AN EXTREMAL MAJORANT FOR THE LOGARITHM AND ITS APPLICATIONS

by

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Supervisor: ______________________________

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Dedicated to my immediate family, i.e., my father Miguel, my sister Isabel, my wife Denise, and my cats Pumby, Blanquita, Chispa, Poquito and Pulga, whose lives fill our lives with joy.

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Miguel A. Lerma

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This research deals with the problem of finding extremal majorants for given functions, a subject started by Beurling and Selberg in the 1930's. We prove some general results useful in finding extremal majorants and minorants for a wide class of even functions, extending results previously found by Graham and Vaaler in 1985, and we apply them to the particular case of $\log|x|$. Next we study in some detail the properties of the extremal majorant for $\log|x|$, and use it to prove an "Erdős-Turán"-type inequality useful to estimate the sup norm of polynomials on the unit circle. We also prove an analogue of Montgomery and Vaughan's inequality. Next we state several conjectures and suggest some possible directions to continue the research. Also, we prove a few theorems concerning harmonic majorants.

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Chapter 1

Introduction

One subject of interest in the theory of uniform distribution modulo one is the estimation of the discrepancy of a sequence $x_1, x_2, \ldots, x_M$ by an expression depending on trigonometric sums of the form $\sum_{m=1}^{M} e^{2\pi i n x_m}$. The discrepancy gives a measure of how much a given sequence gets apart from uniform distribution, and is defined as follows:

$$\Delta^*(x_1, \ldots, x_M) \;=\; \Delta^*_M \;=\; \sup_{0<t-s<1} \left| \sum_{m=1}^{M} \chi_{s,t}(x_m) - M(t-s) \right|, \tag{1.1}$$

where

$$\chi_{s,t}(x) \;=\; \begin{cases} 1 & \text{if } s < x - n < t \;\text{ for some } n \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } s - x \in \mathbb{Z} \;\text{ or } t - x \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

It is interesting to note that an estimation of the discrepancy could have practical applications, such as the estimation of the error of algorithms for numerical integration, particularly those based on computing means of values of the integrand at points of the interval of integration. This is accomplished by the Koksma's inequality: Let f be a function on the interval $[0, 1]$ of bounded variation $V(f)$, and suppose we are given M points in $[0, 1]$ with discrepancy

Δ^*_M. Then:

$$\left| \sum_{n=1}^{M} f(x_n) - M \int_0^1 f(t)\, dt \right| \;\le\; V(f)\, \Delta^*_M. \tag{1.3}$$

One well known upper bound for the discrepancy is given by the Erdős-Turán inequality [3], which has the form

$$\Delta^*_M \;\le\; c_1 \frac{M}{N} \;+\; c_2 \sum_{n=1}^{N} \frac{1}{n} \left| \sum_{m=1}^{M} e^{2\pi i n x_m} \right|, \tag{1.4}$$

where c_1 and c_2 are positive constants and N is an integer that can be chosen so as to minimize the right hand side of (1.4).

That result has been refined by Vaaler [11], using a slightly different definition for the discrepancy:

$$\Delta(x_1, \ldots, x_M) \;=\; \Delta_M \;=\; \sup_{y \in \mathbb{R}} \left| \sum_{m=1}^{M} \psi(y - x_m) \right|, \tag{1.5}$$

where

$$\psi(x) \;=\; \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{1.6}$$

It can be verified that $\Delta_M \le \Delta^*_M \le 2\Delta_M$, so the two discrepancies differ insignificantly. The refinement was made by using some extremal functions, as discussed in [10]. A short exposition of the main results will be given in the next chapter.

1.1 Notations, Definitions, Conventions and Basic Results

Here we give some miscellaneous definitions and notations that will be used later.

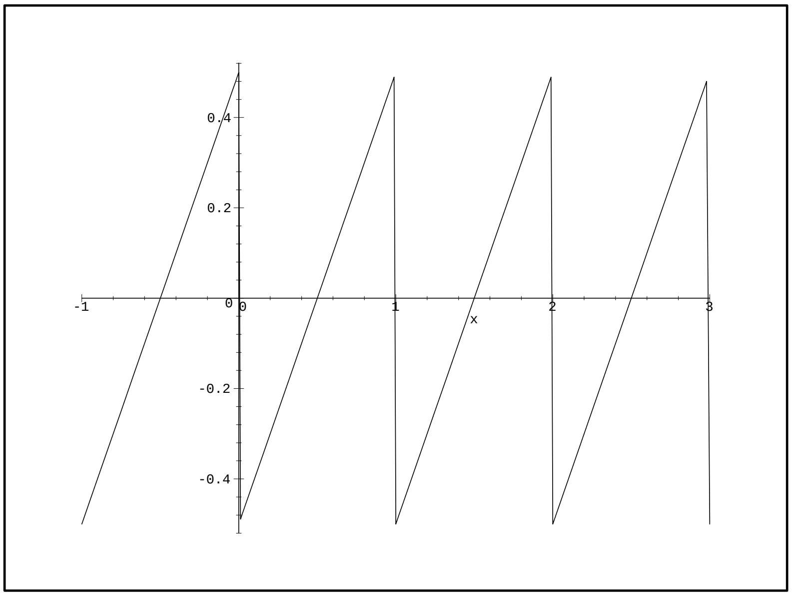

Figure 1.1: The function $\psi(x)$

We use the standard notations for number sets such as the *integers* ($\mathbb{Z}$), the *real numbers* ($\mathbb{R}$) and the *complex numbers* ($\mathbb{C}$). The *real part* of a complex number z is represented as $\Re(z)$, and its *imaginary part* is $\Im(z)$. We denote the *right half plane* $\mathcal{R} = \{z \in \mathbb{C} : \Re(z) > 0\}$, and the *upper half plane* $\mathcal{H}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$.

An entire function $F : \mathbb{C} \to \mathbb{C}$ is said to be of *exponential type* if

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} \;=\; \tau(F) \;<\; \infty \tag{1.7}$$

Then the nonnegative number $\tau(F)$ is the *exponential type* of F. A function is said the be *real entire* if it is entire and takes real values on $\mathbb{R}$.

Following [8], the definition of *Fourier transform* $\widehat{f}$ of a function f in

$L^1(\mathbb{R})$ will be the following:

$$\widehat{f}(t) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i t x}\, dx. \tag{1.8}$$

The Fourier transform can be extended to $L^2(\mathbb{R})$ as shown in [8, II.2]. Also, for certain functions that are not absolutely integrable, the Fourier transform can be defined by

$$\widehat{f}(t) \;=\; \lim_{M\to\infty} \int_{-M}^{M} f(x)\, e^{-2\pi i t x}\, dx. \tag{1.9}$$

The following expression converges even for a wider class of functions:

$$\widehat{f}(t) \;=\; \lim_{M\to\infty} \int_{-M}^{M} f(x) \left(1 - \frac{|x|}{M}\right) e^{-2\pi i t x}\, dx. \tag{1.10}$$

An infinite sum of the form $\sum_{n=-\infty}^{\infty} a_n$, which is unambiguous when the series is absolutely convergent, will be interpreted as the limit of the symmetric partial sums $\lim_{T\to\infty} \sum_{n=-T}^{T} a_n$ in other cases, as long as the limit exists.

We mention here the Poisson summation formula, which is used often in this work.

Theorem (Poisson Summation Formula). *If f is absolutely integrable over $\mathbb{R}$, of bounded variation and normalized in the sense that for every x,*

$$f(x) \;=\; \frac{1}{2} \lim_{h\to 0} \{f(x+h) + f(x-h)\},$$

then

$$\sum_{n=-\infty}^{\infty} f(n) \;=\; \lim_{T\to\infty} \sum_{k=-T}^{T} \widehat{f}(k). \tag{1.11}$$

See [12, II.13] for a more general formulation that includes the case in which f is not absolutely integrable. See also lemma 8.2 for a particular case in which the function is not of bounded variation.

Chapter 2

Some Extremal Functions in Fourier Analysis

2.1 An Extremal Majorant for the Signum Function

In the late 1930's A. Beurling observed that the entire function

$$B(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} \;-\; \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} \;+\; \frac{2}{z} \right\} \qquad (2.1)$$

has the following properties:

1. It is real entire of exponential type 2π.

2. It majorizes $\operatorname{sgn}(x)$ (the signum of x) along the real axis:

$$\operatorname{sgn}(x) \;\le\; B(x) \qquad \text{for every } x \in \mathbb{R}.$$

3. It satisfies:

$$\int_{-\infty}^{\infty} \{\, B(x) - \operatorname{sgn}(x) \,\}\, dx \;=\; 1. \qquad (2.2)$$

4. It is extremal, in the sense that among all functions satisfying 1 and 2, it is the one that minimizes integral (2.2) in 3.

In general, given a real function $f : \mathbb{R} \to \mathbb{R}$, if $F : \mathbb{C} \to \mathbb{C}$ is a real entire function of exponential type at most 2π which majorizes f along the

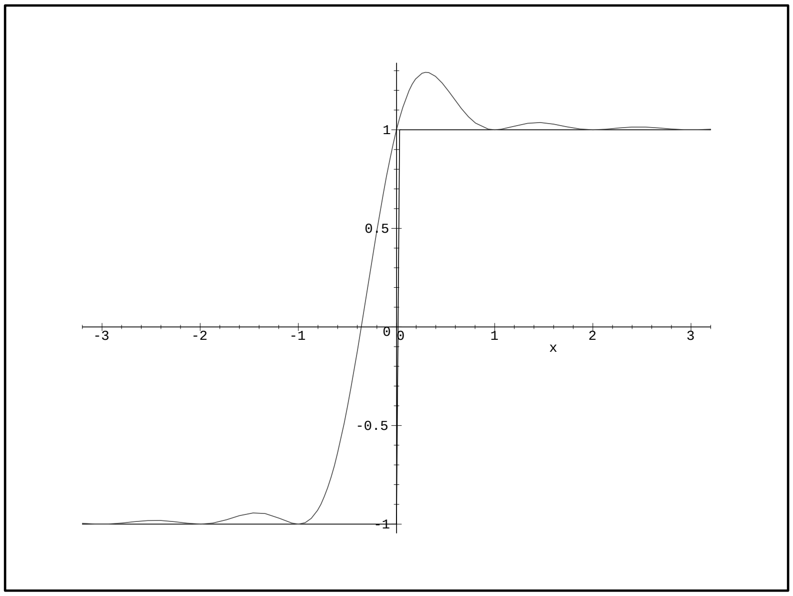

Figure 2.1: Beurling's function $B(x)$ majorizing $\operatorname{sgn}(x)$

real axis:

$$f(x) \ \leq \ F(x) \qquad \text{for every } x \in \mathbb{R}.$$

we say that F is a *majorant* of f. If among all functions with those properties F minimizes the integral

$$\int_{-\infty}^{\infty} \{ \, F(x) - f(x) \, \} \, dx,$$

then we say that F is an *extremal majorant* of f. The definitions of *minorant* and *extremal minorant* are analogous, but with $F(x) \leq f(x)$, and F minimizing the integral

$$\int_{-\infty}^{\infty} \{ \, f(x) - F(x) \, \} \, dx.$$

In 1974 Selberg found an extremal majorant for the characteristic function χ_E of an interval $E = [\alpha, \beta]$, where $\beta - \alpha$ is an integer (see [7]), and obtained a sharp form of the large sieve inequality. Selberg's function coincides with

$$C_E(z) \;=\; \frac{1}{2}\{B(\beta - z) + B(z - \alpha)\}, \tag{2.3}$$

so that

$$\chi_E(x) \;=\; \frac{1}{2}\{\text{sgn}(\beta - z) + \text{sgn}(z - \alpha)\} \;\le\; C_E(x). \tag{2.4}$$

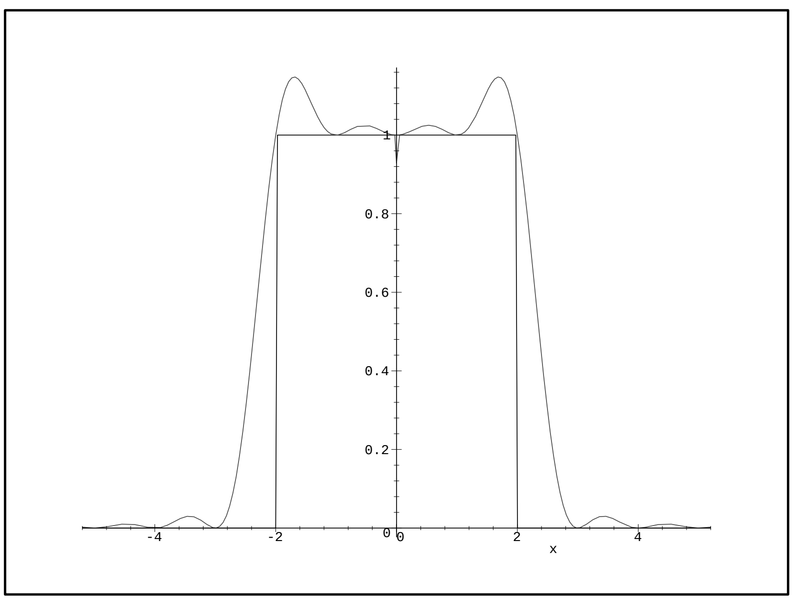

Figure 2.2: Selberg's function $C_E(x)$ majorizing the characteristic function of an interval

Beurling's function is a particular case of a more general class of functions. If $F(z)$ is an entire function of exponential type 2π, bounded on $\mathbb{R}$, and odd, then it can be represented by the interpolation formula:

$$F(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{F(n)}{(z-n)^2} \;+\; \lim_{T\to\infty} \sum_{n=-T}^{T} \frac{F'(n)}{z-n} \right\}. \tag{2.5}$$

So, by giving suitable values to the numerators of the terms of that series it is possible to "force" F and its derivative to take prescribed values at the integers.

A similar result follows if F is in E^p for some finite p, where E^p is the space of entire functions of exponential type at most 2π such that

$$\int_{-\infty}^{\infty} |F(x)|^p \, dx \;<\; \infty \tag{2.6}$$

(see [10, theorem 9]).

Note that $B(z)$ can be interpreted as a function that majorizes $\operatorname{sgn}(x)$, and interpolates that function and its derivative at the nonzero integers, i.e., $B(n) = \operatorname{sgn}(n)$ and $B'(n) = \operatorname{sgn}'(n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. Since we are interested in a majorizing function, $B(0)$ must be 1. On the other hand $\operatorname{sgn}(x)$ has no derivative at zero, so $B'(0)$ can be left as a parameter to be determined later. It turns out that the "right" value for $B'(0)$ is precisely 2. Modifying the definition of $B(z)$ by giving it a value of zero at zero yields a slightly different (and more symmetric) function (note that $\operatorname{sgn}(0) = 0$):

$$H(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} \;+\; \frac{2}{z} \right\}. \tag{2.7}$$

Now $H(z)$ interpolates $\operatorname{sgn}(x)$ at the integers and its derivative at the nonzero integers, but it is not a majorizing function of $\operatorname{sgn}(x)$ any more. However we recover $B(z)$ just by adding the following function:

$$K(z) \;=\; \left(\frac{\sin \pi z}{\pi z}\right)^2, \tag{2.8}$$

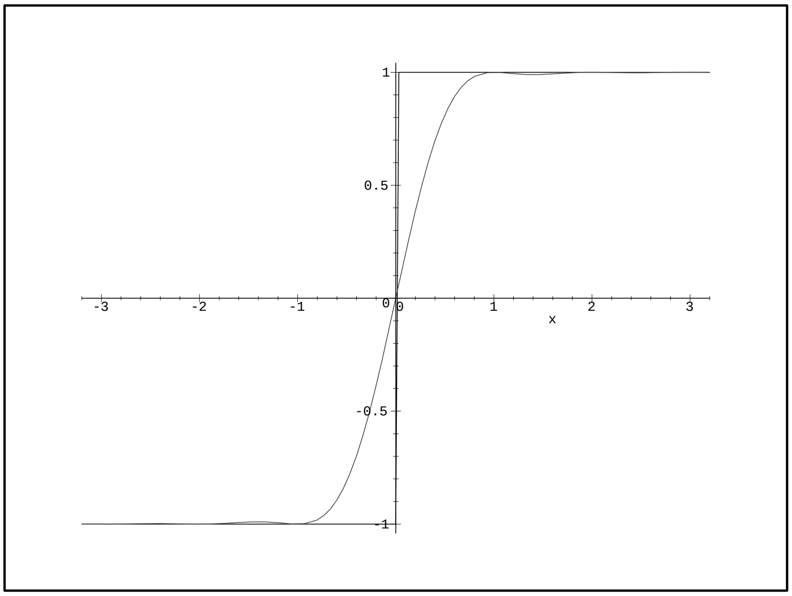

Figure 2.3: Function $H(x)$

i.e.:

$$B(z) \;=\; H(z) \;+\; K(z). \tag{2.9}$$

By *subtracting* $K(z)$ from $H(z)$ we get

$$-B(-z) = H(z) \;-\; K(z), \tag{2.10}$$

which has the property of *minorizing* $\operatorname{sgn}(x)$ along the real axis:

$$-B(-x) \;\leq\; \operatorname{sgn}(x) \qquad \text{for every } x \in \mathbb{R}. \tag{2.11}$$

The facts that $B(z)$ is an extremal function in the sense of minimizing integral (2.2), and is also an interpolating function for $\operatorname{sgn}(x)$ at the integers,

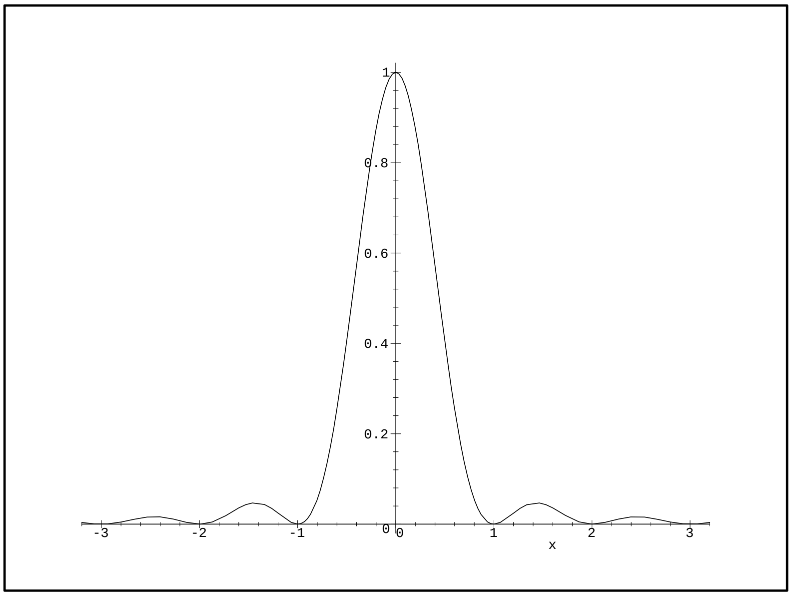

Figure 2.4: Function $K(x)$

are connected. In fact, if $F(z)$ is some other function with the same properties 1 and 2 as $B(z)$, writing $D(x) = B(x) - \operatorname{sgn}(x)$ and using the Poisson summation formula we get:

$$\sum_{l=-\infty}^{\infty} D(x+l) \; = \; \lim_{T\to\infty} \sum_{m=-T}^{T} \widehat{D}(m)\, e^{2\pi i m x}, \tag{2.12}$$

where

$$\widehat{D}(t) \; = \; \int_{-\infty}^{\infty} D(x)\, e^{-2\pi i t x}\, dx \tag{2.13}$$

is the Fourier transform of $D(x)$.

It can be proven that the Fourier transform of $F'(x)$ is supported in $[-1, 1]$, and this implies that for $|t| \geq 1$, $\widehat{D}(t)$ equals the Fourier transform of

$B(x) - \operatorname{sgn}(x)$, which is $\frac{1}{\pi i t}$ (for $|t| \geq 1$), hence:

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \widehat{D}(0) \;+\; \lim_{T\to\infty} \sum_{\substack{m=-T \\ m\neq 0}}^{T} \frac{1}{\pi i m}\, e^{2\pi i m x} \;=\; \widehat{D}(0) + 2\,\psi(x), \tag{2.14}$$

where $\psi(x)$ is the function defined in (1.6). Note that $\psi(0^+) = -\frac{1}{2}$, hence:

$$\sum_{l=-\infty}^{\infty} D(l^+) \;=\; \widehat{D}(0) \;-1. \tag{2.15}$$

Since $D(x)$ is nonnegative, we get that:

$$\int_{-\infty}^{\infty} D(x)\,dx \;=\; \widehat{D}(0) \;\geq\; 1, \tag{2.16}$$

so that B is in fact extremal. Furthermore, if we want F to be extremal, then we need $\widehat{D}(0) = 1$; but this implies:

$$\sum_{l=-\infty}^{\infty} D(l^+) \;=\; 0, \tag{2.17}$$

so that $D(l^+) = 0$, i.e., $F(l^+) = \operatorname{sgn}(l^+)$ for every integer l. Since $F(x) \geq \operatorname{sgn}(x)$, also $F'(l) = 0$ for every nonzero integer l (the derivative at zero can be determined by a slightly more refined argument). Hence F is an interpolating function for $\operatorname{sgn}(x)$ at the integers. Actually, by using the expansion (2.5) we get that F is precisely B, i.e., B is the only function possessing properties 1–3.

A few examples of application of the properties of B are shown next.

Theorem 2.1 (The Large Sieve). *Let*

$$S(x) \;=\; \sum_{n=M+1}^{M+N} a_n\, e^{2\pi n x} \tag{2.18}$$

be a trigonometric polynomial with period 1, and let $\xi_1, \xi_2, \ldots, \xi_R$ *be real numbers which are well spaced modulo 1 in the sense that* $\|\xi_r - \xi_s\| \geq \delta > 0$ *for* $r \neq s$*, where* $\|x\|$ = *distance from* x *to the nearest integer. Then the large sieve inequality*

$$\sum_{r=1}^{R} |S(\xi_r)|^2 \;\leq\; \Delta(N,\delta) \sum_{n=M+1}^{M+N} |a_n|^2 \tag{2.19}$$

holds with $\Delta(N,\delta) = N - 1 + \frac{1}{\delta}$*, which is sharp.*

Proof. See [10, pp.185–186]. The proof actually uses Selberg's function $C_E(z) = \frac{1}{2}\{B(\beta - z) + B(z - \alpha)\}$. □

Theorem 2.2 (Montgomery and Vaughan). *Let* $\lambda_1, \ldots, \lambda_N$ *be real numbers satisfying* $|\lambda_m - \lambda_n| \geq \delta > 0$ *whenever* $m \neq n$*, and let* $a(1), \ldots, a(N)$ *be arbitrary complex numbers. Then*

$$\left| \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{\lambda_m - \lambda_n} \right| \;\leq\; \frac{\pi}{\delta} \sum_{n=1}^{N} |a(n)|^2. \tag{2.20}$$

Proof. See [10, theorem 16]. □

Theorem 2.3 (Erdős-Turán inequality). *If* $x_1, x_2, \ldots, x_M$ *are real numbers and if* N *is a positive integer, then*

$$\Delta_M \;\leq\; \frac{M}{2N} \;+\; \left\{1 + \frac{1}{\pi}\right\} \sum_{n=1}^{N} \frac{1}{n} \left| \sum_{m=1}^{M} e^{2\pi n x_m} \right|. \tag{2.21}$$

Hence the Erdős-Turán inequality (1.4) holds with $c_1 = \frac{1}{2}$ *and* $c_2 = 1 + \frac{1}{\pi}$.

