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On Quadratic Distinction of Automorphic Sheaves

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We prove a geometric version of a classical result on the characterization of an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ being the base change of a stable cuspidal representation of the quasi-split unitary group associated to the quadratic extension E/F, via the nonvanishing of certain period integrals, called being distinguished. We show that certain cohomology of an automorphic sheaf of $\mathrm{GL}_{n,X'}$ is nonvanishing if and only if the corresponding local system E on X' is conjugate self-dual with respect to an étale double cover X'/X of curves, which directly relates to the base change from the associated unitary group. In particular, the geometric setting makes sense over any base field.

1 Introduction

1.1 Classical point of view: distinguished representations

Let E/F be a separable quadratic extension of global fields with the Galois group $\{1, \sigma\}$, and Π an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. We consider the following two period integrals:

$$\mathbf{P}^{+}(\phi) = \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^0} \phi(h)\mathrm{d}h;$$

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$$\mathbf{P}^{-}(\phi)=\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^0}\phi(h)\omega_{E/F}(\det h)\mathrm{d}h,$$

where ϕ is a cusp form in the space of Π, $\mathrm{GL}_n(\mathbb{A}_F)^0$ is the subgroup of $\mathrm{GL}_n(\mathbb{A}_F)$ consisting of elements whose determinant has norm 1 and $\omega_{E/F}$ is the quadratic character of $\mathbb{A}_F^\times$ associated with E/F via global class field theory. If $\mathbf{P}^+$ (resp. $\mathbf{P}^-$) is not identically zero on $\phi\in\Pi$, then we say Π is distinguished (resp. $\omega_{E/F}$-distinguished) by $\mathrm{GL}_{n,F}$, or simply distinguished (resp. $\omega_{E/F}$-distinguished). Assuming that the central character of Π is distinguished, it is proved in [9, Proposition 1] that $\Pi^\vee\cong\Pi^\sigma$ if and only if Π is distinguished (resp. distinguished or $\omega_{E/F}$-distinguished) when n is odd (resp. even). Here, $\Pi^\vee$ and Π^σ denote the contragredient representation and the σ-twisted representation, respectively. The proof is based on previous results of Flicker [3, 4]. From the point of view of L-functions, Π is distinguished if and only if the Asai L-function $L(s,\Pi,\mathrm{As})$ has a (simple) pole at $s=1$, and Π is $\omega_{E/F}$-distinguished if and only if $L(s,\Pi\otimes\Omega,\mathrm{As})$ has a (simple) pole at $s=1$, where Ω is any automorphic character of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ equals $\omega_{E/F}$, cf. [3].

The theory has a perfect geometric counterpart, in the framework of Geometric Langlands Program. Instead of automorphic representations, we work on automorphic sheaves and consider only the everywhere unramified case. Then we will replace the period integrals $\mathbf{P}^+$ and $\mathbf{P}^-$ by certain complexes of ℓ-adic sheaves on the base field, which now makes sense for any field k. The isomorphism $\Pi^\vee\cong\Pi^\sigma$ has a perfect Galois counterpart, namely the isomorphism $E^\vee\simeq E^\sigma$ for the corresponding local system E of Π. The use of poles of L-functions will implicitly appear in our proof as cohomology. Now we are going to state our context and Main Theorem in more details.

1.2 Geometrization and Main Theorem

Let k be a field (which will be assumed algebraically closed) and ℓ a prime number invertible in k. Let X be a connected smooth proper curve over k and $\mu: X'\to X$ a proper étale morphism of degree 2 with X' connected. Let σ be the unique nontrivial automorphism of X' such that $\mu\circ\sigma=\mu: X'\to X$.

For a positive integer n, we denote Bun_n (resp. Bun'_n) the moduli stack of vector bundles on X (resp. X') of rank n. Then the pullback of vector bundles under μ induces a morphism $\mu_n:\mathrm{Bun}_n\to\mathrm{Bun}'_n$. The stack Bun_n (resp. Bun'_n) is a disjoint union of its connected components Bun_n^d (resp. $\mathrm{Bun}_n'^d$) parameterizing those bundles of normalized degree d for $d\in\mathbb{Z}$, then we have $\mu_n:\mathrm{Bun}_n^d\to\mathrm{Bun}_n'^{2d}$. Moreover, we have an isomorphism

$\sigma_n : \mathrm{Bun}'_n \to \mathrm{Bun}'_n$, induced by the pullback under σ. When $n=1$, we write $\mathrm{Pic}_X = \mathrm{Bun}_1$ (resp. $\mathrm{Pic}_{X'} = \mathrm{Bun}'_1$) which is the Picard stack of X (resp. X') and also $\mathrm{Pic}^d_X = \mathrm{Bun}^d_1$ (resp. $\mathrm{Pic}^d_{X'} = \mathrm{Bun}'^d_1$).

For a local system E on X' of rank n, we let $E^\vee$ be its dual system and $E^\sigma = \sigma^* E$. If E is irreducible, then the geometric Langlands correspondence (proved by Drinfeld [2] for $n=2$, formulated by Laumon [11, 12] and proved by Frenkel et al. [6], Gaitsgory [7] for general n) associates to E a $\bar{\mathbb{Q}}_\ell$-perverse sheaf on Bun'_n, denoted by Aut_E, which is irreducible and nontrivial on each Bun'^d_n and satisfies the Hecke property with respect to E. If we denote $\bar{\mathbb{Q}}_\ell$ the trivial local system of rank 1, then we have a canonical decomposition $\mu_* \bar{\mathbb{Q}}_\ell = \bar{\mathbb{Q}}_\ell \oplus L_\mu$ for a local system L_μ on X of rank 1 and degree 2. There is a morphism $\det : \mathrm{Bun}_n \to \mathrm{Bun}_1 = \mathrm{Pic}_X$ induced by sending $\mathscr{M}$ to $\wedge^n \mathscr{M} \otimes \Omega_X^{-\frac{n(n-1)}{2}}$, which preserves degree. For a local system L on X of rank 1, we define $\mathsf{T}_L = \det^* \mathsf{A}_L$, where A_L is the local system placed at degree zero shifted from Aut_L. For simplicity, we let $\mathsf{T}_\mu := \mathsf{T}_{L_\mu}$.

We will define in Section 2 one of the primary objects in this article, the Asai local system $\mathrm{As}(E)$, which is a local system on X of rank n^2. It is constructed by the geometric analog of the classical method, called multiplicative induction or twisted tensor product (cf. [17, Section 7]) for the Asai representation of the Galois group. The following is our Main Theorem.

Main Theorem. Let notations be as above and E an irreducible local system on X' of rank n. Fix any integer δ and consider the following statements:

- (a) $E^\vee \simeq E^\sigma$;
- (b) $\mathrm{D}\mathsf{Aut}_E \simeq \sigma_n^* \mathsf{Aut}_E$, where D denotes the Verdier duality;
- (c) $\mathrm{R}\Gamma_c(\mathrm{Bun}'^\delta_n, \mathsf{Aut}_E \otimes \sigma_n^* \mathsf{Aut}_E) \neq 0$;
- (d^+) $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)$;
- (e^+) $\mathrm{R}\Gamma_c(\mathrm{Bun}^\delta_n, \mu_n^* \mathsf{Aut}_E) \neq 0$;
- (d^-) $L_\mu \subset \mathrm{As}(E)$;
- (e^-) $\mathrm{R}\Gamma_c(\mathrm{Bun}^\delta_n, \mu_n^* \mathsf{Aut}_E \otimes \mathsf{T}_\mu) \neq 0$.

Then we have the following equivalence relations

$$(a) \Longleftrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d^+) \text{ or } (d^-)$$

and

$$(d^+) \Longleftrightarrow (e^+); \qquad (d^-) \Longleftrightarrow (e^-).$$

□

The theorem has the following corollaries on the (compactly supported) total cohomology complexes that are not easy to see directly.

Corollary 1.1. Let E be an irreducible local system on X' of rank n as mentioned earlier, then

(1) The vanishing of the complexes $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n'^{\delta}, \mathsf{Aut}_E \otimes \sigma_n^* \mathsf{Aut}_E)$, $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E)$, or $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E \otimes \mathsf{T}_\mu)$ does not depend on δ;

(2) We have for any $\delta \in \mathbb{Z}$

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E) = 0 \Longleftrightarrow \mathrm{R}\Gamma(\mathrm{Bun}_n^{\delta}, \mu_n^! \mathsf{Aut}_E) = 0,$$

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E \otimes \mathsf{T}_\mu) = 0 \Longleftrightarrow \mathrm{R}\Gamma(\mathrm{Bun}_n^{\delta}, \mu_n^! \mathsf{Aut}_E \otimes \mathsf{T}_\mu) = 0;$$

(3) Between $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E)$ and $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E \otimes \mathsf{T}_\mu)$, there is at most one that is nontrivial and there is one, if and only if $E^\vee \simeq E^\sigma$. □

The Hecke property of Aut_E with respect to the Hecke operator H_n^n (cf. [6, Proposition 1.5]) will imply that, for example, the complex $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^* \mathsf{Aut}_E) = 0$ if and only if $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta+n}, \mu_n^* \mathsf{Aut}_E) = 0$. In general, (1) in Corollary 1.1. is not obvious without knowing the Main Theorem.

1.3 Mirabolic calculation

The proof of the Main Theorem relies on the computation of the direct image of certain complexes on the moduli stack corresponding to the mirabolic subgroup of GL_n. Let $\mathcal{M}_n$ (resp. $\mathcal{M}_n'$) be the moduli stack classifying pairs $(\mathscr{M}, s_1)$ (resp. $(\mathscr{M}', s_1')$), where $\mathscr{M}$ (resp. $\mathscr{M}'$) is an object in Bun_n (resp. Bun_n') and $s_1 : \Omega_X^{n-1} \hookrightarrow \mathscr{M}$ (resp. $s_1' : \Omega_{X'}^{n-1} \hookrightarrow \mathscr{M}'$) is an inclusion of $\mathscr{O}_X$-modules (resp. $\mathscr{O}_{X'}$-modules). Here, $\Omega_\bullet$ and $\mathscr{O}_\bullet$ stand for the sheaf of differentials and the structure sheaf, respectively and $\Omega_\bullet^{n-1}$ means $\Omega_\bullet^{\otimes n-1}$. Again, the pullback of pairs $(\mathscr{M}, s_1)$ under μ induces a morphism $\ddot{\mu}_n : \mathcal{M}_n \to \mathcal{M}_n'$ and we have obvious morphism $\pi : \mathcal{M}_n \to \mathrm{Bun}_n$ (resp. $\pi' : \mathcal{M}_n' \to \mathrm{Bun}_n'$) by forgetting s_1 (resp. s_1'). Let $\mathcal{M}_n^d$ (resp. $\mathcal{M}_n'^d$) be the inverse image of Bun_n^d (resp. $\mathrm{Bun}_n'^d$) under π (resp. π'). For any local system E on X' of rank n, there is a complex W_E on $\mathcal{M}_n'$ as defined in (3.2). By [6, 7.9], we know that if E is irreducible, then $\mathsf{W}_E \simeq \pi'^* \mathsf{Aut}_E[d-c]$ over $\mathcal{M}_n'^d$ for some integer c, which depends only on n and g, but *not* on d. We have the following theorem.

Theorem 1.2. Let notations be as mentioned earlier and E a local system on X' of rank n. Let L be a local system on X of rank 1, then we have for $d \geq 0$ a canonical isomorphism

$$\mathrm{R}\Gamma_c(\mathcal{M}_n^d, \ddot{\mu}_n^* \mathsf{W}_E \otimes \pi^* \mathsf{T}_L) \xrightarrow{\sim} \mathrm{R}\Gamma(X^{(d)}, (\mathrm{As}(E) \otimes L)^{(d)})[2d],$$

where $X^{(d)}$ and $(\mathrm{As}(E) \otimes L)^{(d)}$ denote the dth symmetric product of the curve X and its local system $\mathrm{As}(E) \otimes L$, respectively. □

The proof of the above theorem follows the same line in [14] for the geometrized Rankin–Selberg method due to Lysenko, which can be viewed as the geometrization of the well-known classical way treating the Rankin–Selberg integral of Jacquet, Piatetskii-Shapiro, and Shalika. We modify the argument in [14] to our situation, just as the modification of the classical argument in [3, 9]. It is interesting and useful to make these methodological comparison, which provides certain hint for the proof in the geometric counterpart. There is one technical issue when geometrizing the original argument of Flicker in [3]. In order to get the correct Euler product, one needs to twist the additive character ψ of $\mathbb{A}_E$ (non-canonically) such that it is trivial on $\mathbb{A}_F$. But in the geometric case, any such twist of ψ will ramify at some place, hence is not good for the geometric argument. Instead, we will not twist the additive character, but twist the underlying space where the "period integral" is taken, that is, the stack ${}_{\mu}\bar{\mathcal{Q}}_n^d$ introduced in Section 3. Moreover, this twist is canonical.

1.4 Connection with functoriality

Let us go back to the classical situation. As pointed out in [4, 9], the isomorphism $\Pi^\vee \cong \Pi^\sigma$ has important meaning on the Langlands functoriality. Let $\mathrm{U}_{n,E/F}$ denote the quasi-split unitary group of n variables with respect to the quadratic extension E/F. When n is odd, the representation Π with central character being trivial on $\mathbb{A}_F^\times$ that satisfies $\Pi^\vee \cong \Pi^\sigma$ should be a (stable) base change of a stable cuspidal L-packet of $\mathrm{U}_{n,E/F}$. When n is even, the representation Π satisfying $\Pi^\vee \cong \Pi^\sigma$ should also be a base change of a stable cuspidal L-packet of $\mathrm{U}_{n,E/F}$, either an unstable base change or a stable base change, according to whether Π is distinguished or $\omega_{E/F}$-distinguished.

In the geometric situation, let $\mathrm{U}_{n,X'/X}$ be the quasi-split unitary group with respect to X'/X, which is a reductive group over X and whose Langlands dual group ${}^L\mathrm{U}_{n,X'/X} \simeq \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell) \rtimes \pi_1(X)$ (the action defining the semi-product is trivial on $\pi_1(X')$). Since we have local systems parameterizing automorphic sheaves, it is easy to see the following fact. When n is odd, an irreducible local system E on X' of rank n is a (stable) base change (in fact, restriction) of a ${}^L\mathrm{U}_{n,X'/X}$-local system on X if and only if $E^\vee \simeq E^\sigma$ and $\mathrm{As}(\wedge^n E) = \bar{\mathbb{Q}}_\ell$, cf. 2. When n is even, an irreducible local system E on X' of rank n is an unstable or stable base change of a ${}^L\mathrm{U}_{n,X'/X}$-local system on X if and only if $E^\vee \simeq E^\sigma$. It is a dichotomy of being unstable or stable that corresponds to (d^+) or (d^-) in the Main Theorem. Hence, our Main Theorem provides a geometric/cohomological characterization of automorphic sheaves of $\mathrm{GL}_{n,X'}$ which should correspond to the (conjectural) automorphic sheaves of $\mathrm{U}_{n,X'/X}$ via geometric Langlands functoriality.

