1. (Logic) Determine the truth value of each of the following statements:

S1: $\exists x \exists y \exists z \ (x = y + z)$
S2: $\exists x \exists y \forall z \ (x = y + z)$
S3: $\exists x \forall y \exists z \ (x = y + z)$
S4: $\exists x \forall y \forall z \ (x = y + z)$
S5: $\forall x \exists y \exists z \ (x = y + z)$
S6: $\forall x \exists y \forall z \ (x = y + z)$
S7: $\forall x \forall y \exists z \ (x = y + z)$
S8: $\forall x \forall y \forall z \ (x = y + z)$

in the universe of discourse indicated by the header of each column of the following table (write your answers in the table):

Solution:

<table>
<thead>
<tr>
<th></th>
<th>${0, 1, 2}$</th>
<th>N</th>
<th>Z</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>S2</td>
<td>F</td>
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<tr>
<td>S3</td>
<td>T</td>
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<tr>
<td>S4</td>
<td>F</td>
<td>F</td>
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<tr>
<td>S5</td>
<td>T</td>
<td>T</td>
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</tr>
<tr>
<td>S6</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>S7</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>S8</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
2. (Functions) Let \( f : \mathbb{R} \rightarrow \mathbb{R}^3 \) and \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \) the following functions:

\[
 f(x) = (x, x, x) \\
g(x, y, z) = x + y + z
\]

1. Find \( g \circ f \).
2. Find \( f \circ g \).
3. Determine if \( g \circ f \) is one-to-one, onto or bijective—and in the latter case, find its inverse.
4. Same question for \( f \circ g \).

**Solution:**

1. \( (g \circ f)(x) = g(f(x)) = g(x, x, x) = x + x + x = 3x \).
2. \( (f \circ g)(x, y, z) = f(g(x, y, z)) = f(x + y + z) = (x + y + z, x + y + z, x + y + z) \).
3. \( g \circ f \) is bijective and its inverse is \( (g \circ f)^{-1}(x) = x/3 \).
4. \( f \circ g \) is not one-to-one because for instance \( (f \circ g)(0, 0, 0) = (0, 0, 0) \) and \( (f \circ g)(1, -1, 0) = (0, 0, 0) \), so \((0, 0, 0)\) and \((1, -1, 0)\) are two different elements with the same image. It is not onto either because the image contains only elements of the form \((t, t, t)\), so for instance \((0, 1, 2)\) is not in the image.
3. (Algorithms) Consider the following algorithm:

1: procedure proc(n)
2:   if n = 0 then
3:     return(1)
4:   else
5:     return(proc(n-1) + proc(n-1))
6: end proc

(a) Find the output of proc(n) for any $n \geq 0$.

(b) Assume the complexity of this algorithm is given by the number of times the return commands are executed. Find its complexity in $\Theta$ notation.

(c) Replace the statement in line 5 with a different statement that yields an equivalent algorithm (same output for every $n \geq 0$) of complexity $\Theta(n)$.

Solution:

(a) $proc(n) = 2^n$.

(b) If $a_n =$ number of times the return commands are executed for a given value of $n$, then $a_0 = 1$ and $a_{n+1} = 2a_n + 1$ for $n > 0$. Solving this recurrence we get $a_n = 2^{n+1} - 1$, hence the complexity is $\Theta(2^n)$.

(c) 1: procedure proc(n)
2:   if n = 0 then
3:     return(1)
4:   else
5:     return(2*proc(n-1))
6: end proc
4. (Combinatorics) Find the number of integer solutions of

\[ x_1 + x_2 + x_3 = 15 \]

subject to the conditions:

(a) \( x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \)

(b) \( x_1 \geq 1, x_2 \geq 1, x_3 \geq 1. \)

(c) \( x_1 \geq 1, x_2 \geq 2, x_3 \geq 3. \)

(d) \( 0 \leq x_1 \leq 6, x_2 \geq 0, x_3 \geq 0. \)

**Solution:**

(a) \[ \binom{3 + 15 - 1}{15} = \binom{17}{15} = 136. \]

(b) After the change of variables \( x_1 = y_1 + 1, x_2 = y_2 + 1, x_3 = y_3 + 1, \) the equation becomes \( y_1 + y_2 + y_3 = 12, \) subject to the conditions \( y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \) Its number of solutions is

\[ \binom{3 + 12 - 1}{12} = \binom{14}{12} = 91. \]

(c) After the change of variables \( x_1 = y_1 + 1, x_2 = y_2 + 2, x_3 = y_3 + 3, \) the equation becomes \( y_1 + y_2 + y_3 = 9, \) subject to the conditions \( y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \) Its number of solutions is

\[ \binom{3 + 9 - 1}{12} = \binom{11}{9} = 55. \]