Proof. The result can be easily derived from [11, theorem 1]. □

2.2 An Extremal Majorant for the Logarithm

A way to study a sequence $x_1, x_2, \ldots$ modulo 1 is by looking at the sequence $e^{2\pi i x_m}$ on the unit circle. An analogue of the concept of "discrepancy" of M points $x_1, x_2, \ldots, x_M$ can be obtained by considering the following expression:

$$\Gamma_M \;=\; \frac{1}{\pi} \log \sup_{y\in\mathbb{R}} \left| \prod_{m=1}^{M} (e^{2\pi i y} - e^{2\pi i x_m}) \right| \;=\; \sup_{y\in\mathbb{R}} \sum_{m=1}^{M} \varphi(y - x_m), \tag{2.22}$$

where

$$\varphi(x) \;=\; \begin{cases} \frac{1}{\pi} \log|2\sin \pi x| & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ -\infty & \text{if } x \in \mathbb{Z}. \end{cases} \tag{2.23}$$

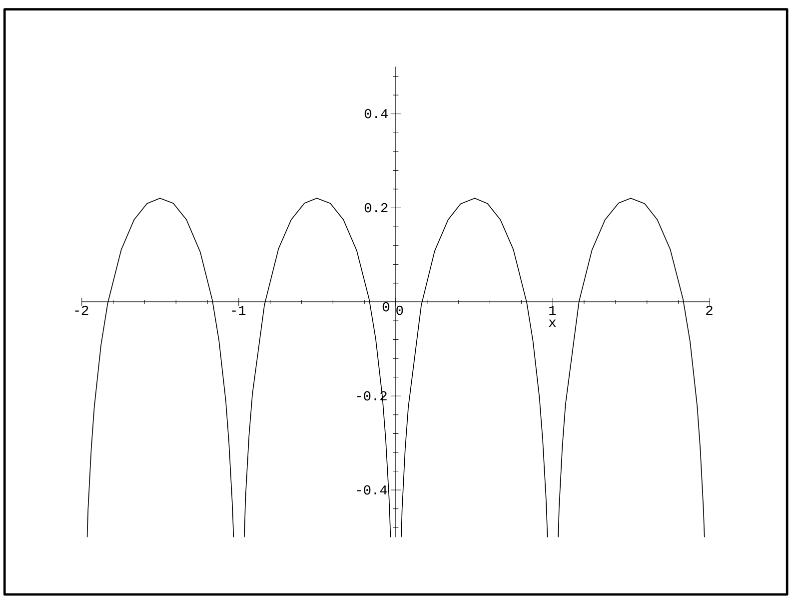

Figure 2.5: The function $\varphi(x)$

Note the similarity with the definition (1.5) of the discrepancy Δ_M. Also note that ψ and φ are (Fourier) conjugate functions, as can be deduced from their Fourier expansions:[1]

$$\widehat{\psi}(n) \;=\; \begin{cases} -\frac{1}{2\pi i n} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases} \tag{2.24}$$

and

$$\widehat{\varphi}(n) \;=\; \begin{cases} -\frac{1}{2\pi |n|} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases} \tag{2.25}$$

so that

$$\psi(x) \;=\; -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n x), \tag{2.26}$$

and

$$\varphi(x) \;=\; -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos(2\pi n x). \tag{2.27}$$

Another interesting remark is that Γ_M can be interpreted as $\frac{1}{\pi}$ times the supremum in the unit disk of the absolute value of a polynomial whose roots all lie on the unit circle:

$$\Gamma_M \;=\; \Gamma(x_1, x_2, \ldots, x_M) \;=\; \frac{1}{\pi} \sup \{|P_M(z)| : |z| \leq 1\}, \tag{2.28}$$

where

$$P_M(z) \;=\; \prod_{m=1}^{M} (z - e^{2\pi i x_m}). \tag{2.29}$$

Recall that $\psi(x)$ appeared naturally in (2.14) after applying the Poisson summation formula to $\sum_{l=-\infty}^{\infty} \{B(x+l) - \operatorname{sgn}(x+l)\}$. This justifies to pose an analogous problem for the the conjugate function of $\operatorname{sgn}(x)$, namely $\frac{2}{\pi} \log |x|$.

[1]See [12] for the relation between Fourier and harmonic conjugate functions.

If we get an entire function $F(z)$ of exponential type 2π that majorizes $\log|x|$ along the real axis and proceed analogously to the Beurling function, then we get (see lemma 8.2):

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \widehat{D}(0) \;+\; \lim_{T\to\infty} \sum_{\substack{m=-T \\ m\neq 0}}^{T} \frac{1}{2|m|}\, e^{2\pi i m x} \;=\; \widehat{D}(0) - \pi\,\varphi(x), \tag{2.30}$$

now with $D(x) = F(x) - \log|x|$. Since $D(x) \geq 0$ then

$$\widehat{D}(0) \geq \pi\,\varphi(x). \tag{2.31}$$

The maximum of $\varphi(x)$ is $\frac{1}{\pi}\log 2$, attained at $n + \frac{1}{2}$ $(n \in \mathbb{Z})$. If we want F to be extremal in the same sense as the Beurling's function, then:

$$\int_{-\infty}^{\infty} D(x)\,dx \;=\; \widehat{D}(0) \;=\; \log 2, \tag{2.32}$$

which implies $D(\frac{1}{2} + l) = 0$ for every $l \in \mathbb{Z}$. This implies that $F(x)$ must interpolate $\log|x|$ and its derivative at the integers plus a half, so its expansion must be:

$$F(z) \;=\; \left(\frac{\cos\pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\log|n+\frac{1}{2}|}{(z-(n+\frac{1}{2}))^2} \;+\; \sum_{n=-\infty}^{\infty} \frac{\frac{1}{n+\frac{1}{2}}}{z-(n+\frac{1}{2})} \right\}. \tag{2.33}$$

It turns out that this function is in fact an extremal majorant for $\log|x|$:

Theorem 2.4 (Main Theorem). *The function defined by 2.33 has the following properties:*

1. *It is real entire of exponential type 2π.*

2. *It majorizes $\log|x|$ along the real axis:*

$$\log|x| \;\leq\; F(x) \qquad \textit{for every } x \in \mathbb{R}.$$

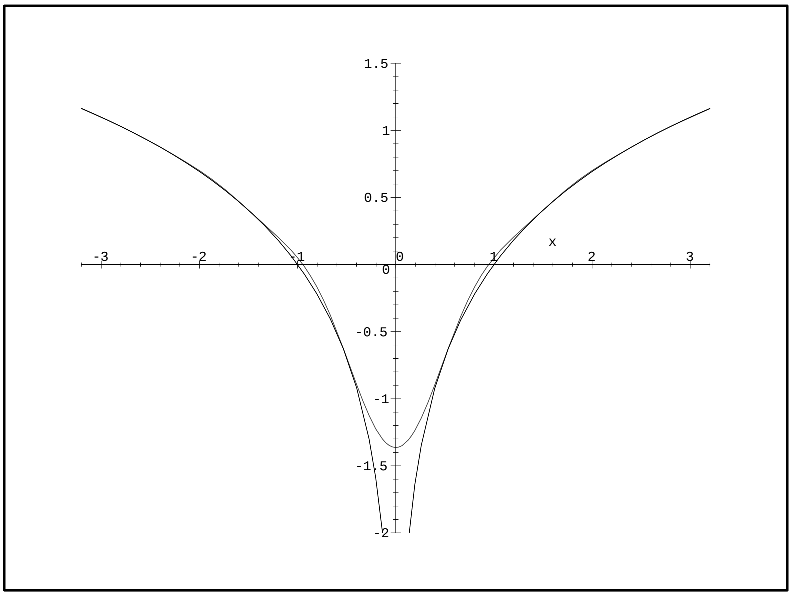

Figure 2.6: Function $F(x)$ majorizing $\log|x|$

3. It verifies:

$$\int_{-\infty}^{\infty} \{\, F(x) - \log|x| \,\}\, dx \;=\; \log 2. \tag{2.34}$$

4. It is extremal, in the sense that among all functions satisfying 1 and 2, it is the one that minimizes integral (2.34) in 3.

In the next chapter we prove some general results that apply to a certain class of even functions. Then the fact that $F(x)$ is a majorant of $\log|x|$ is obtained as a consequence.

Chapter 3

Extremal Minorants for a Class of Even Functions

In this section we generalize some results first obtained by Graham and Vaaler in [4]. Then we use them to prove that the function F defined by (2.33) is an extremal majorant for $\log|x|$.

3.1 The Extremal Minorant for $e^{-\lambda|x|}$

Let K and L be the following entire functions:

$$K(z) \;=\; \left(\frac{\sin \pi z}{\pi z}\right)^2 \qquad \text{and} \qquad L(z) \;=\; z\,K(z). \tag{3.1}$$

Then, for $0 < \lambda$ define an entire function $M_\lambda(z)$ by

$$\begin{aligned} M_\lambda(z) \;=\; & \sum_{m=-\infty}^{\infty} e^{-\lambda|m+\frac{1}{2}|}\, K(z-m-\tfrac{1}{2}) \\ & -\; \lambda \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n+\tfrac{1}{2})\, e^{-\lambda|n+\frac{1}{2}|}\, L(z-n-\tfrac{1}{2}). \end{aligned} \tag{3.2}$$

Graham and Vaaler have shown in [4] that

(i) $M_\lambda(z)$ is a real entire function of exponential type 2π,

(ii) $M_\lambda(x) \le e^{-\lambda|x|}$ for all real x, and

(iii) among all real entire functions which satisfy (i) and (ii) the function $M_\lambda(x)$ minimizes the value of

$$\int_{-\infty}^{\infty} \left\{e^{-\lambda|x|} - M_\lambda(x)\right\} dx. \tag{3.3}$$

Also we find that

$$\int_{-\infty}^{\infty} \left\{e^{-\lambda|x|} - M_\lambda(x)\right\} dx = \frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right). \tag{3.4}$$

For $\lambda > 0$ the entire function $M_\lambda(z)$ defined above has exponential type 2π and is integrable on $\mathbb{R}$, hence

$$\widehat{M}_\lambda(t) = \int_{-\infty}^{\infty} M_\lambda(x)\, e^{-2\pi itx}\, dx \tag{3.5}$$

is supported on $[-1, 1]$, and then

$$M_\lambda(z) = \int_{-1}^{1} \widehat{M}_\lambda(t)\, e^{2\pi itz}\, dt. \tag{3.6}$$

for all complex z.

The following theorem gives the value of $\widehat{M}_\lambda(z)$.

Theorem 3.1. *The Fourier transform of $M_\lambda(z)$ is*

$$\begin{aligned}\widehat{M}_\lambda(t) &= \Re\left\{\frac{1 - |t| - \frac{\lambda}{2\pi i}\operatorname{sgn}(t)}{\sinh\left(\frac{\lambda}{2} + \pi it\right)}\right\} \\ &= \frac{(1 - |t|)\sinh\frac{\lambda}{2}\cos\pi t + \frac{\lambda}{2\pi}\cosh\frac{\lambda}{2}\,|\sin\pi t|}{\sinh^2\frac{\lambda}{2} + \sin^2\pi t}\end{aligned} \tag{3.7}$$

for $|t| \le 1$. Also, $\widehat{M}_\lambda(t) \ge 0$ for all real t.

Proof. For $0 < \lambda$ we define (as in [4])

$$A_\lambda(z) = \left(\frac{\sin\pi z}{\pi}\right)^2 \sum_{n=0}^{\infty} e^{-\lambda n}\left\{\frac{1}{(z-n)^2} - \frac{\lambda}{z-n}\right\}. \tag{3.8}$$

Then $A_\lambda(z)$ has exponential type 2π and its restriction to $\mathbb{R}$ is in $L^2(\mathbb{R})$. By [10, theorem 9], we have

$$A_\lambda(z) \;=\; \int_{-1}^{1} \widehat{A}_\lambda(t)\, e^{2\pi i t z}\, dt. \tag{3.9}$$

for all complex z, where

$$\widehat{A}_\lambda(t) \;=\; (1-|t|)\, u_\lambda(t) \;+\; \frac{\mathrm{sgn}(t)}{2\pi i}\, v_\lambda(t), \tag{3.10}$$

$$u_\lambda(t) \;=\; \sum_{m=0}^{\infty} e^{-\lambda m - 2\pi i m t} \;=\; \frac{1}{1-e^{-\lambda-2\pi i t}}, \tag{3.11}$$

$$v_\lambda(t) \;=\; -\lambda \sum_{m=0}^{\infty} e^{-\lambda m - 2\pi i m t} \;=\; -\lambda\, u_\lambda(t). \tag{3.12}$$

Thus

$$\widehat{A}_\lambda(t) \;=\; \frac{1-|t|-\frac{\lambda}{2\pi i}\,\mathrm{sgn}(t)}{1-e^{-\lambda-2\pi i t}} \tag{3.13}$$

for $|t| \le 1$. Next we observe that

$$\begin{aligned} M_\lambda(z) \;&=\; e^{-\frac{\lambda}{2}}\left\{A_\lambda(z-\tfrac{1}{2}) + A_\lambda(-z-\tfrac{1}{2})\right\} \\ &=\; e^{-\frac{\lambda}{2}}\left\{\int_{-1}^{1} \widehat{A}_\lambda(t)\, e^{-\pi i t + 2\pi i z t}\, dt \;+\; \int_{-1}^{1} \widehat{A}_\lambda(-t)\, e^{\pi i t + 2\pi i z t}\, dt\right\}, \end{aligned} \tag{3.14}$$

and therefore

$$\widehat{M}_\lambda(t) \;=\; e^{-\frac{\lambda}{2}}\left\{\widehat{A}_\lambda(t)\, e^{-\pi i t} + \widehat{A}_\lambda(-t)\, e^{\pi i t}\right\} \tag{3.15}$$

and from here we obtain (3.7).

In the proof that $\widehat{M}_\lambda(t) \ge 0$, by symmetry we can assume that $0 \le t \le 1$. The denominator of the last expression of (3.7) is positive, and the numerator is obviously nonnegative for $0 \le t \le \frac{1}{2}$. On the other hand, for $\frac{1}{2} \le t < 1$ it is enough to take into account that

$$\frac{\tanh\frac{\lambda}{2}}{\frac{\lambda}{2}} \;\le\; \frac{\tan \pi(1-t)}{\pi(1-t)}, \tag{3.16}$$

which is true since the left hand side is not greater than 1, and the right hand side is not less than 1. □

Next we derive some identities for the entire function $z \mapsto e^{-\lambda z} - M_\lambda(z)$, where $\lambda > 0$. In doing so we write

$$C(\omega) \;=\; \frac{\omega}{2\sinh\frac{\omega}{2}} \tag{3.17}$$

for all real ω, with $C(0) = 1$.

Theorem 3.2. *In the half plane $-\frac{1}{2} < \Re(z)$ we have*

$$e^{-\lambda z} - M_\lambda(z) \;=\; \left(\frac{\cos \pi z}{\pi}\right) \int_0^\infty \{C(\lambda-\omega) - C(\lambda+\omega)\}\, e^{-z\omega}\, d\omega. \tag{3.18}$$

Proof. This occurs already in the proof of theorem 8 of [4] for real z. The argument when z is complex is essentially the same. □

Theorem 3.3. *For $-\frac{1}{2} < \alpha < \frac{1}{2}$ we have*

$$C(\lambda-\omega) - C(\lambda+\omega) \;=\; \int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 (2\sinh s\omega)\, e^{-s\lambda}\, ds. \tag{3.19}$$

Proof. In the infinite strip $-\frac{1}{2} < \Re(z) < \frac{1}{2}$ we have the Laplace identity

$$\int_{-\infty}^{\infty} C(\omega)\, e^{s\omega}\, d\omega \;=\; \left(\frac{\pi}{\cos \pi s}\right)^2 \tag{3.20}$$

and its inverse

$$C(\omega) \;=\; \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 e^{-s\omega}\, ds, \tag{3.21}$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$. Then from (3.21) we get (3.19). □

Theorem 3.4. *If* $|\alpha| < \min\{\frac{1}{2}, \Re(z)\}$ *then*

$$e^{-\lambda z} - M_\lambda(z) = \left(\frac{\cos\pi z}{\pi}\right)^2 \left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{\pi}{\cos\pi s}\right)^2 \left(\frac{2s}{z^2-s^2}\right) e^{-s\lambda}\,ds\right\}. \tag{3.22}$$

Proof. Combining (3.17) and (3.19) we find that

$$\begin{aligned} e^{-\lambda z} - M_\lambda(z) &= \left(\frac{\cos\pi z}{\pi}\right)^2 \int_0^\infty \left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{\pi}{\cos\pi s}\right)^2 \right. \\ &\qquad\qquad \left. (2\sinh s\omega)\, e^{-s\lambda}\,ds\right\} e^{-z\omega}\,d\omega \\ &= \left(\frac{\cos\pi z}{\pi}\right)^2 \left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{\pi}{\cos\pi s}\right)^2 \left(\frac{2s}{z^2-s^2}\right) e^{-s\lambda}\,ds\right\}. \end{aligned} \tag{3.23}$$

□

3.2 Minorants for a Class of Even Functions

In this section we generalize the previous results to a wider class of functions.

Theorem 3.5. *Let* ν *be a (not identically zero) measure on the Borel subsets of* $(0,\infty)$ *and define* $f:\mathbb{R}\to[0,\infty)$ *by*

$$f(x) = \int_0^\infty e^{-\lambda|x|}\,d\nu(\lambda) \tag{3.24}$$

(then $f(x) > 0$ *for all real* x*), and assume that* $f(x) < \infty$ *for all* $x > 0$*.*

(i) *If* ν *is finite then* $f(x) < \infty$ *for all real* x*, and* f *has continuous derivatives of all orders in* $\mathbb{R}\setminus\{0\}$*. In particular*

$$f'(x) = -\operatorname{sgn}(x)\int_0^\infty \lambda e^{-\lambda|x|}\,d\nu(\lambda) \tag{3.25}$$

for every $x \neq 0$*.*

(ii) *The function f belongs to $L^1(\mathbb{R})$ if and only if*

$$\int_0^\infty \frac{1}{\lambda}\, d\nu(\lambda) \ < \ \infty. \tag{3.26}$$

(iii) *The function f belongs to $L^2(\mathbb{R})$ if and only if*

$$\int_0^\infty \int_0^\infty \frac{1}{\lambda_1 + \lambda_2}\, d\nu(\lambda_1)\, d\nu(\lambda_2) \ < \ \infty. \tag{3.27}$$

(iv) *The function f' belongs to $L^1(\mathbb{R})$ if and only if ν is a finite measure.*

Proof. The function

$$\varphi(z) \ = \ \int_0^\infty e^{-\lambda z}\, d\nu(\lambda) \tag{3.28}$$

defines an analytic function in the right half plane $\mathcal{R} = \{z \in \mathbb{C} : 0 < \Re(z)\}$ and the restriction of this analytic function to the positive real axis is $f(x)$. Hence f has continuous derivatives of all orders there. A similar remark applies to $f(x)$ on the negative real axis and these can be combined to establish (i). The rest can be proven by using Tonelli's theorem. □

In the following ν will represent a *finite measure* on the Borel subsets of $(0, \infty)$.

Theorem 3.6. *For each $z \in \mathbb{C}$, the function $\lambda \mapsto M(\lambda, z) = M_\lambda(z)$ is ν-integrable on $(0, \infty)$, and the $\mathbb{C}$-valued function*

$$F(z) \ = \ \int_0^\infty M(\lambda, z)\, d\nu(\lambda) \tag{3.29}$$

is an entire function which satisfies the inequality

$$|F(z)| \ \le \ \nu\{(0, \infty)\}\, e^{2\pi|y|} \tag{3.30}$$

for all $z = x + iy$. Thus F is an entire function of exponential type at most 2π.

Proof. We have (here $\widehat{M}(\lambda, t)$ is the function $(\lambda, t) \mapsto \widehat{M_\lambda}(t)$)

$$\begin{aligned} \int_0^\infty |M(\lambda, z)|\, d\nu(\lambda) &= \int_0^\infty \left| \int_{-1}^1 \widehat{M}(\lambda, t)\, e^{2\pi i t z}\, dt \right| d\nu(\lambda) \\ &\le \int_0^\infty \int_{-1}^1 \widehat{M}(\lambda, t)\, e^{-2\pi t y}\, dt\, d\nu(\lambda) \\ &\le e^{2\pi |y|} \int_0^\infty M(\lambda, 0)\, d\nu(\lambda) \\ &\le \nu\{(0, \infty)\}\, e^{2\pi |y|}. \end{aligned} \tag{3.31}$$

This shows that $\lambda \mapsto M(\lambda, z)$ is ν-integrable on $(0, \infty)$. That $F(z)$ is an entire function follows from Morera's theorem. Its bound (3.30) follows from (3.31). This proves the result. □

Next theorems 3.7 and 3.8 show that the function F defined by (3.29) is an extremal minorant for f.

Theorem 3.7. *The function F defined by (3.29) is a minorant of f, i.e.*

$$F(x) \le f(x) \tag{3.32}$$

for all real x. Furthermore F interpolates f and its derivative at the integers plus a half, i.e.

$$F(m + \tfrac{1}{2}) = f(m + \tfrac{1}{2}), \tag{3.33}$$

$$F'(m + \tfrac{1}{2}) = f'(m + \tfrac{1}{2}), \tag{3.34}$$

for all $m \in \mathbb{Z}$. Also, the nonnegative function $f(x) - F(x)$ is integrable, and its Fourier transform is

$$\int_{-\infty}^{\infty} \{f(x) - F(x)\}\, e^{-2\pi i t x}\, dx \;=\; \int_0^{\infty} \left\{ \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2} \;-\; \widehat{M}(\lambda, t) \right\} d\nu(\lambda). \tag{3.35}$$

Proof. The fact that F is a majorizing interpolating function for f follows from results in [4]. On the other hand:

$$\int_{-\infty}^{\infty} \{f(x) - F(x)\}\, dx \;=\; \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ e^{-\lambda |x|} - M(\lambda, x) \right\} d\nu(\lambda)\, dx. \tag{3.36}$$

Since the integrand is nonnegative we can interchange the order of integration:

$$\begin{aligned} \int_{-\infty}^{\infty} \{f(x) - F(x)\}\, dx \;&=\; \int_0^{\infty} \int_{-\infty}^{\infty} \left\{ e^{-\lambda |x|} - M(\lambda, x) \right\} dx\, d\nu(\lambda) \\ &=\; \int_0^{\infty} \left\{ \frac{2}{\lambda} - \frac{1}{\sinh \frac{\lambda}{2}} \right\} d\nu(\lambda). \end{aligned} \tag{3.37}$$

The last integral is finite because ν is a finite measure and the integrand is bounded. In the same manner we can compute the Fourier transform. □

Theorem 3.8. *Let $G(z)$ be a real entire function of exponential type at most 2π. If*

$$G(x) \;\leq\; f(x) \qquad \text{for all real } x, \tag{3.38}$$

then

$$\int_{-\infty}^{\infty} \{f(x) - F(x)\}\, dx \;\leq\; \int_{-\infty}^{\infty} \{f(x) - G(x)\}\, dx. \tag{3.39}$$

Moreover, there is equality in (3.39) if and only if $G(z) = F(z)$.