1.5 Structure of the article

In Section 2, we define the so-called Asai local system, which is the geometric analog of the classical Asai representation of the Galois group. These local systems are the main objects studied in this article. In Section 3, we prove the Main Theorem and its Corollary assuming Theorem 1.2. The proof implicitly uses the notion of poles that appear as cohomology on higher symmetric products of the base curve. The rest sections are devoted to proving Theorem 1.2.

In Section 4, after recalling the definition of Laumon's sheaf and the Whittaker sheaf, we reduce Theorem 1.2 to certain formula (see Proposition 4.2) relating direct images of Whittaker sheaves and symmetric products of constructible sheaves on the base curve. Section 5 and Section 6 are responsible for the proof of this formula.

1.6 Notations and conventions

We fix an algebraically closed field k and a rational prime ℓ that is invertible in k. Fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We work with Artin stacks—that is, algebraic stacks in the smooth topology (cf. [13])—locally of finite type over k. In the main body of the article, we will assume k has positive characteristic p and work in the derived category of unbounded complexes of $\bar{\mathbb{Q}}_\ell$-sheaves with constructible cohomology on an Artin stack $\mathcal{X}$ locally of finite type over k, in the sense of [10], which we denote by $\mathrm{D}(\mathcal{X})$. The operations f^*, $f^!$, f_*, $f_!$, $\otimes$ are applied only to locally bounded complexes and understood in the derived sense, where f is a morphism of finite type between stacks. For $K \in \mathrm{D}(\mathcal{X})$, we let $\mathrm{H}^i K$ be its ith cohomology sheaf with respect to the *usual* t-structure. In particular, we denote $\bar{\mathbb{Q}}_\ell$ the constant sheaf placed in degree 0. We fix a nontrivial additive character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^\times$ and denote AS_ψ the corresponding Artin–Schreier sheaf on the k-affine line $\mathbf{G}_{\mathrm{a},k}$. All the results and proof work perfectly when k has characteristic 0, after replacing $\mathrm{D}(\mathcal{X})$ by the derived category of $\mathscr{D}$-modules with holonomic cohomology and the Artin–Schreier sheaf AS_ψ by the $\mathscr{D}$-module "e^x".

When we say a stack $\mathcal{X}$ classifies something, we always mean that it is an Artin stack locally of finite type over k such that for any k-scheme S and morphism $S \to S'$, the groupoid $\mathrm{Hom}(S, \mathcal{X})$ and the functor $\mathrm{Hom}(S', \mathcal{X}) \to \mathrm{Hom}(S, \mathcal{X})$ are clearly understood. For example, for the stack $\mathcal{M}_n$ defined earlier, objects in $\mathrm{Hom}(S, \mathcal{M}_n)$ are the pairs $(\mathscr{M}_S, s_{1,S})$, where $\mathscr{M}_S$ is a vector bundle on $S \times X$ of rank n and $s_{1,S} : \mathscr{O}_S \boxtimes \Omega_X^{n-1} \hookrightarrow \mathscr{M}_S$ is an inclusion of $\mathscr{O}_{S\times X}$-modules such that the quotient $\mathscr{M}_S/\mathrm{Im} s_{1,S}$ is S-flat; morphisms between $(\mathscr{M}_S, s_{1,S})$ and $(\mathscr{N}_S, t_{1,S})$ are isomorphisms $f : \mathscr{M}_S \xrightarrow{\sim} \mathscr{N}_S$, such that $f \circ s_{1,S} = t_{1,S}$.

We identify Λ_m, the lattice of coweights of GL_m, with $\mathbb{Z}^m$. Let

$$\Lambda_{m,\mathrm{pos}} = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m | \lambda_1 + \cdots + \lambda_i \geq 0, i = 1, \ldots, m\};$$

$$\Lambda_{m,\mathrm{eff}} = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m | \lambda_i \geq 0, i = 1, \ldots, m\} \subset \Lambda_{m,\mathrm{pos}};$$

$$\Lambda_{m,+} = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0\} \subset \Lambda_{m,\mathrm{eff}};$$

$$\Lambda_{m,-} = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m | 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m\} \subset \Lambda_{m,\mathrm{eff}}.$$

For $d \in \mathbb{Z}$, we let $\Lambda_{m,?}^d = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_{m,?} | \lambda_1 + \cdots + \lambda_m = d\}$ for $? = \emptyset$, pos, eff, $+$, $-$. Finally, we denote $\mathfrak{S}_m$ the permutation group of m elements.

For a connected smooth scheme S over k and $d \geq 0$, we define its dth symmetric product $S^{(d)}$ as the quotient of S^d by the action of $\mathfrak{S}_d$ ($S^{(0)} = \mathrm{Spec} k$). For a local system E on S of rank m, we let $E^{(d)} = (\mathrm{sym}_! E^{\boxtimes d})^{\mathfrak{S}_d}$ be the dth symmetric product of E, where $\mathrm{sym} : S^d \to S^{(d)}$ is the natural projection. Hence $E^{(d)}[d \cdot \dim S]$ is a perverse sheaf on $S^{(d)}$. It is irreducible if and only if E is. We denote by $\det E = \wedge^m E$ the determinant of E, where m is the rank of E.

We fix X to be a connected smooth proper curve over k, then $X^{(d)}$ is the variety of effective divisors of degree d on X—that is, collections of d closed points. For $\lambda \in \Lambda_{m,\mathrm{pos}}^d$, we let $X^\lambda = \prod_{i=1}^m X^{(\lambda_1 + \cdots + \lambda_i)}$. Let X_+^λ (resp. X_-^λ) be the closed subscheme of X^λ such that $(D_1, D_1 + D_2, \ldots, D_1 + \cdots + D_m) \in X^\lambda$ belongs to X_+^λ (resp. X_-^λ) if and only if $D_1 \geq \cdots \geq D_m \geq 0$ (resp. $0 \leq D_1 \leq \cdots \leq D_m$), which is nonempty if and only if $\lambda \in \Lambda_{m,+}^d$ (resp. $\lambda \in \Lambda_{m,-}^d$). They are both contained in the subscheme $X_{\mathrm{eff}}^\lambda := \prod_{i=1}^m X^{(\lambda_i)} \hookrightarrow X^\lambda$ (when $\lambda_i \geq 0$). We denote by $\mathrm{sym}^\lambda : X^\lambda \to X^{(d)}$ the projection to the last factor, and also for its restrictions to X_{eff}^λ, X_+^λ, or X_-^λ.

Let X be as above and $\mathscr{M}$ a coherent sheaf on X; its degree $\deg \mathscr{M}$ is normalized to be its usual degree minus $m(m-1)(g-1)$, where $m = \mathrm{rk} \mathscr{M}$ is the rank of $\mathscr{M}$ and g is the genus of X. Hence $\deg \bigoplus_{i=0}^{m-1} \Omega_X^i = 0$.

2 Asai Local Systems and Conjugate Self-Duality

2.1 Asai local systems

Consider an étale morphism $\mu : X' \to X$ of degree $l \geq 1$ with X' being connected, and a local system E on X' of rank $m \geq 1$. Since μ is étale, we get a morphism $\mu^{(1)} : X \to X'^{(l)} - \Delta =: X'^{(l)}_{\mathrm{rss}}$ sending a point x to the divisor $\mu^{-1}(x)$, where we abuse the notation Δ for the union of all possible diagonals and hence $X'^{(l)}_{\mathrm{rss}}$ is the open subscheme of reduced divisors.

Definition 2.1. We define the *Asai local system* $\mathrm{As}(E)$ by

$$\mathrm{As}(E) := \mu^{(1)*}(\mathrm{sym}_! E^{\boxtimes l})^{\mathfrak{S}_l} = \mu^{(1)*}(E^{(l)}).$$

Since the symmetrization morphism $\mathrm{sym} : X'^l \to X'^{(l)}$ is étale with Galois group $\mathfrak{S}_l$ away from Δ, $\mathrm{As}(E)$ is a local system on X of rank m^l. □

Remark.

(1) The construction of Asai local systems is canonical, not like the construction of the classical Asai representation. For the later one, we need to choose a set of representatives for the left coset of $\pi_1(X', x')$ in $\pi_1(X, x)$ and show that the isomorphism class of the representation obtained from the process of multiplicative induction is independent of the choice, cf. [17, Section 7].

(2) As pointed out by the referee, there is an equivalent definition of $\mathrm{As}(E)$ when $\mu : X' \to X$ is an étale Galois covering with Galois group G. We know that the functor μ^* from the category of local systems on X to the category of local systems on X' equipped with a descent datum for the G-action is an equivalence. We define $\mathrm{As}(E)$ such that, under the above equivalence, it corresponds to $\bigotimes_{\sigma \in G} \sigma^* E$ equipped with the natural descent datum with respect to μ. With this definition, the proof of Lemma 2.3(2) is straightforward. □

From now on, we will only consider the case $l = 2$. Let $\sigma : X' \to X'$ be the unique nontrivial automorphism such that $\mu = \mu \circ \sigma$, and $E^\sigma = \sigma^* E$. Since $l = 2$, we have $\mu_* \bar{\mathbb{Q}}_\ell = \bar{\mathbb{Q}}_\ell \oplus L_\mu$ for some local system L_μ on X of rank 1.

Lemma 2.2.

(1) We have $\mathrm{As}(E^\vee) \simeq \mathrm{As}(E)^\vee$ and $\mathrm{As}(E) \simeq \mathrm{As}(E^\sigma)$.

(2) There is a canonical isomorphism $\mu_*(E \otimes E^\sigma) \xrightarrow{\sim} \mathrm{As}(E) \oplus \mathrm{As}(E) \otimes L_\mu$. □

Proof. (1) is straightforward. For (2), consider the following diagram:

$$\begin{array}{ccc} X' & \xrightarrow{(1,\sigma)} & X' \times X' - \Delta \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle \mathrm{sym}} \\ X & \xrightarrow{\mu^{(1)}} & X'^{(2)} - \Delta \end{array}$$

which is Cartesian. Over $X' \times X' - \Delta$, we have a canonical isomorphism $\mathrm{sym}^* E^{(2)} \xrightarrow{\sim} E^{\boxtimes 2}$. Hence

$$\mu^* \mathrm{As}(E) \simeq \mu^* \mu^{(1)*} E^{(2)} \simeq (1, \sigma)^* \mathrm{sym}^* E^{(2)} \xrightarrow{\sim} (1, \sigma)^* E^{\boxtimes 2} \simeq E \otimes E^{\sigma}.$$

By the projection formula, we have

$$\mu_*(E \otimes E^{\sigma}) \simeq \mathrm{As}(E) \otimes \mu_* \bar{\mathbb{Q}}_\ell \simeq \mathrm{As}(E) \oplus \mathrm{As}(E) \otimes L_\mu.$$ ■

2.2 Conjugate self-duality

We make the following definition as the geometric analog from the classical representation theory, cf. [8].

Definition 2.3. Let the situation be as above, we say E is *conjugate self-dual* if there is a nontrivial map $b : E \otimes E^{\sigma} \to \bar{\mathbb{Q}}_\ell$ between local systems, which is equivalent to $E^{\vee} \simeq E^{\sigma}$ when E is irreducible.

Let $b^{\sigma} : E \otimes E^{\sigma} \simeq E^{\sigma} \otimes E = \sigma^*(E \otimes E^{\sigma}) \xrightarrow{\sigma^* b} \sigma^* \bar{\mathbb{Q}}_\ell = \bar{\mathbb{Q}}_\ell$ be the composition of $\sigma^* b$ with the transposition of two factors. Then we say E is *conjugate orthogonal* (resp. *conjugate symplectic*) if moreover, we have $b^{\sigma} = b$ (resp. $b^{\sigma} = -b$) and denote $c(E) = 1$ (resp. $c(E) = -1$). □

It is easy to see that if E is irreducible and conjugate self-dual, then it is either conjugate orthogonal or conjugate symplectic, and $c(E) = c(E^{\vee}) = c(E^{\sigma})$. Its determinant $\det E$ is also conjugate self-dual and $c(\det E) = c(E)^m$, where m is the rank of E. We have the following criterion.

Lemma 2.4. For irreducible conjugate self-dual local system E, $c(E) = 1$ (resp. $c(E) = -1$) if and only if $\mathrm{As}(E)$ contains $\bar{\mathbb{Q}}_\ell$ (resp. L_μ). □

Proof. It is obvious that we have dichotomy on both sides; hence we only need to show, for example, that $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)$ (resp. $L_\mu \subset \mathrm{As}(E)$) implies $c(E)=1$ (resp. $c(E)=-1$). We will prove the first case and the second is similar. By duality and adjunction, we have a map

$$(\mathrm{sym}_! E^{\vee\boxtimes 2})^{\mathfrak{S}_2} = E^{\vee(2)} \to \mu_*^{(1)}\bar{\mathbb{Q}}_\ell$$

over $X'^{(2)} - \Delta$. By the pullback under sym, we get a nontrivial map $b' : E^{\vee\boxtimes 2} \simeq \mathrm{sym}^* E^{\vee(2)} \to \mathrm{sym}^*\mu_*^{(1)}\bar{\mathbb{Q}}_\ell$, which makes the following diagram commutative

$$\begin{array}{ccc} E^\vee \boxtimes E^\vee & \xrightarrow{\sim} & E^\vee \boxtimes E^\vee \\ & {\scriptstyle b'}\searrow \quad \swarrow{\scriptstyle b'} & \\ & \mathrm{sym}^*\mu_*^{(1)}\bar{\mathbb{Q}}_\ell & \end{array}$$

where the upper arrow is the transposition. By the pullback under $(1,\sigma)$, we get $b^\vee : E^\vee \otimes E^{\vee\sigma} \to \bar{\mathbb{Q}}_\ell$ such that $c(E^\vee)=1$, where $b^\vee = (1,\sigma)^* b'$. But this implies that $c(E)=1$. ■

2.3 Symmetrization

For $d \geq 0$, we identify $\mathfrak{S}_d$ with the set of d-by-d permutation matrices. Let $\mathfrak{T}_{2d}$ be the subset of $\mathfrak{S}_{2d}$ consisting of those symmetric permutation matrices whose diagonal entries are 0. For $t=(t_{ij}) \in \mathfrak{T}_{2d}$, we define $1 = i_1(t) < \cdots < i_d(t)$ and $i_a(t) < j_a(t)$ $(a = 1,\ldots,d)$ by the condition that $t_{i_a(t) j_a(t)} = 1$. Then they are uniquely determined and $\{i_a(t), j_a(t) | a = 1,\ldots,d\} = \{1,\ldots,2d\}$. Let

$$\mu^{(d)} : X^{(d)} \to (X'^{(2)})^{(d)} \to X'^{(2d)}$$

be the composed map. For any $t=(t_{ij}) \in \mathfrak{T}_{2d}$, we define a morphism

$$\mathfrak{n}_t : X'^{d} \to X'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

such that the $i_a(t)$th component of $\mathfrak{n}_t(x_1,\ldots,x_d)$ is x_a and the $j_a(t)$th component of $\mathfrak{n}_t(x_1,\ldots,x_d)$ is $\sigma(x_a)$. It is obvious that $\mathfrak{n}_t(x_1,\ldots,x_d)$ locates in $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$. Define

$$\mathfrak{n} := \coprod_{\mathfrak{T}_{2d}} \mathfrak{n}_t : \mathbf{X}_d := \coprod_{\mathfrak{T}_{2d}} X'^{d}_t \to X'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

as the disjoint union over all $t \in \mathfrak{T}_{2d}$ with $X'^{d}_{t} = X'^{d}$. Moreover, let pr denote the projection $X'^{2d} \times_{X'^{(2d)}} X^{(d)} \to X^{(d)}$. Then we have the following lemma.

