3.4. The Logistic Equation

3.4.1. The Logistic Model. In the previous section we discussed a model of population growth in which the growth rate is proportional to the size of the population. In the resulting model the population grows exponentially. In reality this model is unrealistic because environments impose limitations to population growth. A more accurate model postulates that the relative growth rate P'/P decreases when P approaches the *carrying capacity* K of the environment. The corresponding equation is the so called *logistic differential equation*:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) .$$

3.4.2. Analytic Solution. The logistic equation can be solved by separation of variables:

$$\int \frac{dP}{P(1 - P/K)} = \int k \, dt .$$

In order to evaluate the left hand side we write:

$$\frac{1}{P(1 - P/K)} = \frac{K}{P(K - P)} = \frac{1}{P} + \frac{1}{K - P} ,$$

hence

$$\int \frac{dP}{P} + \int \frac{dP}{K - P} = \int k \, dt,$$
$$\ln |P| - \ln |K - P| = kt + C,$$
$$\ln \left|\frac{K - P}{P}\right| = -kt - C,$$
$$\left|\frac{K - P}{P}\right| = e^{-kt-C},$$
$$\frac{K - P}{P} = Ae^{-kt} \qquad (A = \pm e^{-C}) .$$

From here we get:

$$P = \frac{K}{1 + Ae^{-kt}} \quad \text{where } A = \frac{K - P_0}{P_0} .$$

Example: The population of the US in 1800 and 1850 was 5.3 and 23.1 million people respectively. Predict its population in 1900 and in 1950 using the exponential model of population growth. Then considering that the population of the US in 1900 was actually 76 million people

correct your prediction for 1950 using the logistic model of population growth (help: with this data $k = 0.031476$ in the logistic model). What is the carrying capacity of the US according to this model?

Answer: Since we start with observations in 1800 it makes sense to choose the variable t as time elapsed since 1800. According to the exponential model the population at time t is

$$P(t) = P_0 e^{kt},$$

where $P_0 = P(0)$. In our problem we have $P_0 = 5.3$. Next we determine the value of k from $P(50) = 5.3 e^{k \cdot 50} = 23.1 \Rightarrow k = (\log 23.1 - \log 5.3)/50 = 0.029443$. Hence, the population at time t according to the exponential model will be $P(t) = 5.3 e^{0.0294 \cdot t}$, and for 1900 ($t = 100$) and 1950 ($t = 150$) we get respectively:

$$P(100) = 5.3 e^{0.029443 \cdot 100} = 100.7,$$

$$P(150) = 5.3 e^{0.029443 \cdot 150} = 438.8.$$

Now we are told that the population in 1900 was actually $P(100) = 76$ million people and are asked to correct the prediction for 1950 using the logistic model. The logistic model is given by the formula

$$P(t) = \frac{K}{1 + A e^{-kt}},$$

where $A = (K - P_0)/P_0$. The given data tell us that

$$P(50) = \frac{K}{1 + (K - 5.3)e^{-50k}/5.3} = 23.1,$$

$$P(100) = \frac{K}{1 + (K - 5.3)e^{-100k}/5.3} = 76.$$

We can obtain K and k from these system of two equations, but we are told that $k = 0.031476$, so we only need to obtain K (the carrying capacity) from one of the equations, say the first one. The result is $K = 189.4$. From here we get $A = 34.74$ and

$$P(t) = \frac{189.4}{1 + 34.74 e^{-0.031476t}}.$$

hence in 1950, $P(150) = 144.7$ million people (the actual figure was 150.7 million people, slightly higher than expected due to the beginning of the so called "baby boom"). In this model the carrying capacity of the US is $K = 189.4$ million people.