Proof. Clearly it suffices to consider only those functions $G(z)$ such that the integral on the right of (3.39) is finite. Then the difference $F(x) - G(x)$ is integrable and (3.39) is equivalent to

$$0 \le \int_{-\infty}^{\infty} \{F(x) - G(x)\}\,dx. \tag{3.40}$$

From (3.33) and (3.38) we have

$$G(m + \tfrac{1}{2}) \le f(m + \tfrac{1}{2}) = F(m + \tfrac{1}{2}) \tag{3.41}$$

for all $m \in \mathbb{Z}$. By lemma 4 in [4]:

$$\begin{aligned} 0 &\le \lim_{M\to\infty} \sum_{m=-M}^{M} \left(1 - \frac{|m|}{M+1}\right) \left\{F(m + \tfrac{1}{2}) - G(m + \tfrac{1}{2})\right\} \\ &= \int_{-\infty}^{\infty} \{F(x) - G(x)\}\,dx, \end{aligned} \tag{3.42}$$

and this is (3.40). It is obvious from (3.42) that the integral on the right is zero if and only if

$$G(m + \tfrac{1}{2}) = F(m + \tfrac{1}{2}) \tag{3.43}$$

for all $m \in \mathbb{Z}$. Then (3.38), (3.41) and (3.43) imply that

$$G'(m + \tfrac{1}{2}) = F'(m + \tfrac{1}{2}) \tag{3.44}$$

for all $m \in \mathbb{Z}$. A second application of lemma 4 in [4] shows there is equality in (3.42) if and only if $G(z) = F(z)$. □

3.3 The Extremal Majorant for the Logarithm

Here we establish some general results from which the main properties of the function F defined in (2.33) can be derived.

Lemma 3.9. *Let* ν *be a finite measure on the Borel subsets of* $(0,\infty)$. *Define* f *as in (3.24),* F *as in (3.29), and* φ *as in (3.28). Let* $0 < \alpha < \min\{\frac{1}{2}, \Re(z)\}$. *Then* φ *is analytic and bounded in the right half plane* $\mathcal{R} = \{\, z \in \mathbb{C} : 0 < \Re(z)\,\}$, *and satisfies*

$$0 \;\le\; \varphi(x) \;-\; F(x) \qquad \text{for all } x > 0. \tag{3.45}$$

Furthermore

$$\varphi(z) \;-\; F(z) \;=\; \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \varphi(s)\, ds \right\}. \tag{3.46}$$

Proof. We already know that φ is analytic and bounded in $\mathcal{R}$, and that $0 \le \varphi(x) \;-\; F(x)$. Also, equation (3.46) can be obtained from

$$\varphi(z) \;-\; F(z) \;=\; \int_0^\infty \left\{ e^{-\lambda z} - M_\lambda(z) \right\} d\nu(\lambda) \tag{3.47}$$

and (3.22). □

Theorem 3.10. *Let* $\Phi(s)$ *be a function such that*

(i) $\Phi(s)$ *is analytic in* $\mathcal{R}$.

(ii) $|\Phi(s)| << (1+|s|)^{\theta_1}\, e^{\theta_2 |s|}$ *in* $\mathcal{R}$, *where* $\theta_1 < 1$ *and* $\theta_2 < 2\pi$.[1]

For each nonnegative integer m *we define*

$$\Psi_m(z) \;=\; \left\{ \frac{\Phi(m+\frac{1}{2})}{(z-m-\frac{1}{2})^2} + \frac{\Phi'(m+\frac{1}{2})}{z-m-\frac{1}{2}} + \frac{\Phi(m+\frac{1}{2})}{(z+m+\frac{1}{2})^2} - \frac{\Phi'(m+\frac{1}{2})}{z+m+\frac{1}{2}} \right\}. \tag{3.48}$$

[1]The growth condition can be slightly weakened, but what we use here will easily suffice for our purposes. In particular, we note that $\Phi(s) = -\log s$ satisfies (i) and (ii).

Then

$$\lim_{M\to\infty} \left(\frac{\cos \pi z}{\pi}\right)^2 \sum_{m=0}^{M-1} \Psi_m(z) \tag{3.49}$$

converges uniformly on compact subsets of $\mathcal{R}$. *Moreover, if* $0 < \alpha < \min\{\frac{1}{2}, \Re(z)\}$ *then*

$$\begin{aligned} \Phi(z) \;&-\; \lim_{M\to\infty} \left(\frac{\cos \pi z}{\pi}\right)^2 \sum_{m=0}^{M-1} \Psi_m(z) \\ &= \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \varphi(s)\, ds \right\}. \end{aligned} \tag{3.50}$$

Proof. Suppose that $K \subseteq \mathcal{R}$ is compact, $z \in K$, and consider the meromorphic function

$$s \;\mapsto\; \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \Phi(s). \tag{3.51}$$

Also we assume that $z - \frac{1}{2}$ is not an integer. Then (3.51) has a pole at $s = z$ of order 1 and residue

$$-\left(\frac{\pi}{\cos \pi z}\right)^2 \Phi(z). \tag{3.52}$$

For each nonnegative integer m, (3.51) has a pole at $s = m + \frac{1}{2}$ of order 2 and residue $\Psi_m(z)$. Plainly, (3.51) has no other poles in $\mathcal{R}$. Assume $0 < \alpha < \frac{1}{2} < M$, where M is a positive integer, and let $0 < T$. Then write $\Gamma_{M,T}$ for the simple, closed, positively oriented, piece-wise linear path connecting $\alpha - iT$, $M - iT$, $M + iT$, $\alpha + iT$ and $\alpha - iT$. We further assume that α is so small, and M and T are so large, that K is contained in the unique bounded component of $\mathbb{C} \setminus \Gamma_{M,T}$. Then we have

$$\frac{1}{2\pi i} \int_{\Gamma_{M,T}} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \Phi(s)\, ds$$
$$= -\left(\frac{\pi}{\cos \pi z}\right)^2 \Phi(z) + \sum_{m=0}^{M-1} \Psi_m(z), \quad (3.53)$$

by the residue theorem. On the left hand side of (3.53) we let $T \to \infty$ and use the estimate (ii) for $|\Phi(s)|$. We find that:

$$\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \Phi(s)\, ds$$
$$= \left(\frac{\pi}{\cos \pi z}\right)^2 \Phi(z) - \sum_{m=0}^{M-1} \Psi_m(z)$$
$$+ \frac{1}{2\pi i} \int_{M - i\infty}^{M + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \Phi(s)\, ds \quad (3.54)$$

Clearly the integrals in (3.54) determine functions of z which are analytic in $\alpha < \Re(z) < M$. Hence our assumption that $z - \frac{1}{2}$ is not an integer can be dropped. The singularities of

$$\left(\frac{\pi}{\cos \pi s}\right)^2 \Phi(s) - \sum_{m=0}^{M-1} \Psi_m(z) \quad (3.55)$$

at $z = m + \frac{1}{2}$, $0 \le m \le M - 1$, are removable.

Finally, we let $M \to \infty$ on the right hand side of (3.54). We use (ii) again to estimate $|\Phi(s)|$ and find that

$$\lim_{M \to \infty} \left| \frac{1}{2\pi i} \int_{M - i\infty}^{M + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \Phi(s)\, ds \right| = 0 \quad (3.56)$$

uniformly for $z \in K$. From here the desired result follows. □

Corollary 3.11. *If $\Phi(z)$ is constant in $\mathcal{R}$ then the integral on the right hand side of (3.50) is identically zero.*

Proof. This is immediate from the identity

$$\left(\frac{\pi}{\cos \pi z}\right)^2 = \sum_{m=0}^{\infty} \left\{ \frac{1}{\left(z - m - \frac{1}{2}\right)^2} + \frac{1}{\left(z + m + \frac{1}{2}\right)^2} \right\}. \tag{3.57}$$

□

Theorem 3.12. *The function F defined by (2.33) is a majorant of* $\log|x|$.

Proof. For each $l = 1, 2, \ldots$ let ν_l denote the measure defined on Borel subsets E of $(0, \infty)$ by

$$\nu_l(E) = \int_E \left(\int_{\frac{1}{l}}^{l} e^{-\lambda u}\, du \right) d\lambda. \tag{3.58}$$

Let

$$\begin{aligned} \varphi_l(z) &= \int_0^\infty e^{-\lambda z}\, d\nu_l(\lambda) \\ &= \log(l + z) - \log(\tfrac{1}{l} + z) \end{aligned} \tag{3.59}$$

in the right half plane $\mathcal{R}$, where log denotes the principal branch which is real on the positive real axis. Let

$$F_l(z) = \int_0^\infty M(\lambda, z)\, d\nu_l(\lambda) \tag{3.60}$$

so that F_l is a real entire function of exponential type at most 2π and (as in (3.45))

$$0 \le \varphi_l(x) - F_l(x) \qquad \text{for all } x > 0. \tag{3.61}$$

From (3.46) we have

$$\varphi_l(z) - F_l(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) \varphi_l(s)\, ds \right\}. \tag{3.62}$$

provided $0 < \alpha < \min\{\frac{1}{2}, \Re(z)\}$. And the corollary 3.11 allows us to write this as

$$\begin{aligned}\varphi_l(z) \;-\; F_l(z) \\ = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) (\varphi_l(s) - \log l)\, ds\right\}.\end{aligned} \tag{3.63}$$

As

$$|\varphi_l(\alpha + it) - \log l| \;\; <<_\alpha \;\; \log(2 + |t|) \tag{3.64}$$

and

$$\lim_{l \to \infty} \{\varphi_l(s) - \log l\} \;\; = \;\; -\log s, \tag{3.65}$$

we find that

$$\begin{aligned}\lim_{l \to \infty} \{\varphi_l(z) - F_l(z)\} \\ = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \left(\frac{\pi}{\cos \pi s}\right)^2 \left(\frac{2s}{z^2 - s^2}\right) (-\log l)\, ds\right\}\end{aligned} \tag{3.66}$$

whenever $0 < \alpha < \min\{\frac{1}{2}, \Re(z)\}$. By theorem 3.10 we have

$$\lim_{l \to \infty} \{\varphi_l(z) - F_l(z)\} \;\; = \;\; -\log z \;\; - \;\; \left(\frac{\cos \pi z}{\pi}\right)^2 \sum_{m=0}^{\infty} \Psi_m(z) \tag{3.67}$$

where $0 < \Re(z)$ and Ψ_m is defined by (3.48) with $\Phi(z) = -\log z$. In view of (3.61) we also get the inequality

$$0 \;\; \leq \;\; -\log x \;\; - \;\; \left(\frac{\cos \pi x}{\pi}\right)^2 \sum_{m=0}^{\infty} \Psi_m(x) \tag{3.68}$$

for all $x > 0$. In this case the expression on the right of (3.68) defines an *even*, real entire function. Hence

$$0 \;\; \leq \;\; -\log |x| \;\; - \;\; \left(\frac{\cos \pi x}{\pi}\right)^2 \sum_{m=0}^{\infty} \Psi_m(x) \tag{3.69}$$

for all real $x \neq 0$. Finally we note that the function F defined by (2.33) is

$$F(z) \;=\; -\left(\frac{\cos \pi z}{\pi}\right)^2 \sum_{m=0}^{\infty} \Psi_m(z). \tag{3.70}$$

From here the desired result follows. □

Chapter 4

Properties of the Majorant for the Logarithm

Here we present an alternative way to get the main properties of F and obtain some additional results involving also the functions D and S defined below, as well as a number of properties and special values of $\widehat{D}$.

4.1 Properties of F

Let F be the function defined in (2.33). In the following we denote for every real $x \neq 0$:

$$D(x) \; = \; F(x) \; - \; \log|x|, \tag{4.1}$$

and

$$S(x) \; = \; \left(\frac{\pi\, x}{\cos \pi x}\right)^2 D(x). \tag{4.2}$$

Proposition 4.1. *For every $x > 0$, $S'(x) > 0$. Furthermore,*

$$S'(x) = O(x^{-3})$$

for $x \to \infty$.

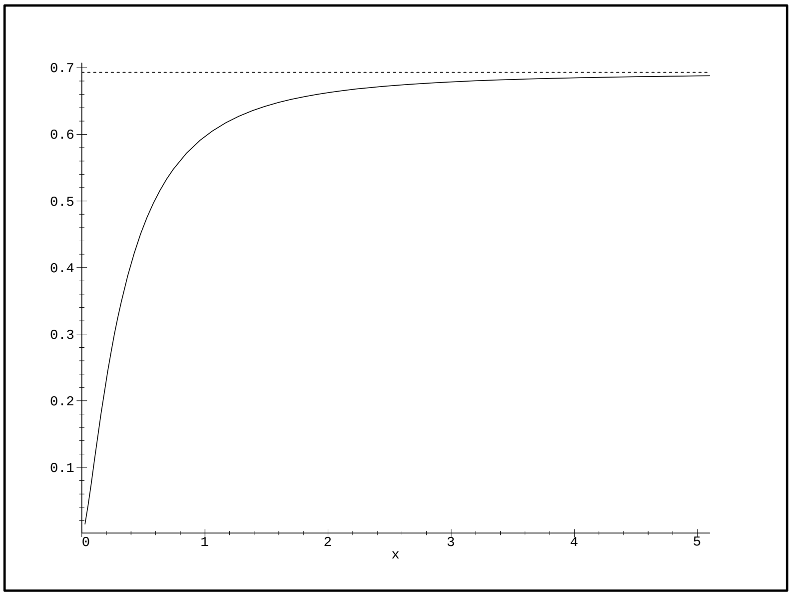

Figure 4.1: The function $S(x)$

In order to prove this proposition we write

$$S(x) = \sum_{n=0}^{\infty} h\left(\frac{n+\frac{1}{2}}{x}\right), \tag{4.3}$$

where

$$h(t) \;=\; \left\{\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2}\right\} \log t \;+\; \frac{2}{(1-t^2)}, \tag{4.4}$$

and

$$S'(x) \;=\; -\frac{1}{x} \sum_{n=0}^{\infty} j\left(\frac{n+\frac{1}{2}}{x}\right), \tag{4.5}$$

where

$$j(t) \;=\; t\, h'(t). \tag{4.6}$$

The proof requires the following lemma:

Lemma 4.2. *Let N be any positive integer. Then, for every $x > 0$*

$$-x\,S'(x) \;=\; \sum_{n=0}^{N-1} j(t_n) \;+\; x\int_{\frac{N}{x}}^{\infty} j(t)\,dt \;+\; \frac{1}{24\,x}\,j'\left(\tfrac{N}{x}\right) \;+\; R_N(x), \qquad (4.7)$$

where $t_n = \frac{n+\frac{1}{2}}{x}$, and

$$|R_N(x)| \;\leq\; \frac{21}{640\,x^2}\sum_{n=N}^{\infty}\frac{1}{n^2}. \qquad (4.8)$$

In particular, for $N = 1$:

$$-x\,S'(x) \;=\; j\left(\tfrac{1}{2x}\right) \;+\; x\int_{\frac{1}{x}}^{\infty} j(t)\,dt \;+\; \frac{1}{24\,x}\,j'\left(\tfrac{1}{x}\right) \;+\; R_1(x), \qquad (4.9)$$

where

$$|R_1(x)| \;\leq\; \frac{7\,\pi^2}{1280\,x^2} \;<\; \frac{7}{128\,x^2}. \qquad (4.10)$$

Proof. By writing

$$-x\,S'(x) \;=\; \sum_{n=0}^{N-1} j(t_n) \;+\; \sum_{N-\frac{1}{2}<n<\infty} j(t_n), \qquad (4.11)$$

and using the Euler-MacLaurin summation formula in the second sum (see lemma 8.1) we get (4.7) with

$$R_N(x) \;=\; \frac{7}{5760\,x^4}\sum_{n=N}^{\infty} j^{(4)}(\theta_n), \qquad (4.12)$$

where $\frac{n}{x} < \theta_n < \frac{n+1}{x}$. Next we use lemma 8.3 in order to get the bound for $|R_N(x)|$. □

Proof of proposition 4.1. Consider the function

$$f(x) \;=\; j\left(\tfrac{1}{2x}\right) \;+\; x\int_{\frac{1}{x}}^{\infty} j(t)\,dt \;+\; \frac{1}{24\,x}\,j'\left(\tfrac{1}{x}\right). \qquad (4.13)$$

According to equation (4.9):

$$-x\,S'(x) \;=\; f(x) \;+\; R_1(x). \tag{4.14}$$

The function $f(x)$ can be explicitly computed (see lemma 8.4), and checked that $f(x) = O(x^{-2})$ for $x \to \infty$. Since also $R_1(x) = O(x^{-2})$, we get that $S'(x) = O(x^{-3})$.

It remains to prove that $S'(x) > 0$ for $x > 0$. To do so, we consider the function $g(x) = -x^2\,f(x) - \frac{7}{128}$, which is positive in the interval (δ, ∞) for some $0 < \delta < \frac{1}{2}$ (see lemma 8.5). From here we get $S'(x) > 0$ for $x > \delta$. Concerning the interval $0 < x < \frac{1}{2}$, we write:

$$\frac{S'(x)}{x} \;=\; -\sum_{n=0}^{\infty} \frac{t_n^3\,h'(t_n)}{(n+\frac{1}{2})^2} \tag{4.15}$$

and check that $t^3\,h'(t)$ is decreasing for $t > 1$ (see lemma 8.6), which implies $\frac{S'(x)}{x}$ is decreasing for $0 < x < \frac{1}{2}$. Since we already know that it is positive for $x > \delta$, the same will happen for $0 < x < \frac{1}{2}$. □

Proposition 4.3. *For every $x > 0$, $S(x)$ is positive and increasing, and its limit $L = \lim_{x\to\infty} S(x)$ exists. Furthermore*

$$L - S(x) \;=\; O(x^{-2}) \tag{4.16}$$

for $x \to \infty$.

Proof. Since $S(0) = 0$, and according to proposition 4.1 $S'(x) > 0$ for $x > 0$, then $S(x)$ is positive and increasing for $x > 0$. Since $S'(x) = O(x^{-3})$, then $S(x)$ is bounded, hence it has a limit L for $x \to \infty$. Finally

$$|L - S(x)| \;=\; \left|\int_x^{\infty} S'(y)\,dy\right| \;\leq\; \left|\int_x^{\infty} A\,y^{-3}\,dy\right| \;=\; \frac{A}{2}\,x^{-2} \tag{4.17}$$

for some $A > 0$, hence $S(x) = O(x^{-2})$. □

Corollary 4.4. *The function F defined by (2.33) is a majorizing function for $\log|x|$, i.e.,*

$$F(x) \;\geq\; \log|x| \qquad \textit{for every } x \in \mathbb{R}\setminus\{0\}. \tag{4.18}$$

Proof. We have

$$D(x) \;=\; \left(\frac{\cos \pi x}{\pi x}\right)^2 S(x). \tag{4.19}$$

From proposition 4.3, we get that $D(x) \geq 0$ for every $x \neq 0$. □

Next we give a result useful to compute numerical approximations of $S(x)$.

Proposition 4.5. *Let N be any positive integer. Then, for every $x > 0$,*

$$S(x) \;=\; \sum_{n=0}^{N-1} h(t_n) \;-\; \frac{2\,N\,x^2\,\log(\frac{N}{x})}{x^2 - N^2} \;+\; \frac{1}{24\,x}\,h'\left(\tfrac{N}{x}\right) \;+\; R_N(x), \tag{4.20}$$

where $t_n = \frac{n+\frac{1}{2}}{x}$, and

$$|R_N(x)| \;\leq\; \frac{49}{2880}\sum_{n=N}^{\infty}\frac{1}{n^4} \;<\; \frac{49}{1080}\,\frac{1}{(2N-1)^3}. \tag{4.21}$$

Proof. By writing

$$S(x) \;=\; \sum_{n=0}^{N-1} h(t_n) \;+\; \sum_{N-\frac{1}{2}<n<\infty} h(t_n) \tag{4.22}$$

and using the Euler-MacLaurin summation formula in the second sum we get

$$\begin{aligned} S(x) \;&=\; \sum_{n=0}^{N-1} h(t_n) \;+\; x\int_{\frac{N}{x}}^{\infty} h(t)\,dt \;+\; \frac{1}{24\,x}\,h'\left(\tfrac{N}{x}\right) \;+\; R_N(x) \\ &=\; \sum_{n=0}^{N-1} h(t_n) \;-\; \frac{2\,N\,x^2\,\log(\frac{N}{x})}{x^2 - N^2} \;+\; \frac{1}{24\,x}\,h'\left(\tfrac{N}{x}\right) \;+\; R_N(x), \end{aligned} \tag{4.23}$$

where

$$R_N(x) \;=\; \frac{7}{5760\, x^4} \sum_{n=N}^{\infty} h^{(4)}(\theta_n), \tag{4.24}$$

and $\frac{n}{x} < \theta_n < \frac{n+1}{x}$. Next we use lemma 8.7 in order to get the bound for $|R_N(x)|$. □

Example 4.6. The following values have been computed by using formula (4.20) with $N = 200$, so that the error is less than 10^{-9}:

$$\begin{aligned} S(\tfrac{1}{2}) &= 0.46165805\ldots \\ S(\tfrac{3}{2}) &= 0.64277314\ldots \\ S(\tfrac{5}{2}) &= 0.67287718\ldots \\ S(\tfrac{7}{2}) &= 0.68242204\ldots \\ S(\tfrac{9}{2}) &= 0.68655437\ldots \\ S(\tfrac{11}{2}) &= 0.68869668\ldots \\ S(\tfrac{13}{2}) &= 0.68994504\ldots \\ S(\tfrac{15}{2}) &= 0.69073451\ldots \\ S(\tfrac{17}{2}) &= 0.69126486\ldots \\ S(\tfrac{19}{2}) &= 0.69163805\ldots \\ S(\tfrac{21}{2}) &= 0.69191048\ldots \end{aligned} \tag{4.25}$$

Proposition 4.7. *The derivative of D verifies $D'(x) = O(x^{-2})$ for $x \to \infty$. As a consequence, $F' \in L^p(\mathbb{R})$ for every $p > 1$.*

Proof. We have:

$$D'(x) \;=\; -\frac{1}{x^3}\left(\frac{\cos \pi x}{\pi}\right)^2 S(x) \;-\; \frac{1}{x^2}\,\frac{\sin 2\pi x}{\pi}\, S(x) \;+\; \frac{1}{x^2}\left(\frac{\cos \pi x}{\pi}\right)^2 S'(x). \tag{4.26}$$

Recall that $S(x)$ is bounded, hence the first two terms of the right hand side are $O(x^{-2})$. Proposition 4.1 allows us to conclude the same for the third term, hence $D'(x) = O(x^{-2})$. From here we get that $F'(x) = \frac{1}{x} + O(x^{-2})$, hence $F'(x) \in L^p(\mathbb{R})$ for $p > 1$. $\square$

4.2 Properties of $\widehat{D}$

Proposition 4.8. *The Fourier transform $\widehat{D}$ of D is the following:*

$$\widehat{D}(t) \;=\; \begin{cases} \log 2 & \text{if } t = 0 \\ \frac{1}{2} \;-\; \frac{1}{4\pi^2|t|}\, v_{F'}(t) & \text{if } 0 < |t| < 1 \\ \frac{1}{2|t|} & \text{if } |t| \geq 1, \end{cases} \tag{4.27}$$

where

$$v_{F'}(t) \;=\; 2\sum_{n=0}^{\infty} F''(n+\tfrac{1}{2})\,\cos\{2\pi\,(n+\tfrac{1}{2})\,t\}. \tag{4.28}$$

Proof. Integration by parts (for $t \neq 0$) shows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} D(x)\, e^{-2\pi i t x}\, dx \\
&= \frac{1}{2\pi i\, t} \lim_{T\to\infty} \left\{ \int_{-T}^{-1/T} D'(x)\, e^{-2\pi i t x}\, dx + \int_{1/T}^{T} D'(x)\, e^{-2\pi i t x}\, dx \right\} \\
&= \frac{1}{2\pi i\, t} \int_{-\infty}^{\infty} F'(x)\, e^{-2\pi i t x}\, dx \\
&\quad - \lim_{T\to\infty} \left\{ \frac{1}{2\pi i\, t} \int_{-T}^{-1/T} \frac{1}{x}\, e^{-2\pi i t x}\, dx + \frac{1}{2\pi i\, t} \int_{1/T}^{T} \frac{1}{x}\, e^{-2\pi i t x}\, dx \right\}.
\end{aligned} \tag{4.29}$$

The second term is well known:

$$\lim_{T\to\infty} \left\{ -\frac{1}{2\pi i\, t} \int_{-T}^{-1/T} \frac{1}{x}\, e^{-2\pi i t x}\, dx + \frac{1}{2\pi i\, t} \int_{1/T}^{T} \frac{1}{x}\, e^{-2\pi i t x}\, dx \right\} \;=\; \frac{1}{2\,|t|}. \tag{4.30}$$

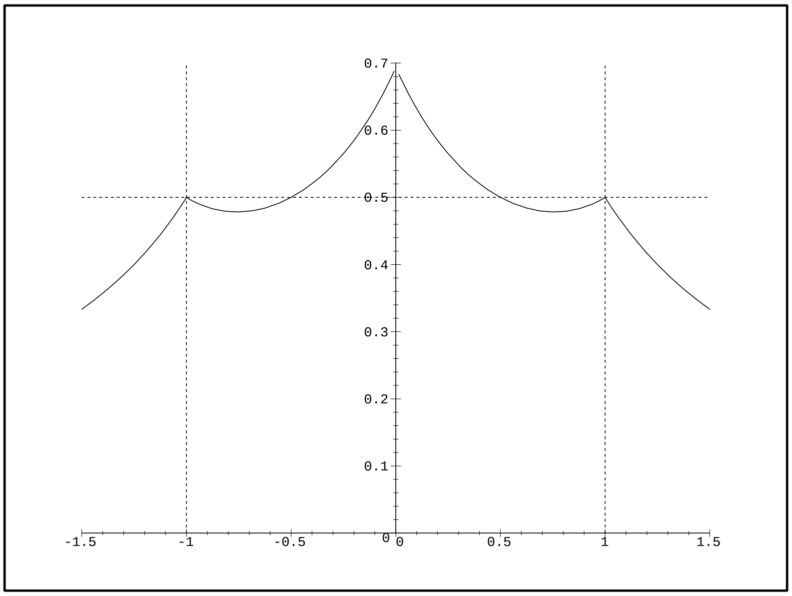

Figure 4.2: The function $\widehat{D}(t)$

The first term requires to compute $F'(x)$ first. To do that, we use Vaaler's method [9].