Lemma 2.5.

(1) The scheme $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ has pure dimension d and its irreducible components one-to-one correspond to schemes X'^{d}_{t} for $t \in \mathfrak{T}_{2d}$;

(2) The morphism $\mathfrak{n}$ is finite and isomorphic over an open dense subscheme of $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$. Since $\mathbf{X}_d$ is smooth, we have $\mathfrak{n}_! \bar{\mathbb{Q}}_\ell[d] \simeq \mathrm{IC}$, where IC denotes the intersection cohomology complex on $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$;

(3) We have a canonical isomorphism $\mathrm{pr}_!((E^{\boxtimes 2d} \boxtimes \bar{\mathbb{Q}}_\ell) \otimes \mathfrak{n}_! \bar{\mathbb{Q}}_\ell)^{\mathfrak{S}_{2d}} \xrightarrow{\sim} \mathrm{As}(E)^{(d)}$ on $X^{(d)}$, where $\mathfrak{S}_{2d}$ acts on X'^{2d} naturally by permuting $2d$ factors. □

The proof of (3) is suggested by the referee.

Proof. For any $t \in \mathfrak{T}_{2d}$, we define $\mathbf{X}_t$ to be the subscheme of $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ consisting of points $(x_1, \dots, x_{2d})$ such that if $j = \min\{j' | x_i = \sigma(x_{j'}), i < j'\}$, then $t_{ij} = 1$. Then the collection of $\mathbf{X}_t$ form a stratification of $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$. Moreover, define $\mathfrak{n}^t : \mathbf{X}_t \to X'^{d}_{t}$ by $(x_1, \dots, x_{2d}) \mapsto (x_{i_1(t)}, \dots, x_{i_d(t)})$. Then $\mathfrak{n}_t \circ \mathfrak{n}^t = \mathrm{id}$. Hence $\mathfrak{n}_t$ is an isomorphism onto $\mathbf{X}_t$ away from the diagonal, and $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ has pure dimension d. Since $\mathfrak{n}_t$ is a closed immersion for each t and X'^{d}_{t} is smooth, (1) and (2) follow.

For (3), since both perverse sheaves are intermediate extensions from $X^{(d)}_{\mathrm{rss}}$, which is the open subscheme of reduced divisors, we only need to prove the isomorphism over $X^{(d)}_{\mathrm{rss}}$. This is equivalent to prove the isomorphism

$$\mu^{(d)*} E^{(2d)} \xrightarrow{\sim} \mathrm{As}(E)^{(d)}$$

over $X^{(d)}_{\mathrm{rss}}$. Consider the diagram

$$\begin{array}{ccc} X^d & \xrightarrow{(\mu^{(2)})^d} & (X'^{(2)})^d \\ \downarrow{\scriptstyle \mathrm{sym}} & & \downarrow{\scriptstyle s'} \\ X^{(d)} & \xrightarrow{\mu^{(d)}} & X'^{(2d)} \end{array}$$

and we only need to establish a $\mathfrak{S}_d$-equivariant isomorphism after restricting by sym over $X^{d}_{\mathrm{rss}} := \mathrm{sym}^{-1}(X^{(d)}_{\mathrm{rss}})$. Over $(X'^{(2)})^d$ there is a canonical morphism $s'^{*} E^{(2d)} \to (E^{(2)})^{\boxtimes d}$, which is an isomorphism (at least) over $s'^{-1}(X'^{(2d)}_{\mathrm{rss}})$. Restricting by $(\mu^{(2)})^d$, we get a $\mathfrak{S}_d$-equivariant isomorphism

$$\mathrm{sym}^*\mu^{(d)*}E^{(2d)} \simeq (\mu^{(2)})^{d*}s'^*E^{(2d)} \xrightarrow{\sim} \mathrm{As}(E)^{\boxtimes d} \simeq \mathrm{sym}^*\mathrm{As}(E)^{(d)}$$

over X^d_{rss}. Hence (3) is proved. ■

3 Steps for the Proof

Proof of $(d^{\pm}) \Longleftrightarrow (e^{\pm})$. We first prove the equivalence between $(d^?)$ and $(e^?)$ for $? = +, -$ assuming Theorem 1.2. Since two cases are similar, we prove only the first case. By the Hecke property of Aut_E (cf. [6, Proposition 1.5]), for a fixed integer δ, the nonvanishing of $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta}, \mu_n^*\mathsf{Aut}_E)$ is equivalent to the nonvanishing of $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^{\delta+n\delta'}, \mu_n^*\mathsf{Aut}_E)$ for any $\delta' \in \mathbb{Z}$.

Recall that a vector bundle $\mathscr{M}$ on X is called *very unstable* if it can be written as $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ with $\mathscr{M}_i \neq 0$ and $\mathrm{Ext}^1(\mathscr{M}_1, \mathscr{M}_2) = 0$. By [6, Lemma 3.3], for any line bundle $\mathscr{L}$ on X, there exists an integer $c_n(\mathscr{L})$ such that if $d \geq c_n(\mathscr{L})$ and $\mathscr{M} \in \mathrm{Bun}_n^d(k)$ with $\mathrm{Hom}(\mathscr{M}, \mathscr{L}) \neq 0$, then $\mathscr{M}$ is very unstable. Define $\mathcal{U}$ (resp. $\mathcal{U}'$) to be the open substack of Bun_n (resp. Bun'_n) given by the condition $\mathrm{Hom}(\mathscr{M}, \Omega_X^n) = 0$ (resp. $\mathrm{Hom}(\mathscr{M}', \Omega_{X'}^n) = 0$). Then it is well-known that $\mathcal{U} \cap \mathrm{Bun}_n^d$ and $\mathcal{U}' \cap \mathrm{Bun}_n'^d$ are of finite type for any $d \in \mathbb{Z}$. Now we fix an even integer $2d \geq c_n(\Omega_{X'}^n)$ and let us consider W_E defined in (3.2). We know that by [6, 7.9], $\mathsf{W}_E \simeq \pi'^*\mathsf{Aut}_E[2d-c]$ over $\mathrm{Bun}_n'^{2d}$ for an integer c, which only depends on n and g, but not on d. Let $d_1 = 2d - c$ for simplicity.

Since E is irreducible, Aut_E is cuspidal and hence $\mathsf{Aut}_E|_{\mathrm{Bun}_n'^{2d}}$ is supported on $\mathcal{U}' \cap \mathrm{Bun}_n'^{2d}$ by [6, Lemma 9.4]. Since $\mu_n^{-1}(\mathcal{U}') \subset \mathcal{U}$, we have

$$\pi_!\ddot{\mu}_n^*\mathsf{W}_E \simeq \pi_!\ddot{\mu}_n^*\pi'^*\mathsf{Aut}_E[d_1] \simeq \pi_!\pi^*\mu_n^*\mathsf{Aut}_E[d_1] \simeq \pi_!(\bar{\mathbb{Q}}_\ell|_{\pi^{-1}(\mathcal{U}\cap\mathrm{Bun}_n^d)}) \otimes \mu_n^*\mathsf{Aut}_E[d_1]$$

over Bun_n^d. By Serre duality, $\pi^{-1}(\mathcal{U} \cap \mathrm{Bun}_n^d)$ is a vector bundle of some rank $d_2 > 0$ over $\mathcal{U} \cap \mathrm{Bun}_n^d$ with zero section removed. Hence over $\mathcal{U} \cap \mathrm{Bun}_n^d$, we have a distinguished triangle

$$\pi_!\bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell[-2d_2] \to \bar{\mathbb{Q}}_\ell \xrightarrow{+1}.$$

Tensoring with $\mu_n^*\mathsf{Aut}_E[d_1]$ and applying $\mathrm{R}\Gamma_{\mathrm{c}}$, we have a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{M}_n^d, \ddot{\mu}_n^*\mathsf{W}_E) \to \mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^d, \mu_n^*\mathsf{Aut}_E)[d_1 - 2d_2] \to \mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^d, \mu_n^*\mathsf{Aut}_E)[d_1] \xrightarrow{+1}.$$

Hence $\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{M}_n^d, \ddot{\mu}_n^*\mathsf{W}_E) = 0$ is equivalent to $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^d, \mu_n^*\mathsf{Aut}_E) = 0$, since the later complex is bounded from above.

Now if $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)$, then there exists $d \geq 0$ such that $d = \delta + n\delta'$ with $\delta' \in \mathbb{Z}$, $\mathrm{R}\Gamma(X^{(d)}, \mathrm{As}(E)^{(d)}) \neq 0$ and $2d \geq c_n(\Omega_{X'}^n)$. Hence by Theorem 1.2 and the above argument, $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^d, \mu_n^*\mathsf{Aut}_E) \neq 0$ which implies $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^\delta, \mu_n^*\mathsf{Aut}_E) \neq 0$. Conversely, if $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^\delta, \mu_n^*\mathsf{Aut}_E) \neq 0$, then pick up some δ' such that $d := \delta + n\delta' > \max\{\frac{1}{2}c_n(\Omega_{X'}^n), n^2(2g-2)\}$. We have $\mathrm{R}\Gamma(X^{(d)}, \mathrm{As}(E)^{(d)}) \neq 0$ for such d which forces $\mathrm{H}^0(X, \mathrm{As}(E))$ or $\mathrm{H}^2(X, \mathrm{As}(E))$ to be nonzero. Otherwise, since we know that the direct sum of cohomology of $\mathrm{R}\Gamma(X^{(d)}, \mathrm{As}(E)^{(d)})$ is isomorphic to

$$\bigoplus_{d_0+d_1+d_2=d} \mathrm{Sym}^{d_0}\mathrm{H}^0(X, \mathrm{As}(E)) \otimes \wedge^{d_1}\mathrm{H}^1(X, \mathrm{As}(E)) \otimes \mathrm{Sym}^{d_2}\mathrm{H}^2(X, \mathrm{As}(E))$$

by Künneth formula, the dth wedge product of $\mathrm{H}^1(X, \mathrm{As}(E))$ is nonzero. By our assumption that $d > n^2(2g-2)$ which is the Euler characteristic of $\mathrm{As}(E)$ and also the dimension of $\mathrm{H}^1(X, \mathrm{As}(E))$ (if both $\mathrm{H}^0(X, \mathrm{As}(E))$ and $\mathrm{H}^2(X, \mathrm{As}(E))$ are zero), we get a contradiction. Hence $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)$ or $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)^\vee \simeq \mathrm{As}(E^\vee)$. By Lemma 2.5, $E^\vee$ is conjugate orthogonal in the later case. Hence E is also conjugate orthogonal which, again by the same lemma, implies that $\bar{\mathbb{Q}}_\ell \subset \mathrm{As}(E)$. ■

Proof of $(a) \Longleftrightarrow (c)$**: Rankin–Selberg.** The proof of the equivalence of (a) and (c) follows the same line as above but replacing Theorem 1.2 by the following main result of Lysenko, which can be viewed as the split case (i.e., X' is disconnected).

Theorem 3.1 ([14]). For any local systems E_1, E_2 on X of rank n and any $d \geq 0$, there is a canonical isomorphism

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{M}_n^d, \mathsf{W}_{E_1} \otimes \mathsf{W}_{E_2}) \xrightarrow{\sim} \mathrm{R}\Gamma(X^{(d)}, (E_1 \otimes E_2)^{(d)})[2d]. \qquad \square$$

Here, W_{E_i} $(i = 1, 2)$ are complexes on $\mathcal{M}_n$ similarly defined by (3.2) but with respect to X instead of X'. The restriction of W_E on $\mathcal{M}_n^d$ is denoted by ${}_n\mathscr{K}_E^d$ in [14]. See [14, Remark 1(ii), 3(i); Definition 2] for more details. By the same argument in the above proof, we have the following lemma.

Lemma 3.2. Let E_1 and E_2 be irreducible local systems on X of rank n. Then for any integer δ, $\mathrm{R}\Gamma_{\mathrm{c}}(\mathrm{Bun}_n^\delta, \mathsf{Aut}_{E_1} \otimes \mathsf{Aut}_{E_2}) \neq 0$ if and only if $E_2 \simeq E_1^\vee$. □

Applying this lemma to $X=X'$, $E_1=E$ and $E_2=E^\sigma$, we get the desired equivalence since it is easy to see that $\sigma_n^*\mathsf{Aut}_E\simeq\mathsf{Aut}_{E^\sigma}$ by the construction. ■

Proof of the rest. The equivalence of (a) and (b) is due to the above lemma and the fact that $\mathsf{DAut}_E\simeq\mathsf{Aut}_{E^\vee}$. Lemma 2.5 implies that $(a)\Longleftrightarrow(d^+)$ or (d^-). Hence the Main Theorem has been proved.

For Corollary 1.1, (1) follows from the Main Theorem; (2) is due to the fact that $\mathrm{As}(E)^\vee\simeq\mathrm{As}(E^\vee)$ and $c(E)=c(E^\vee)$; (3) is due to Lemma 2.5. ■

3.1 Laumon's sheaf and Whittaker sheaf

Let us briefly recall the definition of Laumon's sheaf Lau_E^d. Let E be a local system on X' of rank n and $d\geq 0$. Denote Coh_n^d the stack classifying coherent sheaves $\mathscr{F}$ on X of generic rank n and degree d. Inside Coh_0^d, there is an open substack $\mathrm{Coh}_{0,\leq m}^d$ defined by the additional condition that the restriction of $\mathscr{F}$ at any geometric point has dimension $\leq m$. Denote Fl_0^d the stack classifying complete flags of torsion subsheaves:

$$0=\mathscr{F}_0\subset\mathscr{F}_1\subset\cdots\subset\mathscr{F}_d$$

such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ is in Coh_0^1. We have morphisms $\mathfrak{p}:\mathrm{Fl}_0^d\to\mathrm{Coh}_0^d$ by remembering $\mathscr{F}_d$ and $\mathfrak{q}:\mathrm{Fl}_0^d\to(\mathrm{Coh}_0^1)^d$ by remembering $(\mathscr{F}_i/\mathscr{F}_{i-1})_{i=1}^d$. Define $\mathrm{Fl}_{0,\leq m}^d=\mathrm{Fl}_0^d\times_{\mathrm{Coh}_0^d}\mathrm{Coh}_{0,\leq m}^d$. Similarly, we define for X' the stacks $\mathrm{Coh}_n'^d$, $\mathrm{Coh}_{0,\leq m}'^d$, $\mathrm{Fl}_0'^d$, $\mathrm{Fl}_{0,\leq m}'^d$ and $\mathfrak{p}'$, $\mathfrak{q}'$. Denote $\mathrm{div}_?^d:\mathrm{Coh}_{0,?}^d\to X^{(d)}$ or $\mathrm{Coh}_{0,?}'^d\to X'^{(d)}$ the norm morphism and $\mathrm{div}_?=\mathrm{div}_?^1$ for $?=\emptyset,\leq m$. Moreover, the pullback under μ induces a morphism $\mu_0:\mathrm{Coh}_0^d\to\mathrm{Coh}_0'^{2d}$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Coh}_0'^d & \xleftarrow{\mathfrak{p}'} & \mathrm{Fl}_0'^d & \xrightarrow{\mathfrak{q}'} & \mathrm{Coh}_0'^1\times\cdots\times\mathrm{Coh}_0'^1 \\
\Big\downarrow{\scriptstyle \mathrm{div}^d} & & & & \Big\downarrow{\scriptstyle \mathrm{div}^{\times d}} \\
X'^{(d)} & & \xleftarrow{\quad\mathrm{sym}\quad} & & X'^d
\end{array}$$

and define *Springer's sheaf* $\mathsf{Spr}_E^d=\mathfrak{p}'_!\mathfrak{q}'^*(\mathrm{div}^{\times d})^*E^{\boxtimes d}$ with a natural action by $\mathfrak{S}_d$ and *Laumon's sheaf* $\mathsf{Lau}_E^d=\mathrm{Hom}_{\mathfrak{S}_d}(\mathrm{triv},\mathsf{Spr}_E^d)$ on $\mathrm{Coh}_0'^d$.

