2.3. Arc Length, Parametric Curves

2.3.1. Parametric Curves. A parametric curve can be thought of as the trajectory of a point that moves trough the plane with coordinates $(x, y) = (f(t), g(t))$, where $f(t)$ and $g(t)$ are functions of the *parameter* t. For each value of t we get a point of the curve. Example: A parametric equation for a circle of radius 1 and center $(0, 0)$ is:

$$x = \cos t, \qquad y = \sin t\,.$$

The equations $x = f(t)$, $y = g(t)$ are called *parametric equations*.

Given a parametric curve, sometimes we can eliminate t and obtain an equivalent non-parametric equation for the same curve. For instance t can be eliminated from $x = \cos t$, $y = \sin t$ by using the trigonometric relation $\cos^2 t + \sin^2 t = 1$, which yields the (non-parametric) equation for a circle of radius 1 and center $(0, 0)$:

$$x^2 + y^2 = 1\,.$$

Example: Find a non-parametric equation for the following parametric curve:

$$x = t^2 - 2t, \qquad y = t + 1\,.$$

Answer: We eliminate t by isolating it from the second equation:

$$t = (y - 1)\,,$$

and plugging it in the first equation:

$$x = (y - 1)^2 - 2(y - 1)\,.$$

i.e.:

$$x = y^2 - 4y + 3\,,$$

which is a parabola with horizontal axis.

2.3.2. Arc Length. Here we describe how to find the length of a *smooth arc*. A smooth arc is the graph of a continuous function whose derivative is also continuous (so it does not have corner points).

If the arc is just a straight line between two points of coordinates (x_1, y_1), (x_2, y_2), its length can be found by the Pythagorean theorem:

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}\,,$$

where $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$.

More generally, we approximate the length of the arc by inscribing a polygonal arc (made up of straight line segments) and adding up the lengths of the segments. Assume that the arc is given by the parametric functions $x = f(x)$, $y = g(x)$, $a \le t \le b$.

We divide the interval into n subintervals of equal length. The corresponding points in the arc have coordinates $(f(t_i), g(t_i))$, so two consecutive points are separated by a distance equal to

$$L_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}\,.$$

We have $\Delta t = t_i - t_{i-1} = (b-a)/n$. On the other hand, by the mean value theorem

$$\begin{aligned} f(t_i) - f(t_{i-1}) &= f'(t_i^*)\,\Delta t \\ g(t_i) - f(t_{i-1}) &= g'(t_i^*)\,\Delta t \end{aligned}$$

for some t_i^* in $[t_{i-1}, t_i]$. Hence

$$L_i = \sqrt{[f'(x_i^*)\,\Delta t]^2 + [g'(t_i^*)\,\Delta t]^2} = \sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\,\Delta t\,.$$

The total length of the arc is

$$L \approx \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\,\Delta t\,,$$

which converges to the following integral as $n \to \infty$:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\;dt\,.$$

This formula can also be expressed in the following (easier to remember) way:

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt}$$

The last formula can be obtained by integrating the length of an "infinitesimal" piece of arc

$$ds = \sqrt{(dx)^2 + (dy)^2} = dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.$$

Example: Find the arc length of the curve $x = t^2$, $y = t^3$ between $(1,1)$ and $(4,8)$.

Answer: The given points correspond to the values $t = 1$ and $t = 2$ of the parameter, so:

$$\begin{aligned}
L &= \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\
&= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt \\
&= \int_1^2 t\sqrt{4 + 9t^2}\, dt \\
&= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du \qquad\qquad (u = 4 + 9t^2) \\
&= \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \\
&= \boxed{\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})}\,.
\end{aligned}$$

In cases when the arc is given by an equation of the form $y = f(x)$ or $x = f(x)$ the formula becomes:

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

or

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy}$$

Example: Find the length of the arc defined by the curve $y = x^{3/2}$ between the points $(0, 0)$ and $(1, 1)$.

Answer:

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+[(x^{3/2})']^2}\, dx \\
&= \int_0^1 \sqrt{1+\left(\frac{3x^{1/2}}{2}\right)^2}\, dx = \int_0^1 \sqrt{1+\frac{9x}{4}}\, dx \\
&= \left[\frac{1}{27}(4+9x)^{3/2}\right]_0^1 = \boxed{\frac{1}{27}(13^{3/2}-8)}.
\end{aligned}
$$