Let K and L be

$$K(z) = \left(\frac{\cos \pi z}{\pi z}\right)^2, \tag{4.31}$$

$$L(z) = z\,K(z). \tag{4.32}$$

If

$$F(z) = \sum_{n=-\infty}^{\infty} F(n)\,K(z-n) + \sum_{n=-\infty}^{\infty} F'(n)\,L(z-n) \tag{4.33}$$

and the sum converges uniformly in compact subsets, then:

$$F'(z) = \sum_{n=-\infty}^{\infty} F'(n)\,K(z-n) + \sum_{n=-\infty}^{\infty} F''(n)\,L(z-n), \tag{4.34}$$

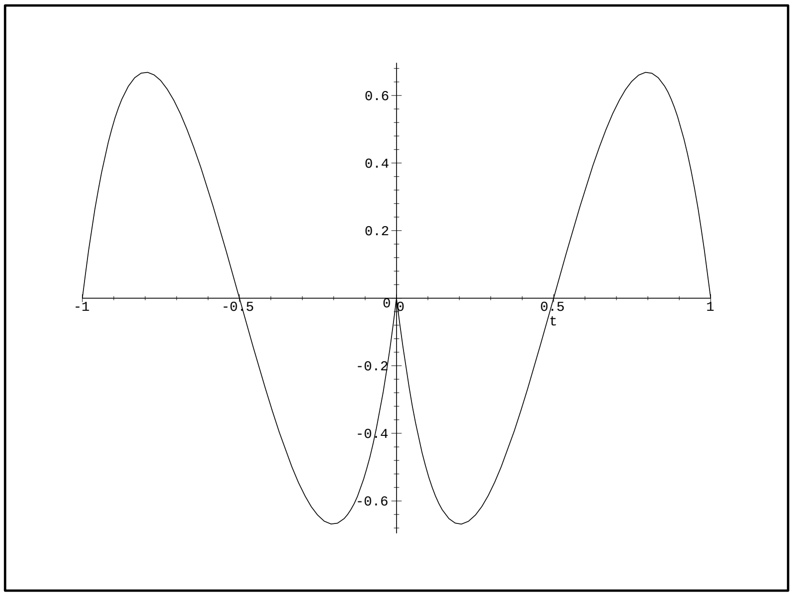

Figure 4.3: The function $\widehat{v}_{F'}(t)$

where $F''(n)$ can be computed like this:

$$\begin{aligned} F''(n) &= -\frac{2\pi^2}{3}\,F(n) \;+\; 2\sum_{j\neq 0}\frac{1}{j^2}\,F(n-j) \;+\; 2\sum_{j\neq 0}\frac{1}{j}\,F'(n-j) \\ &= 2\sum_{j\neq 0}\frac{1}{j^2}\,\{F(n-j)-F(n)\} \;+\; 2\sum_{j\neq 0}\frac{1}{j}\,F'(n-j). \end{aligned} \tag{4.35}$$

Since we are interpolating at the integers plus a half, we apply the results to the function $z \mapsto F(z+\frac{1}{2})$, where F is our function (2.33). So, we get:

$$F'(x) \;=\; \left(\frac{\cos \pi x}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{1}{n+\frac{1}{2}}}{(x-(n+\frac{1}{2}))^2} \;+\; \sum_{n=-\infty}^{\infty} \frac{F''(n+\frac{1}{2})}{x-(n+\frac{1}{2})} \right\}, \tag{4.36}$$

where

$$F''(n+\tfrac{1}{2}) \;=\; 2\sum_{j\neq 0}\frac{1}{j}\,\frac{1}{n+\frac{1}{2}-j} \;+\; 2\sum_{j\neq 0}\frac{1}{j^2}\,\log\left|1-\frac{j}{n+\frac{1}{2}}\right| \\ \;=\; -\,\frac{2}{(n+\frac{1}{2})^2} \;+\; 2\sum_{j=1}^{\infty}\frac{1}{j^2}\,\log\left|1-\frac{j^2}{(n+\frac{1}{2})^2}\right|. \tag{4.37}$$

Another convenient expression for $F''(n+\frac{1}{2})$ can be obtained by differentiating directly in

$$F(x) \;=\; \log|x| \;+\; D(x) \;=\; \log|x| \;+\; \left(\frac{\cos\pi x}{\pi x}\right)^2 S(x) \tag{4.38}$$

and substituting x with $n+\frac{1}{2}$:

$$F''(n+\tfrac{1}{2}) \;=\; \frac{2\,S(n+\frac{1}{2})-1}{(n+\frac{1}{2})^2}. \tag{4.39}$$

Finally we compute the Fourier transform of F' by using formula (3.6) from [10] (adjusted for interpolation at the integers plus a half):

$$\widehat{F'}(t) \;=\; \chi_{[-1,1]}(t)\left\{(1-|t|)\,u_{F'}(t) \;+\; \frac{1}{2\pi i}\;\mathrm{sgn}(t)\,v_{F'}(t)\right\}, \tag{4.40}$$

where

$$\chi_{[a,b]}(t) \;=\; \begin{cases} 1 & \text{if } a<t<b, \\ \frac{1}{2} & \text{if } t=a \text{ or } t=b, \\ 0 & \text{otherwise.} \end{cases} \tag{4.41}$$

$\mathrm{sgn}(t)$ = signum of t, and:

$$u_{F'}(t) \;=\; \lim_{T\to\infty}\sum_{n=-T}^{T} F'(n+\frac{1}{2})\,e^{-2\pi i\,(n+\frac{1}{2})\,t} \\ \;=\; \lim_{T\to\infty}\sum_{n=-T}^{T}\frac{1}{n+\frac{1}{2}}\,e^{-2\pi i\,(n+\frac{1}{2})\,t} \\ \;=\; -\pi i\;\mathrm{sgn}(t), \tag{4.42}$$

$$
\begin{aligned}
v_{F'}(t) &= \sum_{n=-\infty}^{\infty} F''(n+\tfrac{1}{2})\, e^{-2\pi i\,(n+\frac{1}{2})\,t} \\
&= 2\sum_{n=0}^{\infty} F''(n+\tfrac{1}{2})\, \cos\{(2n+1)\,\pi t\}.
\end{aligned} \tag{4.43}
$$

Hence:

$$
\begin{aligned}
\widehat{D}(t) &= \frac{1}{2|t|} + \chi_{[-1,1]}(t)\,\frac{1}{2\pi i\, t}\left\{(1-|t|)\,(-\pi i\,\mathrm{sgn}(t)) \right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{1}{2\pi i}\,\mathrm{sgn}(t)\, v_{F'}(t)\right\} \\
&= \chi_{[-1,1]}(t)\left\{\frac{1}{2} - \frac{1}{4\pi^2|t|}\, v_{F'}(t)\right\} + (1-\chi_{[-1,1]}(t))\,\frac{1}{2|t|}
\end{aligned} \tag{4.44}
$$

for $t \neq 0$.

We have that for every t

$$
|\widehat{D}(t)| \le |\widehat{D}(0)| = \int_{-\infty}^{\infty} D(x)\,dx < \infty, \tag{4.45}
$$

hence

$$
v_{F'}(0) = \sum_{n=-\infty}^{\infty} F''(n+\tfrac{1}{2}) = 0, \tag{4.46}
$$

otherwise $D(t)$ would be unbounded for $t \to 0$. Also:

$$
v_{F'}(k) = (-1)^k \sum_{n=-\infty}^{\infty} F''(n+\tfrac{1}{2}) = 0 \tag{4.47}
$$

for k integer, so

$$
\widehat{D}(t) = \begin{cases} \frac{1}{2} - \frac{1}{4\pi^2|t|}\, v_{F'}(t) & \text{if } 0 < |t| < 1, \\ \frac{1}{2|t|} & \text{if } |t| \ge 1, \end{cases} \tag{4.48}
$$

In particular, $\widehat{D}(k) = \frac{1}{2|k|}$ for integer $k \neq 0$.

Finally we compute $\widehat{D}(0)$ by using the Poisson summation formula (see lemma 8.2) in the following series, whose terms are all zero:

$$0 \;=\; \sum_{n=-\infty}^{\infty} D\left(n+\tfrac{1}{2}\right) \;=\; \lim_{T\to\infty} \sum_{k=-T}^{T} e^{\pi i k}\, \widehat{D}(k) \;=\; \lim_{T\to\infty} \sum_{k=-T}^{T} (-1)^k\, \widehat{D}(k) \;=\; \widehat{D}(0) \;+\; 2\sum_{k=1}^{\infty}(-1)^k \frac{1}{2k} \;=\; \widehat{D}(0) \;-\; \log 2. \tag{4.49}$$

Hence $\widehat{D}(0) = \log 2$. □

Corollary 4.9. *The limit of $S(x)$ as $x \to \infty$ is:*

$$\lim_{x\to\infty} S(x) \;=\; \log 2. \tag{4.50}$$

Furthermore

$$\log 2 - S(x) \;=\; O(x^{-2}). \tag{4.51}$$

Proof. By rearranging the series (4.43) that defines $v_{F'}(t)$ and using proposition 4.3 we get:

$$v_{F'}(t) \;=\; -4\pi^2 \left(L-\tfrac{1}{2}\right)\left(|t|-\tfrac{1}{2}\right) \;-\; 4\sum_{n=0}^{\infty} \frac{L-S(n+\frac{1}{2})}{(n+\frac{1}{2})^2} \cos\{(2n+1)\,\pi t\}, \tag{4.52}$$

where $L = \lim_{x\to\infty} S(x)$. Its derivative can be computed by differentiating termwise:

$$v'_{F'}(t) \;=\; -4\pi^2 \left(L-\tfrac{1}{2}\right) \operatorname{sgn}(t) \;+\; 8\pi \sum_{n=0}^{\infty} \frac{L-S(n+\frac{1}{2})}{n+\frac{1}{2}} \sin\{(2n+1)\,\pi t\} \tag{4.53}$$

for $t \neq 0$. Since D is integrable, its Fourier transform $\widehat{D}$ is continuous, hence:

$$\begin{aligned}\widehat{D}(0) \;&=\; \lim_{t\to 0^+} \widehat{D}(t) \;=\; \frac{1}{2} \;-\; \frac{1}{4\pi^2}\, v'_{F'}(0^+) \\ &=\; \frac{1}{2} \;-\; \frac{1}{4\pi^2}\left\{-4\pi^2\left(L-\tfrac{1}{2}\right)\right\} \;=\; L.\end{aligned} \tag{4.54}$$

Since we already know that $\widehat{D}(0) = \log 2$, we get that $L = \log 2$.

Finally (4.51) is just equation (4.16). □

Proposition 4.10. *The integral of $D(x) = F(x) - \log|x|$ is*

$$\int_{-\infty}^{\infty} \{F(x) - \log|x|\}\, dx \;=\; \log 2. \tag{4.55}$$

The function F is extremal, in the sense that if $G(z)$ is another entire function of exponential type 2π such that

$$G(x) \;\geq\; \log|x| \tag{4.56}$$

for every $x \neq 0$, then

$$\int_{-\infty}^{\infty} \{G(x) - \log|x|\}\, dx \;\geq\; \log 2. \tag{4.57}$$

Furthermore, if there is equality in (4.57) then $G = F$.

Proof. The first claim (4.55) is just $\widehat{D}(0) = \log 2$, already proven in proposition 4.8.

To prove (4.56) we proceed as in theorem 8 of [10]. Let D_G be

$$D_G(x) \;=\; G(x) - \log|x| \tag{4.58}$$

and assume

$$\int_{-\infty}^{\infty} D_G(x)\, dx \;<\; +\infty. \tag{4.59}$$

Since $D(x)$ and $D_G(x)$ are integrable, then $G(x) - F(x) = D_G(x) - D(x)$ is also integrable. Also $F'(x)$ is integrable, hence $G'(x)$ is integrable. Next, we have:

$$\widehat{D_G}(t) \;=\; \frac{1}{2\pi i t}\,\widehat{D'_G}(t) \;=\; \frac{1}{2\pi i t}\left\{\widehat{G'}(t) + \pi i\;\mathrm{sgn}(t)\right\}. \tag{4.60}$$

Since G' is of exponential type 2π, its Fourier transform is supported in $[-1, 1]$, hence

$$\widehat{D_G}(t) \;=\; \frac{1}{2|t|} \tag{4.61}$$

for $|t| \geq 1$. Next, by applying the Poisson summation formula, and using that G is a majorizing function of $\log|x|$:

$$\begin{aligned} 0 \;\leq\; \sum_{n=-\infty}^{\infty} D_G(x+n) \;&=\; \widehat{D_G}(0) \;+\; \sum_{m=1}^{\infty} \frac{1}{m}\cos 2\pi x \\ &=\; \widehat{D_G}(0) \;-\; \log|2\,\sin\pi x|. \end{aligned} \tag{4.62}$$

Since $\max\limits_{x} \log|2\,\sin\pi x| = \log 2$, we get that

$$\int_{-\infty}^{\infty} \{G(x) - \log|x|\}\,dx \;=\; \widehat{D_G}(0) \;\geq\; \log 2. \tag{4.63}$$

Finally, assume $\widehat{D_G}(0) = \log 2$. Then (4.62) implies

$$\sum_{n=-\infty}^{\infty} D_G\left(\tfrac{1}{2} + n\right) \;=\; 0, \tag{4.64}$$

hence $D_G\left(\frac{1}{2} + n\right) = 0$ for every integer l. This implies that $G(n + \frac{1}{2}) = \log|n + \frac{1}{2}|$ and $G'(n + \frac{1}{2}) = \frac{1}{n+\frac{1}{2}}$ for every integer n. By applying theorem 10 from [10], we get

$$G(z) \;=\; F(z) \;+\; \left(\frac{\cos\pi z}{\pi}\right)^2 C, \tag{4.65}$$

where C is a constant equal to $\frac{1}{2}\left\{G''(n+\frac{1}{2}) - F''(n+\frac{1}{2})\right\}$ for any integer n. Since $D_G(x)$ is assumed to be integrable, we get that $C = 0$, hence

$$G(z) \;=\; F(z). \tag{4.66}$$

□

Next we record some useful formulas.

Proposition 4.11. *The function $v_{F'}(t)$ is continuous in $[-1,1]$ and twice continuously differentiable in $(-1,1)\setminus\{0\}$. Furthermore, for every $t \in [-1,1]$,*

$$v_{F'}(t) \;=\; 4\sum_{n=0}^{\infty} \frac{S(n+\frac{1}{2}) - \frac{1}{2}}{(n+\frac{1}{2})^2}\cos\{(2n+1)\,\pi t\} \tag{4.67}$$

and

$$\begin{aligned} v_{F'}(t) \;=\; & -4\pi^2\left(\log 2 - \frac{1}{2}\right)\left(|t| - \frac{1}{2}\right) \\ & -\; 4\sum_{n=0}^{\infty} \frac{\log 2 - S(n+\frac{1}{2})}{(n+\frac{1}{2})^2}\cos\{(2n+1)\,\pi t\}, \end{aligned} \tag{4.68}$$

and for every $t \in (-1,1)\setminus\{0\}$

$$\begin{aligned} v'_{F'}(t) \;=\; & -4\pi^2\left(\log 2 - \frac{1}{2}\right)\operatorname{sgn}(t) \\ & +\; 8\pi\sum_{n=0}^{\infty} \frac{\log 2 - S(n+\frac{1}{2})}{n+\frac{1}{2}}\sin\{(2n+1)\,\pi t\} \end{aligned} \tag{4.69}$$

and

$$v''_{F'}(t) \;=\; 16\pi^2\sum_{n=0}^{\infty}\left\{\log 2 - S(n+\tfrac{1}{2})\right\}\cos\{(2n+1)\,\pi t\}. \tag{4.70}$$

Also,

$$v_{F'}(1-t) \;=\; -v_{F'}(t) \tag{4.71}$$

for $0 \le t \le 1$.

Proof. Equation (4.67) is just the definition of $v_{F'}$. Equation (4.68) is the result of a rearrangement of the series and of using corollary 4.9 and

$$\sum_{n=0}^{\infty} \frac{\cos\{(2n+1)\,\pi t\}}{(n+\frac{1}{2})^2} \;=\; -\pi^2 \left(|t| - \frac{1}{2} \right) \tag{4.72}$$

for $-1 \le t \le 1$.

Equations (4.69) and (4.70) are the result of differentiating termwise in (4.68). Note that since $\log 2 - S(n+\frac{1}{2}) = O(x^{-2})$, and $\cos\{(2n+1)\,\pi t\}$ is continuous and uniformly bounded, the series in (4.70) converges uniformly to a continuous function. This justifies the first claim that $v_{F'}(t)$ is twice continuously differentiable.

Finally (4.71) can be deduced from (4.67) and

$$\cos\{(2n+1)\,\pi\,(1-t)\} \;=\; -\cos\{(2n+1)\,\pi t\}. \tag{4.73}$$

□

Next we record a few particular values of some expressions and functions.

Proposition 4.12.

$$\sum_{n=0}^{\infty} \left\{\log 2 - S(n+\tfrac{1}{2})\right\} \;=\; \frac{1}{2}\,\log 2 \;=\; 0.34657359\ldots \tag{4.74}$$

$$\sum_{n=0}^{\infty} (-1)^n \, \frac{\log 2 - S(n+\frac{1}{2})}{n+\frac{1}{2}} \;=\; \frac{\pi}{2}\left(\log 2 - \frac{1}{2}\right) \;+\; \frac{\pi}{32}\left\{\psi'(-\tfrac{1}{4}) - \psi'(\tfrac{1}{4})\right\} \;=\; 0.43539581\ldots, \tag{4.75}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ *is the digamma function.*

$$\sum_{n=0}^{\infty} \frac{\log 2 - S(n+\frac{1}{2})}{(n+\frac{1}{2})^2} \;=\; \frac{\pi^2}{2}\left(\log 2 - \frac{1}{2}\right) \;=\; 0.95314313\ldots \tag{4.76}$$

$$\sum_{n=0}^{\infty} \frac{S(n+\frac{1}{2})}{(n+\frac{1}{2})^2} \;=\; \frac{\pi^2}{4} \;=\; 2.46740110\ldots \tag{4.77}$$

$$v_{F'}(0) \;=\; v_{F'}(\tfrac{1}{2}) \;=\; v_{F'}(1) \;=\; 0. \tag{4.78}$$

$$v'_{F'}(0^+) \;=\; v'_{F'}(1^-) \;=\; -4\pi^2\left(\log 2 - \frac{1}{2}\right) \;=\; -7.62514505\ldots \tag{4.79}$$

$$v'_{F'}(\tfrac{1}{2}) \;=\; \frac{\pi^2}{4}\left\{\psi'(-\tfrac{1}{4}) - \psi'(\tfrac{1}{4})\right\} \;=\; 3.31754536\ldots \tag{4.80}$$

$$v''_{F'}(0^+) \;=\; -v''_{F'}(1^-) \;=\; 8\pi^2 \log 2 \;=\; 54.72870771\ldots \tag{4.81}$$

$$v''_{F'}(\tfrac{1}{2}) \;=\; 0. \tag{4.82}$$

$$\widehat{D}(0) \;=\; \log 2 \;=\; 0.69314718\ldots \tag{4.83}$$

$$\widehat{D}(\tfrac{1}{2}) \;=\; \widehat{D}(1) \;=\; \frac{1}{2}. \tag{4.84}$$

$$\widehat{D}'(0^+) \;=\; -\log 2 \;=\; -0.69314718\ldots \tag{4.85}$$

$$\widehat{D}'(1^-) \;=\; \log 2 - \frac{1}{2} \;=\; 0.19314718\ldots \tag{4.86}$$

$$\widehat{D}'(\tfrac{1}{2}) \;=\; \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^2} \;=\; -\frac{1}{2}\,\zeta(2,\tfrac{3}{2}) \;+\; \frac{1}{4}\,\zeta(2,\tfrac{5}{4}) \\ \;=\; -\frac{1}{8}\,\left\{\psi'(-\tfrac{1}{4}) - \psi'(\tfrac{1}{4})\right\} \;=\; -0.16806881\ldots \tag{4.87}$$

where $\zeta(s,q)$ is the Hurwitz zeta function.

$$\widehat{D}''(0^+) \;=\; -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} \;=\; \frac{3}{2}\,\zeta(3) \;=\; 1.80308535\ldots \tag{4.88}$$

$$\widehat{D}''(\tfrac{1}{2}) \;=\; \frac{1}{2}\,\left\{\psi'(-\tfrac{1}{4}) - \psi'(\tfrac{1}{4})\right\} \;=\; 0.67227524\ldots \tag{4.89}$$

$$\widehat{D}''(1^-) \;=\; 1. \tag{4.90}$$

Proof. First note that for $t \in (-1,1)\setminus\{0\}$:

$$\widehat{D}(t) \;=\; \frac{1}{2} \;-\; \frac{v_{F'}(t)}{4\pi^2|t|}, \tag{4.91}$$

$$\widehat{D}'(t) \;=\; \frac{1}{4\pi^2}\left\{-\frac{v'_{F'}(t)}{|t|} \;+\; \frac{v_{F'}(t)}{t^2}\,\mathrm{sgn}(t)\right\}, \tag{4.92}$$

$$\widehat{D}''(t) \;=\; \frac{1}{4\pi^2}\left\{-\frac{v''_{F'}(t)}{|t|} \;+\; \frac{2\,v'_{F'}(t)}{t^2}\,\mathrm{sgn}(t) \;-\; \frac{2\,v_{F'}(t)}{|t|^3}\right\}, \tag{4.93}$$

and

$$v_{F'}(t) \;=\; 4\pi^2|t|\left\{\frac{1}{2} \;-\; \widehat{D}(t)\right\}, \tag{4.94}$$

$$v'_{F'}(t) \;=\; 4\pi^2\left\{\left(\frac{1}{2} - \widehat{D}(t)\right)\mathrm{sgn}(t) \;-\; |t|\,\widehat{D}'(t)\right\}, \tag{4.95}$$

$$v''_{F'}(t) \;=\; -4\pi^2\left\{2\,\mathrm{sgn}(t)\,\widehat{D}'(t) \;+\; |t|\,\widehat{D}''(t)\right\}. \tag{4.96}$$

We already know that $\widehat{D}(0) = \log 2$, which is (4.83). From (4.94) and $\widehat{D}(t)$ bounded we get $v_{F'}(0) = 0$. By (4.71) we get that also $v_{F'}(1) = 0$. The value of $v_{F'}(\frac{1}{2})$ can be obtained by evaluating (4.67) at $t = \frac{1}{2}$. This proves (4.78), and by plugging in (4.91) we get (4.84). Similarly, by using (4.69) we easily obtain (4.79) and from (4.70) we get (4.82).