For $d\geq 0$ and $n\geq 1$, we introduce the stack $\bar{\mathfrak{Q}}_n^d$ classifying the data $(\mathscr{M},(s_i))$, where $\mathscr{M}$ is a vector bundle on X of rank n and degree d, and s_i are injective homomorphisms of coherent sheaves

$$s_i : \Omega_X^{(n-1)+\cdots+(n-i)} \to \wedge^i \mathscr{M}, \qquad i = 1, \ldots, n$$

such that they satisfy the Plücker relations; see [1, Section 1; 5, Section 2; 6, Section 4] or [14, Section 4] for details. Similarly, we define the stack $\bar{\mathcal{Q}}_n'^d$ classifying the data $(\mathscr{M}', (s_i'))$ but now on X'. For our purpose, we need to introduce a twisted version of $\bar{\mathcal{Q}}_n^d$ as follows.

Since μ is étale of degree 2, we have a canonical decomposition $\mu_*\mathscr{O}_{X'} = \mathscr{O}_X \oplus \mathscr{L}_\mu$ for a line bundle $\mathscr{L}_\mu$ on X. We also have canonical isomorphisms $\mu^*\Omega_X \simeq \Omega_{X'}$ and $\mu^*\mathscr{L}_\mu \simeq \mathscr{O}_{X'}$. Let ${}_\mu\bar{\mathcal{Q}}_n^d$ be the stack classifying the data $(\mathscr{M}, (s_i))$ similar to the previous one, but now s_i are injective homomorphisms of coherent sheaves

$$s_i : \Omega_X^{(n-1)+\cdots+(n-i)} \otimes \mathscr{L}_\mu^{0+\cdots+(i-1)} \to \wedge^i \mathscr{M}, \qquad i = 1, \ldots, n$$

still satisfying the Plücker relations. In fact, the stack ${}_\mu\bar{\mathcal{Q}}_n^d$ is nothing but the stack $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ defined in [5], where $\mathcal{F}_T$ is the $T \simeq \mathrm{GL}_1^n$-bundle on X corresponding to the n-tuple of line bundles:

$$(\Omega_X^{n-1}, \Omega_X^{n-2} \otimes \mathscr{L}_\mu, \ldots, \Omega_X \otimes \mathscr{L}_\mu^{n-2}, \mathscr{L}_\mu^{n-1}).$$

The pullback under μ induces a natural morphism $\bar{\mu}_n : {}_\mu\bar{\mathcal{Q}}_n^d \to \bar{\mathcal{Q}}_n'^{2d}$. Moreover, we have the natural morphism $\pi_n : {}_\mu\bar{\mathcal{Q}}_n^d \to \mathcal{M}_n^d$ (resp. $\pi_n' : \bar{\mathcal{Q}}_n'^d \to \mathcal{M}_n'^d$) by forgetting $s_2, \ldots, s_n$ (resp. $s_2', \ldots, s_n'$). Let $\mathfrak{p}_n = \pi \circ \pi_n : {}_\mu\bar{\mathcal{Q}}_n^d \to \mathrm{Bun}_n^d$ (resp. $\mathfrak{p}_n' = \pi' \circ \pi_n' : \bar{\mathcal{Q}}_n'^d \to \mathrm{Bun}_n'^d$). We have morphisms $\mathfrak{c}_n : {}_\mu\bar{\mathcal{Q}}_n^d \to \mathrm{Coh}_0^d$ (resp. $\mathfrak{c}_n' : \bar{\mathcal{Q}}_n'^d \to \mathrm{Coh}_0'^d$) sending $(\mathscr{M}, (s_i))$ (resp. $(\mathscr{M}', (s_i'))$) to $\wedge^n\mathscr{M}/\mathrm{Im} s_n$ (resp. $\wedge^n\mathscr{M}'/\mathrm{Im} s_n'$) and $\mathfrak{d}_n = \mathrm{div}^d \circ \mathfrak{c}_n : {}_\mu\bar{\mathcal{Q}}_n^d \to X^{(d)}$ (resp. $\mathfrak{d}_n' = \mathrm{div}^d \circ \mathfrak{c}_n' : \bar{\mathcal{Q}}_n'^d \to X'^{(d)}$). Finally, let $\bar{\mathcal{Q}}_n = \coprod_{d\geq 0} \bar{\mathcal{Q}}_n^d$, $\bar{\mathcal{Q}}_n' = \coprod_{d\geq 0} \bar{\mathcal{Q}}_n'^d$ and ${}_\mu\bar{\mathcal{Q}}_n = \coprod_{d\geq 0} {}_\mu\bar{\mathcal{Q}}_n^d$.

Inside ${}_\mu\bar{\mathcal{Q}}_n^0$, there is an open substack $j : {}_\mu\mathcal{Q}_n^0 \hookrightarrow {}_\mu\bar{\mathcal{Q}}_n^0$ classifying the data $(\mathscr{M}_i, r_i)$ where

$$0 = \mathscr{M}_0 \subset \mathscr{M}_1 \subset \cdots \subset \mathscr{M}_n$$

is a complete flag of sub-vector bundles and

$$r_i : \Omega_X^{n-i} \otimes \mathscr{L}_\mu^{i-1} \xrightarrow{\sim} \mathscr{M}_i/\mathscr{M}_{i-1}$$

are isomorphisms for $i = 1, \ldots, n$. Similarly, one has $\mathcal{Q}_n'^0$ and $j' : \mathcal{Q}_n'^0 \hookrightarrow \bar{\mathcal{Q}}_n'^0$. We define the evaluation map $\mathrm{ev} : \mathcal{Q}_n'^0 \to \mathbf{G}_{a,k}$ to be sum of the classes in $\mathrm{Ext}^1(\Omega_{X'}^{n-i-1}, \Omega_{X'}^{n-i}) \simeq \mathbf{G}_{a,k}$

of the extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{M}'_i/\mathscr{M}'_{i-1} & \longrightarrow & \mathscr{M}'_{i+1}/\mathscr{M}'_{i-1} & \longrightarrow & \mathscr{M}'_{i+1}/\mathscr{M}'_i & \longrightarrow & 0 \\ && s'_i \uparrow \wr &&&& \wr \downarrow {s'_{i+1}}^{-1} && \\ && \Omega^{n-i}_{X'} &&&& \Omega^{n-i-1}_{X'} && \end{array}$$

for $i = 1, \dots, n-1$. Define $\overline{\mathsf{AS}}^0_\psi := j'_! \mathrm{ev}^* \mathsf{AS}_\psi$ as a sheaf on $\bar{\mathcal{Q}}'^0_n$.

Let Mod^d_n be the stack classifying modifications $(\mathscr{M} \hookrightarrow \mathscr{N})$ of vector bundles on X of rank n such that $\deg(\mathscr{N}/\mathscr{M}) = d$. We have morphisms $\mathfrak{h}_\leftarrow : \mathrm{Mod}^d_n \to \mathrm{Bun}_n$ (resp. $\mathfrak{h}_\rightarrow : \mathrm{Mod}^d_n \to \mathrm{Bun}_n$) sending $(\mathscr{M} \hookrightarrow \mathscr{N})$ to $\mathscr{M}$ (resp. $\mathscr{N}$) and $\mathfrak{c} : \mathrm{Mod}^d_n \to \mathrm{Coh}^d_0$ sending $(\mathscr{M} \hookrightarrow \mathscr{N})$ to $\mathscr{N}/\mathscr{M}$. Similarly, we define Mod'^d_n, $\mathfrak{h}'_\leftarrow$, $\mathfrak{h}'_\rightarrow$ and $\mathfrak{c}'$. Consider the following commutative diagram (cf. [6, Section 4.3]):

$$\begin{array}{ccccc} \bar{\mathcal{Q}}'^0_n & \xleftarrow{\bar{\mathfrak{h}}'_\leftarrow} & \bar{\mathcal{Z}}'^d_n & \xrightarrow{\bar{\mathfrak{h}}'_\rightarrow} & \bar{\mathcal{Q}}'^d_n \\ \mathfrak{p}'_n \downarrow && \mathfrak{q}'_n \downarrow && \downarrow \mathfrak{p}'_n \\ \mathrm{Bun}'^0_n & \xleftarrow{\mathfrak{h}'_\leftarrow} & \mathrm{Mod}'^d_n & \xrightarrow{\mathfrak{h}'_\rightarrow} & \mathrm{Bun}'^d_n \end{array}$$

where $\bar{\mathcal{Z}}'^d_n = \bar{\mathcal{Q}}'^0_n \times_{\mathrm{Bun}'^0_n} \mathrm{Mod}'^d_n$ is the fibre product of $\mathfrak{p}'_n$ and $\mathfrak{h}'_\leftarrow$, and hence the left square is Cartesian. The *Whittaker sheaf* $\mathsf{W}_{E,n}$ on $\bar{\mathcal{Q}}'_n$ is defined by the formula

$$\mathsf{W}^d_{E,n} := \mathsf{W}_{E,n}|_{\bar{\mathcal{Q}}'^d_n} := \bar{\mathfrak{h}}'_{\rightarrow !}(\bar{\mathfrak{h}}'^*_\leftarrow(\overline{\mathsf{AS}}^0_\psi) \otimes (\mathfrak{c}' \circ \mathfrak{q}'_n)^*(\mathsf{Lau}^d_E))[\dim \mathcal{Q}'^0_m + dn], \tag{3.1}$$

where $\dim \mathcal{Q}'^0_m = 2\dim_\mu \mathcal{Q}^0_n := 2q_n$. Finally, we define W_E on $\mathcal{M}'_n$ by the formula

$$\mathsf{W}_E = \pi'_{n!} \mathsf{W}_{E,n}. \tag{3.2}$$

Notice that we have the following commutative diagram:

$$\begin{array}{ccc} \bar{\mathcal{Q}}'^d_n & \xrightarrow{\mathfrak{d}'_n} & X'^{(d)} \\ \pi'_n \downarrow && \downarrow \alpha \\ \mathcal{M}'^d_n & \xrightarrow{\mathfrak{d}'} & \mathrm{Pic}^d_{X'} \end{array}$$

where α is the Abel–Jacobi map and $\mathfrak{d} : \mathcal{M}_n^d \to \mathrm{Pic}_X^d$ (resp. $\mathfrak{d}' : \mathcal{M}_n'^d \to \mathrm{Pic}_{X'}^d$) sends $(\mathscr{M}, s_1)$ (resp. $(\mathscr{M}', s_1')$) to the line bundle $\det \mathscr{M} \otimes \Omega_X^{-\frac{n(n-1)}{2}}$ (resp. $\det \mathscr{M}' \otimes \Omega_{X'}^{-\frac{n(n-1)}{2}}$). Hence there is a morphism

$$\delta_n' := \mathfrak{d}_n' \times \mathfrak{d} : \bar{\mathcal{Q}}_n'^{2d} \times_{\mathcal{M}_n'^{2d}} \mathcal{M}_n^d \to X'^{(2d)} \times_{\mathrm{Pic}_{X'}^{2d}} \mathrm{Pic}_X^d$$

and a natural morphism $\iota := (\mu^{(d)}, \alpha) : X^{(d)} \to X'^{(2d)} \times_{\mathrm{Pic}_{X'}^{2d}} \mathrm{Pic}_X^d$. The rest sections are devoted to proving the following proposition.

Proposition 3.3. For any local systems E on X' of rank n and L on X of rank 1, we have a canonical isomorphism

$$\delta_{n!}'(\mathsf{W}_{E,n}^{2d} \boxtimes \mathfrak{d}^* \mathsf{A}_L) \xrightarrow{\sim} \iota_!(\mathrm{As}(E) \otimes L)^{(d)}[2d]$$

over $X'^{(2d)} \times_{\mathrm{Pic}_{X'}^{2d}} \mathrm{Pic}_X^d$ for all $d \geq 0$. □

Proof of Theorem 1.2. Assuming Proposition 3.3, the proof is immediate once realizing the following commutative diagram:

$$\begin{array}{ccc}
\bar{\mathcal{Q}}_n'^{2d} \times_{\mathcal{M}_n'^{2d}} \mathcal{M}_n^d & \xrightarrow{\pi_n' \times \mathrm{id}} & \mathcal{M}_n'^{2d} \times_{\mathcal{M}_n'^{2d}} \mathcal{M}_n^d \simeq \mathcal{M}_n^d \\
\Big\downarrow {\scriptstyle \delta_n' = \mathfrak{d}_n' \times \mathfrak{d}} & & \Big\downarrow {\scriptstyle \mathfrak{d}} \\
X'^{(2d)} \times_{\mathrm{Pic}_{X'}^{2d}} \mathrm{Pic}_X^d & \xrightarrow{\alpha \times \mathrm{id}} & \mathrm{Pic}_{X'}^{2d} \times_{\mathrm{Pic}_{X'}^{2d}} \mathrm{Pic}_X^d \simeq \mathrm{Pic}_X^d
\end{array}$$

and the definition (3.2). ■

4 Restriction of the Whittaker Sheaf

Let $m \geq 1$ be an integer and define the Whittaker sheaf $\mathsf{W}_{E,m}$ by the same formula (3.1) but replacing n by m (where E is still of rank n). Consider the following commutative diagram:

$$\begin{array}{ccc} {}_{\mu}\bar{\mathcal{Q}}^{d}_{m} & \xrightarrow{(\bar{\mu}_m,\pi_m)} & \bar{\mathcal{Q}}'^{2d}_{m} \times_{\mathcal{M}'^{2d}_{m}} \mathcal{M}^{d}_{m} \\ \downarrow{\scriptstyle \mathfrak{d}_m} & \searrow{\scriptstyle \delta_m} & \downarrow{\scriptstyle \delta'_m} \\ X^{(d)} & \xrightarrow{\iota} & X'^{(2d)} \times_{\mathrm{Pic}^{2d}_{X'}} \mathrm{Pic}^{d}_{X} \end{array}$$

and let $\nu_m = (\bar{\mu}_m, \pi_m)$, which is proper and representable since π_m and π'_m are, $\delta_m = \delta'_m \circ \nu_m$. Proposition 3.3 will follow from the following two propositions and this section is dedicated to proving the first one.

