varphi_- \circ \mu^k$ is the constant $\chi^{(i)}(t_+^1 t_-^1)$, and

$$
\int_{X_i(C)} \frac{\Theta^{k-1}_i f^*_+ \omega_+ \otimes c^\ast \Theta^{k-1}_i f^*_- \omega_-}{\mu^k} \ dx = (\Delta^{k-1}_{+,+}(R(j^+_i)\phi_i(f^*_+\omega_+)), \Delta^{k-1}_{-,+}(R(j^-_i)\phi_i(f^*_+\omega_-)))_{\text{Pet}}.
$$

Thus (5.5) holds and the proposition follows.

The proposition has the following corollary.

**Corollary 4.3.5.** For $\chi \in \Xi(\omega_A, K)^n$ with $k \geq 1$, the ratio

$$
\frac{\mathcal{P}_{\text{univ}}^+(f_+)(\chi) \mathcal{P}_{\text{univ}}^-(f_-)(\chi)}{\mathcal{D}(f_+, f_-)(\chi)} \in (\text{Lie } A^+ \otimes F^M \text{ Lie } A^-) \otimes F^M K,
$$

if the denominator is nonzero, is independent of the choice of $n$-admissible stable vectors $f_\pm \in (\Pi^\pm)^\otimes_K$. Moreover, for $\iota: \mathbb{C}_p \rightarrow \mathbb{C}$, we have

$$
\iota \left( \frac{\mathcal{P}_{\text{univ}}^+(f_+)(\chi) \mathcal{P}_{\text{univ}}^-(f_-)(\chi)}{\mathcal{D}(f_+, f_-)(\chi)} \right)
$$

$$
\quad = L(1/2, \rho_A^{(i)} \chi^{(i)}). \frac{2^{g-1/2}d_1^{1/2}(2)^{\zeta_F(2)}\mathcal{P}_i(A, \chi)}{L(1, \rho_A^{(i)} \chi^{(i)} \Ad)} \cdot \frac{t \left( \epsilon(1/2, \psi, \rho_{A,p} \otimes \chi_{\mathbb{Q}}) \right)}{\left( L(1/2, \rho_{A,p} \otimes \chi_{\mathbb{Q}})^2 \right)}.
$$

**Proposition 4.3.6.** For $n$-admissible stable vectors $f_\pm \in (\Pi^\pm)^\otimes_K$ and a character $\chi \in \Xi(\omega_A, K)^n$ of weight 0 and depth $n$, we have

$$
\mathcal{P}_{\text{univ}}^\pm(f_\pm)(\chi) = \log A^\pm P_X^\pm(f_\pm).
$$
Proof. We may choose a tame level $U^p \in \mathfrak{U}$ that fixes both $f_+$ and $f_-$, and such that $\chi$ is fixed by $U^p \cap A_E^{\infty}$. We may realize $f_\pm$ as $K$-linear combinations of morphisms from $X_{U^p U_{p,\pm}}$ to $A^\pm$, respectively, for some sufficiently large integer $m \geq n$. By linearity, we may assume that $f_\pm$ are just morphisms from $X_{U^p U_{p,\pm}}$ to $A^\pm$, respectively.

For $\omega_\pm \in H^0(A^\pm, \Omega^1_A)$, we have by Theorem 2.3.17 (3,4) that
\[
d\mathbb{M}((f_\pm^* \omega_\pm)_{ord})(\chi|_{O_{E^p}^{\infty}}) = \Theta_{ord} \mathbb{M}((f_\pm^* \omega_\pm)_{ord})(\chi|_{O_{E^p}^{\infty}}) = (f_\pm^* \omega_\pm)_{ord}.
\]
On the other hand, by Proposition A.0.8, we know that $(f_\pm^* \log \omega_\pm)_{ord}$ are Coleman integrals of $(f_\pm^* \omega_\pm)_{ord}$ on (the generic fiber of) $\mathfrak{X}(m, U^p)$, respectively. Therefore, we have
\[
(4.6) \quad \mathbb{M}((f_\pm^\ast \omega_\pm)_{ord})(\chi|_{O_{E^p}^{\infty}}) = (f_\pm^* \log \omega_\pm)_{ord}
\]
on $\mathfrak{X}(m, U^p)$ since both of them are Coleman integrals of $f_\pm^* \omega_\pm$ on $\mathfrak{X}(m, U^p)$ that belong to $\mathcal{M}_\emptyset^0(m, K)^\circ$, respectively. By Definition 3.3.1, we have
\[
\log \omega_\pm P_\chi^\pm(f_\pm) = \int_{E^x \setminus A_E^{\infty}} \log \omega_\pm(f_\pm(T_t P^\pm) \cdot \chi(t)^{\pm1} dt
\]
\[
= \int_{E^x \setminus A_E^{\infty}} f_\pm^* \log \omega_\pm(T_t P^\pm) \cdot \chi(t)^{\pm1} dt
\]
\[
= \int_{E^x \setminus A_E^{\infty}} (f_\pm^* \log \omega_\pm)_{ord}(Y_t T_t P^\pm) \cdot \chi(t)^{\pm1} dt,
\]
which by (4.6) is equal to
\[
\int_{E^x \setminus A_E^{\infty}} \mathbb{M}((f_\pm^* \omega_\pm)_{ord})(\chi|_{O_{E^p}^{\infty}})(Y_t T_t P^\pm) \cdot \chi(t)^{\pm1} dt
\]
\[
= \int_{E^x \setminus A_E^{\infty}} \mathfrak{w}(\mathbb{M}((f_\pm^* \omega_\pm)_{ord})(\chi)(Y_t T_t P^\pm) \cdot \chi(t)^{\pm1} dt,
\]
respectively. Then the proposition follows from Remark 4.2.6.

\[\square\]

4.4. Proof of main theorems. Let $K$ be a complete field extension of $MF_\emptyset$.

For $V^p \in \mathfrak{U}$, denote by $\mathcal{C}(\omega, K, V^p)$ the (closed) subspace of $\mathcal{C}(\omega, K)$ (Definition 3.2.5) of functions that are invariant under the right translation of $V^p$. It is also a closed subspace of $C(E^x \setminus A_E^{\infty}/V^p, K)$. The strong dual of $\mathcal{C}(\omega, K, V^p)$ is canonically isomorphic to $\mathcal{D}(\omega, K, V^p)$ (Notation 4.2.2).

We consider totally definite (not necessarily incoherent) quaternion algebras $\mathbb{B}$ over $\mathbb{A}$ such that for a finite place $v$ of $F$, $\epsilon(\mathbb{B}_v) = 1$ if $v$ is split in $E$ or the Galois representation $\rho_{A,v}$ corresponds to a principal series.

For such (a representative in the isomorphism class of) $\mathbb{B}$, we choose an $E$-embedding as (1.9), which is possible. We define representations
\[
\Pi(\mathbb{B})^{tame}_{A^\pm} = \bigotimes_M \Pi_{v,A^\pm},
\]
where the restricted tensor products (over $M$) are taken over all finite places $v \neq p$ of $F$; and $\Pi_{v,A^\pm}$ are $M$-representations of $\mathbb{B}^\infty_v$ determined by $\rho_{A^\pm,v}$, respectively. In particular, if $\mathbb{B}$ is incoherent (that is, $\mathbb{B} \in \mathcal{B}(A)$ in Notation 3.2.1), then $\Pi(\mathbb{B})^{tame}_{A^\pm}$ are isomorphic to the away-from-$p$ components of $\Pi(\mathbb{B})_{A^\pm}$ (Notation 3.1.2), respectively.

