CHAPTER 4

Infinite Sequences and Series

4.1. Sequences

A sequence is an infinite ordered list of numbers, for example the sequence of odd positive integers:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29...

Symbolically the terms of a sequence are represented with indexed letters:

\[ a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots, a_n, \ldots \]

Sometimes we start a sequence with \( a_0 \) (index zero) instead of \( a_1 \).

Notation: the sequence \( a_1, a_2, a_3, \ldots \) is also denoted by \( \{ a_n \} \) or \( \{ a_n \}_{n=1}^\infty \).

Some sequences can be defined with a formula, for instance the sequence 1, 3, 5, 7, \ldots of odd positive integers can be defined with the formula \( a_n = 2n - 1 \).

A recursive definition consists of defining the next term of a sequence as a function of previous terms. For instance the Fibonacci sequence starts with \( f_1 = 1, f_2 = 1 \), and then each subsequent term is the sum of the two previous ones: \( f_n = f_{n-1} + f_{n-2} \); hence the sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots

4.1.1. Limits. The limit of a sequence is the value to which its terms approach indefinitely as \( n \) becomes large. We write that the limit of a sequence \( a_n \) is \( L \) in the following way:

\[ \lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty. \]

For instance

\[ \lim_{n \to \infty} \frac{1}{n} = 0, \]
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\[ \lim_{n \to \infty} \frac{n + 1}{n} = 1, \]

etc.

If a sequence has a (finite) limit then it is said to be convergent, otherwise it is divergent.

If the sequence becomes arbitrarily large then we write

\[ \lim_{n \to \infty} a_n = \infty. \]

For instance

\[ \lim_{n \to \infty} n^2 = \infty. \]

4.1.2. Theorem. Let \( f \) be a function defined in \( [1, \infty] \). If \( \lim_{x \to \infty} f(x) = L \) and \( a_n = f(n) \) for integer \( n \geq 1 \) then \( \lim_{n \to \infty} a_n = L \) (i.e., we can replace the limit of a sequence with that of a function.)

Example: Find \( \lim_{n \to \infty} \frac{\ln n}{n} \).

Answer: According to the theorem that limit equals \( \lim_{x \to \infty} \frac{\ln x}{x} \), where \( x \) represents a real (rather than integer) variable. But now we can use L'Hôpital’s Rule:

\[ \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{(\ln x)'}{(x)'} = \lim_{x \to \infty} \frac{1/x}{1} = 0, \]

hence

\[ \lim_{n \to \infty} \frac{\ln n}{n} = 0. \]

Example: Find \( \lim_{n \to \infty} r^n \) \((r > 0)\).

Answer: This limit is the same as that of the exponential function \( r^x \), hence

\[ \lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } 0 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases} \]
4.1.3. Operations with Limits. If \( a_n \to a \) and \( b_n \to b \) then:

\[
(a_n + b_n) \to a + b.
\]

\[
(a_n - b_n) \to a - b.
\]

\( ca_n \to ca \) for any constant \( c \).

\( a_n b_n \to ab \).

\( \frac{a_n}{b_n} \to \frac{a}{b} \) if \( b \neq 0 \).

\( (a_n)^p \to a^p \) if \( p > 0 \) and \( a_n > 0 \) for every \( n \).

**Example:** Find \( \lim_{n \to \infty} \frac{n^2 + n + 1}{2n^2 + 3} \).

**Answer:** We divide by \( n^2 \) on top and bottom and operate with limits inside the expression:

\[
\lim_{n \to \infty} \frac{n^2 + n + 1}{2n^2 + 3} = \lim_{n \to \infty} \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{2 + \frac{3}{n^2}} = \frac{1 + 0 + 0}{2 + 0} = \frac{1}{2}.
\]

4.1.4. Squeeze Theorem. If \( a_n \leq b_n \leq c_n \) for every \( n \geq n_0 \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \).

**Consequence:** If \( \lim_{n \to \infty} |a_n| = 0 \) then \( \lim_{n \to \infty} a_n = 0 \).

**Example:** Find \( \lim_{n \to \infty} \frac{\cos n}{n} \).

**Answer:** We have \( -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \), and \( \frac{1}{n} \to 0 \) as \( n \to \infty \), hence by the squeeze theorem

\[
\lim_{n \to \infty} \frac{\cos n}{n} = 0.
\]

4.1.5. Other definitions.

4.1.5.1. **Increasing, Decreasing, Monotonic.** A sequence is increasing if \( a_{n+1} > a_n \) for every \( n \). It is decreasing if \( a_{n+1} < a_n \) for every \( n \). It is called monotonic if it is either increasing or decreasing.

**Example:** Prove that the sequence \( a_n = \frac{n + 1}{n} \) is decreasing.
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Answer: \[ a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0, \]
hence \( a_{n+1} < a_n \) for all positive \( n \).

4.1.5.2. Bounded. A sequence is bounded above if there is a number \( M \) such that \( a_n \leq M \) for all \( n \). It is bounded below if there is a number \( m \) such that \( m \leq a_n \) for all \( n \). It is called just bounded if it is bounded above and below.

Example: Prove that the sequence \( a_n = \frac{n+1}{n} \) is bounded.

Answer: It is in fact bounded below because all its terms are positive: \( a_n > 0 \). To prove that it is bounded above note that

\[ a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \leq 2, \]
since \( 1/n \leq 1 \) for all positive integer \( n \).

4.1.6. Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

For instance, we proved that \( a_n = \frac{n+1}{n} \) is bounded and monotonic, so it must be convergent (in fact \( \frac{n+1}{n} \to 1 \) as \( n \to \infty \)).

Next example shows that sometimes in order to find a limit you may need to make sure that the limits exists first.

Example: Prove that the following sequence has a limit. Find it:

\[ \sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots \]

Answer: The sequence can be defined recursively as \( a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \) for \( n \geq 1 \). First we will prove by induction that \( 0 < a_n < 2 \), so the sequence is bounded.

We start (base of induction) by noticing that \( 0 < a_1 = \sqrt{2} < 2 \). Next the induction step. Assume (induction hypothesis) that for a given value of \( n \) it is true that \( 0 < a_n < 2 \). From here we must prove that the same is true for the next value of \( n \), i.e. that \( 0 < a_{n+1} < 2 \). In fact \( (a_{n+1})^2 = 2 + (a_n) < 2 + 2 = 4 \), hence \( 0 < a_{n+1} < \sqrt{4} = 2 \), q.e.d. So by the induction principle all terms of the sequence verify that \( 0 < a_n < 2 \).
Now we prove that $a_n$ is increasing:

\[(a_{n+1})^2 = 2 + a_n > a_n + a_n = 2a_n > a_n \cdot a_n = (a_n)^2 ,\]

hence $a_{n+1} > a_n$.

Finally, since the given sequence is bounded and increasing, by the monotonic sequence theorem it has a limit $L$. We can find it by taking limits in the recursive relation:

\[a_{n+1} = \sqrt{2 + a_n} .\]

Since $a_n \to L$ and $a_{n+1} \to L$ we have:

\[L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L \Rightarrow L^2 - L - 2 = 0 .\]

That equation has two solutions, $-1$ and $2$, but since the sequence is positive the limit cannot be negative, hence $L = 2$.

Note that the trick works only when we know for sure that the limit exists. For instance if we try to use the same trick with the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ ($f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$), calling $L$ the “limit” we get from the recursive relation that $L = L + L$, hence $L = 0$, so we “deduce” $\lim_{n \to \infty} f_n = 0$. But this is wrong, in fact the Fibonacci sequence is divergent.