CHAPTER 3

Differential Equations

3.1. Differential Equations and Separable Equations

3.1.1. Population Growth. The growth of a population is usually modeled with an equation of the form

\[
\frac{dP}{dt} = kP,
\]

where \( P \) represents the number of individuals at a given time \( t \). This model assumes that the rate of growth of population is proportional to the population size.

A solution to this equation is the exponential function:

\[
P(t) = Ce^{kt}.
\]

Check: \( P'(t) = kCe^{kt} = kP(t) \).

A more realistic model takes into account that any environment has a limited carrying capacity \( K \), so if \( P \) reaches \( K \) the population stops growing. The model in this case is the following:

\[
\frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right).
\]

This is called the logistic differential equation.

3.1.2. Motion of a Spring. Consider an object of mass \( m \) at the end of a vertical spring. According to Hook’s law the restoring force of a spring stretched (or compressed) a distance \( x \) from its natural length is

\[
F = -kx,
\]

where \( k \) is a positive constant (the spring constant) and the negative sign expresses that the sense of the force is opposite to the sense of the stretching. By Newton’s Second Law (force equals mass times
acceleration):

\[ m \frac{d^2x}{dt^2} = -kx , \]

or equivalently:

\[ \frac{d^2x}{dt^2} = - \frac{k}{m} x . \]

This is an example of a second order differential equation because it involves second order derivatives.

3.1.3. General Differential Equations. A differential equation is an equation that contains one or more unknown functions and one or more of its derivatives. The order of the differential equation is the order of the highest derivative that occurs in the equation.

3.1.4. First-order Differential Equations. A first-order differential equation is an equation of the form

\[ \frac{dy}{dx} = F(x, y) , \]

where \( F(x, y) \) is a function of \( x \) and \( y \). A solution of the differential equation is a function \( y(x) \) such that \( y'(x) = F(x, y(x)) \) for all \( x \) in some appropriate interval.

Example: Consider the following differential equation:

\[ \frac{dy}{dx} = \frac{2y}{x} . \]

A possible solution for that equation is, for instance, \( y = x^2 \), because

\[ \frac{dy}{dx} = y'(x) = (x^2)' = 2x , \]

and

\[ \frac{2y}{x} = 2 \frac{x^2}{x} = 2x , \]

hence \( y'(x) = \frac{2y(x)}{x} \) for all \( x \neq 0 \).
3.1.5. **Separable Differential Equations.** A differential equation is said to be *separable* if it can be written in the form

\[ f(y) \, dy = g(x) \, dx, \]

so that the left hand side depends on \( y \) only and the right hand side depends on \( x \) only. In particular this is true if the equation is of the form

\[ \frac{dy}{dx} = g(x) \phi(y), \]

where the right hand side is a product of a function of \( x \) and a function of \( y \). In this case we get:

\[ \frac{1}{\phi(y)} \, dy = g(x) \, dx. \]

Given the equation

\[ f(y) \, dy = g(x) \, dx, \]

we can solve it by integrating both sides. Since the antiderivatives of a function differ in a constant, we get:

\[ \int f(y) \, dy = \int g(x) \, dx + C, \]

If \( F(y) = \int f(y) \, dy \) and \( G(x) = \int g(x) \, dx \) then the solution takes the form

\[ F(y) = G(x) + C'. \]

Next we will try to solve this equation algebraically in order to either write \( y \) as a function of \( x \), or \( x \) as a function of \( y \).

*Example:* Consider the equation

\[ \frac{dy}{dx} = y^2 x. \]

The right hand side is the product of a function of \( x \) and a function of \( y \), so it is separable:

\[ \frac{1}{y^2} \, dy = x \, dx. \]

Integrating both sides we get:

\[ -\frac{1}{y} = \frac{x^2}{2} + C, \]

hence

\[ y = -\frac{2}{x^2 + 2C} = -\frac{2}{x^2 + C'}, \]

where \( C' \) is a new constant equal to \( 2C \).
3.1.6. **Initial Value Problems.** A differential equation together with an initial condition

\[
\begin{align*}
\frac{dy}{dx} &= F(x, y) \\
y(x_0) &= y_0
\end{align*}
\]

is called an *initial value problem*.

The initial condition can be used to determine the value of the constant in the solution of the equation.

*Example:* Solve the following initial value problem:

\[
\begin{align*}
\frac{dy}{dx} &= y^2 x \\
y(0) &= 1
\end{align*}
\]

*Solution:* We already found the general solution to the differential equation:

\[ y = -\frac{2}{x^2 + C}. \]

Next we let \( x = 0 \) and \( y = 1 \), and solve for \( C \):

\[ 1 = -\frac{2}{C} \implies C = -2. \]

So the solution is

\[ y = -\frac{2}{x^2 - 2} = \frac{2}{2 - x^2}. \]