THE BERNOULLI PERIODIC FUNCTIONS

MIGUEL A. LERMA

ABSTRACT. We study slightly modified versions of the Bernoulli periodic functions with nicer structural properties, and use them to give a very simple proof of the Euler-McLaurin summation formula.

1. DEFINITIONS

The Bernoulli polynomials $\mathrm{B}_n^*(x)$ can be defined in various ways.[1] The following are two of them ([4, ch. 1], [1]):

(1) By a generating function:

$$\frac{t\,e^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \mathrm{B}_n^*(x)\,\frac{t^n}{n!} \tag{1.1}$$

(2) By the following recursive formulas ($n \geq 1$):

$$\mathrm{B}_0^*(x) = 1\,, \tag{1.2}$$

$$\mathrm{B}_n^{*\prime}(x) = n\,\mathrm{B}_{n-1}^*(x)\,, \tag{1.3}$$

$$\int_0^1 \mathrm{B}_n^*(x)\,dx = 0\,. \tag{1.4}$$

The first Bernoulli polynomials are:

$\mathrm{B}_0^*(x) = 1$
$\mathrm{B}_1^*(x) = x - \frac{1}{2}$
$\mathrm{B}_2^*(x) = x^2 - x + \frac{1}{6}$
$\mathrm{B}_3^*(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

Date: June 19, 2002.

[1]Here we use the notation B_n^* for the Bernoulli polynomials, and reserve the notation B_n for the Bernoulli periodic functions.

The Bernoulli numbers are $B_n = \mathrm{B}_n^*(0)$, and the Bernoulli periodic functions are usually defined $\mathrm{B}_n(x) = \mathrm{B}_n^*(\langle x \rangle)$. However here we normalize B_1 defining $\mathrm{B}_1(k) = 0$ instead of $-1/2$ for k integer, so that B_1 coincides with the normalized sawtooth function:

$$\mathrm{B}_1(x) = \begin{cases} \langle x \rangle - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} \tag{1.5}$$

where $\langle x \rangle = x - \lfloor x \rfloor$ = fractional part of x, $\lfloor x \rfloor$ = integer part of x. Also we will leave $B_0(k)$ undefined for k integer—in fact B_0 should be defined as the distribution $B_0(x) = 1 - \delta_{per}(x)$, where $\delta_{per}(x) = \sum_{k=-\infty}^{\infty} \delta(x-k)$ is the periodic Dirac's delta.

1.0.1. *Properties of the Bernoulli Periodic Functions.* ($n \geq 1$):

1. $\mathrm{B}_1(x)$ = sawtooth function (eq. 1.5).
2. $\mathrm{B}_n'(x) = n\,\mathrm{B}_{n-1}(x)$ for $n > 2$ or $x \notin \mathbb{Z}$.
3. $\displaystyle\int_0^1 \mathrm{B}_n(x)\,dx = 0.$

1.0.2. *Fourier expansions.* The Fourier expansion for the Bernoulli periodic functions is ($n \geq 1$):

$$\mathrm{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^n}\,, \tag{1.6}$$

so:

$$\widehat{\mathrm{B}}_n(k) = \begin{cases} 0 & \text{if } k = 0, \\ -\dfrac{n!}{(2\pi i k)^n} & \text{otherwise.} \end{cases} \tag{1.7}$$

This result also holds in the distributional sense for $n = 0$.

1.1. **Polylogarithms.** The Bernoulli periodic functions appear naturally in expressions involving polylogarithms, together with the so called Clausen functions (see [3]):

$$\mathrm{Cl}_{2n-1}(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2n-1}}, \tag{1.8}$$

$$\mathrm{Cl}_{2n}(\theta) = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2n}}, \tag{1.9}$$

for $n \geq 1$.