By plugging $v_{F'}(1)$ and $v'_{F'}(1^-)$ in (4.92) we get (4.86).

By the Poisson summation formula we have:

$$\lim_{T\to\infty} \sum_{n=-T}^{T} (-1)^n \, \widehat{D}(n+t) \;=\; \sum_{n=-\infty}^{\infty} e^{-2\pi i(n+\frac{1}{2})t} \, D(n+\tfrac{1}{2}) \;=\; 0. \tag{4.97}$$

Hence the derivatives $\widehat{D}^{(k)}$ of $\widehat{D}$ verify:

$$\lim_{T\to\infty} \sum_{n=-T}^{T} (-1)^n \, \widehat{D}^{(k)}(n+t) \;=\; 0 \tag{4.98}$$

From here we easily get for k odd:

$$\widehat{D}^{(k)}(\tfrac{1}{2}) \;=\; \frac{k!}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^{k+1}} \tag{4.99}$$

and

$$\widehat{D}^{(k)}(0^+) \;+\; \widehat{D}^{(k)}(1^-) \;=\; \widehat{D}^{(k)}(1^+) \;=\; -\frac{k!}{2}, \tag{4.100}$$

and for k even:

$$\begin{aligned} \widehat{D}^{(k)}(0^+) \;-\; D^{(k)}(1^-) \;&=\; -k! \left\{ \frac{1}{2} \;+\; \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{k+1}} \right\} \\ &=\; k! \left\{ -\frac{1}{2} \;+\; \left(1 - \frac{1}{2^k}\right) \zeta(k+1) \right\}. \end{aligned} \tag{4.101}$$

From (4.99) for $k = 1$ we get (4.87), and plugging in (4.95) we get (4.80). From (4.100) and (4.86) we get (4.85). From (4.101) we see that $\widehat{D}''(0^+)$ is finite, hence from (4.96) and (4.85) we get (4.81).

By using (4.93) we compute (4.89) and (4.90). Finally from (4.101) and (4.90) we get (4.88).

The relations (4.74), (4.75), (4.76) and (4.77) can be obtained by using proposition 4.11 and the values already computed for $v_{F'}$ and its derivatives.

□

Next we give a new convenient representation of $v_{F'}(t)$, useful to compute numerical approximations.

Proposition 4.13. *The function $v_{F'}$ can be written like this:*

$$v_{F'}(t) \;=\; -4\pi^2\left(\log 2-\frac{1}{2}\right)\left\{|t|-\frac{1}{2}+\frac{1}{2}\cos\pi t\right\} \;+\; 16\,b(t)\,\cos\pi t, \tag{4.102}$$

where

$$b(t) \;=\; \sum_{l=0}^{\infty}(-1)^l\sum_{k=1}^{\infty}\frac{\log 2-S(l+k+\frac{1}{2})}{(l+k+\frac{1}{2})^2}\sin^2 k\pi t. \tag{4.103}$$

Proof. Using the relation:

$$\frac{\cos\{(2n+1)\pi t\}}{\cos\pi t} \;=\; 1 \;-\; 4\,(-1)^n\sum_{k=1}^{n}(-1)^k\sin^2 k\pi t, \tag{4.104}$$

and rearranging the series (4.68) we get (4.102) with

$$b(t) \;=\; \sum_{\substack{n\geq 1\\ 1\leq k\leq n}}(-1)^{n+k}\,\frac{\log 2-S(n+\frac{1}{2})}{(n+\frac{1}{2})^2}\sin^2 k\pi t. \tag{4.105}$$

We get the announced result by changing $n=l+k$. □

Proposition 4.14. *Let b_L be*

$$b_L(t) \;=\; \sum_{l=0}^{L-1}(-1)^l\sum_{k=1}^{\infty}\frac{\log 2-S(l+k+\frac{1}{2})}{(l+k+\frac{1}{2})^2}\sin^2 k\pi t \tag{4.106}$$

and

$$v_L(t) \;=\; -4\pi^2 \left(\log 2 - \frac{1}{2}\right) \left\{ |t| - \frac{1}{2} + \frac{1}{2}\cos \pi t \right\} \;+\; 16\, b_L(t)\, \cos \pi t. \tag{4.107}$$

Then for $L \geq 0$

$$b_{2L}(t) \;\leq b(t) \;\leq\; b_{2L+1}(t) \qquad \textit{for every } t, \tag{4.108}$$

and

$$\begin{aligned} v_{2L}(t) \;&\leq v(t) \;\leq\; v_{2L+1}(t) \qquad \textit{if} \quad 0 \leq |t| \leq \frac{1}{2}, \\ v_{2L}(t) \;&\geq v(t) \;\geq\; v_{2L+1}(t) \qquad \textit{if} \quad \frac{1}{2} \leq |t| \leq 1. \end{aligned} \tag{4.109}$$

Proof. Inequalities (4.109) are a simple consequence of (4.108). On the other hand, (4.108) can be proven by writing $b_L(t)$ like this

$$b_L(t) \;=\; \sum_{k=1}^{\infty} c_{L,k} \sin^2 k\pi t \tag{4.110}$$

and

$$b(t) \;=\; \sum_{k=1}^{\infty} c_k \sin^2 k\pi t, \tag{4.111}$$

where

$$c_{L,k}(t) \;=\; \sum_{l=0}^{L-1} (-1)^l \, \frac{\log 2 - S(l+k+\frac{1}{2})}{(l+k+\frac{1}{2})^2} \tag{4.112}$$

and

$$c_k \;=\; \lim_{L\to\infty} c_{L,k} \;=\; \sum_{l=0}^{\infty} (-1)^l \, \frac{\log 2 - S(l+k+\frac{1}{2})}{(l+k+\frac{1}{2})^2}. \tag{4.113}$$

Since (4.113) is an alternating series of decreasing terms, we have $c_{2L,k} \leq c_k \leq c_{2L+1,k}$, and from here (4.108) follows. □

Remark. Since the functions $v_L(t)$ can be easily approximated by numerical methods, proposition 4.14 provides a way to compute numerical approximations for $v_{F'}(t)$.

Proposition 4.15. *The second derivative of the function $v_{F'}(t)$ verifies*

$$\frac{v''_{F'}(t)}{\cos \pi t} \;\geq\; 8\pi^2 \log 2 \qquad \text{for } |t| \in (0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1). \tag{4.114}$$

Hence $v_{F'}(t)$ is convex in $(0, \frac{1}{2})$ and concave in $(\frac{1}{2}, 1)$. As a consequence:

$$\begin{aligned} v_{F'}(t) &\;<\; 0 \qquad \text{for } 0 < |t| < \tfrac{1}{2}, \\ v_{F'}(t) &\;>\; 0 \qquad \text{for } \tfrac{1}{2} < |t| < 1. \end{aligned} \tag{4.115}$$

Proof. By using expression (4.70), and relations (4.104) and (4.74), we get:

$$\begin{aligned} &\frac{v''_{F'}(t)}{\cos \pi t} \;= \\ &16\,\pi^2 \sum_{n=0}^{\infty} \left\{ \log 2 - S(n + \tfrac{1}{2}) \right\} \left\{ 1 \;-\; 4\,(-1)^n \sum_{k=0}^{n} (-1)^k \sin^2 k\pi t \right\} = \\ &16\,\pi^2 \left\{ \frac{1}{2} \log 2 \;+\; 4 \sum_{l,k \geq 1} (-1)^{l+1} \left\{ \log 2 - S(l + k + \tfrac{1}{2}) \right\} \sin^2 k\pi t \right\}. \end{aligned} \tag{4.116}$$

For any fixed k, the last sum is an alternating series of decreasing terms, so it has the sign of its first term, which is positive. Hence:

$$\frac{v''_{F'}(t)}{\cos \pi t} \;\geq\; 16\,\pi^2 \,\frac{1}{2} \log 2 \;=\; 8\,\pi^2 \log 2. \tag{4.117}$$

Since $v_{F'}(0) = v_{F'}(\frac{1}{2}) = v_{F'}(1) = 0$, and since $v_{F'}(0)$ is even, (4.115) follows.

□

Corollary 4.16. *The function $\widehat{D}(t)$ verifies*

$$\frac{1}{2} \;\leq \widehat{D}(t) \;\leq\; \log 2 \qquad \text{for } |t| \in [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{4.118}$$

$$1 - \log 2 \;\leq \widehat{D}(t) \;\leq\; \frac{1}{2} \qquad \text{for } |t| \in [-1, 1] \setminus (-\tfrac{1}{2}, \tfrac{1}{2}). \tag{4.119}$$

Hence, in particular $0 < \widehat{D}(t) \le \frac{1}{2|t|}$ *for every* t*.*

Proof. For $t \in [-1, 1]$ we have:

$$\widehat{D}(t) \ = \ \frac{1}{2} \ - \frac{v_{F'}(t)}{4\pi^2|t|}. \tag{4.120}$$

We get the first inequality from $v_{F'}(t) \le 0$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, as proven in proposition 4.15.

The second inequality can be obtained from (4.109) with $L = 0$:

$$\begin{aligned} v_{F'}(t) \ \le \ v_0(t) \ &= \ -4\pi^2 \left(\log 2 - \frac{1}{2}\right) \left\{ |t| - \frac{1}{2} + \frac{1}{2}\cos \pi t \right\} \\ &\le \ 4\pi^2 \left(\log 2 - \frac{1}{2}\right) (1 - |t|) \end{aligned} \tag{4.121}$$

for $\frac{1}{2} \le |t| \le 1$. Hence:

$$\begin{aligned} \widehat{D}(t) \ &\ge \ \frac{1}{2} \ - \ \left(\log 2 - \frac{1}{2}\right) \left(\frac{1}{|t|} - 1\right) \\ &\ge \ \frac{1}{2} \ - \ \left(\log 2 - \frac{1}{2}\right) \ = \ 1 - \log 2 \end{aligned} \tag{4.122}$$

for $\frac{1}{2} \le |t| \le 1$. □

4.3 Other Results

We record here another result of interest.

Proposition 4.17. *The value of* F *at zero is:*

$$\begin{aligned} F(0) \ &= \ -\frac{\zeta'(2)}{\zeta(2)} \ - \ \frac{4}{3}\log 2 \ - \ 1 \\ &= \ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} \ - \ \frac{4}{3}\log 2 \ - \ 1, \end{aligned} \tag{4.123}$$

where

$$\Lambda(n) \;=\; \begin{cases} \log p & \textit{if } n = p^{\alpha},\ p \textit{ prime}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.124}$$

Proof. We have:

$$\begin{aligned} F(0) \;&=\; \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{\log|n+\frac{1}{2}| - 1}{(n+\frac{1}{2})^2} \\ &=\; \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\log(2n+1) - \log 2 - 1}{(2n+1)^2} \\ &=\; \frac{8}{\pi^2} \left\{ \sum_{n=0}^{\infty} \frac{\log(2n+1)}{(2n+1)^2} \;-\; (\log 2 + 1) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \right\}. \end{aligned} \tag{4.125}$$

On the other hand

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \;=\; (1 - 2^{-2})\,\zeta(2) \;=\; \frac{\pi^2}{8}, \tag{4.126}$$

and for $s > 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\log(2n+1)}{(2n+1)^s} \;&=\; -\frac{\partial}{\partial s} \sum_{n=0}^{\infty} (2n+1)^{-s} \\ &=\; -\frac{\partial}{\partial s}\,(1 - 2^{-s})\,\zeta(s) \\ &=\; -\zeta(s) \left\{ 2^{-s} \log 2 + (1 - 2^{-s})\,\frac{\zeta'(s)}{\zeta(s)} \right\}, \end{aligned} \tag{4.127}$$

hence, for $s = 2$,

$$\sum_{n=0}^{\infty} \frac{\log(2n+1)}{(2n+1)^2} \;=\; -\frac{1}{8}\,\frac{\pi^2}{3} \left\{ \log 2 + 3\,\frac{\zeta'(2)}{\zeta(2)} \right\}. \tag{4.128}$$

Substituting above:

$$\begin{aligned} F(0) \;&=\; -\frac{\zeta'(2)}{\zeta(2)} \;-\; \frac{4}{3} \log 2 \;-\; 1 \\ &=\; \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} \;-\; \frac{4}{3} \log 2 \;-\; 1 \\ &=\; -1.3542352476\ldots \end{aligned} \tag{4.129}$$

□

Chapter 5

Applications

5.1 A Majorizing Trigonometric Polynomial for $\varphi(x)$

Lemma 5.1. *For every real x and every $\delta > 0$*

$$\sum_{m=-\infty}^{\infty} D\left(\tfrac{x+m}{\delta}\right) = \delta \log 2 + \sum_{0<k<\frac{1}{\delta}} \left\{2\delta\, \widehat{D}(\delta k) - \frac{1}{k}\right\} \cos 2\pi x k - \log|2 \sin \pi x|. \tag{5.1}$$

Proof. We apply the Poisson summation formula:

$$\begin{aligned}\sum_{m=-\infty}^{\infty} D\left(\tfrac{x+m}{\delta}\right) &= \delta \lim_{T\to\infty} \sum_{k=-T}^{T} e^{2\pi i x k}\, \widehat{D}(\delta k) \\ &= \delta\, \widehat{D}(0) + 2\delta \sum_{0<\delta k<1} \widehat{D}(\delta k) \cos 2\pi x k \\ &\quad + 2\delta \sum_{\delta k \geq 1} \frac{1}{2\delta k} \cos 2\pi x k \\ &= \delta \log 2 + \sum_{0<k<\frac{1}{\delta}} \left\{2\delta\, \widehat{D}(\delta k) - \frac{1}{k}\right\} \cos 2\pi x k \\ &\quad - \log|2 \sin \pi x|.\end{aligned} \tag{5.2}$$

□

Corollary 5.2. *Let φ be the function defined by (2.23), let H be a positive integer, and let T_H be the following trigonometric polynomial:*

$$T_H(x) = \frac{\log 2}{\pi\,(H+1)} + \frac{1}{\pi} \sum_{k=1}^{H} \left\{ \frac{2\widehat{D}\left(\frac{k}{H+1}\right)}{H+1} - \frac{1}{k} \right\} \cos 2\pi x k. \tag{5.3}$$

Then

$$\varphi(x) \;\leq\; T_H(x) \tag{5.4}$$

for every $x \in \mathbb{R}$.

Proof. Substitute $\frac{1}{H+1}$ for δ in lemma 5.1, and use that $D(x) \geq 0$ for every real x. □

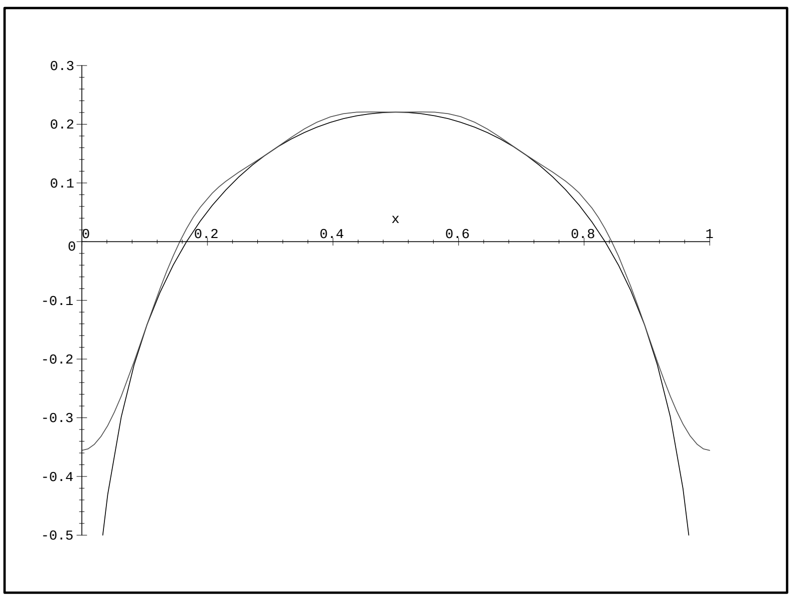

Figure 5.1: The trigonometric polynomial $T_5(x)$ majorizing $\varphi(x)$

5.2 An Erdős-Turán-Type Inequality

Theorem 5.3. *Let* $P(z)$ *be a polynomial whose roots lie all on the unit circle:*

$$P(z) \;=\; \prod_{m=1}^{M} (z - e^{2\pi i x_m}), \tag{5.5}$$

where $x_1, \ldots, x_M$ are real numbers. Let H be a positive integer. Then:

$$
\begin{aligned}
&\sup\{\log|P(z)| : |z| \leq 1\} \leq \\
&\frac{M\log 2}{H+1} + \sum_{k=1}^{H}\left\{\frac{1}{k} - \frac{2}{H+1}\widehat{D}\left(\tfrac{k}{H+1}\right)\right\}\sup_{x\in\mathbb{R}}\sum_{m=1}^{M} -\cos\{2\pi(x_m - x)k\} \\
&\leq \frac{M\log 2}{H+1} + \sum_{k=1}^{H}\left\{\frac{1}{k} - \frac{2}{H+1}\widehat{D}\left(\tfrac{k}{H+1}\right)\right\}\left|\sum_{m=1}^{M} e^{2\pi i x_m k}\right| \\
&\leq \frac{M\log 2}{H+1} + \sum_{k=1}^{H}\left\{\frac{1}{k} - \frac{2\,(1-\log 2)}{H+1}\right\}\left|\sum_{m=1}^{M} e^{2\pi i x_m k}\right| \\
&\leq \frac{M\log 2}{H+1} + \sum_{k=1}^{H}\frac{1}{k}\left|\sum_{m=1}^{M} e^{2\pi i x_m k}\right|, \qquad (5.6)
\end{aligned}
$$

where $D(x) = F(x) - \log x$, F is the function defined by (2.33), and $\widehat{D}$ is the Fourier transform of D.

Proof. By the maximum modulus principle the maximum of $|P(z)|$ in $|z| \leq 1$ is attained at some point of $|z| = 1$, hence it is enough to find a bound for $\log|P(e^{2\pi i x})|$. We have:

$$
\begin{aligned}
\log|P(e^{2\pi i x})| &= \log\left|\prod_{m=1}^{M}(e^{2\pi i x} - e^{2\pi i x_m})\right| \\
&= \sum_{m=1}^{M}\log|e^{2\pi i x} - e^{2\pi i x_m}| \\
&= \sum_{m=1}^{M}\log|1 - e^{2\pi i(x_m - x)}| \\
&= \sum_{m=1}^{M}\log|2\sin\pi(x_m - x)|.
\end{aligned} \qquad (5.7)
$$

From lemma 5.1 and taking into account that $D(x) \geq 0$ we get:

$$
\log|2\,\sin\pi x| \leq \delta\,\log 2 + \sum_{0<k<\frac{1}{\delta}}\left\{2\delta\,\widehat{D}(\delta k) - \frac{1}{k}\right\}\cos 2\pi x k \qquad (5.8)
$$

for any $\delta > 0$, hence:

$$\log|P(e^{2\pi i x})| \le$$
$$\sum_{m=1}^{M} \left\{ \delta \log 2 + \sum_{0<k<\frac{1}{\delta}} \left\{ 2\delta\, \widehat{D}(\delta k) - \frac{1}{k} \right\} \cos\{2\pi\,(x_m - x)\,k\} \right\} =$$
$$\delta\, M \log 2 + \sum_{0<k<\frac{1}{\delta}} \left\{ \frac{1}{k} - 2\delta\, \widehat{D}(\delta k) \right\} \sum_{m=1}^{M} -\cos\{2\pi\,(x_m - x)\,k\}. \quad (5.9)$$

Note that since $\widehat{D}(t) \le \frac{1}{2\,|t|}$, we have that $\frac{1}{k} - 2\delta\, \widehat{D}(\delta k) \ge 0$. Next, letting $\delta = \frac{1}{H+1}$ we get the desired result. Finally we use

$$\sum_{m=1}^{M} -\cos\{2\pi(x_m - x)k\} \le \left| \sum_{m=1}^{M} e^{2\pi i (x_m - x)k} \right| = \left| \sum_{m=1}^{M} e^{2\pi i x_m k} \right|, \quad (5.10)$$

and apply corollary 4.16. □

Corollary 5.4. *Let $x_1, \ldots, x_M$ be real numbers. Let H be a positive integer. Assume that Γ_M is defined as in (2.22). Then*

$$\Gamma_M \le \frac{M \log 2}{\pi\,(H+1)} + \frac{1}{\pi} \sum_{k=1}^{H} \frac{1}{k} \left| \sum_{m=1}^{M} e^{2\pi k x_m} \right|. \quad (5.11)$$

Proof. Use theorem 5.3 and (2.28). □

Next, a couple of particular cases.