Proposition 4.1. For any local systems E on X' of rank n and L on X of rank 1, $m \geq 1$, $d \geq 0$, the natural map

$$\delta'_{m!}(\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^{*}\mathsf{A}_{L}) \to \delta_{m!}\nu^{*}_{m}(\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^{*}\mathsf{A}_{L})$$

is an isomorphism. □

Proposition 4.2. Let notations be as mentioned already, there is a canonical isomorphism

$$\mathfrak{d}_{m!}\nu^{*}_{m}(\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^{*}\mathsf{A}_{L}) \xrightarrow{\sim} (\mathrm{As}(E) \otimes L)^{(d)}_{m}[2d]$$

on $X^{(d)}$, where

$$0 = (\mathrm{As}(E) \otimes L)^{(d)}_{0} \subset (\mathrm{As}(E) \otimes L)^{(d)}_{1} \subset \cdots \subset (\mathrm{As}(E) \otimes L)^{(d)}_{m} \subset \cdots$$

is a filtration of $(\mathrm{As}(E) \otimes L)^{(d)}$, which will be introduced in Section 6, such that $(\mathrm{As}(E) \otimes L)^{(d)}_{m} = (\mathrm{As}(E) \otimes L)^{(d)}$ for $m \geq n$. □

4.1 Stratifications-I

To proceed, we recall certain stratifications defined on $\bar{\mathcal{Q}}'^{d}_{m}$ and ${}_{\mu}\bar{\mathcal{Q}}^{d}_{m}$, respectively.

For any $\lambda \in \Lambda^{d}_{m,\mathrm{pos}}$, let $\bar{\mathcal{Q}}'^{\lambda}_{m}$ be the stack classifying the data $(\mathscr{M}', (s'_i), (D'_i))$, where $\mathscr{M}'$ is a vector bundle on X' of rank m, $(D'_1, D'_1 + D'_2, \ldots, D'_1 + \cdots + D'_m) \in X'^{\lambda}$, and s'_i ($i = 1, \ldots, m$) is an inclusion of *vector bundles*

$$s'_i : \Omega^{(n-1)+\cdots+(n-i)}_{X'}(D'_1 + \cdots + D'_i) \to \wedge^{i}\mathscr{M}'$$

such that $(s'_1, \dots, s'_m)$ satisfies the Plücker relations. We have a natural morphism $\bar{\mathcal{Q}}'^{\lambda}_m \to \bar{\mathcal{Q}}'^{d}_m$. It is shown in [1] that each $\bar{\mathcal{Q}}'^{\lambda}_m$ is a locally closed substack and they together form a stratification of $\bar{\mathcal{Q}}'^{d}_m$ for all $\lambda \in \Lambda^d_{m,\mathrm{pos}}$. We have a natural morphism $\mathfrak{s}' : \bar{\mathcal{Q}}'^{\lambda}_m \to X'^{\lambda}$ by remembering $(D'_1, D'_1 + D'_2, \dots, D'_1 + \cdots + D'_m)$ and define $\bar{\mathcal{Q}}'^{\lambda}_{m,?} = \bar{\mathcal{Q}}'^{\lambda}_m \times_{X'^{\lambda}} X'^{\lambda}_?$ which are closed substacks of $\bar{\mathcal{Q}}'_m$, $\mathfrak{s}'_? = \mathfrak{s}'|_{\bar{\mathcal{Q}}'^{\lambda}_{m,?}}$ for $? = \mathrm{eff}, +, -$.

For any $\lambda \in \Lambda^d_{m,\mathrm{pos}}$, let ${}_\mu\bar{\mathcal{Q}}^{\lambda}_m$ be the stack classifying the data $(\mathscr{M}, (s_i), (D_i))$, where $\mathscr{M}$ is a vector bundle on X of rank m, $(D_1, \dots, D_1 + \cdots + D_m) \in X^{\lambda}$, and s_i is an inclusion of vector bundles

$$s_i : \Omega_X^{(n-1)+\cdots+(n-i)} \otimes \mathscr{L}_\mu^{0+\cdots+(i-1)}(D_1 + \cdots + D_i) \to \wedge^i \mathscr{M}$$

such that $(s_1, \dots, s_m)$ satisfies the Plücker relations. As in the previous case, we have a natural morphism ${}_\mu\bar{\mathcal{Q}}^{\lambda}_m \to {}_\mu\bar{\mathcal{Q}}^{d}_m$ such that ${}_\mu\bar{\mathcal{Q}}^{\lambda}_m$ becomes a locally closed substack, and they together form a stratification of ${}_\mu\bar{\mathcal{Q}}^{d}_m$. We have a natural morphism $\mathfrak{s} : {}_\mu\bar{\mathcal{Q}}^{\lambda}_m \to X^{\lambda}$ and define ${}_\mu\bar{\mathcal{Q}}^{\lambda}_{m,?} = {}_\mu\bar{\mathcal{Q}}^{\lambda}_m \times_{X^{\lambda}} X^{\lambda}_?$ which are closed substacks of ${}_\mu\bar{\mathcal{Q}}_m$, $\mathfrak{s}_? = \mathfrak{s}|_{{}_\mu\bar{\mathcal{Q}}^{\lambda}_{m,?}}$ for $? = \mathrm{eff}, +, -$.

The pullback under μ again induces a natural morphism $\bar{\mu}^{\lambda}_m : {}_\mu\bar{\mathcal{Q}}^{\lambda}_{m,?} \to \bar{\mathcal{Q}}'^{2\lambda}_{m,?}$, and hence $\nu^{\lambda}_m : {}_\mu\bar{\mathcal{Q}}^{\lambda}_{m,?} \to \bar{\mathcal{Q}}'^{2\lambda}_{m,?} \times_{\mathcal{M}'^{2d}_m} \mathcal{M}^d_m$ for $? = \emptyset, \mathrm{eff}, +, -$, which are proper and representable such that the following diagram commutes:

$$\begin{array}{ccccc}
 & {}_\mu\bar{\mathcal{Q}}^{\lambda}_{m,?} & \xrightarrow{\nu^{\lambda}_m} & \bar{\mathcal{Q}}'^{2\lambda}_{m,?} \times_{\mathcal{M}'^{2d}_m} \mathcal{M}^d_m & \\
 & \downarrow{\scriptstyle \mathfrak{s}} & & \downarrow{\scriptstyle \mathfrak{s}'\times\mathfrak{d} = \mathfrak{s}'\times_{\mathfrak{d}'}\mathfrak{d}} & \\
\mathfrak{d}_m & X^{\lambda}_? & \xrightarrow{(\mu^{\lambda}, \alpha\circ\mathrm{sym}^{\lambda})} & X'^{2\lambda}_? \times_{\mathrm{Pic}^{2d}_{X'}} \mathrm{Pic}^d_X & \mathfrak{d}'_m\times\mathfrak{d} = \mathfrak{d}'_m\times_{\mathfrak{d}'}\mathfrak{d} \\
 & \downarrow{\scriptstyle \mathrm{sym}^{\lambda}} & & \downarrow{\scriptstyle \mathrm{sym}^{2\lambda}\times\mathrm{id}} & \\
 & X^{(d)} & \xrightarrow{\iota} & X'^{(2d)} \times_{\mathrm{Pic}^{2d}_{X'}} \mathrm{Pic}^d_X. &
\end{array}$$

For $\lambda \in \Lambda^d_{m,-}$, the stack $\bar{\mathcal{Q}}'^{\lambda}_{m,-}$ will be nonempty and it equivalently classifies the data $((\mathscr{M}'_i), (r'_i), (D'_i))$, where

$$0 = \mathscr{M}'_0 \subset \mathscr{M}'_1 \subset \cdots \subset \mathscr{M}'_m = \mathscr{M}'$$

is a complete flag of sub-vector bundles of $\mathscr{M}'$, D'_i is as above but with $0 \le D'_1 \le \cdots \le D'_m$ and r'_i is an isomorphism

$$r'_i : \Omega_{X'}^{n-i}(D'_i) \xrightarrow{\sim} \mathscr{M}'_i/\mathscr{M}'_{i-1}$$

for $i = 1, \ldots, m$. Let $\mathrm{ev}^{\lambda} : \bar{\mathcal{Q}}'^{\lambda}_{m,-} \to \mathbf{G}_{\mathrm{a},k}$ be the morphism sending the above data to the sum of $m-1$ classes in $\mathrm{Ext}^1(\Omega_{X'}^{n-i-1}(D'_i), \Omega_{X'}^{n-i}(D'_i)) \simeq \mathbf{G}_{\mathrm{a},k}$ corresponding to the pullbacks of the extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{M}'_i/\mathscr{M}'_{i-1} & \longrightarrow & \mathscr{M}'_{i+1}/\mathscr{M}'_{i-1} & \longrightarrow & \mathscr{M}'_{i+1}/\mathscr{M}'_i & \longrightarrow & 0 \\ & & s''_i \uparrow \wr & & & & \wr \downarrow {s''_{i+1}}^{-1} & & \\ & & \Omega_{X'}^{n-i}(D'_i) & & & & \Omega_{X'}^{n-i-1}(D'_{i+1}) & & \end{array}$$

under the inclusion $\Omega_{X'}^{n-i-1}(D'_i) \hookrightarrow \Omega_{X'}^{n-i-1}(D'_{i+1})$ for $i = 1, \ldots, m-1$. Finally, let $\overline{\mathsf{AS}}_{\psi}^{\lambda} = \mathrm{ev}^{\lambda *}\mathsf{AS}_{\psi}$ be a local system on $\bar{\mathcal{Q}}'^{\lambda}_{m,-}$.

4.2 Pullback of Laumon's sheaf

For $\lambda \in \Lambda^d_{m,-}$, the scheme X^{λ}_{-} classifies $(D_1, \ldots, D_m)$, where D_i is a divisor on X of degree λ_i and $0 \le D_1 \le \cdots \le D_m$. Let $\mathfrak{c}^{\lambda} : X^{\lambda}_{-} \to \mathrm{Coh}^d_{0,\le m}$ be the morphism sending $(D_1, \ldots, D_m)$ to

$$\Omega_X^{m-1}(D_1)/\Omega_X^{m-1} \oplus \Omega_X^{m-2} \otimes \mathscr{L}_{\mu}(D_2)/\Omega_X^{m-2} \otimes \mathscr{L}_{\mu} \oplus \cdots \oplus \mathscr{L}_{\mu}^{m-1}(D_m)/\mathscr{L}_{\mu}^{m-1}.$$

Similarly, we have a morphism $\mathfrak{c}'^{\lambda} : X'^{\lambda}_{-} \to \mathrm{Coh}'^d_{0,\le m}$ sending $(D'_1, \ldots, D'_m)$ to

$$\Omega_{X'}^{m-1}(D'_1)/\Omega_{X'}^{m-1} \oplus \Omega_{X'}^{m-2}(D'_2)/\Omega_{X'}^{m-2} \oplus \cdots \oplus \mathscr{O}_{X'}^{m-1}(D'_m)/\mathscr{O}_{X'}^{m-1}.$$

By [11, Théorème 3.3.8], the restriction $\mathfrak{c}'^{\lambda *}\mathsf{Lau}^d_E$ of Laumon's sheaf has maximal (possible) cohomological degree (with respect to the usual t-structure) $2d(\lambda)$, where $d(\lambda) := \sum_{i=1}^{m}(m-i)\lambda_i$. Its highest cohomology sheaf

$$E^{\lambda}_{-} := \mathrm{H}^{2d(\lambda)}\mathfrak{c}'^{\lambda *}\mathsf{Lau}^d_E$$

does not vanish if and only if $\lambda_1 = \cdots = \lambda_{m-n} = 0$ (which is an empty condition if $m \le n$). See [11] for a description of the stalks of E^{λ}_{-}. The following proposition proved by

Lysenko [14, Proposition 2] will play a key role later, of which the proof uses the geometric Casselman–Shalike formula for general linear groups, cf. [5, 15, 16].

Proposition 4.3. The restriction $\mathsf{W}^d_{E,m}|_{\bar{\mathbb{Q}}'^{\lambda}_m}$ is supported on $\bar{\mathbb{Q}}'^{\lambda}_{m,-}$, and its further restriction to the later one is isomorphic to

$$\overline{\mathsf{AS}}^{\lambda}_{\psi} \otimes \mathfrak{s}'_{-}{}^{*}E^{\lambda}_{-}[2q_m + md - 2d(\lambda)] \qquad \square$$

(A similar result on the restriction of Whittaker sheaves on different strata is proved in [6, Proposition 4.12]). The following simple lemma will be important in the later proof.

Lemma 4.4. Let $\mathscr{F}$ be a coherent sheaf on X and consider the paring

$$\mathrm{ev}_{\mathscr{F}}: \mathrm{Ext}^1(\mathscr{F}, \Omega_X) \times \mathrm{H}^0(X', \mu^*\mathscr{F}) \xrightarrow{i^1_+ \times \mathrm{id}} \mathrm{Ext}^1(\mu^*\mathscr{F}, \Omega_{X'}) \times \mathrm{H}^0(X', \mu^*\mathscr{F})$$
$$\to \mathrm{H}^1(X', \Omega_{X'}) \simeq \mathbf{G}_{\mathrm{a},k}.$$

Then

$$\mathrm{pr}_{2!}\mathrm{ev}_{\mathscr{F}}{}^{*}\mathsf{AS}_{\psi} = i^0_{-!}\bar{\mathbb{Q}}_{\ell}[-2\dim\mathrm{H}^0(X, \mathscr{F})],$$

where pr_i is the projection to the ith factor,

$$i^1_+ : \mathrm{Ext}^1(\mathscr{F}, \Omega_X) \hookrightarrow \mathrm{Ext}^1(\mu^*\mathscr{F}, \Omega_{X'}); \quad i^0_- : \mathrm{Hom}(\mathscr{L}_{\mu}, \mathscr{F}) \hookrightarrow \mathrm{H}^0(X', \mu^*\mathscr{F})$$

are closed embeddings induced by μ. $\square$

Proof. We only need to show that the orthogonal complement of $i^1_+\mathrm{Ext}^1(\mathscr{F}, \Omega_X) \subset \mathrm{Ext}^1(\mu^*\mathscr{F}, \Omega_{X'})$ inside $\mathrm{H}^0(X', \mu^*\mathscr{F})$ under the pairing $\mathrm{ev}_{\mathscr{F}}$ is $i^0_-\mathrm{Hom}(\mathscr{L}_{\mu}, \mathscr{F})$. The paring between these two subspaces is zero due to the fact that $\mathrm{H}^0(X, \mathscr{L}_{\mu}) = 0$, and they are orthogonal complement of each other because they have the complimentary dimensions since $\mathrm{H}^0(X', \mu^*\mathscr{F}) = \mathrm{H}^0(X, \mathscr{F}) \oplus \mathrm{Hom}(\mathscr{L}_{\mu}, \mathscr{F})$. ■

Proof of Proposition 4.1. By Proposition 4.3, it amounts to prove the following assertion. For $\rho \in \Lambda^{2d}_{m,-}$, the direct image $(\mathfrak{s}' \times \mathfrak{d})_!\overline{\mathsf{AS}}^{\rho}_{\psi}$ vanishes unless $\rho = 2\lambda$ and then the natural map

$$(\mathfrak{s}' \times \mathfrak{d})_!\mathrm{pr}^{2\lambda *}_m\overline{\mathsf{AS}}^{2\lambda}_{\psi} \to (\mathfrak{s}' \times \mathfrak{d})_!\nu^{\lambda}_{m!}\nu^{\lambda}_m{}^{*}\mathrm{pr}^{2\lambda *}_m\overline{\mathsf{AS}}^{2\lambda}_{\psi}$$

is an isomorphism, where $\mathrm{pr}^{\rho}_{m} : \bar{\mathcal{Q}}'^{\rho}_{m,-} \times_{\mathcal{M}'^{2d}_{m}} \mathcal{M}^{d}_{m} \to \bar{\mathcal{Q}}'^{\rho}_{m,-}$ is the projection to the first factor.