Notation 4.4.1. Let $\mathcal{D}_+(\omega_A, K, V^p)$ be the closed ideal of $\mathcal{D}(\omega_A, K, V^p)$ generated by
\[
\{ \mathcal{D}(f_+, f_-) \mid f_{\pm} \in (\Pi(\mathbb{B})^{tame}_{A^\pm})^{V^p} \otimes_M K, \epsilon(\mathbb{B}) + 1 \},
\]
and $\mathcal{I}_-(\omega_A, K, V^p)$ the closed ideal of $\mathcal{D}(\omega_A, K, V^p)$ generated by

$$\{ \mathcal{D}(f_+, f_-) \mid f_\pm \in (\Pi(\mathbb{B})_{A^{\pm}}) V^p \otimes_M K, \epsilon(\mathbb{B}) = -1 \},$$

where $\mathcal{D}(f_+, f_-)$ is similarly defined (as the product) in Lemma 4.1.8.

Let $\mathcal{C}_+(\omega_A, K, V^p)$ (resp. $\mathcal{C}_-(\omega_A, K, V^p)$) be the subspace of $\mathcal{C}(\omega_A, K, V^p)$ consisting of functions lying in the kernel of every element in $\mathcal{I}_-(\omega_A, K, V^p)$ (resp. $\mathcal{I}_+(\omega_A, K, V^p)$).

Put $\Xi(A, K, V^p) = \Xi(A, K) \cap \mathcal{C}(\omega_A, K, V^p)$ and $\Xi(\omega_A, K, V^p)$, where $\Xi(A, K)$ and $\Xi(\omega_A, K)$ are introduced in Definition 3.2.5.

**Remark 4.4.2.** The ideals $\mathcal{I}_+(\omega_A, K, V^p)$ are topologically finitely generated. The subspace $\mathcal{C}_+(\omega_A, K, V^p)$ are closed in $\mathcal{C}(\omega_A, K, V^p)$.

The following lemma concerns some algebraic properties of objects introduced above.

**Lemma 4.4.3.** Suppose that $V^p \in \mathfrak{Q}$ is sufficiently small. We have

1. $\mathcal{I}_+(\omega_A, K, V^p) \cap \mathcal{I}_-(\omega_A, K, V^p) = 0$;
2. $\mathcal{I}_+(\omega_A, K, V^p) + \mathcal{I}_-(\omega_A, K, V^p) = \mathcal{D}(\omega_A, K, V^p)$;
3. $\mathcal{C}(\omega_A, K, V^p) = \mathcal{C}_+(\omega_A, K, V^p) \oplus \mathcal{C}_-(\omega_A, K, V^p)$;
4. the subset $\Xi(A, K, V^p)$ is contained in and generates a dense subspace of $\mathcal{C}_-(\omega_A, K, V^p)$;
5. $\mathcal{I}_+(\omega_A, K, V^p)$ is the closed ideal generated by elements that vanish on $\Xi(A, K, V^p)$.

**Proof.** We first realize that $\Xi(\omega_A, K, V^p)$ generates a dense subspace of $\mathcal{C}(\omega_A, K, V^p)$. Thus (1) follows from the dichotomy theorem of Saito–Tunnell [Tun83, Sai93]. For (2), assume the converse and suppose that $\mathcal{I}_+(\omega_A, K, V^p) + \mathcal{I}_-(\omega_A, K, V^p)$ is contained in a (closed) maximal ideal with the residue field $K'$. Then all local period distributions $\mathcal{D}(f_+, f_-)$ will vanish on the character

$$E^\times \backslash A^\times_E / V^p \xrightarrow{\delta} \mathcal{D}(\omega_A, K, V^p) \rightarrow K',$

which contradicts the theorem of Saito–Tunnell. Part (3) is a direct consequence of (1) and (2). It is clear that $\Xi(A, K, V^p)$ is contained in $\mathcal{C}_-(\omega_A, K, V^p)$ and by Saito–Tunnell, $\Xi(\omega_A, K, V^p) \setminus \Xi(A, K, V^p) \subset \mathcal{C}_+(\omega_A, K, V^p)$, which together imply (4). Finally, (5) follows from (4). \qed

**Remark 4.4.4.** If we put $\mathcal{D}(A, K, V^p) = \mathcal{D}(\omega_A, K, V^p) / \mathcal{I}(\omega_A, K, V^p)$, then we obtain a canonical isomorphism

$$\mathcal{D}(A, K) \xrightarrow{\sim} \lim_{V^p \in \mathfrak{Q}} \mathcal{D}(A, K, V^p).$$

Moreover, we have $\mathcal{D}(A, K) \simeq \prod_{B \in \mathfrak{B}(A)} \mathcal{D}(A, B, K)$ (Definition 3.2.5).

We have $\mathcal{D}(A, K, V^p) \otimes_K K' \simeq \mathcal{D}(A, K', V^p)$ and $\mathcal{D}(A, K) \otimes_K K' \simeq \mathcal{D}(A, K')$ for a complete field extension $K'/K$.

**Remark 4.4.5.** In fact, for sufficiently small $V^p \in \mathfrak{Q}$, the morphism $w$ (4.3) is injective with the quotient which is a finite étale $K$-algebra. We also have $D(O^\text{anti}_p, K) \cap \mathcal{I}_+(\omega_A, K, V^p) = \{0\}$. Thus if $K$ is discretely valued, $\mathcal{D}(A, K, V^p)$ is a (commutative) nuclear Fréchet–Stein $K$-algebras (defined for example in [Eme, Definition 1.2.10]). Moreover, it is not hard to see that the transition homomorphism $\mathcal{D}(A, K, V^p) \rightarrow \mathcal{D}(A, K, V^p)$ is finite étale for $V^p \subset V^p$. The rigid analytic variety $\mathcal{C}_-(V^p)$ associated to $\mathcal{D}(A, MF_p, V^p)$ is a smooth rigid curve over $MF_p$, which may be regarded as an eigencurve for the group $U(1)_{E/F}$ of tame level $V^p$, twisted by (the cyclotomic character) $\omega_A$ and cut off by the condition that $\epsilon(1/2, \rho_A) = -1$. 
The ind-rigid analytic variety $\mathcal{E}_-$ mentioned in §1.7 is actually $\lim_{V^p \in \mathfrak{B}} \mathcal{E}_-(V^p)$.

**Proof of Theorem 3.2.10.** For the existence, note that the union $\bigcup_{k \geq 1} \Xi(\omega_A, \mathbb{C}_p)_k^0$ already spans a dense subspace of $\mathcal{C}(\omega_A, \mathbb{C}_p)$ by Lemma 2.1.11. By Corollary 4.3.5, Lemma 4.4.3 and (the nonvanishing part of) Lemma 4.1.9, the collection of ratios

$$\frac{\mathcal{P}^+_{\text{univ}}(f_+) \mathcal{P}^-_{\text{univ}}(f_-)}{\mathcal{D}(f_+, f_-)}$$

for $f_\pm$ running over $(\Pi(\mathbb{B})_{A^\pm})^0_K$ with $\epsilon(\mathbb{B}) = -1$ defines an element

$$\mathcal{L}(A) \in (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{D}(A, \mathbb{C}_p).$$

It actually belongs to $(\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{D}(A, MF_p^{\text{ht}})$ by the lemma below. We need to show that the element

$$\mathcal{L}(A) \in (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{D}(A, MF_p^{\text{ht}})$$

introduced in Definition 4.4.7 satisfies (3.4). However, this follows from Corollary 4.3.5.

The uniqueness follows from the fact that $\bigcup_{k \geq 1} \Xi(\omega_A, \mathbb{C}_p)_k$ is dense in $\mathcal{C}(\omega_A, \mathbb{C}_p)$, which we already used in the construction of $\mathcal{L}(A)$. \hfill $\square$

**Lemma 4.4.6.** The element $\mathcal{L}(A)$ belongs to $(\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{D}(A, MF_p^{\text{ht}})$.