Corollary 5.5. *Let α be an irrational number, $x_m = m\alpha$. Then for any positive integer H*

$$\Gamma_M \le \frac{M \log 2}{\pi\,(H+1)} + \frac{1}{2\pi} \sum_{k=1}^{H} \frac{1}{k\,\|k\alpha\|}, \quad (5.12)$$

where $\|x\| = \min\{|x - n| \,:\, n \in \mathbb{Z}\}$.

Proof. Use corollary 5.4 and:

$$\left| \sum_{m=1}^{M} e^{2\pi k x_m} \right| \;=\; \frac{1}{|\sin \pi k\alpha|} \;=\; \frac{1}{\sin \pi \|k\alpha\|} \;\leq\; \frac{1}{2\,\|k\alpha\|}, \tag{5.13}$$

□

Corollary 5.6. *Let H be any positive integer. If $P(z)$ is a cyclotomic polynomial*

$$P(z) \;=\; \prod_{\substack{a=1\\(a,q)=1}}^{q} (z - e^{2\pi i a/q}) \tag{5.14}$$

then

$$\sup\{\log|P(z)| : |z| \leq 1\} \;\leq\; \frac{\varphi(q)\,\log 2}{H+1} \;+\; \sum_{k=1}^{H} \frac{1}{k}\,(k,q), \tag{5.15}$$

where $\varphi(q) = \sum_{(a,q)=1} 1$ is Euler's function.

Remark. Note that if q is a prime, by taking $H = q-1$ we get

$$\sup\{\log|P(z)| : |z| \leq 1\} \;\leq\; C \;+\; \log q \tag{5.16}$$

with $C = 1 + \log 2$, which has the right order, since in this case $\sup\{\log|P(z)| : |z| \leq 1\} = \log q$ exactly.

Proof of corollary 5.6. Use theorem 5.3 and the following result about Ramanujan's sums ([1], sec. 8.4):

$$\left| \sum_{\substack{a=1\\(a,q)=1}}^{q} e^{2\pi i k a/q} \right| \;=\; \left| \frac{\varphi(q)\,\mu(\frac{q}{(k,q)})}{\varphi(\frac{q}{(k,q)})} \right| \;\leq\; (k,q). \tag{5.17}$$

□

5.3 Analogue of Montgomery and Vaughan's Inequality

Theorem 5.7. *Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers satisfying $0 < \delta \le |\lambda_j - \lambda_k|$ for $j \neq k$, and let $a_1, a_2, \dots, a_N$ be arbitrary complex numbers. Then*

$$-\frac{2\,\log 2}{\delta} \sum_{n=1}^{N} |a_n|^2 \;\le\; \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{k=1}^{N} \frac{a_j\,\overline{a}_k}{|\lambda_j - \lambda_k|}. \tag{5.18}$$

The constant $2\,\log 2$ is sharp.

Proof. We use the function $D(x)$, that we know is nonnegative and integrable, $\widehat{D}(0) = \log 2$ and $\widehat{D}(t) = \frac{1}{2\,|t|}$ for $|t| \ge 1$. First assume $\delta = 1$:

$$\begin{aligned}
0 \;&\le\; \int_{-\infty}^{\infty} D(x) \left| \sum_{n=1}^{N} a_n\, e^{-2\pi i\,\lambda_n\, x} \right|^2 dx \\
&= \int_{-\infty}^{\infty} \sum_{j=1}^{N} \sum_{k=1}^{N} a_j\,\overline{a}_k\, e^{-2\pi i\,(\lambda_j - \lambda_k)\,x}\, D(x)\, dx \\
&= \sum_{n=1}^{N} |a_n|^2\, \widehat{D}(0) \;+\; \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{k=1}^{N} a_j\,\overline{a}_k\, \widehat{D}(\lambda_j - \lambda_k) \\
&= \sum_{n=1}^{N} |a_n|^2\, \log 2 \;+\; \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{k=1}^{N} a_j\,\overline{a}_k\, \frac{1}{2\,|\lambda_j - \lambda_k|},
\end{aligned} \tag{5.19}$$

hence:

$$-\log 2 \sum_{n=1}^{N} |a_n|^2 \;\le\; \frac{1}{2} \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{k=1}^{N} \frac{a_j\,\overline{a}_k}{|\lambda_j - \lambda_k|}. \tag{5.20}$$

Finally, for $\delta \neq 1$ a change of variables allows us to complete the proof.

We see that the constant $-2\,\log 2$ is sharp by putting $\lambda_n = n$, $a_n = (-1)^n$, and N arbitrarily large. □

Remark. Note that the expression in the right hand side of (5.18) is not bounded above, as proven in [5], end of section 8.12, p. 214.

5.4 Generalization of the Erdős-Turán-Type Inequality

The "Erdős-Turán"-type inequality given by theorem 5.3 can be substantially generalized. In fact, suppose $\alpha_1, \ldots, \alpha_M$ are complex numbers, and $\omega_1, \ldots, \omega_M$ are non-negative numbers with

$$\sum_{m=1}^{M} \omega_m \;=\; 1. \tag{5.21}$$

Define $f : \mathbb{C} \to [-\infty, \infty)$ by

$$f(z) \;=\; \sum_{m=1}^{M} \omega_m \log|z - \alpha_m|, \tag{5.22}$$

and note that f is subharmonic on any open subset of $\mathbb{C}$. In particular f is subharmonic on $\Delta \;=\; \{\, z \in \mathbb{C} : |z| < 1 \,\}$.

We wish to estimate $\sup\{\, f(z) : z \in \Delta \,\}$ in terms of the power sums

$$\sum_{m=1}^{M} \omega_m \,(\alpha_m)^n \qquad 1 \le n \le N. \tag{5.23}$$

In fact such sums may be dominated by a few large α's, so we make the following modification. Suppose that $0 \le |\alpha_1| \le |\alpha_2| \le \cdots \le |\alpha_L| \le 1 < |\alpha_{L+1}| \le \cdots \le |\alpha_M|$ and then set

$$\beta_m \;=\; \begin{cases} \alpha_m & \text{if} \quad 1 \le m \le L, \\ (\overline{\alpha}_m)^{-1} & \text{if} \quad L+1 \le m \le M. \end{cases} \tag{5.24}$$

Then define the power sums

$$s_n \;=\; \sum_{m=1}^{M} \omega_m \,(\beta_m)^n \qquad n = 1, 2, \ldots \tag{5.25}$$

Next we define $g : \mathbb{C} \to [-\infty, \infty)$ and $h : \mathbb{C} \to [-\infty, \infty)$ by

$$g(z) \;=\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \sum_{m=1}^{M} \omega_m \, \log |z - \beta_m| \tag{5.26}$$

and

$$h(z) \;=\; \sum_{l=1}^{L} \omega_l \, \log |\overline{\alpha}_l \, z - 1| \;+\; \sum_{m=L+1}^{M} \omega_m \, \log |z - \alpha_m|. \tag{5.27}$$

It follows easily that f and g are subharmonic on Δ, h is harmonic on Δ, and $f(e^{2\pi i\theta}) = g(e^{2\pi i\theta}) = h(e^{2\pi i\theta})$ for all real θ. Therefore $f(z) \le h(z)$ and $g(z) \le h(z)$ for all $z \in \Delta$. So we have

$$\begin{aligned} \sup\{\, f(z) : z \in \Delta \,\} \;&\le\; \sup\{\, h(z) : z \in \Delta \,\} \\ &=\; \sup\{\, h(e^{2\pi i\theta}) : \theta \in \mathbb{R}/\mathbb{Z} \,\} \\ &=\; \sup\{\, g(e^{2\pi i\theta}) : \theta \in \mathbb{R}/\mathbb{Z} \,\}. \end{aligned} \tag{5.28}$$

Let $\varphi : \mathbb{R}/\mathbb{Z} \to [-\infty, \infty)$ be defined by

$$\varphi(\theta) \;=\; \pi^{-1} \log |e^{2\pi i\theta} - 1| \;=\; -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (2\pi |n|)^{-1} \, e^{2\pi i n\theta}. \tag{5.29}$$

Then let $T_N(\theta)$ be a trigonometric polynomial of degree at most N such that

$$\varphi(\theta) \;\le\; T_N(\theta) \tag{5.30}$$

for all $\theta \in \mathbb{R}/\mathbb{Z}$. We also assume that $\theta \mapsto T_N(\theta)$ is even (so that $n \mapsto \widehat{T}_N(n)$ is real) and

$$\widehat{T}_N(0) \;=\; \int_0^1 T_N(x)\, dx \;=\; \frac{\log 2}{\pi\,(N+1)}. \tag{5.31}$$

Now let $\beta = r\, e^{2\pi i \xi}$ with $0 \le r < 1$ and $\xi \in \mathbb{R}/\mathbb{Z}$. Then we have

$$\begin{aligned} \log|e^{2\pi i\theta} - \beta| &= \log|1 - \overline{\beta}\, e^{2\pi i\theta}| \\ &= -\Re\left\{ \sum_{n=1}^{\infty} n^{-1}\, (\overline{\beta})^n\, e^{2\pi i n\theta} \right\} \\ &= -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (2|n|)^{-1}\, r^{|n|}\, e^{2\pi i n\{\theta-\xi\}}. \end{aligned} \tag{5.32}$$

Next recall that

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{2\pi i n\theta} = (1-r^2)\,|1 - re^{2\pi i\theta}|^{-2} \ge 0 \tag{5.33}$$

for all θ. It follows from (5.29), (5.30) and (5.32) that

$$\begin{aligned} \log|e^{2\pi i\theta} - \beta| &= \pi \int_0^1 P_r(\theta - \xi - \tau)\, \varphi(\tau)\, d\tau \\ &\le \pi \int_0^1 P_r(\theta - \xi - \tau)\, T_N(\tau)\, d\tau \\ &= \pi \sum_{n=-N}^{N} r^{|n|}\, \widehat{T}_N(n)\, e^{2\pi i n(\theta-\xi)}. \end{aligned} \tag{5.34}$$

In order to use (5.34) (which holds also if $r = 1$) to estimate $g(e^{2\pi i\theta})$, write $\beta_m = r_m\, e^{2\pi i \xi_m}$, where $0 \le r_m \le 1$, $\xi_m \in \mathbb{R}/\mathbb{Z}$, $m = 1, 2, \ldots, M$. Then we find that

$$
\begin{aligned}
g(e^{2\pi i\theta}) \;&=\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \sum_{m=1}^{M} \omega_m \, \log |e^{2\pi i\theta} - \beta_m| \\
&\le\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \pi \sum_{m=1}^{M} \omega_m \sum_{n=-N}^{N} r_m^{|n|} \, \widehat{T}_N(n) \, e^{2\pi i n(\theta - \xi_m)} \\
&=\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \pi \, \widehat{T}_N(0) \\
&\quad\; + \; \pi \sum_{\substack{n=-N \\ n\neq 0}}^{N} \widehat{T}_N(n) \, \left\{ \sum_{m=1}^{M} \omega_m \, r_m^{|n|} \, e^{-2\pi i n \xi_m} \right\} e^{2\pi i n\theta} \\
&=\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \pi \, \widehat{T}_N(0) \\
&\quad\; + \; \pi \sum_{\substack{n=-N \\ n\neq 0}}^{N} \widehat{T}_N(n) \, \{ \, \overline{s}_n \, e^{2\pi i n\theta} + s_n \, e^{-2\pi i n\theta} \, \}.
\end{aligned}
\tag{5.35}
$$

This leads to the following inequality:

$$
\sup \{ \, f(z) : z \in \Delta \, \} \;\le\; \sum_{m=1}^{M} \omega_m \, \log^+ |\alpha_m| \;+\; \frac{\log 2}{N+1} \;+\; 2\pi \sum_{n=1}^{N} \left| \widehat{T}_N(n) \right| \, |s_n|,
\tag{5.36}
$$

where

$$
s_n \;=\; \sum_{m=1}^{M} \omega_m \, (\beta_m)^n \qquad n = 1, 2, \ldots, N.
\tag{5.37}
$$

Chapter 6

Conjectures and Future Research

6.1 Generalization of the Majorant for $\log|x|$

A small variation of the problems studied in the previous sections would be to substitute $\log|x|$ with $\frac{1}{2}\log(x^2+y^2)$ $(y>0)$. Note that

$$\log|x| = \lim_{y\to 0^+} \frac{1}{2}\log(x^2+y^2).$$

Then equation (2.30) becomes

$$\sum_{l=-\infty}^{\infty} D(x+l) \;=\; \widehat{D}(0) \;+\; \lim_{T\to\infty} \sum_{\substack{m=-T\\ m\neq 0}}^{T} \frac{e^{-2\pi y|m|}}{2|m|}\, e^{2\pi i m x} \;=\; \widehat{D}(0) - \pi\,\varphi_y(x), \tag{6.1}$$

where

$$\begin{aligned} \varphi_y(x) \;&=\; -y \;+\; \frac{1}{2\pi}\log(2\,\cosh 2\pi y - 2\,\cos 2\pi x) \\ &=\; \frac{1}{\pi}\log\left|\, e^{2\pi i x} - e^{-2\pi y}\,\right|. \end{aligned} \tag{6.2}$$

That function becomes $\varphi_(x)$ as $y\to 0^+$, but has some extra properties of interest. It is periodic of period 2π and has maxima and minima respectively at the points of the form $n+\frac{1}{2}$ and n for $n\in\mathbb{Z}$, where it takes the values:

$$\begin{aligned} \varphi_y(n+\tfrac{1}{2}) \;&=\; \frac{1}{\pi}\log\left(1+e^{-2\pi y}\right), \\ \varphi_y(n) \;&=\; \frac{1}{\pi}\log\left(1-e^{-2\pi y}\right). \end{aligned} \tag{6.3}$$

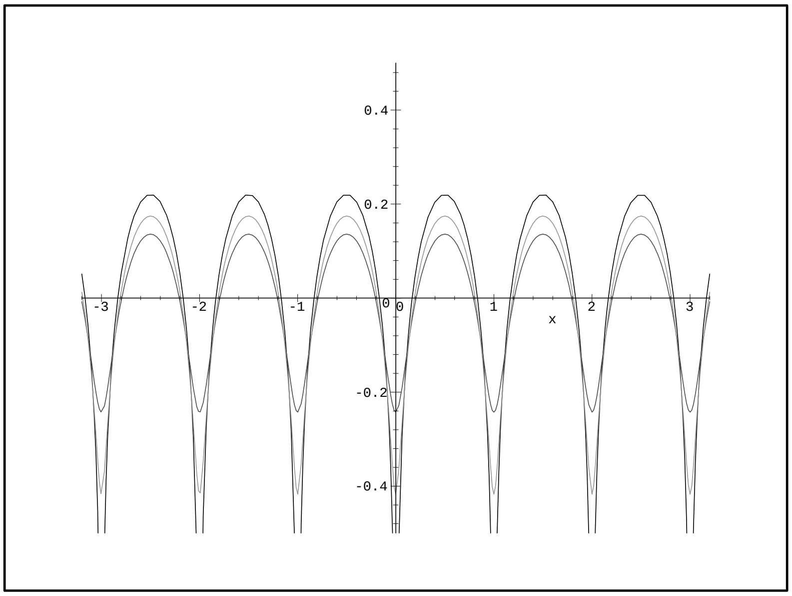

Figure 6.1: The function $\varphi_y(x)$ for several values of y

Reasoning as in the previous chapter we have that the following is a good candidate for an entire function of exponential type 2π that majorizes $\frac{1}{2}\log(x^2+y^2)$ and is extremal in the sense of minimizing the L^1 norm of $F_y^+(x) - \frac{1}{2}\log(x^2+y^2)$:

$$F_y^+(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{1}{2}\log\{(n+\frac{1}{2})^2+y^2\}}{(z-(n+\frac{1}{2}))^2} + \sum_{n=-\infty}^{\infty} \frac{\frac{n+\frac{1}{2}}{(n+\frac{1}{2})^2+y^2}}{z-(n+\frac{1}{2})} \right\}. \quad (6.4)$$

However now we also have a candidate for a *minorizing* extremal function, which will result from interpolating at the integers instead of the integers

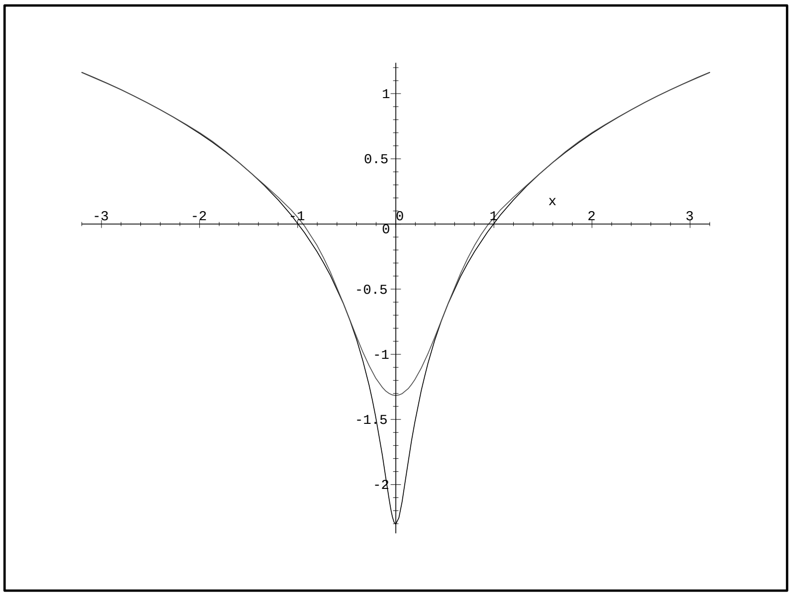

Figure 6.2: Function $F_y^+(x)$ majorizing $\frac{1}{2}\,\log(x^2+y^2)$

plus a half:

$$F_y^-(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{1}{2}\,\log\{n^2+y^2\}}{(z-n)^2} \;+\; \sum_{n=-\infty}^{\infty} \frac{\frac{n}{n^2+y^2}}{z-n} \right\}. \tag{6.5}$$

If we define $D_y^{\pm}(x) = F_y^{\pm}(x) - \frac{1}{2}\,\log(x^2+y^2)$, then:

$$\begin{aligned} \widehat{D}_y^+(0) &= \log\left(1+e^{-2\pi y}\right) \\ \widehat{D}_y^-(0) &= \log\left(1-e^{-2\pi y}\right). \end{aligned} \tag{6.6}$$

At this point we can state the following conjecture:

Conjecture 6.1. *Let $F_y^+(x)$ and $F_y^-(x)$ be the functions defined in (6.4) and (6.5) respectively. Then:*

1. *$F_y^+(z)$ and $F_y^-(z)$ are entire functions of exponential type 2π.*

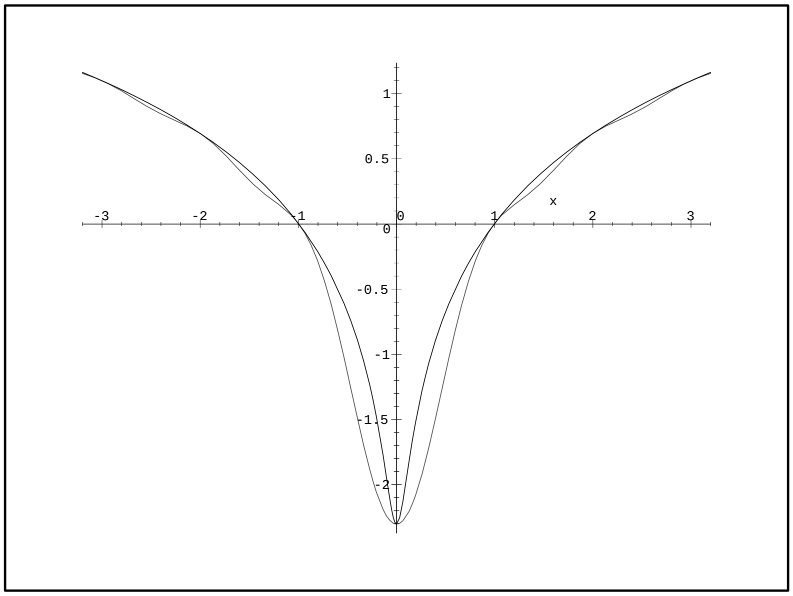

Figure 6.3: Function $F_y^-(x)$ minorizing $\frac{1}{2}\,\log(x^2+y^2)$

2. *For every* $x \in \mathbb{R}$

$$F_y^-(x) \;\le\; \frac{1}{2}\,\log(x^2+y^2) \;\le\; F_y^+(x). \tag{6.7}$$

3. *They satisfy:*

$$\begin{aligned} \int_{-\infty}^{\infty} \{\, F_y^+(x) - \frac{1}{2}\,\log(x^2+y^2)\,\}\, dx \;&=\; \log\left(1+e^{-2\pi y}\right), \\ \int_{-\infty}^{\infty} \{\, \frac{1}{2}\,\log(x^2+y^2) - F_y^-(x)\,\}\, dx \;&=\; \log\left(1-e^{-2\pi y}\right). \end{aligned} \tag{6.8}$$

4. *They are extremal, in the sense that among all pairs of functions satisfying 1 and 2, they are the ones that minimize integrals (6.8) in 3.*

The key property is, of course, the double inequality (6.7). From here several results follow, such as:

Conjecture 6.2. *Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \geq \delta > 0$ whenever $m \neq n$, and let $a(1), \dots, a(N)$ be arbitrary complex numbers. Also let α be any positive real number. Then*

$$-\frac{2}{\delta} \log\left(1 + e^{-2\pi\alpha\delta}\right) \sum_{n=1}^{N} |a(n)|^2 \leq \sum_{m=1}^{N} \sum_{\substack{n=1 \\ m \neq n}}^{N} \frac{a(m)\,\overline{a(n)}}{|\lambda_m - \lambda_n|} e^{-2\pi\alpha|\lambda_m - \lambda_n|} \leq -\frac{2}{\delta} \log\left(1 - e^{-2\pi\alpha\delta}\right) \sum_{n=1}^{N} |a(n)|^2. \quad (6.9)$$

In the proof for this result we will set $y = \alpha\,\delta$, use (6.6) and

$$\widehat{D}_y^{\pm}(t) = \frac{e^{-2\pi y|t|}}{2|t|} \qquad \text{for } |t| \geq 1\ , \quad (6.10)$$

and proceed as in the proof of theorem 5.7.

Note that this result becomes theorem 5.7 as $\alpha \to 0^+$. Note also that the upper bound tends to infinity.