For $j = 1, \dots, m$, we introduce the stack $\mathcal{P}^{\rho}_{m,j}$ classifying the data

$$(\mathscr{M}, (\mathscr{M}_i)_{i=1}^{j}, (\mathscr{M}'_i)_{i=j+1}^{m-1}; (D_i)_{i=1}^{j}, (D'_i)_{i=j+1}^{m}; (r_i)_{i=1}^{j}, (r'_i)_{i=j+1}^{m}),$$

where

(1) $0 \subset \mu^*\mathscr{M}_1 \subset \cdots \subset \mu^*\mathscr{M}_j \subset \mathscr{M}'_{j+1} \subset \cdots \subset \mathscr{M}'_{m-1} \subset \mu^*\mathscr{M}$ is a complete flag of sub-vector bundles of $\mu^*\mathscr{M}$, where $\mathscr{M}$ is a vector bundle on X of rank m;
(2) $0 \le \mu^* D_1 \le \cdots \le \mu^* D_j \le D'_{j+1} \le \cdots \le D'_m$ is in X'^{ρ}_{-} ($\mu^* D := \mu^{-1}(D)$);
(3) $r_i : \Omega_X^{m-i} \otimes \mathscr{L}_{\mu}^{i-1}(D_i) \xrightarrow{\sim} \mathscr{M}_i/\mathscr{M}_{i-1}$ for $i = 1, \dots, j$ ($\mathscr{M}_0 = 0$);
(4) $r'_i : \Omega_{X'}^{m-i}(D'_i) \xrightarrow{\sim} \mathscr{M}'_i/\mathscr{M}'_{i-1}$ for $i = j+1, \dots, m$ ($\mathscr{M}'_m = \mu^*\mathscr{M}$).

Hence $\mathcal{P}^{\rho}_{m,j}$ is empty if $2 \nmid \rho_i$ for some $1 \le i \le j$. We have the following successive closed embeddings

$$\mathcal{P}^{\rho}_{m,m} \hookrightarrow \mathcal{P}^{\rho}_{m,m-1} \hookrightarrow \cdots \hookrightarrow \mathcal{P}^{\rho}_{m,1} = \bar{\mathcal{Q}}'^{\rho}_{m,-} \times_{\mathcal{M}'^{2d}_{m}} \mathcal{M}^{d}_{m}$$

and $\mathcal{P}^{\rho}_{m,m}$ is empty unless $\rho = 2\lambda$ in which case $\mathcal{P}^{2\lambda}_{m,m} =_{\mu} \bar{\mathcal{Q}}^{\lambda}_{m}$. Write $\nu^{\rho}_{m,j}$ for the inclusion $\mathcal{P}^{\rho}_{m,j} \hookrightarrow \bar{\mathcal{Q}}'^{\rho}_{m,-} \times_{\mathcal{M}'^{2d}_{m}} \mathcal{M}^{d}_{m}$. We prove successively that the natural maps

$$(\mathfrak{s}' \times \mathfrak{d})_! \nu^{\rho}_{m,j!} \nu^{\rho\,*}_{m,j} \mathrm{pr}^{\rho*}_{m} \overline{\mathsf{AS}}^{\rho}_{\psi} \to (\mathfrak{s}' \times \mathfrak{d})_! \nu^{\rho}_{m,j+1!} \nu^{\rho\,*}_{m,j+1} \mathrm{pr}^{\rho*}_{m} \overline{\mathsf{AS}}^{\rho}_{\psi}$$

are isomorphisms for $j = 1, \dots, m-1$.

In fact, let $\tilde{\mathcal{R}}^{\rho}_{m,j}$ be the stack classifying $(\mathscr{N}, (\mathscr{N}_i')_{i=j+1}^{m-1}; (D_i)_{i=1}^{j}, (D'_i)_{i=j+1}^{m}; (\tilde{r}'_i)_{i=j+1}^{m})$ where

(1) $0 \subset \mathscr{N}'_{j+1} \subset \cdots \subset \mathscr{N}'_{m-1} \subset \mu^*\mathscr{N}$ is a complete flag of sub-vector bundles of $\mu^*\mathscr{N}$, where $\mathscr{N}$ is a vector bundle on X of rank $m - j$;
(2) $(D_i)_{i=1}^{j}, (D'_i)_{i=j+1}^{m}$ are as above;
(3) $\tilde{r}'_i : \Omega_{X'}^{m-i}(D'_i) \xrightarrow{\sim} \mathscr{N}_i'/\mathscr{N}'_{i-1}$ for $i = j+1, \dots, m$ ($\mathscr{N}'_j = 0$, $\mathscr{N}'_m = \mu^*\mathscr{N}$).

The closed substack $\mathcal{R}^{\rho}_{m,j} \hookrightarrow \tilde{\mathcal{R}}^{\rho}_{m,j}$ is defined by the condition that

$$\tilde{r}'_{j+1} : \Omega_{X'}^{m-j-1}(D'_{j+1}) \xrightarrow{\sim} \mathscr{N}'_{j+1}$$

coincides with the pullback of

$$\tilde{r}_{j+1} : \Omega_X^{m-j-1} \otimes \mathscr{L}_{\mu}^{j}(D_{j+1}) \xrightarrow{\sim} \mathscr{N}_{j+1}$$

for some divisor D_{j+1} on X with $\mu^* D_{j+1} = D'_{j+1}$. There are natural morphisms $\tilde{\mathfrak{f}} : \mathcal{P}^{\rho}_{m,j} \to \tilde{\mathcal{R}}^{\rho}_{m,j}$ and $\mathfrak{f} : \mathcal{P}^{\rho}_{m,j+1} \to \mathcal{R}^{\rho}_{m,j}$ via taking the quotient by $\mathscr{M}_j$, which are generalized affine fibrations; see [14, 0.1.1] for the convention. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{P}^{\rho}_{m,j+1} & \hookrightarrow & \mathcal{P}^{\rho}_{m,j} & & \\ \downarrow{\scriptstyle \mathfrak{f}} & & \downarrow{\scriptstyle \tilde{\mathfrak{f}}} & \searrow^{\mathfrak{s}' \times \mathfrak{d}} & \\ \mathcal{R}^{\rho}_{m,j} & \hookrightarrow & \tilde{\mathcal{R}}^{\rho}_{m,j} & \longrightarrow & X'^{\rho}_{-} \times_{\mathrm{Pic}^{2d}_{X'}} \mathrm{Pic}^d_X \end{array}$$

in which the square is Cartesian. Applying Lemma 4.4 to the vector bundle $\mathscr{N} \otimes \Omega_X^{-m+j+1} \otimes \mathscr{L}_{\mu}^{-j+1}$, one easily see that $\tilde{\mathfrak{f}}_! \nu^{\rho}_{m,j}{}^* \mathrm{pr}^{\rho *}_m \overline{\mathsf{AS}}^{\rho}_{\psi}$ is supported on $\mathcal{R}^{\rho}_{m,j}$, and our assertion follows. Hence Proposition 4.1 is proved. Moreover, it has the following corollary. ∎

Corollary 4.5. Let E and L be as in Proposition 4.1, the complex $\mathfrak{d}_{m!} \nu_m^* (\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d]$ on $X^{(d)}$ is placed in degree zero. It has a canonical filtration by constructible subsheaves such that the direct sum of all graded terms

$$\mathrm{gr}\, \mathfrak{d}_{m!} \nu_m^* (\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d] \simeq \bigoplus_{\lambda \in \Lambda^d_{m,-}} (\mathrm{sym}^{\lambda}_! \mu^{\lambda *} E^{2\lambda}_{-}) \otimes L^{(d)}.$$

In particular,

$$\mathrm{gr}\, \mathfrak{d}_{m!} \nu_m^* (\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d] \simeq \mathrm{gr}\, \mathfrak{d}_{n!} \nu_n^* (\mathsf{W}^{2d}_{E,n} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d]$$

for $m \geq n$. □

Proof. Since the morphism $\mathfrak{s} :_{\mu} \bar{\mathcal{Q}}^{\lambda}_{m,-} \to X^{\lambda}_{-}$ is a generalized affine fibration of rank $q_m + md - d - 2d(\lambda)$, the assertion follows from Proposition 4.3, the proof of Proposition 4.1, Lemma 4.4 and the Cousin spectral sequence for the computation of the !-direct image via stratifications that provides the canonical filtration. ∎

5 Reduction to the Moduli of Torsion Flags

We prove Proposition 4.2 in the remaining two sections.

5.1 More stacks

Recall that we have stacks $\mathcal{Q}'^0_m \hookrightarrow \bar{\mathcal{Q}}'^0_m$ and

$$\bar{\mathfrak{h}}'_{\rightarrow} : \bar{\mathcal{Z}}'^{2d}_m = \bar{\mathcal{Q}}'^0_m \times_{\mathrm{Bun}'^0_m} \mathrm{Mod}'^{2d}_m \to \bar{\mathcal{Q}}'^{2d}_m.$$

We define

$$\mathcal{Z}'^{2d}_m = \mathcal{Q}'^0_m \times_{\mathrm{Bun}'^0_m} \mathrm{Mod}'^{2d}_m \hookrightarrow \bar{\mathcal{Z}}'^{2d}_m$$

to be the fibre product of $\mathfrak{p}'_m \circ j' : \mathcal{Q}'^0_m \to \mathrm{Bun}'^0_m$ and $\mathfrak{h}'_{\leftarrow} : \mathrm{Mod}'^{2d}_m \to \mathrm{Bun}'^0_m$, and similarly,

$$_\mu\mathcal{Z}^d_m = _\mu \mathcal{Q}^0_m \times_{\mathrm{Bun}^0_m} \mathrm{Mod}^d_m$$

to be the fibre product of $\mathfrak{p}_m \circ j :_\mu \mathcal{Q}^0_m \to \mathrm{Bun}^0_m$ and $\mathfrak{h}_{\leftarrow} : \mathrm{Mod}^d_m \to \mathrm{Bun}^0_m$.

Define $\mathcal{C}^{2d}_m$ to be the stack classifying the data $(\mathscr{N}, (\mathscr{M}'_i), (r'_i))$, where

(1) $\mathscr{N}$ is in Coh^d_m;
(2) $0 = \mathscr{M}'_0 \subset \mathscr{M}'_1 \subset \cdots \subset \mathscr{M}'_m \subset \mu^*\mathscr{N}$ is a filtration;
(3) $r'_i : \Omega^{m-i}_{X'} \xrightarrow{\sim} \mathscr{M}'_i/\mathscr{M}'_{i-1}$ for $i = 1, \ldots, m$ are isomorphisms.

Define $\mathcal{C}^{\circ 2d}_m$ to be the open substack of $\mathcal{C}^{2d}_m$ given by the condition that $\mathscr{N}$ is locally free. Then $\mathcal{C}^{\circ 2d}_m$ is naturally identified with $\mathcal{Z}'^{2d}_m \times_{\mathrm{Bun}'_m} \mathrm{Bun}_m$. Now consider the subfunctor $\mathcal{Y}^{2d}_m$ of $\mathcal{C}^{2d}_m$ defined in the following way. For any scheme S, $(\mathscr{N}_S, (\mathscr{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{C}^{2d}_m)$ is in $\mathrm{Hom}(S, \mathcal{Y}^{2d}_m)$ if there exists $(\mathscr{M}_{i,S}, (r_{i,S}))$, where

(1) $\mathscr{M}_{i,S}$ is a vector bundle on $S \times X$ of rank i;
(2) $0 = \mathscr{M}_{0,S} \subset \mathscr{M}_{1,S} \subset \cdots \subset \mathscr{M}_{m,S} \subset \mathscr{N}_S$ is a filtration;
(3) $r_{i,S} : \mathscr{O}_S \boxtimes (\Omega^{m-i}_X \otimes \mathscr{L}^{i-1}_\mu) \xrightarrow{\sim} \mathscr{M}_{i,S}/\mathscr{M}_{i-1,S}$ for $i = 1, \ldots, m$ are isomorphisms;
(4) There is a closed subscheme T of $S \times X$ which is finite over S such that over $S \times X' - (\mathrm{id}_S \times \mu)^{-1}T$, $\mu^*\mathscr{M}_{i,S}$ equals $\mathscr{M}'_{i,S}$ and $r'_{i,S} : \mathscr{O}_S \boxtimes \Omega^{m-i}_{X'} \xrightarrow{\sim} \mathscr{M}'_{i,S}/\mathscr{M}'_{i-1,S}$ coincides with $\mu^* r_{i,S}$.

We have the following lemma.