**Proof.** Note that in the definition of $\mathcal{L}(A)$, we only need to consider $f_\pm \in (\Pi(\mathbb{B})_{A^\pm})^0_{FM}$ such that both $f_+$ and $\Pi(\mathbb{B})_{A^-(j)} f_-$ are invariant under $O^\times_{\mathbb{B}}$. Then the lemma follows if we can show that for every $\chi \in \bigcup_{k \geq 1} \Xi(\omega_A, \mathbb{C}_p)_k$ and $\sigma \in \text{Gal}(\mathbb{C}_p/MF_p^{\text{ht}})$, we have

$$\langle \omega_+ \cdot \mathcal{P}^+_{\text{univ}}(f_+) \chi \rangle = \langle \omega_+ \cdot \mathcal{P}^+_{\text{univ}}(f_+) \sigma \circ \chi \rangle.$$  

Without lost of generality, we consider the one for $f^+$. As in the proof of Lemma 4.3.3, we have the equality

$$\langle \omega_+ \cdot \mathcal{P}^+_{\text{univ}}(f_+) \chi \rangle = C \sum_{E^\times \backslash E^\times / V^p \otimes \mathcal{O}^\times_{\mathbb{B}_p}} \Theta^{k-1}_{\text{ord}}(f^+_\omega_+) \gamma(\mathcal{T}_t \mathcal{T}_f P^+) \cdot \omega_k^-(\gamma \mathcal{T}_t \mathcal{T}_f P^+) \cdot \chi(t),$$

where $C$ is a positive rational constant. However, the product $\Theta^{k-1}_{\text{ord}}(f^+_\omega_+) \gamma \omega_k^-$ is naturally an element in $\mathcal{M}^0(\infty, MF_p^{\text{ht}})$ (Definition 2.3.8), and $\chi$ can be viewed as an element in $\mathcal{N}^+(\infty, \mathbb{C}_p)$ (§4.2). Thus (4.7) holds and the lemma follows. \hfill $\square$

**Definition 4.4.7** ($p$-adic $L$-function). We call the element

$$\mathcal{L}(A) \in (\text{Lie } A^+ \otimes_{FM} \text{Lie } A^-) \otimes_{FM} \mathcal{D}(A, MF_p^{\text{ht}})$$

in the proof of Theorem 3.2.10 the **anti-cyclotomic $p$-adic $L$-function** attached to $A$.

**Proof of Theorem 3.3.2.** It follows from Proposition 4.3.6 and Proposition 4.1.6. \hfill $\square$

**Appendix A. Compatibility of logarithm and Coleman integral**

In this appendix, we generalize a result of Coleman in [Col85] about the compatibility of $p$-adic logarithm and Coleman integral. Such result will only be used in the proof of Proposition 4.3.6.

Let $F$ be a local field contained in $\mathbb{C}_p$ with the ring of integers $O_F$ and the residue field $k$. Let $X$ be a quasi-projective scheme over $F$ and $U \subset X^{\text{rig}}$ an affinoid domain with good reduction. We say a closed rigid analytic 1-form $\omega$ on $U$ is **Frobenius proper** if there exits a
Frobenius endomorphism $\phi$ of $U$ and a polynomial $P(X)$ over $\mathbb{C}_p$ such that $P(\phi^*)\omega$ is the differential of a rigid analytic function on $U$ and such that no root of $P(X)$ is a root of unity. Therefore, by [Col85, Theorem 2.1], there exits a locally analytic function $f_\omega$ on $U(\mathbb{C}_p)$, unique up to an additive constant on each geometric connected component, such that

- $df_\omega = \omega$;
- $P(\phi^*)f_\omega$ is rigid analytic.

Such $f_\omega$ is known as a Coleman integral of $\omega$ on $U$, which is independent of the choice of $P$ [Col85, Corollary 2.1b].

**Proposition A.0.8.** Let $X$ and $U$ be as above. Let $A$ be an abelian variety over $F$ which has either totally degenerate reduction or potentially good reduction. Then for a morphism $f: X \to A$ and a differential form $\omega \in \Omega^1(\mathcal{A}/F)$, the form $f^*\omega|_U$ is Frobenius proper which admits $f^*\log_\omega|_U$ as a Coleman integral, where $\log_\omega: \mathcal{A}(\mathbb{C}_p) \to \mathbb{C}_p$ is the $p$-adic logarithm associated to $\omega$.

**Proof.** We may assume that $X$ is projective. Replacing $F$ by a finite extension, we may assume that $A$ has good reduction, or split totally degenerate reduction (that is, the connected neutral component $\mathcal{A}_s^0$ of the special fiber $\mathcal{A}_s$ of the Néron model $\mathcal{A}$ of $A$ is isomorphic to $G^d_{m,k}$, where $d$ is the dimension of $A$). The first case follows from [Col85, Theorem 2.8, Proposition 2.2].

Now we consider the second case. Denote by $\mathcal{A}_s^\circ$ the analytic domain of $\mathcal{A}^{\rig}$ of points whose reduction is in $\mathcal{A}_s^0$. By the well-known result of uniformization ([Mum72, §6]), we have $\mathcal{A}^{\rig} \simeq (G^{\rig}_{m,F})^d/\Lambda$ for a lattice $\Lambda \subset G^d_{m,F}(F)$. Moreover, $\mathcal{A}_s^\circ$ is isomorphic to $\text{Sp} F\langle T_1, \ldots, T_d, T_1^{-1}, \ldots, T_d^{-1} \rangle$, the rigid analytic multi-torus of multi-radius 1.

Choose an admissible covering $\mathcal{U}$ of $X^{\rig}$ containing $U$, which determines a formal model $X_{\mathcal{U}}$ of $X$ over $O_F$. Since $X$ is projective, we may assume that $X_{\mathcal{U}}$ is algebraic. Let $Z$ be the non-smooth locus of $X_{\mathcal{U}}$ over $O_F$. The set of closed points of $X$ whose reduction is not in $Z$ forms an analytic domain $W$ of $X^{\rig}$. Since $U$ has good reduction, we have $U \subset W$. By the Néron mapping property, the morphism $f$ extends uniquely to a morphism $X_{\mathcal{U}} - Z \to \mathcal{A}$, which induces a morphism $f': U \to \mathcal{A}^{\rig}$. Without loss of generality, we assume that $f'(U)$ is contained in $\mathcal{A}_s^0$. By [Col85, Proposition 2.2], we only need to show that $\omega|_{\mathcal{A}_s^0}$ is Frobenius proper and $\log_\omega|_{\mathcal{A}_s^0}$ is a Coleman integral of it.

In fact, we have

$$\{ \omega|_{\mathcal{A}_s^0} : \omega \in \Omega^1(\mathcal{A}/F) \} = \text{Span}_F \left\{ \frac{dT_1}{T_1}, \ldots, \frac{dT_d}{T_d} \right\}.$$  

By linearity, we may assume $\omega^o := \omega|_{\mathcal{A}_s^0} = \frac{dT_1}{T_1}$. We choose the Frobenius endomorphism on $\mathcal{A}_s^0$ to be given by $\phi((T_1, \ldots, T_d)) = (T_1^q, \ldots, T_d^q)$ where $q = |k|$. We have that $P(\phi^*)\omega^o = 0$ for $P(X) = X - q$. On the other hand, the $p$-adic logarithm $\log \text{Sp} F\langle T_1, T_1^{-1} \rangle$ is also killed by $P(\phi^*)$. Therefore, the function $(\log, 1, \ldots, 1)$ on $\text{Sp} F\langle T_1, T_1^{-1} \rangle \times \cdots \times \text{Sp} F\langle T_d, T_d^{-1} \rangle \simeq \mathcal{A}_s^0$ is a Coleman integral of $\omega^o$, which coincides with the restriction of $\log_\omega$. □

**Appendix B. Serre–Tate local moduli for $0$-divisible groups (d’après N. Katz)**

In this appendix, we describe the Kodaira–Spencer isomorphism for ordinary $0$-divisible groups in terms of their Serre–Tate coordinates, generalizing a classical result of Katz [Kat81]...
which is for ordinary $p$-divisible groups. Only Theorem B.1.1 and Theorem B.2.3 will be used in the main part of the article. Some notation in this appendix may be different from those in §1.8.