6.2 Generalization of Beurling's Function

Writing $z = x + yi$ we note that $i\,\log(-iz) = \arctan\frac{x}{y} + i\,\frac{1}{2}\,\log(x^2+y^2)$, hence $\frac{1}{2}\,\log(x^2+y^2)$ is the harmonic conjugate of $\arctan\frac{x}{y}$. This function becomes $\frac{\pi}{2}\,\text{sgn}(x)$ for $y \to 0^+$. If we substitute $\text{sgn}(x)$ by $\frac{2}{\pi}\arctan\frac{x}{y}$ and repeat the work done for Beurling's function, we get:

$$\sum_{l=-\infty}^{\infty} D(x+l) = \widehat{D}(0) + \lim_{T\to\infty} \sum_{\substack{m=-T \\ m\neq 0}}^{T} \frac{e^{-2\pi y|m|}}{\pi i m} e^{2\pi i m x} = \widehat{D}(0) + 2\,\psi_y(x), \quad (6.11)$$

where

$$\psi_y(x) \;=\; -\frac{1}{\pi}\,\arctan\left\{\frac{\sin 2\pi x}{e^{2\pi y}-\cos 2\pi x}\right\}. \tag{6.12}$$

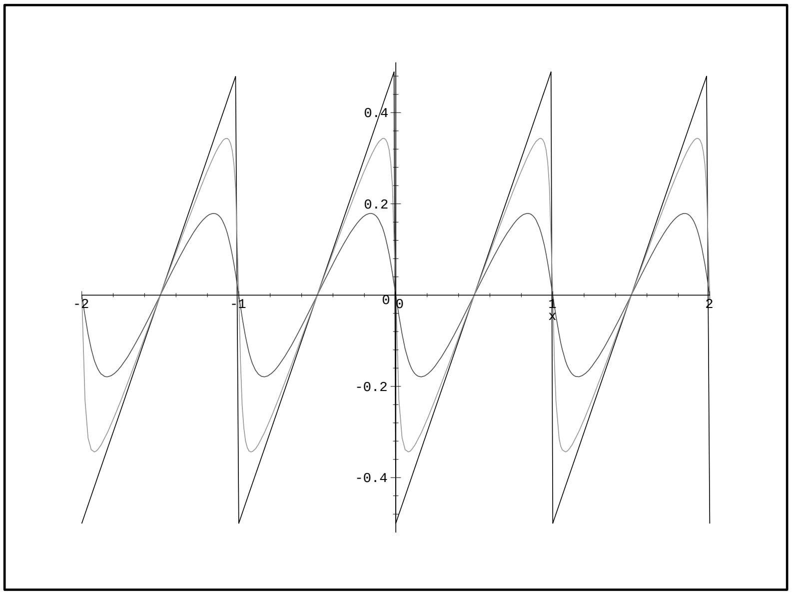

Figure 6.4: The function $\psi_y(x)$ for several values of y

The function $\psi_y(x)$ becomes $\psi(x)$ for $y \to 0^+$, but also has some properties of interest. It is periodic of period 2π and has maxima and minima respectively at the points of the form $n - \delta_y$ and $n + \delta_y$ for $n \in \mathbb{Z}$, where

$$\delta_y \;=\; \frac{1}{2\pi}\,\arccos\left\{\,e^{-2\pi y}\,\right\} \qquad 0 < \delta_y < \frac{1}{2}, \tag{6.13}$$

and it takes the values:

$$\psi_y(n \pm \delta_y) \;=\; \mp\frac{1}{\pi}\,\arctan\left\{(e^{4\pi y}-1)^{-\frac{1}{2}}\right\}. \tag{6.14}$$

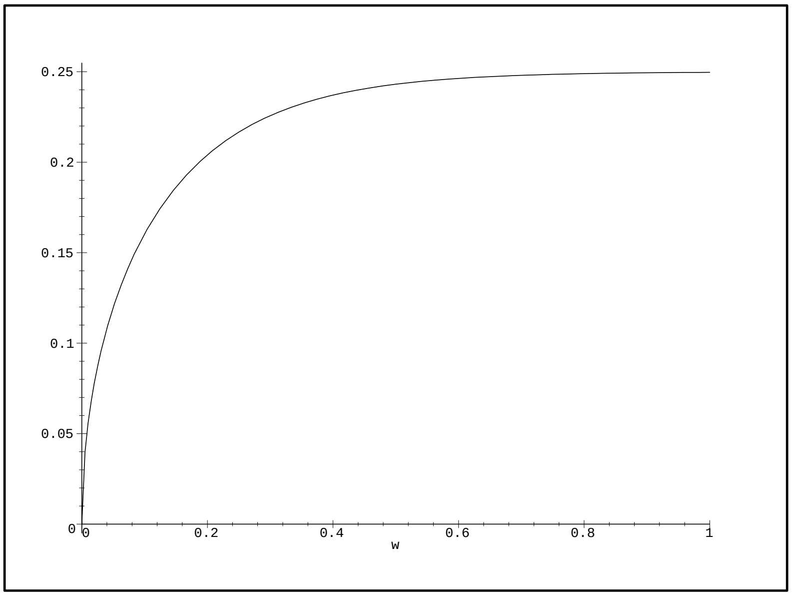

Figure 6.5: δ_y as a function of y

From here we get that the following function is a candidate for an entire extremal function of exponential type 2π that majorizes $\frac{2}{\pi}\arctan\frac{x}{y}$ along the real axis:

$$B_y^+(z) = \left(\frac{\sin\pi(z-\delta_y)}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\frac{2}{\pi}\arctan\frac{n+\delta_y}{y}}{(z-(n+\delta_y))^2} + \sum_{n=-\infty}^{\infty} \frac{\frac{2y/\pi}{(n+\delta_y)^2+y^2}}{z-(n+\delta_y)} \right\}. \quad (6.15)$$

By substituting δ_y with $-\delta_y$ we get an analogous function, $B_y^-(z)$, intended to minorize $\frac{2}{\pi}\arctan\frac{x}{y}$ along the real axis.

Note that by letting $y \to 0^+$, then $\delta_y \to 0^+$, and $B_y^+(z)$ and $B_y^-(z)$ approach $B(z)$ and $-B(-z)$ respectively.

Also, we note that $B_y^{+\prime}(\delta_y) \to 2$ as $y \to 0^+$, which confirms the value

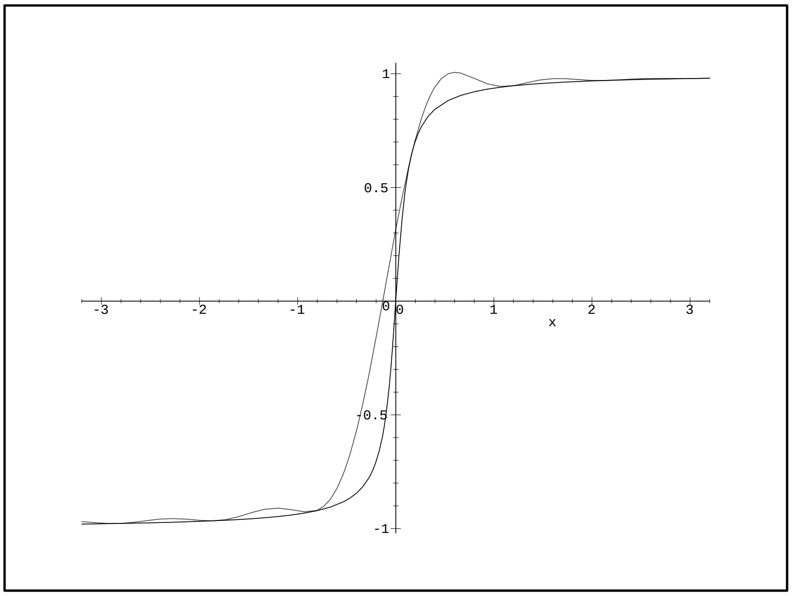

Figure 6.6: Function $B_y^+(x)$ majorizing $\frac{2}{\pi}$ arctan $\frac{x}{y}$

$B'(0) = 2$ for the derivative of the Beurling's function at zero.

If $D_y^{\pm} = B_y^{\pm}(x) - \frac{2}{\pi}$ arctan $\frac{x}{y}$ then

$$\widehat{D}_y^{\pm}(0) \;=\; \pm\frac{2}{\pi}\,\arctan\left\{(e^{4\pi y} - 1)^{-\frac{1}{2}}\right\}. \qquad (6.16)$$

At this point we can state the following conjecture:

Conjecture 6.3. *Let $B_y^+(x)$ be the function defined in (6.15), and $B_y^-(x)$ be the analogous functions obtained by substituting δ_y with $-\delta_y$ in the definition of $B_y^+(x)$. Then:*

1. *$B_y^+(z)$ and $B_y^-(z)$ are entire functions of exponential type 2π.*

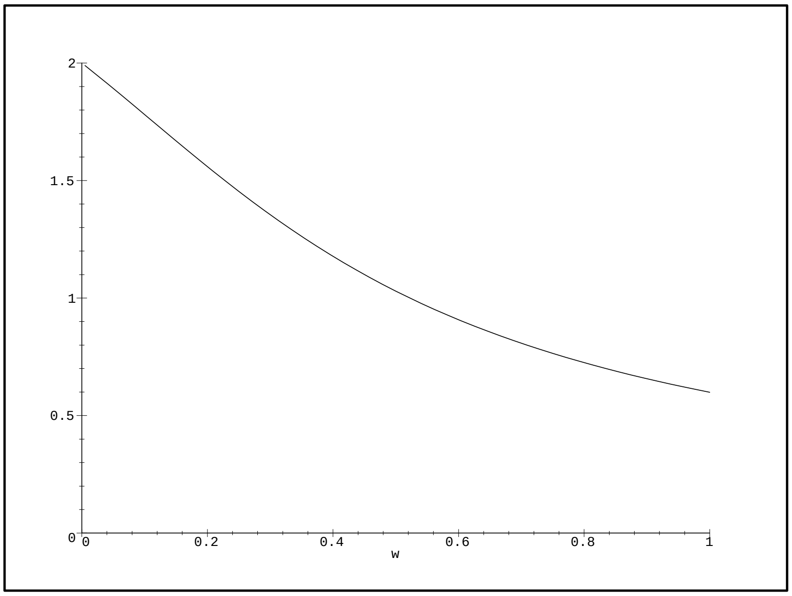

Figure 6.7: ${B_y^+}'(\delta_y)$ as a function of y

2. For every $x \in \mathbb{R}$

$$B_y^-(x) \;\leq\; \frac{2}{\pi}\,\arctan\frac{x}{y} \;\leq\; B_y^+(x). \tag{6.17}$$

3. They satisfy:

$$\begin{aligned}
\int_{-\infty}^{\infty} \{\, B_y^+(x) - \frac{2}{\pi}\,\arctan\frac{x}{y} \,\}\, dx &= \frac{2}{\pi}\,\arctan\left\{(e^{4\pi y}-1)^{-\frac{1}{2}}\right\}, \\
\int_{-\infty}^{\infty} \{\, \frac{2}{\pi}\,\arctan\frac{x}{y} - B_y^-(x) \,\}\, dx &= \frac{2}{\pi}\,\arctan\left\{(e^{4\pi y}-1)^{-\frac{1}{2}}\right\}.
\end{aligned} \tag{6.18}$$

4. They are extremal, in the sense that among all pairs of functions satisfying 1 and 2, they are the ones that minimize integrals (6.18) in 3.

Next, some results that follow from the extremal properties of $B_y^{\pm}(x)$:

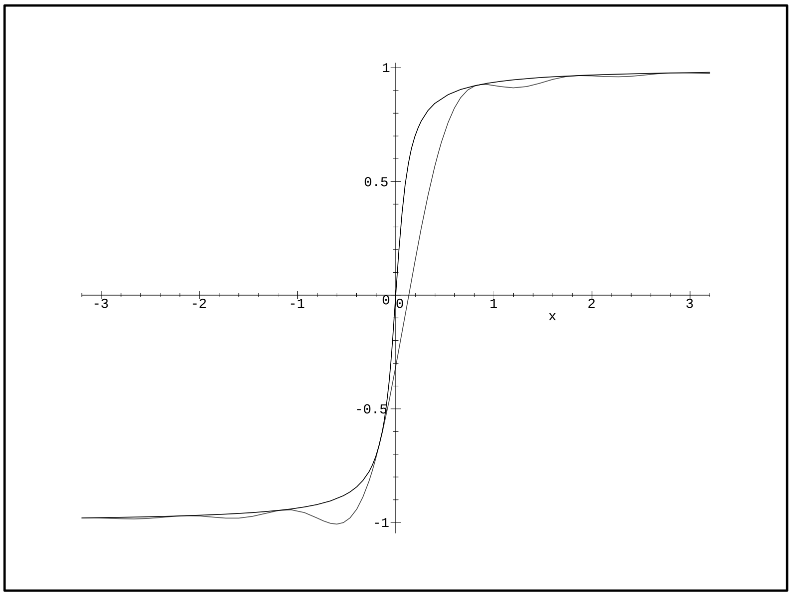

Figure 6.8: Function $B_y^-(x)$ minorizing $\frac{2}{\pi}$ arctan $\frac{x}{y}$

Conjecture 6.4. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \geq \delta > 0$ whenever $m \neq n$, and let $a(1), \ldots, a(N)$ be arbitrary complex numbers. Also let α be any positive real number. Then*

$$\left| \sum_{\substack{m=1 \\ m \neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{\lambda_m - \lambda_n}\, e^{-2\pi\alpha|\lambda_m - \lambda_n|} \right| \leq \frac{2}{\delta} \arctan\left\{(e^{4\pi\alpha\delta} - 1)^{-\frac{1}{2}}\right\} \sum_{n=1}^{N} |a(n)|^2. \quad (6.19)$$

In the proof of this result we will set $y = \alpha\,\delta$, use (6.16) and

$$D_y^{\pm} = \frac{e^{-2\pi y|t|}}{\pi i t}, \quad (6.20)$$

and proceed as in the proof of (2.2), (Montgomery and Vaughan's inequality).

By letting $\alpha \to 0^+$ this result becomes (2.2), so (6.19) can be considered as a generalization of Montgomery and Vaughan's inequality.

6.3 Extremal Majorants and Minorants for $\log^+|x|$

In this section we briefly address the problem of obtaining extremal majorants and minorants for the function

$$\log^+|x| \;=\; \begin{cases} \log|x| & \text{if } |x| \geq 1, \\ 0 & \text{if } |x| < 1. \end{cases} \tag{6.21}$$

Assume that $G(z)$ is an entire function of exponential type at most 2π, and assume that the difference $D_G(x) = G(x) - \log^+|x|$ is integrable. Note that $\log^+|x| = \log|x| - \chi_{[-1,1]}(x)\,\log|x|$, so by using the Poisson summation formula and taking into account equation (2.30) we get

$$\begin{aligned} \sum_{l=-\infty}^{\infty} D_G(x+l) \;&=\; \widehat{D}_G(0) \;+\; 2 \;-\; \pi\,\varphi(x) \;+\; \log(\langle x\rangle) \;+\; \log(1-\langle x\rangle) \\ &=\; \widehat{D}_G(0) \;-\xi(x), \end{aligned} \tag{6.22}$$

where

$$\xi(x) \;=\; \log\left\{ \frac{|2\,\sin\pi x|}{\langle x\rangle(1-\langle x\rangle)} \right\} \;-\; 2, \tag{6.23}$$

$\langle x\rangle$ = fractional part of x.

The function ξ is periodic of period 1, has a minimum $\xi(n) = \log 2\pi - 2 \approx -0.16212293\ldots$ at the integers, and a maximum $\xi(n+1/2) = 3\,\log 2 - 2 \approx 0.07944154\ldots$ at the integers plus a half. Hence, candidates to extremal minorizing and majorizing entire functions of exponential type at most 2π for

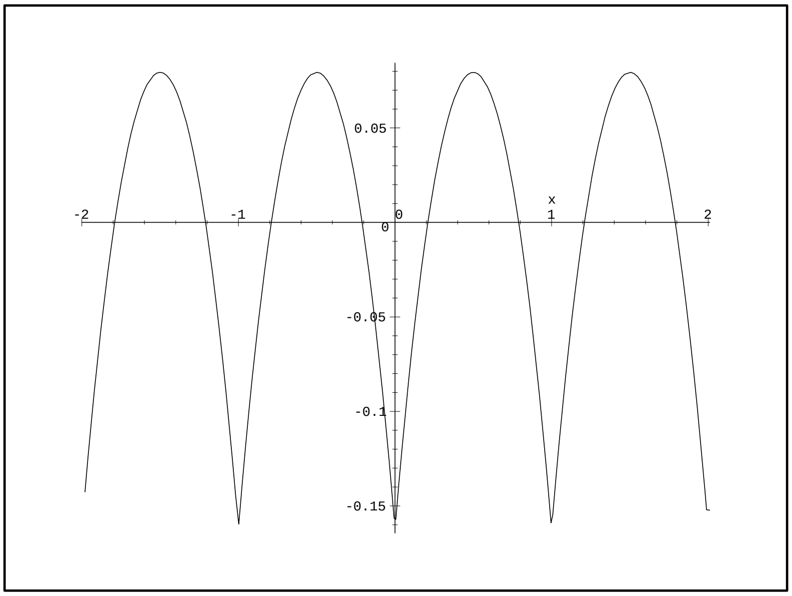

Figure 6.9: The function $\xi(x)$

$\log^+|x|$ are interpolating functions at the integers and at the integers plus a half respectively:

$$F_-(z) \;=\; \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{|n|\geq 1} \frac{\log|n|}{(z-n)^2} \;+\; \sum_{|n|\geq 2} \frac{\frac{1}{n}}{z-n} \;+\; \frac{2F_-'(1)}{x^2-1} \right\}, \quad (6.24)$$

$$F_+(z) \;=\; \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{\substack{n=-\infty \\ n\neq -1,0}}^{\infty} \frac{\log|n+\frac{1}{2}|}{(z-(n+\frac{1}{2}))^2} \;+\; \sum_{\substack{n=-\infty \\ n\neq -1,0}}^{\infty} \frac{\frac{1}{n+\frac{1}{2}}}{z-(n+\frac{1}{2})} \right\}. \quad (6.25)$$

The derivative $F_-'(1)$ of F_- at 1 is a parameter to be determined. This parameter does not affect the value of

$$D_{F_-}(0) \;=\; \int_{-\infty}^{\infty} \{F_-(x) - \log^+(|x|)\}\, dx, \quad (6.26)$$

but it might play a role in getting $F_-(x) \leq \log^+|x|$ for every $x \in \mathbb{R}$.

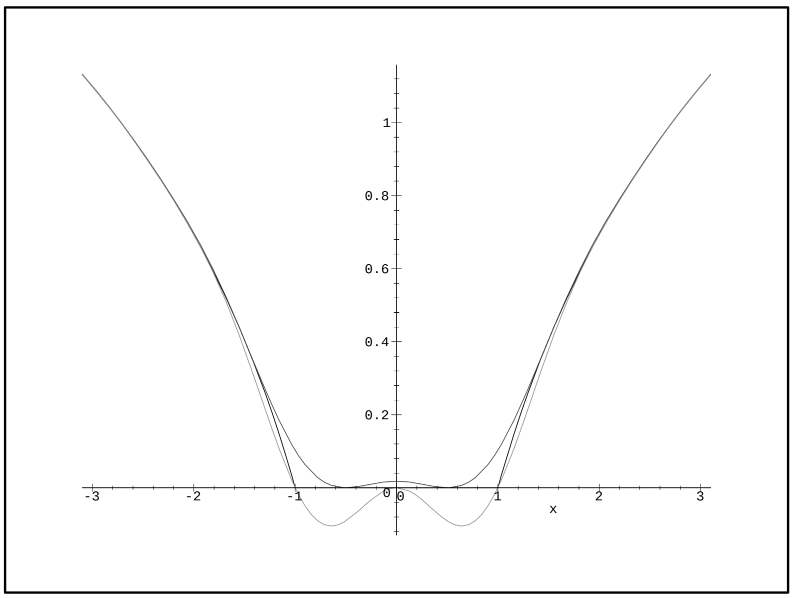

Figure 6.10: The functions $\log^+|x|$, $F_+(x)$ and $F_-(x)$

Chapter 7

Harmonic Majorants

In chapter 6 we pose the problem of generalizing the previous results to the functions $\frac{2}{\pi}\arctan\frac{x}{y}$ and $\frac{1}{2}\log(x^2+y^2)$ $(y>0)$. Since they are respectively the harmonic extensions of $\operatorname{sgn}(x)$ and $\log|x|$ to the upper half complex plane $\mathcal{H}^+ = \{x+yi \,:\, y>0\}$, the harmonic extensions of the Beurling's function and that of the function F defined in (2.33) will be majorants for $\frac{2}{\pi}\arctan\frac{x}{y}$ and $\frac{1}{2}\log(x^2+y^2)$ $(y>0)$ respectively. Of course, they will not be extremal in the sense of the previous chapters, but they do solve a different optimization problem.

We will use the theory in [2, ch. 11], and [6, ch. 8]. In particular we restate the following theorem from [6] (after some changes of notation):

Theorem 7.1. *Let $p \geq 1$ and let f a function in $L^p(\mathbb{R})$. Let F be the harmonic function defined in $\mathcal{H}^+$ by*

$$F(x+yi) \;=\; P_y * f(x) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\,\frac{y}{(x-u)^2+y^2}\,du, \tag{7.1}$$

where

$$P_y(x) \;=\; \frac{1}{\pi}\,\frac{y}{x^2+y^2} \tag{7.2}$$

is the Poisson Kernel for the upper half plane. Then:

1. *For each $y > 0$ the function $F_y(x)$ is in $L^p(\mathbb{R})$.*

2. *The L^p-norms $\|F_y\|_p$ are bounded for $y > 0$. In fact $\|F_y\|_p$ is a decreasing function of y for $y > 0$.*

3. *The functions F_y converge to f in L^p-norm as $y \to 0^+$.*

4. *$F(z)$ tends uniformly to zero as z tends to infinity in any fixed half-plane $\Im(z) \geq \delta > 0$.*

7.1 A Harmonic Majorant for the Arctangent

Theorem 7.2. *Let F be an entire function of exponential type at most 2π such that for every $y > 0$*

$$\Re\{F(x+yi)\} \;\geq\; \frac{2}{\pi}\arctan\frac{x}{y}. \tag{7.3}$$

For $y > 0$ let $I_y(\Re\{F\})$ be the L^1-norm of the difference:

$$I_y(\Re\{F\}) \;=\; \int_{-\infty}^{\infty}\left\{\Re\{F(x+yi)\} - \frac{2}{\pi}\arctan\frac{x}{y}\right\} dx. \tag{7.4}$$

Then $I_y(\Re\{F\})$ is independent of y—so we can drop the subscript y in it: $I(\Re\{F\}) = I_y(\Re\{F\})$. Also:

1. *$I(\Re\{F\}) \geq 1$.*

2. *The Beurling function (2.1) verifies (7.3). Furthermore, $I(\Re\{F\}) = 1$ if and only if $\frac{1}{2}\left\{F(z) + \overline{F(\overline{z})}\right\} = B(z)$.*

Proof. Let D be:

$$D(x+yi) \;=\; \Re\{F(x+yi)\} - \frac{2}{\pi}\arctan\frac{x}{y} \tag{7.5}$$

for $y > 0$. Its limit function for $y \to 0^+$ is $D(x) = \Re\{F(x)\} - \operatorname{sgn}(x)$, and since D is harmonic, we have $D(x+yi) = P_y * D(x)$. By Fubini's theorem we see that

$$I_y(\Re\{F\}) = \|P_y * (\Re\{F\} - \operatorname{sgn})\|_1 = \|\Re\{F\} - \operatorname{sgn}\|_1, \tag{7.6}$$

hence it is independent of y.