To be more precise, the polylogarithms can be defined by the series:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{1.10}$$

for $n \geq 0$, $|z| < 1$, or by the following recursive relations:

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \tag{1.11}$$

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(\xi)}{\xi}\, d\xi \qquad (n \geq 1), \tag{1.12}$$

in $\mathbb{C} \setminus [1, \infty)$. Note that $\mathrm{Li}_1(z) = -\log(1-z)$ is the usual logarithm. $\mathrm{Li}_2(z)$ is the *dilogarithm.*[2]

A generating function is

$$\int_0^{\infty} \frac{z\, e^{(t+1)u}}{(e^u - z)^2}\, du = \sum_{n=0}^{\infty} \mathrm{Li}_n(z)\, t^n. \tag{1.13}$$

The Bernoulli periodic functions and the Clausen functions are related to the polylogarithms in the following way:

$$-\frac{2in!}{(2\pi i)^n}\, \mathrm{Li}_n(e^{2\pi i x}) = \mathrm{A}_n(x) \;+\; i\; \mathrm{B}_n(x), \tag{1.14}$$

for $x \notin \mathbb{Z}$, where

$$\mathrm{A}_n(x) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \frac{2n!}{(2\pi)^n}\, \mathrm{Cl}_n(2\pi x). \tag{1.15}$$

We will call the $\mathrm{A}_n(x)$ *conjugate Bernoulli periodic functions.* The first ones are $A_0(x) = \cot \pi x$, $A_1(x) = \frac{2}{\pi}\, \log\left(2\,|\sin \pi x|\right), \ldots$

The series (1.10) converges for $|z| = 1$ if $n \geq 2$.

[2]For some authors the dilogarithm is $\mathrm{Li}_2(1-z)$.

For $n=1$ both $\mathrm{Li}_1(e^{2\pi ix})$ and $\mathrm{Cl}_1(2\pi x)$ diverge at $x=0$, but

$$-\pi i\,\mathrm{B}_1(x)=\mathrm{Li}_1(e^{2\pi ix})-\mathrm{Cl}_1(2\pi x)=i\sum_{k=1}^{\infty}\frac{\sin 2\pi kx}{k}, \tag{1.16}$$

and the series becomes zero for $x=0$, so our definition $\mathrm{B}_1(0)=0$ allows (1.16) to hold also for $x=0$.

For $n=0$, $x\notin\mathbb{Z}$, we easily compute

$$\mathrm{Cl}_0(x)=-i\ \mathrm{Li}_0(e^{2\pi ix})-\frac{i}{2}=\frac{1}{2}\cot(\pi x). \tag{1.17}$$

Also by definition $\mathrm{Cl}_0(k)=0$ for $k\in\mathbb{Z}$. Hence,

$$\Im\{\mathrm{Li}_0(e^{2\pi ix})\}=\mathrm{Cl}_0(x) \tag{1.18}$$

for every $x\in\mathbb{R}$.

Finally we observe that for $y>0$

$$\Re\left\{\int_{-\frac{1}{2}}^{x}\mathrm{Li}_0(e^{2\pi i(u+yi)})\,du\right\}=-\frac{1}{2\pi}\arg\left\{e^{2\pi y}-e^{2\pi ix}\right\}, \tag{1.19}$$

which tends to $-\frac{1}{2}\ \mathrm{B}_1(x)$ as $y\to 0^+$ for every $x\in\mathbb{R}$, hence

$$\lim_{y\to 0^+}\Re\left\{\mathrm{Li}_0(e^{2\pi i(x+yi)})\right\}=-\frac{1}{2}\ \mathrm{B}_0(x)=-\frac{1}{2}+\frac{1}{2}\,\delta_{per}(x) \tag{1.20}$$

(where δ_{per} is the periodic Dirac's delta) in the distributional sense.

We also note that the Bernoulli periodic functions and their conjugates have harmonic extensions to the upper half plane, given by the formula:

$$-\frac{2in!}{(2\pi i)^n}\ \mathrm{Li}_n(e^{2\pi iz})=\mathrm{A}_n(z)\ +\ i\ \mathrm{B}_n(z), \tag{1.21}$$

for $\Im(z)>0$.