B.1. $\mathcal{O}$-divisible groups and Serre–Tate coordinates. Let $F$ be a finite field extension of $\mathbb{Q}_p$, where $p$ is a rational prime. Denote by $\hat{F}$ the completion of a maximal unramified extension of $F$. The ring of integers of $F$ (resp. $\hat{F}$) is denoted by $\mathcal{O}$ (resp. $\hat{\mathcal{O}}$). Let $k$ be the residue field of $\hat{\mathcal{O}}$, which is algebraic closure of $\mathbb{F}_p$. For a $p$-divisible group $G$ over Spec $R$, we denote by $\Omega(G/R)$ the $R$-module of invariant differentials of $G$ over $R$, which is the dual $R$-module of the tangent space $\text{Lie}(G/R)$ at the identity.

Let $S$ be an $\hat{\mathcal{O}}$-scheme. Recall that an $\mathcal{O}$-divisible group over $S$ is a $p$-divisible group $G$ over $S$ with an action by $\mathcal{O}$ such that the induced action of $\mathcal{O}$ on the sheaf $\text{Lie}(G/S)$ coincides with the natural action as an $\mathcal{O}_S$-module (hence an $\mathcal{O}$-module). Denote by $\text{BT}_S^\mathcal{O}$ the category of $\mathcal{O}$-divisible groups over $S$, which is an abelian category. We omit the superscript $\mathcal{O}$ if it is $\mathbb{Z}_p$. The height $h$ of $G$, as a $p$-divisible group, must be divisible by $[F : \mathbb{Q}_p]$. We define the $\mathcal{O}$-height of $G$ to be $[F : \mathbb{Q}_p]^{-1}h$. An $\mathcal{O}$-divisible group $G$ is connected (resp. étale) if its underlying $p$-divisible group is. We denote by $\mathcal{L}T$ the Lubin–Tate $\mathcal{O}$-formal group over Spec $\mathcal{O}$, which is unique up to isomorphism. We use the same notation for its base change to $S$.

For an $\mathcal{O}$-divisible group $G$ over $S$, there exists an $\mathcal{O}$-formal group $G^0$ over $S$, unique up to isomorphism, such that its associated $p$-divisible group $G^0[p^\infty]$ is the maximal connected subgroup of $G$. In particular, $G^0[p^\infty]$ is an $\mathcal{O}$-divisible group. We define the $\mathcal{O}$-Cartier dual of $G$ to be

$$G^D := \lim_{\overset{\longrightarrow}{n}} \text{Hom}_\mathcal{O}(G[p^n], \mathcal{L}T[p^n])$$

as in [Fal02]. An $\mathcal{O}$-divisible group $G$ is ordinary if $(G^0[p^\infty])^D$ is étale. Denote by $T_pG = \lim_{\overset{\longleftarrow}{n}} G[p^n]$ the Tate module functor. Denote by $\text{Nilp}_\mathcal{O}$ the category of $\hat{\mathcal{O}}$-schemes on which $p$ is locally nilpotent.

**Theorem B.1.1** (Serre–Tate coordinates). Let $G$ be an ordinary $\mathcal{O}$-divisible group over $k$. Consider the moduli functor $\mathfrak{M}_G$ on $\text{Nilp}_\mathcal{O}$ such that for every $\mathcal{O}$-scheme $S$ on which $p$ is locally nilpotent, $\mathfrak{M}_G(S)$ is the set of isomorphism classes of pairs $(G, \varphi)$ where $G$ is an object in $\text{BT}_S^\mathcal{O}$ and $\varphi: G \times_S (S \otimes_\mathcal{O} k) \to G \times_{\text{Spec}k} (S \otimes_\mathcal{O} k)$ is an isomorphism. Then $\mathfrak{M}_G$ is canonically pro-represented by the $\mathcal{O}$-formal scheme $\text{Hom}_\mathcal{O}(T_pG(k) \otimes_\mathcal{O} T_pG^D(k), \mathcal{L}T)$.

In particular, for every Artinian local $\mathcal{O}$-algebra $R$ with the maximal ideal $m_R$ and $G/R$ a deformation of $G$, we have a pairing

$$q(G/R; \ , ) : T_pG(k) \otimes_\mathcal{O} T_pG^D(k) \to \mathcal{L}T(R) = 1 + m_R.$$

It satisfies:

1. For every $\alpha \in T_pG(k)$ and $\alpha_D \in T_pG^D(k)$, we have

$$q(G/R; \alpha, \alpha_D) = q(G^D/R; \alpha_D, \alpha).$$

2. Suppose that we have another ordinary $\mathcal{O}$-divisible groups $H$ over $k$, and its deformation $H$ over $R$. Let $f: G \to H$ be a homomorphism and $f^D$ be its dual. Then $f$ lifts to a (unique) homomorphism $f: G \to H$ if and only if

$$q(G/R; \alpha, f^D\beta_D) = q(H/R; \alpha f, \beta_D).$$
for every $\alpha \in T_pG(k)$ and $\beta_D \in T_pH^D(k)$.

By abuse of notation, we will use $\mathcal{M}_G$ to denote the formal scheme $\text{Hom}_\mathcal{O}(T_pG(k) \otimes_\mathcal{O} T_pG^D(k), \mathcal{L})$. The proof of the theorem follows exactly in the way of [Kat81, Theorem 2.1].

**Proof.** The fact that $\mathcal{M}_G$ is pro-presentable is well-known. Now we determine the representing formal scheme.

Since $G$ is ordinary, we have a canonical isomorphism

$$G \simeq G^0[p^\infty] \times T_pG(k) \otimes_\mathcal{O} F/\mathcal{O}. $$

By the definition of $\mathcal{O}$-Cartier duality, we have a morphism

$$e_{p^n} : G[p^n] \times G^D[p^n] \to \mathcal{L}[p^n].$$

The restriction of the first factor to $G^0[p^n]$ gives rise an isomorphism

$$G^0[p^n] \simeq \text{Hom}_\mathcal{O}(G^D[p^n](k), \mathcal{L}[p^n])$$

of group schemes over $k$ preserving $\mathcal{O}$-actions. Passing to the limit, we obtain an isomorphism

$$G^0 \simeq \text{Hom}_\mathcal{O}(T_pG^D(k), \mathcal{L}),$$

which induces a pairing

$$E_G : G^0 \times T_pG^D(k) \to \mathcal{L}.$$

Let $G/R$ be a deformation of $G$, then we have an extension

$$(B.1) \quad 0 \longrightarrow G^0[p^n] \longrightarrow G \longrightarrow T_pG(k) \otimes_\mathcal{O} F/\mathcal{O} \longrightarrow 0$$

of $\mathcal{O}$-divisible groups. We have pairings

$$E_{G,p^n} : G^0[p^n] \times G^D[p^n] \to \mathcal{L}[p^n],$$

$$E_G : G^0 \times T_pG^D(k) \to \mathcal{L},$$

which lift $e_{p^n}$ and $E_G$, respectively.

Similar to the $p$-divisible group case, the extension (B.1) is obtained from the extension

$$0 \longrightarrow T_pG(k) \longrightarrow T_pG(k) \otimes_\mathcal{O} F \longrightarrow T_pG(k) \otimes_\mathcal{O} F/\mathcal{O} \longrightarrow 0$$

by pushing out along a unique $\mathcal{O}$-linear homomorphism

$$\varphi_{G/R} : T_pG(k) \to G^0(R).$$

The homomorphism $\varphi_{G/R}$ may be recovered from (B.1) in the way described in [Kat81, page 151]. It is the composite

$$T_pG(k) \to T_pG[p^n](k) \xrightarrow{(p^n)} G^0(R)$$

for every $n \geq 1$ such that $m_{R}^{n+1} = 0$. Therefore, from $G/R$, we obtain a pairing

$$q(G/R; \ , ) = E_G(R) \circ (\varphi_{G/R}; \text{id}) : T_pG(k) \otimes_\mathcal{O} T_pG^D(k) \to \mathcal{L}(R) = 1 + m_R.$$ 

This shows that the functor $\mathcal{M}_G$ is canonically pro-represented by the $\mathcal{O}$-formal scheme $\text{Hom}_\mathcal{O}(T_pG(k) \otimes_\mathcal{O} T_pG^D(k), \mathcal{L})$. 