Lemma 5.1. The subfunctor $\mathcal{Y}^{2d}_m \hookrightarrow \mathcal{C}^{2d}_m$ is a closed embedding; hence $\mathcal{Y}^{2d}_m$ is an Artin stack. □

Proof. An object $(\mathscr{N}_S, (\mathscr{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{C}^{2d}_m)$ will induce maps

$$s'_{i,S} : \mathscr{O}_S \boxtimes \Omega_{X'}^{(m-1)+\cdots+(m-i)} \to \wedge^i \mu^* \mathscr{N} = \mu^* \wedge^i \mathscr{N}$$

for $i = 1, \ldots, m$. Let

$$s'^{\sigma}_{i,S} : \mathscr{O}_S \boxtimes \Omega_{X'}^{(m-1)+\cdots+(m-i)} \simeq \mathscr{O}_S \boxtimes \sigma^* \Omega_{X'}^{(m-1)+\cdots+(m-i)} \xrightarrow{\sigma^* s'_{i,S}} \sigma^* \mu^* \wedge^i \mathscr{N} \simeq \mu^* \wedge^i \mathscr{N}.$$

Then $(\mathscr{N}_S, (\mathscr{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{Y}^{2d}_m)$ if and only if the support of $s'_{i,S} - (-1)^{\frac{i(i-1)}{2}} s'^{\sigma}_{i,S}$ is a closed subscheme of $S \times X'$ finite over S for $i = 1, \ldots, m$. Since $\mathscr{N}$ is flat over S, it is locally free outside a closed subscheme of $S \times X$ finite over S. Hence the assertion follows from [14, Sublemma 4]. ■

For $(\mathscr{N}_S, (\mathscr{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{Y}^{2d}_m)$, the induced map

$$s^{\sharp}_S : \mathscr{O}_S \boxtimes \Omega_{X'}^{\frac{m(m-1)}{2}} \xrightarrow{\sim} \det \mathscr{M}'_m \hookrightarrow \det \mu^* \mathscr{N}$$

satisfies that $s^{\sharp}_S = (-1)^{\frac{m(m-1)}{2}} (s^{\sharp}_S)^{\sigma}$. Hence the zero divisor of $s^{\sharp}_S$ locates in the closed subscheme $S \times X^{(d)}$ of $S \times X'^{(2d)}$. It induces a morphism $\mathbf{d}_m : \mathcal{Y}^{2d}_m \to X^{(d)}$. Define $\mathcal{Y}^{\circ 2d}_m = \mathcal{C}^{\circ 2d}_m \cap \mathcal{Y}^{2d}_m \hookrightarrow \mathcal{Y}^{2d}_m$, which is an open substack naturally identified with $\mathcal{Z}'^{2d}_m \times_{\bar{\mathcal{Q}}'^{2d}_m} {}_{\mu}\bar{\mathcal{Q}}^d_m$. Moreover, the restriction of $\mathbf{d}_m$ to $\mathcal{Y}^{\circ 2d}_m = \mathcal{Z}'^{2d}_m \times_{\bar{\mathcal{Q}}'^{2d}_m} {}_{\mu}\bar{\mathcal{Q}}^d_m$ coincides with $\mathrm{id} \times \mathfrak{d}_m$.

For $j = 0, \ldots, m$, we define a subfunctor $\mathcal{Y}^{2d}_{m,j}$ of $\mathcal{Y}^{2d}_m$ by the condition that the partial data $((\mathscr{M}'_i)^j_{i=1}, (r'_i)^j_{i=1})$ is a pullback of $((\mathscr{M}_i)^j_{i=1}, (r_i)^j_{i=1})$ (on the whole $S \times X$) in the above sense. Then we have successive closed embeddings

$$\mathcal{Y}^{2d}_{m,m} \hookrightarrow \mathcal{Y}^{2d}_{m,m-1} \hookrightarrow \cdots \hookrightarrow \mathcal{Y}^{2d}_{m,0} = \mathcal{Y}^{2d}_m$$

and natural morphisms

$$\kappa : \mathcal{Y}^{2d}_{m,j} \to \mathcal{Y}^{2d}_{m-j}$$

via taking the quotient by $\mathscr{M}'_j$, which are generalized affine fibrations. Let $\mathcal{Y}^{\circ 2d}_{m,j} = \mathcal{C}^{\circ 2d}_m \cap \mathcal{Y}^{2d}_{m,j} \hookrightarrow \mathcal{Y}^{2d}_{m,j}$ be the open substack, then $\mathcal{Y}^{\circ 2d}_{m,m}$ is naturally identified with ${}_{\mu}\mathcal{Z}^d_m$.

5.2 Iterated modifications

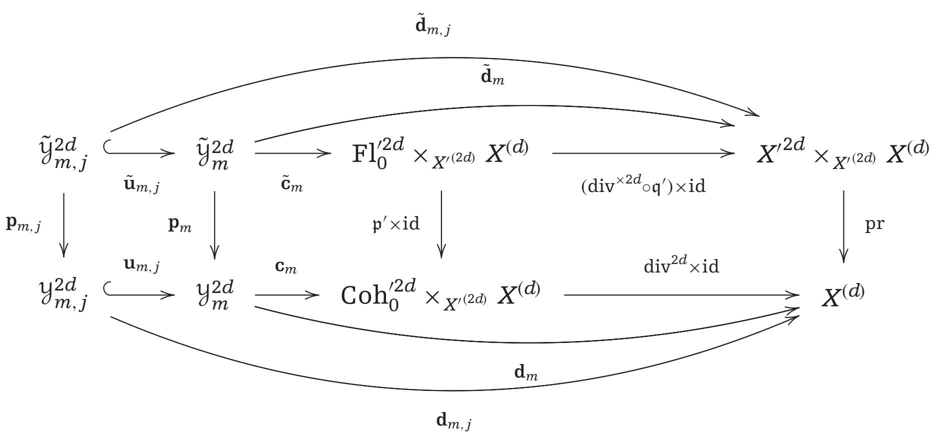
We have a natural morphism $\mathcal{Y}_m^{2d} \hookrightarrow \mathcal{C}_m^{2d} \to \mathrm{Coh}_0'^{2d}$ sending $(\mathcal{N}, (\mathcal{M}_i'), (r_i'))$ to $\mu^*\mathcal{N}/\mathcal{M}_m'$, and define

$$\tilde{\mathcal{Y}}_m^{2d} = \mathcal{Y}_m^{2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Fl}_0'^{2d}; \quad \tilde{\mathcal{Y}}_m^{\circ 2d} = \mathcal{Y}_m^{\circ 2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Fl}_0'^{2d}.$$

Denote

$$\mathbf{c}_m : \mathcal{Y}_m^{2d} \to \mathrm{Coh}_m'^{2d} \times_{X'^{(2d)}} X^{(d)}; \quad \tilde{\mathbf{c}}_m : \tilde{\mathcal{Y}}_m^{2d} \to \mathrm{Fl}_0'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

the induced morphisms from $\mathbf{d}_m$. We have the natural projection $\mathbf{p}_m : \tilde{\mathcal{Y}}_m^{2d} \to \mathcal{Y}_m^{2d}$. Moreover, for $j = 0, \ldots, m$, we define $\tilde{\mathcal{Y}}_{m,j}^{2d} = \mathcal{Y}_{m,j}^{2d} \times_{\mathcal{Y}_m^{2d}} \tilde{\mathcal{Y}}_m^{2d}$ and $\mathbf{p}_{m,j} : \tilde{\mathcal{Y}}_{m,j}^{2d} \to \mathcal{Y}_{m,j}^{2d}$ the natural projections. We have the following commutative diagram:



Similarly, we define $\tilde{\mathcal{Y}}_{m,j}^{\circ 2d} = \mathcal{Y}_{m,j}^{2d} \times_{\mathcal{Y}_m^{2d}} \tilde{\mathcal{Y}}_m^{\circ 2d}$, which is a closed substack of $\tilde{\mathcal{Y}}_m^{\circ 2d}$ and an open substack of $\tilde{\mathcal{Y}}_{m,j}^{2d}$. We have restriction of morphisms $\tilde{\mathbf{u}}_{m,j}^\circ$, $\mathbf{u}_{m,j}^\circ$, $\mathbf{p}_{m,j}^\circ$, $\mathbf{p}_m^\circ$, $\tilde{\mathbf{d}}_{m,j}^\circ$, $\mathbf{d}_{m,j}^\circ$, $\tilde{\mathbf{d}}_m^\circ$ and $\mathbf{d}_m^\circ$. But now the restriction of $\tilde{\mathbf{c}}_m$ (resp. $\mathbf{c}_m$) to $\tilde{\mathcal{Y}}_m^{\circ 2d}$ (resp. ${\mathcal{Y}^{\circ 2d}}_m$) will factor through $\mathrm{Fl}_{0,\leq m}'^{2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Coh}_0^d$ (resp. $\mathrm{Coh}_{0,\leq m}^d$), and we define $\tilde{\mathbf{c}}_m^\circ$ (resp. $\mathbf{c}_m^\circ$) to be the restriction but with this new target. Let $\tilde{\mathbf{c}}_{m,j}^\circ = \tilde{\mathbf{c}}_m^\circ \circ \tilde{\mathbf{u}}_{m,j}^\circ$. We will have a similar diagram as mentioned earlier, which we omit. In particular, the stack $\tilde{\mathcal{Y}}_{m,m}^{\circ 2d}$ is naturally identified with ${}_\mu\mathcal{Z}_m^d \times_{\mathrm{Coh}_0^d} (\mathrm{Coh}_0^d \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Fl}_0'^{2d})$.

As for $\mathcal{Q}_m'^0$, we define in the same way the evaluation map $\mathrm{ev}^{2d} : \mathcal{Y}_m^{2d} \to \mathbf{G}_{\mathrm{a},k}$, and ${\mathsf{AS}_\psi}^{2d} = \mathrm{ev}^{2d*}\mathsf{AS}_\psi$; $\widetilde{\mathsf{AS}}_\psi^{2d} = \mathbf{p}_m^*\mathsf{AS}_\psi^{2d}$ which are local systems of rank 1. The $*$-restrictions of $\widetilde{\mathsf{AS}}_\psi^{2d}$ to $\tilde{\mathcal{Y}}_{m,j}^{2d}$ and $\tilde{\mathcal{Y}}_{m,j}^{\circ 2d}$ are still denoted by $\widetilde{\mathsf{AS}}_\psi^{2d}$. Then we have the following proposition.

Proposition 5.2. The morphism $\tilde{\mathbf{d}}_m^{\circ}: \tilde{\mathfrak{Y}}_m^{\circ 2d} \to X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ is of relative dimension $\le q_m + (m-1)d$, and the natural map between the highest cohomology sheaves

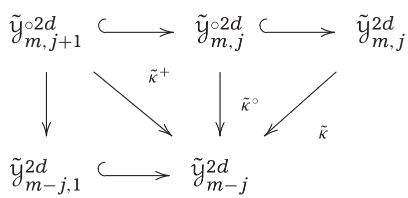
$$\mathrm{H}^{2(q_m+(m-1)d)}\tilde{\mathbf{d}}_{m!}^{\circ}\widetilde{\mathsf{AS}}_{\psi}^{2d} \to \mathrm{H}^{2(q_m+(m-1)d)}\tilde{\mathbf{d}}_{m,m!}^{\circ}\bar{\mathbb{Q}}_{\ell}$$

is an isomorphism. □

Proof. The estimation of the relative dimension will be accomplished in Lemma 5.4. For the second assertion, we inductively prove that the natural maps

$$\mathrm{H}^{2(q_m+(m-1)d)}\tilde{\mathbf{d}}_{m,j!}^{\circ}\widetilde{\mathsf{AS}}_{\psi}^{2d} \to \mathrm{H}^{2(q_m+(m-1)d)}\tilde{\mathbf{d}}_{m,j+1!}^{\circ}\widetilde{\mathsf{AS}}_{\psi}^{2d}$$

are isomorphisms for $j = 0, \ldots, m-1$. Consider the commutative diagram:



where the square is Cartesian. The morphism $\tilde{\kappa}$ induced from κ is a generalized affine fibration of rank $q_m - q_{m-j} + jd$ and hence the natural map

$$\mathrm{H}^{2(q_m-q_{m-j}+jd)}\tilde{\kappa}_{!}^{\circ}\widetilde{\mathsf{AS}}_{\psi}^{2d} \to \mathrm{H}^{2(q_m-q_{m-j}+jd)}\tilde{\kappa}_{!}\widetilde{\mathsf{AS}}_{\psi}^{2d}$$

is an isomorphism over the image of $\tilde{\kappa}^{\circ}$. Applying Lemma 4.4 to the coherent sheaf $(\mathscr{N}/\mathscr{M}_j) \otimes \Omega_X^{-m+j+1} \otimes \mathscr{L}_{\mu}^{-j+1}$, we see that $\mathrm{H}^{2(q_m-q_{m-j}+jd)}\tilde{\kappa}_{!}\widetilde{\mathsf{AS}}_{\psi}^{2d}$ is supported on the closed substack $\tilde{\mathfrak{Y}}_{m-j,1}^{2d}$. Hence, the natural map

$$\mathrm{H}^{2(q_m-q_{m-j}+jd)}\tilde{\kappa}_{!}^{\circ}\widetilde{\mathsf{AS}}_{\psi}^{2d} \to \mathrm{H}^{2(q_m-q_{m-j}+jd)}\tilde{\kappa}_{!}^{+}\widetilde{\mathsf{AS}}_{\psi}^{2d}$$

is an isomorphism. To conclude the desired isomorphism at the beginning of the proof, we need to apply only Lemma 5.5 for $\tilde{\mathbf{d}}_{m-j}$. Now the second assertion immediately follows since $\widetilde{\mathsf{AS}}_{\psi}^{2d}|_{\tilde{\mathfrak{Y}}_{m,m}^{\circ 2d}} = \bar{\mathbb{Q}}_{\ell}$ by Lemma 4.4. ■

Corollary 5.3. Let E and L be as in Proposition 4.1, then we have a canonical isomorphism

$$\mathfrak{d}_{m!}\nu_m^*(\mathsf{W}_{E,m}^{2d}\boxtimes\mathfrak{d}^*\mathsf{A}_L)[-2d]\xrightarrow{\sim}(\mathrm{H}^{-2d}\mathrm{div}_{\leq m!}^d(\mu_0^*\mathsf{Lau}_E^{2d}|_{\mathrm{Coh}_{0,\leq m}^d}))\otimes L^{(d)}$$

of constructible sheaves over $X^{(d)}$, where we recall the morphisms $\mu_0:\mathrm{Coh}_0^d\to\mathrm{Coh}_0^{\prime 2d}$ and $\mathrm{div}_{\leq m}^d:\mathrm{Coh}_{0,\leq m}^d\to X^{(d)}$. □

Proof. By the earlier-mentioned proposition, we have canonical isomorphisms

$$\begin{aligned}
&\mathrm{H}^{2(q_m+md-d)}\mathfrak{d}_{m!}(\bar{\mathfrak{h}}'_{\leftarrow}{}^*\overline{\mathsf{AS}}_\psi^0|_{\tilde{\mathfrak{Y}}_m^{\circ 2d}}\otimes\mathbf{c}_m^*(\mathsf{Spr}_E^{2d}\boxtimes L^{(d)})|_{\tilde{\mathfrak{Y}}_m^{\circ 2d}})\\
&\xrightarrow{\sim}\mathrm{H}^{2(q_m+md-d)}\mathrm{pr}_!\mathfrak{d}_{m!}\mathbf{p}_{m!}^{\circ}(\widetilde{\mathsf{AS}}_\psi^{2d}\otimes\tilde{\mathbf{c}}_m^*((\mathrm{div}^{\times 2d*}E^{\boxtimes 2d})\boxtimes L^{(d)})|_{\tilde{\mathfrak{Y}}_m^{\circ 2d}})\\
&\xrightarrow{\sim}\mathrm{H}^{2(q_m+md-d)}\mathrm{pr}_!\tilde{\mathbf{d}}_{m!}^{\circ}(\widetilde{\mathsf{AS}}_\psi^{2d}\otimes\tilde{\mathbf{c}}_m^*((\mathrm{div}^{\times 2d*}E^{\boxtimes 2d})\boxtimes L^{(d)})|_{\tilde{\mathfrak{Y}}_m^{\circ 2d}})\\
&\xrightarrow{\sim}\mathrm{H}^{2(q_m+md-d)}\mathrm{pr}_!\tilde{\mathbf{d}}_{m,m!}^{\circ}(\tilde{\mathbf{c}}_{m,m}^*((\mathrm{div}^{\times 2d*}E^{\boxtimes 2d})\boxtimes L^{(d)})|_{\tilde{\mathfrak{Y}}_{m,m}^{\circ 2d}}).
\end{aligned}$$