2. The Euler-Maclaurin Summation Formula

Theorem 2.1. *Let* $f : [a, b] \to \mathbb{C}$ *be* q *times differentiable,* $\int_a^b |f^{(q)}(x)|\, dx < \infty$*. Then for* $1 \le m \le q$*:*

$$
\begin{aligned}
\sum_{a \le n \le b}{}' f(n) &= \int_a^b f(x)\, dx \\
&\quad + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \left(\mathrm{B}_k(b)\, f^{(k-1)}(b) - \mathrm{B}_k(a)\, f^{(k-1)}(a)\right) \\
&\quad + \frac{(-1)^{m+1}}{m!} \int_a^b \mathrm{B}_m(x)\, f^{(m)}(x)\, dx,
\end{aligned} \tag{2.1}
$$

where $\sum_{a \le k \le b}{}' f(k)$ *for* $a < b$ *represents a summation modified by taking only half of* $f(k)$ *when* $k = a$ *or* $k = b$*.*

Proof. (See [2]) We have

$$
\begin{aligned}
\sum_{a \le k \le b}{}' f(n) &= \int_a^b f(x)\, d(x - \mathrm{B}_1(x)) \\
&= \int_a^b f(x)\, dx - \int_a^b f(x)\, d\,\mathrm{B}_1(x)
\end{aligned} \tag{2.2}
$$

Next, integrate by parts successively the last integral on the right hand side of (2.2). □

2.0.1. *Sum of Powers.* As an example of application of the Euler-Maclaurin summation formula, we give the sum of the first m rth powers:

$$
S(m, r) = \sum_{n=1}^{m} n^r = 1^r + 2^r + 3^r + \cdots + m^r \,.
$$

Here $f(x) = x^r$, so $f^{(k)}(x) = r!x^{(r-k)}/(r-k)!$ for $k = 0, 1, \ldots, r$, $f^{(k)}(x) = 0$ for $k > r$, and

$$\begin{aligned}
\sum_{0\le n\le m}{}' n^r = \int_0^m x^r\,dx \\
&+ \sum_{k=1}^{r+1} \frac{(-1)^k}{k!}\left(\mathrm{B}_k(m)\,f^{(k-1)}(m) - \mathrm{B}_k(0)\,f^{(k-1)}(0)\right) \\
&= \frac{m^{r+1}}{r+1} + \sum_{k=1}^{r+1} \frac{(-1)^k}{k!}\,B_k(0)\,\frac{r!}{(r-k+1)!}\,m^{r-k+1} - \frac{B_{r+1}(0)}{r+1} \\
&= \frac{m^{r+1}}{r+1} + \frac{1}{r+1}\sum_{k=1}^{r+1}(-1)^k\tbinom{r+1}{k}\,B_k(0)\,m^{r-k+1} - \frac{B_{r+1}(0)}{r+1}
\end{aligned}$$

Hence

$$\begin{aligned}
S(m,r) &= \sum_{0\le n\le m}{}' n^r + \frac{m^r}{2} \\
&= \frac{1}{r+1}\left\{\left(\sum_{k=0}^{r+1}(-1)^k\tbinom{r+1}{k}\,B_k\,m^{r-k+1}\right) - B_{r+1}\right\}
\end{aligned}$$

where B_k are the Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, $B_k = B_k(0)$ for $k > 1$.

References

[1] Tom M. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag, New York, 1976.

[2] Ralph P. Boas, Jr. Partial sums of infinite series, and how they grow. *Amer. Math. Monthly*, 84:237–258, 1977.

[3] Leonard Lewin. *Polylogartihms and Associated Functions*. North Holland, 1981.

[4] Hans Rademacher. *Topics in Analytic Number Theory*. Springer-Verlag, 1973.