

1.5. Integration by Parts

The method of integration by parts is based on the product rule for differentiation:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

which we can write like this:

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x).$$

Integrating we get:

$$\int f(x)g'(x)dx = \int [f(x)g(x)]' dx - \int g(x)f'(x)dx,$$

i.e.:

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}.$$

Writing $u = f(x)$, $v = g(x)$, we have $du = f'(x)dx$, $dv = g'(x)dx$, hence:

$$\boxed{\int u dv = uv - \int v du}.$$

Example: Integrate $\int xe^x dx$ by parts.

Answer: In integration by parts the key thing is to choose u and dv correctly. In this case the “right” choice is $u = x$, $dv = e^x dx$, so $du = dx$, $v = e^x$. We see that the choice is right because the new integral that we obtain after applying the formula of integration by parts is simpler than the original one:

$$\int \underbrace{x}_u \underbrace{e^x dx}_{dv} = \underbrace{x}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{dx}_{du} = \boxed{xe^x - e^x + C}.$$

Usually it is a good idea to check the answer by differentiating it:

$$(xe^x - e^x + C)' = e^x + xe^x - e^x = xe^x.$$

A couple of additional typical examples:

Example: $\int x \sin x dx = \dots$

$u = x$, $dv = \sin x \, dx$, so $du = dx$, $v = -\cos x$:

$$\begin{aligned} \dots &= \int \underbrace{x}_u \underbrace{\sin x \, dx}_{dv} = \underbrace{x}_u \underbrace{(-\cos x)}_v - \int \underbrace{(-\cos x)}_v \underbrace{dx}_{du} \\ &= \boxed{-x \cos x + \sin x + C}. \end{aligned}$$

Example: $\int \ln x \, dx = \dots$

$u = \ln x$, $dv = dx$, so $du = \frac{1}{x} dx$, $v = x$:

$$\begin{aligned} \dots &= \int \underbrace{\ln x}_u \underbrace{dx}_{dv} = \underbrace{\ln x}_u \underbrace{x}_v - \int \underbrace{x}_v \underbrace{\frac{1}{x} dx}_{du} \\ &= x \ln x - \int dx \\ &= \boxed{x \ln x - x + C}. \end{aligned}$$

Sometimes we need to use the formula more than once.

Example: $\int x^2 e^x \, dx = \dots$

$u = x^2$, $dv = e^x \, dx$, so $du = 2x \, dx$, $v = e^x$:

$$\dots = \int \underbrace{x^2}_u \underbrace{e^x \, dx}_{dv} = x^2 e^x - \int e^x 2x \, dx = \dots$$

$u = 2x$, $dv = e^x \, dx$, so $du = 2 \, dx$, $v = e^x$:

$$\begin{aligned} \dots &= x^2 e^x - \int \underbrace{2x}_u \underbrace{e^x \, dx}_{dv} = x^2 e^x - 2x e^x + \int 2e^x \, dx \\ &= \boxed{x^2 e^x - 2x e^x + 2e^x + C}. \end{aligned}$$

In the following example the formula of integration by parts does not yield a final answer, but an equation verified by the integral from which its value can be derived.

Example: $\int \sin x e^x \, dx = \dots$

$u = \sin x$, $dv = e^x dx$, so $du = \cos x dx$, $v = e^x$:

$$\dots = \int \underbrace{\sin x}_u \underbrace{e^x dx}_{dv} = \sin x \cdot e^x - \int e^x \cos x dx = \dots$$

$u = \cos x$, $dv = e^x dx$, so $du = -\sin x dx$, $v = e^x$:

$$\begin{aligned} \dots &= \sin x \cdot e^x - \int \underbrace{\cos x}_u \underbrace{e^x dx}_{dv} \\ &= \sin x \cdot e^x - \cos x \cdot e^x - \int e^x \sin x dx \end{aligned}$$

Hence the integral $I = \int \sin x e^x dx$ verifies

$$I = \sin x \cdot e^x - \cos x \cdot e^x - I,$$

i.e.,

$$2I = \sin x \cdot e^x - \cos x \cdot e^x,$$

hence

$$\boxed{I = \frac{1}{2}e^x(\sin x - \cos x) + C}.$$

1.5.1. Integration by parts for Definite Integrals. Combining the formula of integration by parts with the Evaluation Theorem we get:

$$\boxed{\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x) dx}.$$

Example: $\int_0^1 \tan^{-1} x dx = \dots$

$$\begin{aligned} u &= \tan^{-1} x, dv = dx, \text{ so } du = \frac{1}{1+x^2} dx, v = x: \\ \dots &= \int_0^1 \underbrace{\tan^{-1} x}_u \underbrace{dx}_{dv} = \left[\underbrace{\tan^{-1} x}_u \cdot \underbrace{x}_v \right]_0^1 - \int_0^1 \underbrace{x}_v \underbrace{\frac{1}{1+x^2} dx}_{du} \\ &= [\tan^{-1} 1 \cdot 1 - \tan^{-1} 0 \cdot 0] - \int_0^1 \frac{x}{1+x^2} dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx \end{aligned}$$

The last integral can be computed with the substitution $t = 1 + x^2$, $dt = 2x dx$:

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{t} dt = \frac{1}{2} [\ln t]_1^2 = \frac{\ln 2}{2}.$$

Hence the original integral is:

$$\int_0^1 \tan^{-1} x dx = \boxed{\frac{\pi}{4} - \frac{\ln 2}{2}}.$$

1.5.2. Reduction Formulas. Assume that we want to find the following integral for a given value of $n > 0$:

$$\int x^n e^x dx.$$

Using integration by parts with $u = x^n$ and $dv = e^x dx$, so $v = e^x$ and $du = nx^{n-1} dx$, we get:

$$\boxed{\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx}.$$

On the right hand side we get an integral similar to the original one but with x raised to $n-1$ instead of n . This kind of expression is called a *reduction formula*. Using this same formula several times, and taking into account that for $n = 0$ the integral becomes $\int e^x dx = e^x + C$, we can evaluate the original integral for any n . For instance:

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3 \int x^2 e^x dx \\ &= x^3 e^x - 3(x^2 e^x - 2 \int x e^x dx) \\ &= x^3 e^x - 3(x^2 e^x - 2(xe^x - \int e^x dx)) \\ &= x^3 e^x - 3(x^2 e^x - 2(xe^x - e^x)) + C \\ &= \boxed{x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}. \end{aligned}$$

Another example:

$$\begin{aligned}
 \int \sin^n x \, dx &= \int \underbrace{\sin^{n-1} x}_u \underbrace{\sin x \, dx}_{dv} \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \underbrace{\cos^2 x}_{1-\sin^2 x} \sin^{n-2} x \, dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \\
 &\quad - (n-1) \int \sin^n x \, dx
 \end{aligned}$$

Adding the last term to both sides and dividing by n we get the following reduction formula:

$$\boxed{\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx}$$