But the morphism

$$\tilde{\mathbf{d}}_{m,m}^{\circ}:\tilde{\mathfrak{Y}}_{m,m}^{\circ 2d}\to X'^{2d}\times_{X'^{(2d)}}X^{(d)}$$

is the composition of

$$\tilde{\mathbf{c}}_{m,m}^{\circ}:\tilde{\mathfrak{Y}}_{m,m}^{\circ 2d}\to\mathrm{Fl}_{0,\leq m}^{\prime 2d}\times_{\mathrm{Coh}_0^{\prime 2d}}\mathrm{Coh}_0^d$$

and the natural morphism

$$\widetilde{\mathrm{div}}_{\leq m}^{2d}:\mathrm{Fl}_{0,\leq m}^{\prime 2d}\times_{\mathrm{Coh}_0^{\prime 2d}}\mathrm{Coh}_0^d\to X'^{2d}\times_{X'^{(2d)}}X^{(d)},$$

in which the first one is smooth and surjective with connected fiber of dimension q_m+md and the second one is of relative dimension $\leq -d$ by Lemma 6.1. Hence

$$\begin{aligned}
&\mathrm{H}^{2(q_m+md-d)}\mathrm{pr}_!\tilde{\mathbf{d}}_{m,m!}^{\circ}(\tilde{\mathbf{c}}_{m,m}^*((\mathrm{div}^{\times 2d*}E^{\boxtimes 2d})\boxtimes L^{(d)})|_{\tilde{\mathfrak{Y}}_{m,m}^{\circ 2d}})\\
&\xrightarrow{\sim}(\mathrm{H}^{-2d}\mathrm{div}_{\leq m!}^d(\mu_0^*\mathsf{Spr}_E^{2d}|_{\mathrm{Coh}_{0,\leq m}^d}))\otimes L^{(d)},
\end{aligned}$$

where all these sheaves carry natural actions of $\mathfrak{S}_{2d}$, and the isomorphisms are equivariant under these actions. Taking $\mathfrak{S}_{2d}$ invariants and by Corollary 4.5, we get the desired isomorphism. ■

5.3 Stratifications-II

We recall a stratification on $\tilde{\mathcal{Z}}'^d_m := \mathcal{Z}'^d_m \times_{\mathrm{Coh}'^d_0} \mathrm{Fl}'^d_0$ introduced in [14, Section 4.3]. For $\lambda \in \Lambda^d_{m,\mathrm{eff}}$, the stack $\tilde{\mathcal{Z}}'^d_m \times_{\bar{\mathcal{Q}}'^d_m} \bar{\mathcal{Q}}'^\lambda_m$ is stratified by locally closed substack $\tilde{\mathcal{Z}}'^e_m$ for $e \in {}^d_m J_d$ (in the notation of [14]), where $e = (e^j_i)$ is a $d \times m$ matrix with entries being nonnegative integers such that $\sum_i e^j_i = 1$ for $j = 1, \ldots, d$ and $\sum_j e^j_i = \lambda_i$ for $i = 1, \ldots, m$. Put $X'^e = \prod_{i,j} X'^{e^j_i}$, then there is a natural morphism $\tilde{\mathcal{Z}}'^e_m \to X'^e \times_{X'^\lambda} \bar{\mathcal{Q}}'^\lambda_m$, which is an affine fibration of rank $d(\lambda) = \sum_{i=1}^m (m-i)\lambda_i$. Now it is easy to deduce the following lemma;

Lemma 5.4. The morphisms $\tilde{\mathbf{d}}^\circ_m$ is of relative dimension $\leq q_m + (m-1)d$. □

Proof. The restriction of $\tilde{\mathbf{d}}^\circ_m$ to the stratum $\tilde{\mathcal{Z}}'^e_m \times_{\bar{\mathcal{Q}}'^{2\lambda}_m} {}_\mu\bar{\mathcal{Q}}^\lambda_m$ factors as

$$\tilde{\mathcal{Z}}'^e_m \times_{\bar{\mathcal{Q}}'^{2\lambda}_m} {}_\mu\bar{\mathcal{Q}}^\lambda_m \to (X'^e \times_{X'^{2\lambda}} X^\lambda) \times_{X^\lambda} {}_\mu\bar{\mathcal{Q}}^\lambda_m \to X'^e \times_{X'^{2\lambda}} X^\lambda \to X'^{2d} \times_{X'^{(2d)}} X^{(d)},$$

where the first morphism is an affine fibration of rank $d(2\lambda)$ and ${}_\mu\bar{\mathcal{Q}}^\lambda_m \to X^\lambda$ is a generalized affine fibration of rank $q_m + (m-1)d - 2d(\lambda)$. Hence the lemma follows. ■

Now we estimate the relative dimension of $\tilde{\mathbf{d}}_m$. There is a stratification of $\tilde{\mathcal{Y}}^{2d}_m$ given by the degree of the maximal torsion subsheaf $\mathcal{F}$ of $\mathcal{N}$. We define a locally closed subscheme $\tilde{\mathcal{Y}}^{2d,2d'}_m$ of $\tilde{\mathcal{Y}}^{2d}_m$ for $0 \leq d' \leq d$ classifying the data

(1) $(\mathcal{N}, (\mathcal{M}'_i), (r'_i)) \in \tilde{\mathcal{Y}}^{2d}_m$;
(2) a filtration $\mathcal{M}'_m = \mathcal{N}'_0 \subset \mathcal{N}'_1 \subset \cdots \subset \mathcal{N}'_{2d} = \mu^*\mathcal{N}$ with $\mathcal{N}'_i/\mathcal{N}'_{i-1} \in \mathrm{Coh}'^1_0$ for $i = 1, \ldots, 2d$;
(3) an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow \mathcal{V} \longrightarrow 0$$

of $\mathcal{O}_X$-modules such that $\mathcal{F}$ is a torsion sheaf of degree d' and $\mathcal{V}$ is a vector bundle of rank m.

Let $\mathcal{F}'_i = \mathcal{N}'_i \cap \mu^*\mathcal{F}$ for $i = 0, \ldots, 2d$, then the successive inclusions of torsion sheaves

$$0 = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \cdots \subset \mathcal{F}'_{2d} = \mu^*\mathcal{F}$$

define a point of $\mathrm{Fl}_0'^{2d'} \times_{\mathrm{Coh}_0'^{2d'}} \mathrm{Coh}_0^{d'}$ and an element $c \in \mathfrak{C}_{2d}^{2d'}$, which is the set of subsets of $\{1, \ldots, 2d\}$ of cardinality $2d'$. Hence c determines a locally closed subscheme $\tilde{\mathcal{Y}}_m^c$ of $\tilde{\mathcal{Y}}_m^{2d,2d'}$ and for each c, a morphism $\tilde{\mathcal{Y}}_m^c \to \mathrm{Fl}_0'^{2d'} \times_{\mathrm{Coh}_0'^{2d'}} \mathrm{Coh}_0^{d'}$ on the one hand. On the other hand, the quotient sheaves $\mathscr{V}_i' = \mathscr{N}_i'/\mu^*\mathscr{F}$ and the induced filtration define a point of $\tilde{\mathcal{Y}}_m^{\circ 2d-2d'}$. They together define a morphism

$$\tilde{\mathcal{Y}}_m^c \to (\mathrm{Fl}_0'^{2d'} \times_{\mathrm{Coh}_0'^{2d'}} \mathrm{Coh}_0^{d'}) \times \tilde{\mathcal{Y}}_m^{\circ 2d-2d'},$$

which is a generalized affine fibration of rank md'. By Lemma 5.4, the morphism

$$\tilde{\mathcal{Y}}_m^{\circ 2d-2d'} \to X'^{2d-2d'} \times_{X'^{(2d-2d')}} X^{(d-d')}$$

is of relative dimension $\leq q_m + (m-1)(d-d')$, and by Lemma 6.1, the morphism

$$\mathrm{Fl}_0'^{2d'} \times_{\mathrm{Coh}_0'^{2d'}} \mathrm{Coh}_0^{d'} \to X'^{2d'} \times_{X'^{(2d')}} X^{(d')}$$

is of relative dimension $\leq -d'$, which together imply the following lemma;

Lemma 5.5. The morphisms $\tilde{\mathbf{d}}_m$ is of relative dimension $\leq q_m + (m-1)d$. □

6 Direct Image of Laumon's Sheaf

6.1 Stratifications-III

A point of $\mathrm{Fl}_0'^{2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Coh}_0^{d}$ is given by a torsion sheaf $\mathscr{F}$ on X of degree d and a complete flag of torsion subsheaves

$$0 = \mathscr{F}_0' \subset \mathscr{F}_1' \subset \cdots \subset \mathscr{F}_{2d}' = \mu^*\mathscr{F} \simeq \sigma^*\mu^*\mathscr{F}$$

of $\mu^*\mathscr{F}$. Let $\mathscr{F}_{i,j}' = \mathscr{F}_i' \cap \sigma^*\mathscr{F}_j'$, then $\mathscr{F}_{j,i}' = \sigma^*\mathscr{F}_{i,j}'$ as subsheaves of $\mu^*\mathscr{F}$. As in [14, Section 6.3], define $\mathscr{F}_{i,j}''$ by the following coCartesian square:

$$\begin{array}{ccc} \mathscr{F}_{i-1,j}' & \hookrightarrow & \mathscr{F}_{i,j}'' \\ \uparrow & & \uparrow \\ \mathscr{F}_{i-1,j-1}' & \hookrightarrow & \mathscr{F}_{i,j-1}' \end{array}$$

for $i, j = 1, \dots, 2d$, and $\mathscr{G}_{i,j} = \mathscr{F}'_{i,j}/\mathscr{F}''_{i,j}$. Let $t_{ij} = \deg \mathscr{G}_{i,j}$. Then it is easy to see that $t = (t_{ij})$ is an element in $\mathfrak{T}_{2d}$. In this way, $\mathrm{Fl}_0'^{2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Coh}_0^d$ is stratified by locally closed subschemes $\mathrm{Fl}_0'^t$ for $t \in \mathfrak{T}_{2d}$. For each t, the natural morphism $\mathrm{Fl}_0'^t \to \prod_{i<j} \mathrm{Coh}_0'^{t_{ij}}$ sending $(\mathscr{F}, (\mathscr{F}'_i))$ to $(\mathscr{G}_{i,j})_{i<j}$ is a generalized affine fibration of rank 0. Since $\mathrm{Coh}_0'^1 \to X' \to X$ is of relative dimension -1, we have the following lemma whose second assertion follows from the same argument of [14, Lemma 11].

Lemma 6.1. The morphism

$$\widetilde{\mathrm{div}}^{2d} := (\mathrm{div}^{\times 2d} \circ \mathfrak{q}') \times \mathrm{div}^d : \mathrm{Fl}_0'^{2d} \times_{\mathrm{Coh}_0'^{2d}} \mathrm{Coh}_0^d \to X'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

is of relative dimension $\le -d$ and we have a canonical isomorphism

$$\mathrm{H}^{-2d} \widetilde{\mathrm{div}}^{2d}_! \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathfrak{n}_! \bar{\mathbb{Q}}_\ell,$$

where $\mathfrak{n}$ is defined before Lemma 2.6. □

Proof of Proposition 4.2. By the above lemma, we have an isomorphism

$$(\mathrm{H}^{-2d} \mathrm{div}^d_! \mu_0^* \mathsf{Spr}_E^{2d}) \otimes L^{(d)} \xrightarrow{\sim} \mathrm{pr}_!((E^{\boxtimes 2d} \boxtimes \bar{\mathbb{Q}}_\ell) \otimes \mathfrak{n}_! \bar{\mathbb{Q}}_\ell),$$

which is $\mathfrak{S}_{2d}$-equivariant. Taking invariants on both side and by Lemma 2.6 (3), we have

$$(\mathrm{H}^{-2d} \mathrm{div}^d_! \mu_0^* \mathsf{Lau}_E^{2d}) \otimes L^{(d)} \xrightarrow{\sim} (\mathrm{As}(E) \otimes L)^{(d)}.$$

By Corollary 5.3, we have

$$\begin{aligned}\mathfrak{d}_{m!} \nu_m^* (\mathsf{W}_{E,m}^{2d} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d] &\xrightarrow{\sim} (\mathrm{H}^{-2d} \mathrm{div}^d_{\le m!} (\mu_0^* \mathsf{Lau}_E^{2d}|_{\mathrm{Coh}^{2d}_{0,\le m}})) \otimes L^{(d)} \\ &\hookrightarrow (\mathrm{H}^{-2d} \mathrm{div}^d_! \mu_0^* \mathsf{Lau}_E^{2d}) \otimes L^{(d)} \xrightarrow{\sim} (\mathrm{As}(E) \otimes L)^{(d)}.\end{aligned}$$

Denote $(\mathrm{As}(E) \otimes L)^{(d)}_m \subset (\mathrm{As}(E) \otimes L)^{(d)}$ the image of $\mathfrak{d}_{m!} \nu_m^* (\mathsf{W}_{E,m}^{2d} \boxtimes \mathfrak{d}^* \mathsf{A}_L)[-2d]$. Then we have a filtration

$$0 = (\mathrm{As}(E) \otimes L)^{(d)}_0 \subset (\mathrm{As}(E) \otimes L)^{(d)}_1 \subset \cdots \subset (\mathrm{As}(E) \otimes L)^{(d)}_m \subset \cdots$$

of $(\mathrm{As}(E)\otimes L)^{(d)}$ such that $(\mathrm{As}(E)\otimes L)^{(d)} = \cup_m (\mathrm{As}(E)\otimes L)^{(d)}_m$. By Corollary 4.5, the filtration becomes stable when $m \geq n$. Hence the proposition is proved. ■

Remark. In fact, it is not difficult to see that the inclusion

$$\mathfrak{d}_{m!}\nu_m^*(\mathsf{W}^{2d}_{E,m} \boxtimes \mathfrak{d}^*\mathsf{A}_L)[-2d] \simeq (\mathrm{As}(E)\otimes L)^{(d)}_m \hookrightarrow (\mathrm{As}(E)\otimes L)^{(d)}_{m'} \simeq \mathfrak{d}_{m'!}\nu_{m'}^*(\mathsf{W}^{2d}_{E,m'} \boxtimes \mathfrak{d}^*\mathsf{A}_L)[-2d]$$

for $m \leq m'$ is compatible with the canonical filtration in Corollary 4.5. □

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