For (2), if the given homomorphism \( f : G \to H \) can be lifted to \( f : G \to H \), then we must have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_\varnothing(T_p G^D(k), L[p^\infty]) \\
\otimes \pi T^D_p(k) & \downarrow & f \\
0 & \longrightarrow & \text{Hom}_\varnothing(T_p H^D(k), L[p^\infty])
\end{array}
\]

Conversely, if we may fill \( f \) in the above diagram, then \( f \) lifts.

The existence of the middle arrow is equivalent to that the push-out of the top extension by the left arrow is isomorphic to the pull-back of the lower extension by the right arrow. The above mentioned push-out is an element in

\[ \text{Ext}_{BT^\varnothing_R} (T_p G(k) \otimes F/\varnothing, \text{Hom}_\varnothing(T_p H^D(k), L[p^\infty])) \]

which is isomorphic to

\[ \text{Hom}_\varnothing(T_p G(k) \otimes T_p H^D(k), L(T(R))) \]

by the bilinear pairing

\[ (\alpha, \beta_D) \mapsto q(G/R; \alpha, f^D \beta_D). \]

Similarly, the above mentioned pull-back is an element in

\[ \text{Hom}_\varnothing(T_p G(k) \otimes T_p H^D(k), L(T(R))) \]

defined by the bilinear pairing

\[ (\alpha, \beta_D) \mapsto q(H/R; f \alpha, \beta_D). \]

It remains to prove (1). Choose \( n \) such that \( m_{n+1}^{n+1} = 0 \). Then both \( G^0(R) \) and \( (G^D)^0(R) \) are annihilated by \( p^n \). Denote by \( \alpha(n) \) the image of \( \alpha \) under the canonical projection \( T_p G(k) \to G[p^n](k) \) and similarly for \( \alpha_D(n) \). By construction, we have \( \varphi_{G/R}(\alpha) = (p^n) \alpha(n) \in G^0(R) \) and \( \varphi_{G^D/R}(\alpha_D) = (p^n) \alpha_D(n) \in (G^D)^0(R) \). Therefore, we have

\[ q(G/R; \alpha, \alpha_D) = E_{G,p^n}((p^n) \alpha(n), \alpha_D(n)). \]

Similarly, we have \( q(G^D/R; \alpha_D, \alpha) = E_{G^D,p^n}((p^n) \alpha_D(n), \alpha(n)) \).

The remaining argument is formal and one only needs to replace \( \hat{G}_m \) (resp. abelian varieties) by \( \mathcal{E} \) (resp. \( \varnothing \)-divisible groups) in the proof of [Kat81, Theorem 2.1]. In particular, we have the following: Given an integer \( n \geq 1 \), elements \( x \in G^0[p^n](R) \) and \( y \in G^D[p^n](R) \), there exist an Artinian local ring \( R' \) that is finite and flat over \( R \), and a point \( Y \in G^D[p^n](R') \) lifting \( y \). For every such \( R' \) and \( Y \), we have the equality \( E_{G,p^n}(x, y) = e_{p^n}(x, Y) \) inside \( \mathcal{E}(R') \).

**B.2. Main theorem.** We fix an ordinary \( \varnothing \)-divisible group \( G \) over \( k \). Denote by \( \mathfrak{R} \) the coordinate ring of \( \mathfrak{M}_G \), which is a complete \( \mathcal{O} \)-algebra. We have the universal pairing

\[ q : T_p G(k) \otimes T_p G^D(k) \to \mathcal{E}(\mathfrak{R}) \subset \mathfrak{R}^\times. \]

Therefore, we may regard \( q(\alpha, \alpha_D) \) as a regular function on \( \mathfrak{M}_G \). For each \( \varnothing \)-linear form \( \ell \in \text{Hom}_\varnothing(T_p G(k) \otimes T_p G^D(k), \varnothing) \), denote by \( D(\ell) \) the translation-invariant continuous derivation of \( \mathfrak{R} \) given by

\[ D(\ell) q(\alpha, \alpha_D) = \ell(\alpha \otimes \alpha_D) \cdot q(\alpha, \alpha_D). \]

By abuse of notation, we also denote by \( D(\ell) \) the corresponding map \( \Omega_{\mathfrak{R}/\varnothing} \to \mathfrak{R} \). Denote by \( \mathcal{G} \) the universal \( \varnothing \)-divisible group over \( \mathfrak{M}_G \).
They together define the following factors through the quotient sequence.

58 YIFENG LIU, SHOUWU ZHANG, AND WEI ZHANG

Let $R$ be as in Theorem B.1.1 and $G/R$ a deformation of $G$. We have the canonical isomorphism of $\mathcal{O}$-modules

$$\lambda_G: T_p G^D(k) \sim \text{Hom}_{\text{BT}_R}(G^0[p^\infty], \mathcal{L}[p^\infty]).$$

Define the $\mathcal{O}$-linear map $\omega_G: T_p G^D(k) \to \Omega(G/R)$ by the formula

$$\omega_G(\alpha_D) = \lambda_G(\alpha_D)^* \omega_0 \in \Omega(G^0/R) = \Omega(G/R).$$

Let $L_G: \text{Hom}_0(T_p G^D(k), \mathcal{O}) \to \text{Lie}(G/R)$ be the unique $\mathcal{O}$-linear map such that

$$\omega_G(\alpha_D) \cdot L_G(\alpha_D^\vee) = \alpha_D \cdot \alpha_D^\vee \in \mathcal{O}.$$

In fact, the $R$-linear extensions

$$\omega_G: T_p G^D(k) \otimes_{\mathcal{O}} R \to \Omega(G/R)$$

and

$$L_G: \text{Hom}_0(T_p G^D(k), R) \to \text{Lie}(G/R)$$

are isomorphisms. Similarly, we have an isomorphism

$$\lambda_{G^\vee}: T_p G(k) = T_p G^{\text{ét}}(k) = T_p G^{\text{ét}}(R) \sim \text{Hom}_{\text{BT}_R}((G^{\text{ét}})^\vee, \widehat{G}_m[p^\infty]),$$

which induces an isomorphism

$$T_p G(k) \otimes_{\mathcal{O}} R \sim \Omega((G^{\text{ét}})^\vee/R)$$

by pulling back the differential form $\frac{dT}{T}$ on $\widehat{G}_m$. It further induces an isomorphism

$$\omega_{G^\vee}: T_p G(k) \otimes_{\mathcal{O}} R = (T_p G(k) \otimes_{\mathcal{O}} R)_0 \sim \Omega((G^{\text{ét}})^\vee/R)_0.$$

Here, the subscript $0$ denotes the maximal flat quotient on which $\mathcal{O}$ acts via the structure map. By construction, we have the following lemma of functoriality.

**Lemma B.2.1.** Let $f: G \to H$ be as in Theorem B.1.1 and $f: G \to H$ a homomorphism lifting $f$. Then

1. We have $((f^{\text{ét}})^\vee)^*(\omega_{G^\vee}(\alpha)) = \omega_{H^\vee}(f\alpha)$ for every $\alpha \in T_p G(k)$, where $f^{\text{ét}}: G^{\text{ét}} \to H^{\text{ét}}$ is the induced homomorphism on the étale quotient.
2. We have $f_*(L_H(\alpha_D^\vee)) = L_G(\alpha_D^\vee \circ f^D)$ for every $\alpha_D^\vee \in \text{Hom}_0(T_p G^{D}(k), \mathcal{O})$.

Denote by $D(G)$ the (contra-variant) Dieudonné crystal of $G$. We have the following exact sequence

$$0 \to \Omega(G^\vee/R) \to D(G^\vee)_R \to \text{Lie}(G/R) \to 0$$

and the Gauss–Manin connection

$$\nabla: D(G^\vee)_R \to D(G^\vee)_R \otimes_{\mathcal{O}} \Omega_{R/\mathcal{O}}.$$

They together define the following (universal) Kodaira–Spencer map

$$\text{KS}: \Omega(G^\vee/R) \to \text{Lie}(G/R) \otimes_{\mathcal{O}} \Omega_{R/\mathcal{O}},$$

which factors through the quotient $\Omega(G^\vee/R)_0 \to \Omega((G^{\text{ét}})^\vee/R)_0$. The following lemma is immediate.

