1.3. The Fundamental Theorem of Calculus

1.3.1. The Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus (FTC) connects the two branches of calculus: differential calculus and integral calculus. It says the following:

Suppose f is continuous on $[a, b]$. Then:

(1) The function

$$g(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f, i.e., $g'(x) = f(x)$.

(2) (Evaluation Theorem) If F is an antiderivative of f, i.e. $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

The two parts of the theorem can be rewritten like this:

(1) $\displaystyle \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$

(2) $\displaystyle \int_a^b F'(x)\,dx = F(b) - F(a).$

So the theorem states that integration and differentiation are inverse operations, i.e., the derivative of an integral of a function yields the original function, and the integral of a derivative also yields the function originally differentiated (up to a constant).

Example: Find $\displaystyle \frac{d}{dx}\int_0^{x^2} t^3\,dt$.

Answer: We solve this problem in two ways. First directly:

$$g(x) = \int_0^{x^2} t^3\,dt = \left[\frac{t^4}{4}\right]_0^{x^2} = \frac{(x^2)^4}{4} = \frac{x^8}{4}\,,$$

hence

$$g'(x) = \frac{8x^7}{4} = \boxed{2x^7}\,.$$

Second, using the FTC:

$$h(u) = \int_0^u t^3 \, dt \;\Rightarrow\; h'(u) = u^3 \, .$$

Now we have $g(x) = h(x^2)$, hence (using the chain rule):

$$g'(x) = h'(x^2) \cdot 2x = (x^2)^3 \cdot 2x = \boxed{2x^7} \, .$$