(d) Let \( S_1 \) be the set of solutions verifying \( x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \) \( S_2 \) the set of solutions verifying \( x_1 \geq 7, x_2 \geq 0, \) and \( S_3 \) the set of solutions verifying \( 0 \leq x_1 \leq 6, x_2 \geq 0, x_3 \geq 0. \) Then we have

\[ |S_1| = 136, \quad |S_2| = \binom{3 + 8 - 1}{8} = \binom{10}{8} = 45, \]

and

\[ |S_3| = |S_1| - |S_2| = 136 - 45 = 91. \]
Alternatively, we can write the equation $x_2 + x_3 = 15 - x_1$, so the number of solutions is

$$\sum_{x_1=0}^{6} \binom{2 + 15 - x_1 - 1}{15 - x_1} = \sum_{x_1=0}^{6} (16 - x_1)$$

$$= 16 + 15 + 14 + 13 + 12 + 11 + 10$$

$$= 91.$$
5. (Graphs) In each of the following cases draw a connected simple planar graph with the given characteristics, or prove that none exists:

(a) 4 vertices all of degree 3, 4 regions.
(b) 4 vertices, 6 edges, 5 regions.
(c) 4 vertices all of degree 4.
(d) 6 vertices all of degree 3, 5 regions.

Solution:

(a) The following graph fulfills the requirements:

(b) No such graph exists because we have $v - e + f = 4 - 6 + 5 = 3$, contradicting Euler’s formula.

(c) No such graph exists. There are various ways to prove it:

- Such graph would have $4 \cdot 4/2 = 8$ edges, contradicting the inequality $e \leq 3v - 6$.
- In a simple graph (no parallel edges or loops) with $n$ vertices each vertex has degree at most $n - 1$ (at most one edge connecting that vertex to each of the other $n - 1$ vertices).

(d) The following graph fulfills the requirements:
6. (Shortest Path) Use Dijkstra’s algorithm to find the length of a shortest path, and a shortest path, from $a$ to $z$ in the following weighted graph:

![Graph Diagram]

Show also the final $L$-values of all vertices of the graph.

Solution:

The shortest path is $abcdz$. Its length is 12.

$L(a) = 0, L(b) = 4, L(c) = 7, L(d) = 9, L(e) = 7, L(f) = 5, L(g) = 11, L(h) = 13, L(i) = 8, L(j) = 2, L(z) = 12.$
7. (Binary Trees) Given the following string: \textit{abcde}

(a) Find all full binary trees of which it can be the \textit{inorder} transversal.

(b) Find all full binary trees of which it can be the \textit{preorder} transversal.

(c) Find all full binary trees of which it can be the \textit{postorder} transversal.

\textit{Solution:}

Recall that a \textit{full binary tree} is a binary tree in which each each vertex has either two children or zero children—this reduces the number of possibilities, for instance the following is not a valid answer to part (a) because the tree is not full binary:

\begin{itemize}
  \item \textit{Inorder transversal}
  \item \textit{Preorder transversal}
  \item \textit{Postorder transversal}
\end{itemize}
8. (Combinatorial Circuits and Boolean Algebras) Write the output \( f(x, y, z) \) of the following combinatorial circuit as a Boolean expression involving \( x, y \) and \( z \). Simplify that Boolean expression. Design an equivalent simpler circuit based on the simplified expression using the minimum possible number of gates.

\[ f(x, y, z) = (x \lor y \lor z) \land (x \land y \land z) \]
\[ = ((x \land y) \lor z) \land ((x \land y) \lor \overline{z}) \]
\[ = x \land y. \]

\[ x \]
\[ y \]
\[ z \]
\[ f(x, y, z) \]

\[ x \land y \]

Solution:
9. (Automata)

(a) Design (draw the transition diagram of) a finite-state machine that inputs any string of 0’s and 1’s and outputs the difference between the current and the previous symbol, plus 1 (for the first symbol assume that the “previous” symbol is 0.) For instance input “0011101” would produce output “1121102”.

(b) Design a finite-state automaton that accepts the language

\[ L = \{ab^n c \mid n = 0, 1, 2, \ldots \} \]

Solution:

(a) \[ \begin{array}{c}
\text{start} \\
\sigma_0 \\
\sigma_1
\end{array} \]

(b) \[ \begin{array}{c}
\text{start} \\
\sigma_0 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{array} \]

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10. (Languages) Let $G$ be the grammar with terminal symbols \{a, b\}, non terminal symbols \{\sigma, S\}, productions:

$$
\sigma \rightarrow \sigma b, \quad \sigma \rightarrow Sb, \quad \sigma \rightarrow aS, \quad S \rightarrow a
$$

and starting symbol $\sigma$.

Prove that the language $L = L(G)$ associated to $G$ is regular by finding an equivalent grammar for $L$ that is regular.

**Solution:**

The language associated to $G$ is can be described with the regular expression $a(a + b) b^*$, i.e., one $a$ followed by $a$ or $b$, followed by any number of $b$'s.

The following is an equivalent regular grammar for $L$: terminal symbols \{a, b\}, non terminal symbols \{\sigma, A, B\}, productions:

$$
\sigma \rightarrow aA, \quad A \rightarrow aB, \quad A \rightarrow bB, \quad B \rightarrow bB, \quad B \rightarrow \lambda,
$$

and starting symbol $\sigma$. 