**Lemma B.2.2.** The natural map $\Omega(G^\vee/R)_0 \to \Omega((G^{\text{ét}})^\vee/R)_0$ is an isomorphism.
In particular, we may regard $\omega_G^\vee$ as a map from $T_p G(k)$ to $\Omega(G^\vee/R)_0$. The following result on the compatibility of the Kodaira–Spencer map and the Serre–Tate coordinate is the main theorem of this appendix.

**Theorem B.2.3.** We have the following equality
$$\omega_G(\alpha_D) \cdot \text{KS}(\omega_G^\vee(\alpha)) = d\log(q(\alpha, \alpha_D))$$
in $\Omega_{R/\hat{O}}$ for every $\alpha \in T_p G(k)$ and $\alpha_D \in T_p G^D(k)$.

Note that the definition of $\omega_G$, but not $\omega_G^\vee$, depends on the choice of log, which is compatible with the right-hand side.

**B.3. Frobenius.** Denote by $\sigma$ the Frobenius automorphism of $\hat{O}$ such that $\hat{O} = \hat{O}^\sigma = 1$. Put $X^\sigma = X \otimes_{\hat{O}, \sigma} \hat{O}$ for every $\hat{O}$-(formal) scheme $X$, $\Sigma_X : X^\sigma \to X$ the natural projection, and $F_X : X \to X^\sigma$ the relative Frobenius morphism which is $\hat{O}$-linear. We omit the subscript $X$ if it is $M_G$.

**Lemma B.3.1.** We have

1. There is a natural isomorphism
   $$M_G^\sigma \cong M_G^\sigma$$
   under which the regular function $q(\sigma(\alpha), \sigma(\alpha_D))$ is mapped to $\Sigma^* q(\alpha, \alpha_D)$.
2. Under the map $\Sigma_{(G^\et)^\vee} : (G^\et)^\vee \to (G^\et)^\vee$, we have
   $$\Sigma^* (\omega_G^\et(\alpha)) = \omega_{(G^\et)^\vee}(\sigma \alpha)$$
   for every $\alpha \in T_p G(k)$.
3. Under the map $F_G : G \to G^\sigma$, we have
   $$F_{G^\sigma} L_G(\alpha_D^\vee) = L_G(\alpha^\vee_D \circ \sigma^{-1})$$
   for every $\alpha_D^\vee \in \text{Hom}_0(T_p G^D(k), \hat{O})$.

**Proof.** The proof is the same as [Kat81, Lemma 4.1.1 & 4.1.1.1].

From now on, we choose a uniformizer $\varpi$ of $F$, which gives rise to an isomorphism $\mathcal{L}^{\sigma} \cong \mathcal{L}$. In particular, we may identify $(G^D)^\sigma$ and $(G^\sigma)^D$.

For a deformation $G/R$ of $G$, we denote by $G'/R$ the quotient of $G$ by subgroup $G^0[\varpi]$. The induced projection map
$$\mathcal{F}_G : G \to G'$$
lifts the relative Frobenius morphism
$$F_G : G \to G^\sigma.$$
Define the Verschiebung to be
$$V_G = (F_G^D)^D : G^\sigma \to G^{D\sigma D} \to G.$$ 
Note that the isomorphism depends on $\varpi$.

**Lemma B.3.2.** For $\alpha \in T_p G(k)$ and $\alpha_D \in T_p G^D(k)$, we have formulae

1. $F_G(\alpha) = \sigma \alpha$ and $V_G(\sigma \alpha) = \varpi \alpha_D$;
2. $q(G'/R; \sigma \alpha, \sigma \alpha_D) = \varpi q(G/R; \alpha, \alpha_D)$.

**Proof.** The proof is the same as [Kat81, Lemma 4.1.2], with $V_G \circ F_G = \varpi$. 

□
Lemma B.3.3. For $\alpha \in T_p G(k)$ and $\alpha^\vee \in \hom_\O(T_p G^D(k), \O)$, we have formulae
\begin{enumerate}
\item $((\mathcal{F}_G^\vee)^\ast \omega_{G^\vee})(\alpha) = \omega_{G^\vee}(\sigma \alpha)$;
\item $\mathcal{F}_G \circ L_G(\alpha^\vee) = \varpi L_G(\alpha^\vee \circ \sigma^{-1})$.
\end{enumerate}

Proof. It follows from Lemmas B.2.1 and B.3.2.

If we apply the construction to the universal object $\mathcal{G}$, we obtain a formal deformation $\mathcal{G}'/\mathcal{R}$ of $G^\sigma$. Its classifying map is the unique morphism
\[
\Phi: \mathcal{M}_G \to \mathcal{M}_{G^\sigma} \cong \mathcal{M}_G'
\]
such that $\Phi^* \mathcal{G} \cong \mathcal{G}'$. Therefore, we may regard $\mathcal{F}_G$ as a morphism
\[
\mathcal{F}_G: \mathcal{G} \to \Phi^* \mathcal{G}^\sigma
\]
of $\mathcal{O}$-divisible groups over $\mathcal{M}_G$. Taking dual, we have
\[
\mathcal{F}_G^\vee: \Phi^* \mathcal{G}^\sigma \cong (\Phi^* \mathcal{G})^\vee \to \mathcal{G}^\vee.
\]

Lemma B.3.4. We have
\begin{enumerate}
\item The map $\omega_{\mathcal{G}^\vee}: T_p G(k) \otimes \O \to \Omega(\mathcal{G}^\vee/\mathcal{R})_\O$ induces an isomorphism
\[
T_p G(k) \cong \Omega(\mathcal{G}^\vee/\mathcal{R})_\O^1 := \{\omega \in \Omega(\mathcal{G}^\vee/\mathcal{R})_\O \mid (\mathcal{F}_G^\vee)^\ast \omega = \Phi^* \Sigma_{G^\vee} \omega\}
\]
of $\O$-modules.
\item The map $L_{\mathcal{G}}: \hom_\O(T_p G^D(k), \mathcal{R}) \to \lie(\mathcal{G}/\mathcal{R})$ induces an isomorphism
\[
\hom_\O(T_p G^D(k), \O) \cong \lie(\mathcal{G}/\mathcal{R})^0 := \{\delta \in \lie(\mathcal{G}/\mathcal{R}) \mid \mathcal{F}_G \delta = \varpi \Phi^* \mathcal{F}_{G^\vee} \delta\}
\]
of $\O$-modules.
\end{enumerate}

Proof. It can be proved by the same way as [Kat81, Corollary 4.1.5] by using Lemmas B.3.1 and B.3.3.

Consider the following commutative diagram:
\[(B.2)\]
\[
\begin{array}{ccccccc}
0 & \xrightarrow{a} & \Omega(\mathcal{G}^\vee/\mathcal{R})_\O & \xrightarrow{b} & \lie(\mathcal{G}/\mathcal{R}) & \xrightarrow{0} \\
\downarrow{\mathcal{F}_G^\vee}^\ast & & \downarrow{\mathcal{F}_G^\vee}D(F_G^\vee) & & \downarrow{\mathcal{F}_G} & & \\
0 & \xrightarrow{\Phi^* \Sigma_{G^\vee}} & \Omega(\mathcal{G}^\vee/\mathcal{R})_\O & \xrightarrow{\Phi^* \mathcal{F}_{G^\vee}} & \lie(\mathcal{G}/\mathcal{R}) & \xrightarrow{0} \\
\downarrow{\Phi^* \Sigma_{G^\vee}} & & \downarrow{\mathcal{D}(\Sigma_{G^\vee})} & & \downarrow{\Phi^* \mathcal{F}_{G^\vee}} & & \\
0 & \xrightarrow{\triangleleft} & \Omega(\mathcal{G}^\vee/\mathcal{R})_\O & \xrightarrow{\triangleleft} & \lie(\mathcal{G}/\mathcal{R}) & \xrightarrow{0} \\
\end{array}
\]

For $k \in \mathbb{Z}$, we define $\O$-modules
\[
D(G^\vee)^k_{\mathcal{R}} := \{\xi \in (D(G^\vee))_{\mathcal{R}}_\O \mid D(F_G^\vee)\xi = \varpi^{1-k}D(\Sigma_{G^\vee})\xi\} = \{\xi \in (D(G^\vee))_{\mathcal{R}}_\O \mid D(V_G\vee)D(\Sigma_{G^\vee})\xi = \varpi^k\xi\}.
\]

Lemma B.3.5. The maps $\omega_{\mathcal{G}^\vee}$ and $a$ in (B.2) together induce an isomorphism
\[
a_1: T_p G(k) \xrightarrow{\cong} D(G^\vee)^1_{\mathcal{R}}
\]
of $\O$-modules. The maps $L_{\mathcal{G}}$ and $b$ in (B.2) together induce an isomorphism
\[
b_0: D(G^\vee)^0_{\mathcal{R}} \xrightarrow{\cong} \hom_\O(T_p G^D(k), \O)
\]
of $\O$-modules.
Lemma B.3.4. For the first part, by a similar argument in [Kat81, Lemma 4.2.1], we know that $b(\xi) = 0$ for $\xi \in D(G^\vee)_\emptyset$, that is, $\xi$ is in the image of $a$. The conclusion then follows from Lemma B.3.4 (1).

For the second part, it is easy to see that $\text{Im}(a) \cap D(G^\vee)_\emptyset = \{0\}$ by choosing an $\emptyset$-basis of $T_p G(k)$. Therefore, $b$ restricts to an injective map $D(G^\vee)_\emptyset \to \text{Lie}(\mathfrak{g}/\mathfrak{m})^0$. We only need to show that this map is also surjective. For every $\delta \in \text{Lie}(\mathfrak{g}/\mathfrak{m})^0$, choose an element $\xi_0 \in (D(G^\vee)_\emptyset)_\emptyset$. Put $\xi_{n+1} = D(V_G^\ell)D(\Sigma_G^\vee)\xi_n$ for $n \geq 0$. Then $b(\xi_n) = \delta$ and $\{\xi_n\}$ converge to an element $\xi \in D(G^\vee)_\emptyset$.

□

Lemma B.3.6. For every $\ell \in \text{Hom}_\emptyset(T_p G(k) \otimes_\emptyset T_p G^D(k), \emptyset)$, the action of $D(\ell)$ under the Gauss–Manin connection on $(D(G^\vee)_\emptyset)_\emptyset$ satisfies the formula

$$D(\ell)(\nabla(D(V_G^\ell)D(\Sigma_G^\vee)\xi)) = \varpi D(V_G^\ell)D(\Sigma_G^\vee)(D(\ell)(\nabla\xi))$$

for every $\xi \in (D(G^\vee)_\emptyset)_\emptyset$.

Proof. It is proved in the same way as [Kat81, Lemma 4.3.3]. □

Lemma B.3.7. If $\xi \in (D(G^\vee)_\emptyset)_\emptyset$ satisfies $D(V_G^\ell)D(\Sigma_G^\vee)\xi = \lambda\xi$ for some $\lambda \in \bar{\emptyset}$, then for every $\ell \in \text{Hom}_\emptyset(T_p G(k) \otimes_\emptyset T_p G^D(k), \emptyset)$, the element $D(\ell)(\nabla\xi) \in (D(G^\vee)_\emptyset)_\emptyset$ satisfies

$$\varpi D(V_G^\ell)D(\Sigma_G^\vee)(D(\ell)(\nabla\xi)) = \lambda D(\ell)(\nabla\xi).$$

Proof. It follows immediately from Lemma B.3.6. □

Proposition B.3.8. For $\alpha \in T_p G(k)$ and $\alpha_D \in T_p G^D(k)$, there exists a unique character $Q(\alpha, \alpha_D)$ of $\mathfrak{m}_G$ such that

$$\omega_\delta(\alpha_D) \cdot \text{KS}(\omega_\emptyset(\alpha)) = d\log Q(\alpha, \alpha_D).$$

Proof. Let $\{\alpha_i\}$ (resp. $\{\alpha_{D,i}\}$) be an $\emptyset$-basis of $T_p G(k)$ (resp. $T_p G^D(k)$). Let $\{\ell_{i,j}\}$ be the basis of $\text{Hom}_\emptyset(T_p G(k) \otimes_\emptyset T_p G^D(k), \emptyset)$ dual to $\{\alpha_i \otimes \alpha_{D,j}\}$. Then for every element $\xi \in (D(G^\vee)_\emptyset)_\emptyset$, we have

$$\nabla\xi = \sum_{i,j} D(\ell_{i,j})(\nabla\xi) \otimes d\log q(\alpha_i, \alpha_{D,j}).$$

In particular, for $\xi = \omega_\emptyset(\alpha)$, we have

$$\nabla\omega_\emptyset(\alpha) = \sum_{i,j} D(\ell_{i,j})(\nabla\omega_\emptyset(\alpha)) \otimes d\log q(\alpha_i, \alpha_{D,j}).$$

By Lemmas B.3.4 and B.3.7, $\nabla\omega_\emptyset(\alpha) \in D(G^\vee)_\emptyset$. Therefore, there exist unique elements $\alpha_{D,i,j}^\vee \in \text{Hom}_\emptyset(T_p G^D(k), \emptyset)$ such that

$$\nabla\omega_\emptyset(\alpha) = b_0^{-1}(\alpha_{D,i,j}^\vee)$$

for every $i, j$. By definition,

$$\text{KS}(\omega_\emptyset(\alpha)) = \sum_{i,j} L_\emptyset(\alpha_{D,i,j}^\vee) \otimes d\log q(\alpha_i, \alpha_{D,j}),$$

and

$$\omega_\delta(\alpha_D) \cdot \text{KS}(\omega_\emptyset(\alpha)) = d\log \left( \prod_{i,j} q(\alpha_i, \alpha_{D,j})^{\alpha_D \cdot \alpha_{D,i,j}^\vee} \right).$$

□

The above proposition has the following two corollaries.
Corollary B.3.9. For elements \( \alpha \in T_pG(k) \), \( \alpha_D \in T_pG^D(k) \) and \( \ell \in \text{Hom}_O(T_pG(k) \otimes_O T_pG^D(k), \mathcal{O}) \), we have that \( D(\ell)(\omega_{\mathfrak{G}}(\alpha_D) \cdot KS(\omega_{\mathfrak{G}^\vee}(\alpha))) \) is a constant in \( \mathcal{O} \).

Corollary B.3.10. Suppose for every integer \( n \geq 1 \), we can find a homomorphism \( f_n: \mathcal{A} \to \mathcal{O}/p^n \) such that

\[
f_n(D(\ell)(\omega_{\mathfrak{G}}(\alpha_D) \cdot KS(\omega_{\mathfrak{G}^\vee}(\alpha)))) = \ell(\alpha \otimes \alpha_D)
\]

holds \( W_n \). Then \( Q = q \) and Theorem B.2.3 follows.

The condition of this corollary is fulfilled by Theorem B.4.2. Therefore, we have reduced Theorem B.2.3 to Theorem B.4.2 in the next section.

B.4. Infinitesimal computation. Let \( R \) be an (Artinian) local \( \mathcal{O} \)-algebra with the maximal ideal \( m_R \) satisfying \( m_R^{n+1} = 0 \). Let \( G/R \) be the canonical deformation of \( G \). Let \( \tilde{G} \) be a deformation of \( G \) to \( \tilde{R} : = R[\varepsilon]/(\varepsilon^2) \), which gives rise to a map \( \partial: \Omega(G'/{\tilde{R}}) \to \text{Lie}(G/R) \). Note that the target \( \text{Lie}(G/R) \) may be identified with \( \ker(G^0({\tilde{R}}) \to G^0(R)) \).

Lemma B.4.1. The reduction map \( T_pG(R) \to T_pG(k) \) is an isomorphism.

Proof. It follows from the same argument in [Kat81, Lemma 6.1]. \( \square \)

In particular, we may define maps \( \lambda_{G^\vee}: T_pG(k) \to \text{Hom}_{BT_R}(G^\vee, \tilde{G}_m[p^\infty]) \) and

\[
(\text{B.3}) \quad \omega_{G^\vee}: T_pG(k) \to \Omega(G^\vee/{\tilde{R}}).
\]

Theorem B.4.2. The Serre–Tate coordinate for \( \tilde{G}/\tilde{R} \) satisfies

\[
q(\tilde{G}/\tilde{R}; \alpha, \alpha_D) = 1 + \varepsilon \omega_G(\alpha_D) \cdot \partial(\omega_{G^\vee}(\alpha)).
\]

Lemma B.4.3. For \( \alpha_D \in T_pG^D(k) \) and \( \alpha \in \ker(G^0(\tilde{R}) \to G^0(R)) = \text{Lie}(G/R) \), we have

\[
E_G(\alpha, \alpha_D) = 1 + \varepsilon \omega_G(\alpha_D) \alpha.
\]

Proof. By functoriality, we only need to prove the lemma for the universal object \( \mathfrak{G}/\mathcal{A} \). By definition,

\[
1 + \varepsilon \omega_{\mathfrak{G}}(\alpha_D) \alpha = 1 + \varepsilon(\lambda_{\mathfrak{G}}(\alpha_D),\alpha \cdot \omega_0) \in \mathcal{L}\mathcal{T}(\tilde{R}).
\]

We also have

\[
\lambda_{\mathfrak{G}}(\alpha_D),\alpha \cdot \omega_{\mathfrak{G}^T} = (\log \circ \lambda_{\mathfrak{G}}(\alpha_D)) , \alpha \cdot dT
\]

in \( \mathcal{A}[p^{-1}] \). Therefore, we have the equality

\[
E_{\mathfrak{G}}(\alpha, \alpha_D) = 1 + \varepsilon \omega_{\mathfrak{G}}(\alpha_D) \alpha
\]

in \( \ker(\mathcal{L}\mathcal{T}(\tilde{R}[p^{-1}]) \to \mathcal{L}\mathcal{T}(\mathcal{A}[p^{-1}]))) \). \( \square \)

For an integer \( N > n \), denote by \( \alpha_N \) the image of \( \alpha \) in \( G[p^N](R) \). Let \( \tilde{\alpha}_N \in \tilde{G}(\tilde{R}) \) be an arbitrary lifting of \( \alpha_N \). Then

\[
p^N \tilde{\alpha}_N \in \ker(\tilde{G}(\tilde{R}) \to G(R)) = \ker(\tilde{G}^0(\tilde{R}) \to G^0(R)) \simeq \text{Lie}(G/R).
\]

Such process defines a map \( \varphi_G: T_pG(R) \to \text{Lie}(G/R) \).

Proposition B.4.4. We have \( \partial \omega_{G^\vee}(\alpha) = \varphi_G(\alpha) \) for every \( \alpha \in T_pG(R) \).

Assuming the above proposition, we prove Theorem B.4.2.
Proof of Theorem B.4.2. It is clear that $G^0[p^\infty] \otimes_R \bar{R}$ is the unique, up to isomorphism, deformation of $G^0[p^\infty]$ to $\bar{R}$. Then the deformation $\tilde{G}$ corresponds to the extension

$$0 \longrightarrow G^0[p^\infty] \otimes_R \bar{R} \longrightarrow \tilde{G} \longrightarrow T_pG(k) \otimes_O F/\mathcal{O} \longrightarrow 0.$$ 

In particular, we may identify $\tilde{G}^0$ with $G^0 \otimes_R \bar{R}$. We have

$$\text{Ker}(\tilde{G}(\bar{R}) \rightarrow G(R)) = \text{Ker}(\tilde{G}^0(\bar{R}) \rightarrow G^0(R)) = \text{Ker}(G^0(\bar{R}) \rightarrow G^0(R)) = \text{Lie}(G^0/R).$$

For $D \in \text{Lie}(G^0/R)$, we have

$$E_G(D, -) = E_G(D, -); T_pG^D(k) \simeq T_p(G^0[p^\infty])^D(k) \rightarrow \mathcal{L}(\bar{R}),$$

where in the pairing $E_G$ (resp. $E_G$), we view $D$ as an element in $\tilde{G}^0(\bar{R})$ (resp. $G^0(\bar{R})$). For $\alpha_D \in T_pG^D(k)$, we have

$$E_G(D, \alpha_D) = 1 + \varepsilon_{\omega_G(\omega_\alpha)}D$$

by definition. Therefore, Theorem B.4.2 follows from Proposition B.4.4 and the construction of $q$. \hfill $\square$

The rest of the appendix is devoted to the proof of Proposition B.4.4. We will reduce it to certain statements in [Kat81] about abelian varieties. It is an interest problem to find a proof purely using $\mathcal{O}$-divisible groups.

Recall that ordinary $\mathcal{O}$-divisible groups over $k$ are classified by its dimension and $\mathcal{O}$-height. Let $G_{r,s}$ be an $\mathcal{O}$-divisible group of dimension $r$ and $\mathcal{O}$-height $r + s$ with $r \geq 0, r + s > 0$.

Proof of Proposition B.4.4. Choose a totally real number field $E^+$ such that $F \simeq E^+ \otimes_Q \mathbb{Q}_p$, and an imaginary quadratic field $K$ in which $p = p^+p^-$ splits. Put $E = E^+ \otimes_Q K$. Suppose $\tau_1, \tau_2, \ldots, \tau_h$ are all complex embeddings of $E^+$. Consider the data $(A_{r,s}, \theta, i)$ where

- $A_{r,s}$ is an abelian variety over $k$;
- $\theta: A_{r,s} \rightarrow A_{r,s}^\vee$ is a prime-to-$p$ polarization;
- $i: O_E \rightarrow \text{End}_k A_{r,s}$ is an $O_E$-action which sends the complex conjugation on $O_E$ to the Rosati involution and such that, in the induced decomposition

$$A_{r,s}[p^\infty] = A_{r,s}[p^\infty]^+ \oplus A_{r,s}[p^\infty]^-$$

of the $O_E \otimes \mathbb{Z}_p$-module $A_{r,s}[p^\infty]$, the summand $A_{r,s}[p^\infty]^+$ is isomorphic to $G_{r,s}$ as an $\mathcal{O}$-divisible group.

It is clear that the polarization $\theta$ induces an isomorphism $A_{r,s}[p^\infty]^+ \xrightarrow{\sim} (A_{r,s}[p^\infty]^-)$. By the Serre–Tate theorem, $\mathfrak{M}_{G_{r,s}}$ also parameterizes deformation of the triple $(A_{r,s}, \theta, i)$. In what follows, we fix $r, s$ and suppress them from notation. Let $R$ be as in Theorem B.1.1, $A/R$ be the canonical deformation of $A/k$, and $\tilde{A}$ be a deformation of $A$ to $\bar{R}$ such that $\tilde{G} \simeq \tilde{A}[p^\infty]^+$. There is a similar map (B.3) for $A$ and we have $\omega_G(\alpha) = \omega_{A^\vee}(\alpha)$ for $\alpha \in T_pG(R) \subset T_pA(R)$, where we view $\Omega(G^\vee/R)$ as a submodule of $H^0(A^\vee, \Omega^1_{A^\vee/R})$. Moreover, the map $\varphi_G: T_pG(R) \rightarrow \text{Lie}(G/R)$ can be extended in a same way to a map $\varphi_A: T_pA(R) \rightarrow \text{Lie}(A/R)$. Then Proposition B.4.4 follows from [Kat81, Lemma 5.4 & §6.5], where the argument uses normalized cocycles and does not require $A$ to be ordinary in the usual sense. \hfill $\square$
References


A p-ADIC WALDSPURGER FORMULA


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