On the other hand

$$\frac{1}{2}\left\{F(z) + \overline{F(\overline{z})}\right\} \tag{7.7}$$

is a real entire function of exponential type at most 2π that coincides with $\Re\{F\}$ on the real line. Hence, by the properties of Beurling's function we get (1) and (2). □

Remark. Analogously, the following holds for $y > 0$:

$$\Re\{-B(-x-yi)\} \;\le\; \frac{2}{\pi}\arctan\frac{x}{y}, \tag{7.8}$$

and

$$\int_{-\infty}^{\infty}\left\{\Re\{-B(-x-yi)\} - \frac{2}{\pi}\arctan\frac{x}{y}\right\}dx \;=\; -1. \tag{7.9}$$

Next we give a generalization of Montgomery and Vaughan's inequality that can be obtained as a consequence of theorem 7.2. The result is similar to conjecture 6.4—but somewhat weaker.

Theorem 7.3. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \ge \delta > 0$ whenever $m \ne n$, and let $a(1), \ldots, a(N)$ be arbitrary complex numbers. Also* *let α be any positive real number. Then*

$$\left| \sum_{\substack{m=1\\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{\lambda_m - \lambda_n}\, e^{-2\pi\alpha|\lambda_m-\lambda_n|} \right| \;\leq\; \frac{\pi}{\delta} \sum_{n=1}^{N} |a(n)|^2. \tag{7.10}$$

Proof. Let $D_y(x)$ be equal to

$$D_y(x) \;=\; D(x+yi) \;=\; \Re\{B(x+yi)\} \;-\; \frac{2}{\pi}\,\arctan\frac{x}{y}. \tag{7.11}$$

$D(x+yi)$ is a harmonic function with $D(x) = B(x) - \operatorname{sgn}(x)$ as limit for $y \to 0^+$, hence $D_y(x) = P_y * D(x)$, where P_y is the Poisson kernel (7.2). Then $\widehat{D}_y(t) = \widehat{P}_y(t)\,\widehat{D}(t) = e^{-2\pi y|t|}\,\widehat{D}(t)$ for $y > 0$. Hence $\widehat{D}_y(0) = 1$ and $\widehat{D}_y(t) = -e^{-2\pi y|t|}/\pi i t$ for $|t| \geq 1$. Next set $y = \alpha\,\delta$, and proceed as in the proof of theorem 16 of [10] (the Montgomery and Vaughan's inequality). □

7.2 A Harmonic Majorant for the Logarithm

Theorem 7.4. *Let G be an entire function of exponential type at most 2π such that for every $y > 0$*

$$\Re\{G(x+yi)\} \;\geq\; \frac{1}{2}\,\log\,(x^2+y^2). \tag{7.12}$$

For $y > 0$ let $I_y(\Re\{G\})$ be the L^1-norm of the difference:

$$I_y(\Re\{G\}) \;=\; \int_{-\infty}^{\infty} \left\{ \Re\{G(x+yi)\} - \frac{1}{2}\,\log\,(x^2+y^2) \right\} dx. \tag{7.13}$$

Then $I_y(\Re\{G\})$ is independent of y—so we can drop the subscript y in it: $I(\Re\{G\}) = I_y(\Re\{G\})$. Also:

1. $I(\Re\{G\}) \geq \log 2$.

2. *The function F defined by (2.33) verifies (7.12). Furthermore, $I(\Re\{G\}) = \log 2$ if and only if $\frac{1}{2}\left\{G(z) + \overline{G(\overline{z})}\right\} = F(z)$.*

Proof. Let D be:

$$D(x+yi) \;=\; \Re\{G(x+yi)\} - \frac{1}{2}\log\left(x^2+y^2\right) \tag{7.14}$$

for $y > 0$. Its limit function for $y \to 0^+$ is $D(x) = \Re\{G(x)\} - \log|x|$, and since D is harmonic, we have $D(x+yi) = P_y * D(x)$. By Fubini's theorem we see that

$$I_y(\Re\{G\}) = \|P_y * (\Re\{G\} - \log|\cdot|)\|_1 = \|\Re\{G\} - \log|\cdot|\|_1, \tag{7.15}$$

hence it is independent from y.

On the other hand

$$\frac{1}{2}\left\{G(z) + \overline{G(\overline{z})}\right\} \tag{7.16}$$

is a real entire function of exponential type at most 2π that coincides with $\Re\{G\}$ on the real line. Hence, by proposition 4.10 we get (1) and (2). □

As a consequence of theorem 7.4 we give a result that generalizes theorem 5.7. The result is analogous to conjecture 6.2, but somewhat weaker.

Theorem 7.5. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \geq \delta > 0$ whenever $m \neq n$, and let $a(1), \ldots, a(N)$ be arbitrary complex numbers. Also let α be any positive real number. Then*

$$-\frac{2}{\delta}\log 2 \sum_{n=1}^{N} |a(n)|^2 \;\leq\; \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \frac{a(m)\,\overline{a(n)}}{|\lambda_m - \lambda_n|}\, e^{-2\pi\alpha|\lambda_m-\lambda_n|}. \tag{7.17}$$

Proof. Let $D_y(x)$ be equal to

$$D_y(x) \;=\; D(x+yi) \;=\; \Re\{F(x+yi)\} \;-\; \frac{1}{2}\log{(x^2+y^2)}, \qquad (7.18)$$

where F is function (2.33). $D(x+yi)$ is a harmonic function with $D(x) = F(x) - \log|x|$ as limit for $y \to 0^+$, hence $D_y(x) = P_y * D(x)$, where P_y is the Poisson kernel (7.2). Then $\widehat{D}_y(t) = \widehat{P}_y(t)\,\widehat{D}(t) = e^{-2\pi y|t|}\,\widehat{D}(t)$ for $y > 0$. Hence $\widehat{D}_y(0) = \log 2$ and $\widehat{D}_y(t) = e^{-2\pi y|t|}/2|t|$ for $|t| \geq 1$. Next set $y = \alpha\,\delta$, and proceed as in the proof of theorem 5.7. $\square$

Chapter 8

Lemmas

Here we give some details about results used in some proofs.

We begin with a version of the Euler-MacLaurin summation formula.

Lemma 8.1. *Let $N \leq M$ integers, and let $f : [N - \frac{1}{2}, M + \frac{1}{2}] \to \mathbb{R}$ be $2m$ times continuously differentiable, $m \geq 2$, and such that*

$$\int_{N-\frac{1}{2}}^{M+\frac{1}{2}} |f^{(2m)}(x)|\, dx < \infty,$$

where $f^{(2m)}$ is the $2m$-th derivative of f. Then

$$\begin{aligned} \sum_{n=N}^{M} f(n) \;&=\; \int_{N-\frac{1}{2}}^{M+\frac{1}{2}} f(x)\, dx \\ &\quad + \sum_{k=1}^{m-1} \frac{\mathrm{B}_{2k}(\frac{1}{2})}{(2k)!} \left\{ f^{(2k-1)}(M + \tfrac{1}{2}) - f^{(2k-1)}(N - \tfrac{1}{2}) \right\} \\ &\quad + \frac{\mathrm{B}_{2m}(\frac{1}{2})}{(2m)!} \sum_{n=N}^{M} f^{(2m)}(\theta_n). \end{aligned} \tag{8.1}$$

where $\mathrm{B}_k(x) =$ k-th Bernoulli polynomial, and $n - \frac{1}{2} < \theta_n < n + \frac{1}{2}$.

Proof. Using the general Euler-MacLaurin formula and taking into account that $B_{2k-1}(\frac{1}{2}) = 0$:

$$\sum_{N-\frac{1}{2}<n<M+\frac{1}{2}} f(n) = \int_{N-\frac{1}{2}}^{M+\frac{1}{2}} f(x)\,dx + \sum_{k=1}^{m-1} \frac{B_{2k}(\frac{1}{2})}{(2k)!} \left\{ f^{(2k-1)}(M+\tfrac{1}{2}) - f^{(2k-1)}(N-\tfrac{1}{2}) \right\} + \frac{1}{(2m-1)!} \int_{N-\frac{1}{2}}^{M+\frac{1}{2}} B_{2m-1}(\langle x \rangle)\, f^{(2m-1)}(x)\,dx, \quad (8.2)$$

where $\langle x \rangle$ = fractional part of x.

By integrating by parts and using the mean value theorem for integrals (justified by the fact that $B_{2k}(\frac{1}{2}) - B_{2m}(\langle x \rangle)$ never changes its sign) the last integral becomes:

$$\begin{aligned}
\int_{N-\frac{1}{2}}^{M+\frac{1}{2}} & B_{2m-1}(\langle x \rangle)\, f^{(2m-1)}(x)\,dx \\
&= \frac{1}{m} \int_{N-\frac{1}{2}}^{M+\frac{1}{2}} \left\{ B_{2m}(\tfrac{1}{2}) - B_{2m}(\langle x \rangle) \right\} f^{(2m)}(x)\,dx \\
&= \frac{1}{m} \sum_{n=N}^{M} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \left\{ B_{2m}(\tfrac{1}{2}) - B_{2m}(\langle x \rangle) \right\} f^{(2m)}(x)\,dx \\
&= \frac{1}{m} \sum_{n=N}^{M} f^{(2m)}(\theta_n) \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \left\{ B_{2m}(\tfrac{1}{2}) - B_{2m}(\langle x \rangle) \right\} dx \\
&= \frac{B_{2m}(\frac{1}{2})}{m} \sum_{n=N}^{M} f^{(2m)}(\theta_n). \quad (8.3)
\end{aligned}$$

From here the announced result follows. □

Next, we justify the use of the Poisson summation formula for the function $D(x)$.

Lemma 8.2. *Let D be the function defined in (4.1). Then*

$$\sum_{n=-\infty}^{\infty} D(x+n) \;=\; \lim_{T\to\infty} \sum_{k=-T}^{T} \widehat{D}(k)\, e^{2\pi ikx} \tag{8.4}$$

for every real $x \notin \mathbb{Z}$.

Proof. For small $\varepsilon > 0$, let D_ε be the function

$$D_\varepsilon(x) \;=\; \begin{cases} D(x) = F(x) - \log|x| & \text{if } |x| > \varepsilon \\ F(x) - \log|\varepsilon| & \text{if } |x| \le \varepsilon \end{cases} \tag{8.5}$$

Now $D_\varepsilon(x)$ is a continuous function of bounded variation and absolutely integrable on $\mathbb{R}$, so we can apply the Poisson summation formula to it:

$$\sum_{n=-\infty}^{\infty} D_\varepsilon(x+n) \;=\; \lim_{T\to\infty} \sum_{k=-T}^{T} \widehat{D}_\varepsilon(k)\, e^{2\pi ikx} \tag{8.6}$$

for every real $x \notin \mathbb{R}$. Note that if $\varepsilon < \|x\|$ = distance from x to the nearest integer, then the left hand sides of (8.4) and (8.6) are the same. In order to compare the right hand sides, we compute:

$$\begin{aligned} \widehat{D}(t) - \widehat{D}_\varepsilon(t) \;&=\; \int_{-\varepsilon}^{\varepsilon} \log\frac{\varepsilon}{|y|}\, e^{-2\pi iyt}\, dy \\ &=\; \frac{1}{\pi t}\, \mathrm{Si}(2\pi\varepsilon t), \end{aligned} \tag{8.7}$$

where

$$\mathrm{Si}(x) \;=\; \int_0^x \frac{\sin u}{u}\, du \tag{8.8}$$

is the sine integral. So:

$$\begin{aligned} &\sum_{k=-T}^{T} \widehat{D}(k)\, e^{2\pi ikx} \;-\; \sum_{k=-T}^{T} \widehat{D}_\varepsilon(k)\, e^{2\pi ikx} \\ &\qquad =\; \widehat{D}(0) \;-\; \widehat{D}_\varepsilon(0) \;+\; \frac{2}{\pi} \sum_{k=1}^{T} \frac{1}{k}\, \mathrm{Si}(2\pi\varepsilon k)\, \cos 2\pi kx \end{aligned} \tag{8.9}$$

Obviously $\widehat{D}(0) \;-\; \widehat{D}_\varepsilon(0) \to 0$ as $\varepsilon \to 0^+$, hence we only need to worry about the last sum, which is the real part of

$$\frac{2}{\pi} \sum_{k=1}^{T} \frac{1}{k} \,\mathrm{Si}(2\pi\varepsilon k)\, e^{2\pi i k x}. \tag{8.10}$$

Summation by parts shows that the absolute value of (8.10) is bounded by

$$\frac{4/\pi}{|1 - e^{2\pi i x}|} \left\{ \mathrm{Si}(2\pi\varepsilon) \;+\; \sum_{k=1}^{T} \left| \frac{\mathrm{Si}(2\pi\varepsilon(k+1))}{k+1} - \frac{\mathrm{Si}(2\pi\varepsilon k)}{k} \right| \right\}. \tag{8.11}$$

We have $\mathrm{Si}(2\pi\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Also we notice that by the mean value theorem:

$$\begin{aligned} &\frac{\mathrm{Si}(2\pi\varepsilon(k+1))}{k+1} - \frac{\mathrm{Si}(2\pi\varepsilon k)}{k} \\ &\qquad = \frac{1}{(k+\theta_k)^2} \left\{ \sin\{2\pi(k+\theta_k)\varepsilon\} - \mathrm{Si}(2\pi(k+\theta_k)\varepsilon) \right\}, \end{aligned} \tag{8.12}$$

where $0 < \theta_k < 1$. Hence the sum in (8.11) converges uniformly as $T \to \infty$, and each of its terms tends to zero as $\varepsilon \to 0^+$, so that

$$\lim_{\varepsilon \to 0^+} \sum_{k=1}^{\infty} \left| \frac{\mathrm{Si}(2\pi\varepsilon(k+1))}{k+1} - \frac{\mathrm{Si}(2\pi\varepsilon k)}{k} \right| \;=\; 0. \tag{8.13}$$

It follows that

$$\lim_{\varepsilon \to 0^+} \lim_{T \to \infty} \left| \sum_{k=-T}^{T} \widehat{D}(k)\, e^{2\pi i k x} \;-\; \sum_{k=-T}^{T} \widehat{D}_\varepsilon(k)\, e^{2\pi i k x} \right| \;=\; 0, \tag{8.14}$$

and the announced result follows. □

Lemma 8.3. *Let j be the function defined in (4.6). Then*

$$-\frac{27}{t^2} \;\le\; j^{(4)}(t) \;\le\; \frac{6}{t^2} \qquad \text{for every } t > 0. \tag{8.15}$$

As a consequence:

$$|j^{(4)}(t)| \;\le\; \frac{27}{t^2} \qquad \text{for every } t > 0. \tag{8.16}$$

Proof. Let f_a be the function:

$$f_a(t) \;=\; j^{(4)}(t) - \frac{a}{t^2}. \tag{8.17}$$

After some lengthly but elementary computations we can write $f_a(t)$ like this:

$$f_a(t) \;=\; \frac{-480\,p(t)}{(t^2-1)^7}\left\{\log t \;+\; \frac{g_a(t)}{t^2\,p(t)}\right\}, \tag{8.18}$$

where

$$p(t) \;=\; t^8 + 105t^4 + 35t^6 + 49t^2 + 2, \tag{8.19}$$

and $g_a(t)$ is a certain polynomial of 13th degree whose coefficients depend on a. The factor outside the curly brackets is negative for $t > 1$, and positive for $0 < t < 1$ (note that $f_a(t)$ has a removable discontinuity at $t = 1$), and the expression inside has the following derivative:

$$\frac{d}{dt}\left\{\log t \;+\; \frac{g_a(t)}{t^2\,p(t)}\right\} \;=\; \frac{(t^2-1)^6\,q_a(t)}{120\,t^3\,p(t)^2}, \tag{8.20}$$

where $q_a(t)$ is a 10th-degree polynomial. By counting the zeros of $q_a(t)$ in $(0,\infty)$ (e.g., by the Sturm method) we find that $q_6(t)$ and $q_{-27}(t)$ have no zeros in that interval. Also we have that $q_a(1) = 672a - 3456$, hence $q_6(1) = 576$ and $q_{-27}(1) = -21600$, which implies $q_6(t) > 0$ and $q_{-27}(t) < 0$ for $t > 0$. Thus, the expression inside the curly brackets is increasing for $a = 6$ and decreasing for $a = -27$. Since it vanishes at $t = 1$, this allows us to determine its sign along the intervals $(0,1)$ and $(1,\infty)$, which together with that of the factor outside the curly brackets, gives us the sign of $f_a(t)$ in $(0,\infty)\setminus\{1\}$. At $t = 1$ the sign of f_a can be checked by directly computing $f_a(1) = \frac{36}{7} - a$. From here we get for every $t > 0$

$$f_6(t) \;<\; 0 \;<\; f_{-27}(t), \tag{8.21}$$

and the desired result follows. $\square$

Lemma 8.4. *Let f be the function*

$$f(x) \;=\; j\left(\tfrac{1}{2x}\right) \;+\; x\int_{\frac{1}{x}}^{\infty} j(t)\,dt \;+\; \frac{1}{24\,x}\,j'\left(\tfrac{1}{x}\right), \tag{8.22}$$

where j is the function defined in (4.6). Then

$$f(x) \;=\; \frac{\frac{11}{6} \;-\; 3\,\log 2}{x^2} \;+\; O\left(\frac{\log x}{x^4}\right) \tag{8.23}$$

for $x \to \infty$.

Proof. The proof is just a lengthly but elementary computation. We have:

$$\begin{aligned} j\left(\tfrac{1}{2x}\right) \;&=\; \frac{x^2}{(x^2-\frac{1}{4})^3}\left\{\left(-3x^2-\frac{1}{4}\right)\log(2x)+2x^4+x^2-\frac{3}{8}\right\} \\ &=\; 2 \;-\; \frac{3\,\log x}{x^2} \;+\; \frac{\frac{5}{2}-3\,\log 2}{x^2} \;+\; O\left(\frac{\log x}{x^4}\right), \end{aligned} \tag{8.24}$$

$$\begin{aligned} x\int_{\frac{1}{x}}^{\infty} j(t)\,dt \;&=\; \frac{2\,x^2\,(2\,\log x - x^2+1)}{(x^2-1)^2} \\ &=\; -2 \;+\; \frac{4\,\log x}{x^2} \;-\; \frac{2}{x^2} \;+\; O\left(\frac{\log x}{x^4}\right), \end{aligned} \tag{8.25}$$

$$\begin{aligned} \frac{1}{24\,x}\,j'\left(\tfrac{1}{x}\right) \;&=\; -\frac{x^2}{3\,(x^2-1)^4}\left\{(3x^4+8x^2+1)\,\log x+4x^4-2x^2-2\right\} \\ &=\; -\frac{\log x}{x^2} \;+\; \frac{4/3}{x^2} \;+\; O\left(\frac{\log x}{x^4}\right). \end{aligned} \tag{8.26}$$

By adding up the three expressions we get the announced result. $\square$

Lemma 8.5. *Let g be*

$$g(x) \;=\; -x^2\,f(x) - \frac{7}{128}, \tag{8.27}$$

where f is defined in (8.22). Then $g(x) > 0$ in (δ, ∞) for some $0 < \delta < \frac{1}{2}$.

Proof. The function $g(x)$ can be written like this:

$$g(x) \;=\; \frac{x^4\,p(x)\,b(x)}{3\,(x^2-1)^4\,(4x^2-1)^3}, \tag{8.28}$$

where

$$p(x) \;=\; 224x^8 + 916x^6 - 1095x^4 + 220x^2 + 59, \tag{8.29}$$

$$b(x) \;=\; \log x \;+\; \frac{m(x)}{x^4\,p(x)}, \tag{8.30}$$

and $m(x)$ is a certain polynomial of 14th degree.

We note that $p(x)$ has no zeros, which can be checked by Sturm's method. The derivative of $b(x)$ is the following:

$$b'(x) \;=\; \frac{(x^2-1)^3\,(4x^2-1)^2\,q(x)}{32\,x^5\,p(x)^2}, \tag{8.31}$$

where

$$\begin{aligned} q(x) \;=\;& \\ &(-324800 + 516096\,\log 2)\,x^{12} \;+\; (-3692848 + 6027264\,\log 2)\,x^{10} \\ &+\; (-885648 - 18432\,\log 2)\,x^8 \;+\; (1140416 - 774144\,\log 2)\,x^6 \\ &-\; 167462\,x^4 \;+\; 10647\,x^2 \;+\; 1239. \end{aligned} \tag{8.32}$$

Using Sturm's method again we check that $q(x)$ has exactly 2 zeros[1] in the interval $(0,\infty)$, one at some $x_1 \in (0,\frac{1}{2})$ and the other one at $x_2 \in (\frac{1}{2},1)$. We easily check the sign of $q(x)$ to be negative in (x_1,x_2) and positive in $(0,\infty)\setminus(x_1,x_2)$.

[1] By numerical methods we get: $x_1 = 0.47657357\ldots$, $x_2 = 0.74127575\ldots$

Next we determine the sign of $b(x)$ at several intervals along (x_1, ∞). First we note that $b'(x)$ has a double zero at $x = \frac{1}{2}$ and a triple zero at $x = 1$, hence $b(x)$ has a triple zero at $x = \frac{1}{2}$ and a quadruple zero at $x = 1$. Since the derivative of $b(x)$ vanishes only once in $(\frac{1}{2}, 1)$, then $b(x)$ cannot have any zeros in that interval, so its sign remains constant there, and it can be checked to be positive. Since the zero of $b(x)$ at $x = 1$ is even, then $b(x)$ will have the same sign (positive) to the right of that point. Since $b'(x) > 0$ in $(1, \infty)$, $b(x)$ will remain positive along the whole interval $(1, \infty)$. Finally the sign in $(x_1, \frac{1}{2})$ can be determined by observing that the zero of $b(x)$ at $x = \frac{1}{2}$ is even, so $b(x)$ will be negative to the left of that point, and its derivative is positive in $(x_1, \frac{1}{2})$, hence $b(x)$ must be negative in that interval. We summarize the result as: $b(x) < 0$ in $(x_1, \frac{1}{2})$ and $b(x) > 0$ in $(\frac{1}{2}, 1) \cup (1, \infty)$. From here we get that $b(x)$ has no other zeros in (x_1, ∞) but those at $x = \frac{1}{2}$ and $x = 1$.

Finally we note that the zeros of $b(x)$ in (x_1, ∞) cancel out exactly the poles of the denominator of $g(x)$, so that $g(x)$ is a non-vanishing continuous function in that interval, where it can be easily checked to be positive. Finally we end the proof by setting $\delta = x_1$. □

Lemma 8.6. *Let h defined as in (4.4). Then the function $t \mapsto t^3\, h'(t)$, is increasing for $t > 1$.*

Proof. The derivative of $t^3\, h'(t)$ is

$$\frac{d}{dt}\{t^3\, h'(t)\} \;=\; \frac{48\, t^3\, (t^2+1)}{(t^2-1)^4} \left\{ \log t - \frac{t^6 + 9t^4 - 9t^2 - 1}{12\, t^2\, (t^2+1)} \right\}. \qquad (8.33)$$

The factor outside the curly brackets is positive for $t > 1$, and the expression inside has the following derivative:

$$\frac{d}{dt}\left\{\log t - \frac{t^6+9t^4-9t^2-1}{12\,t^2\,(t^2+1)}\right\} \;=\; -\frac{(t^2-1)^4}{6\,t^3\,(t^2+1)^2}, \tag{8.34}$$

which is negative for $t > 1$, so the expression inside the curly brackets is decreasing. Since its value is zero at $t = 1$, it will be negative for $t > 1$. This proves that the derivative of $t^3\,h'(t)$ is negative for $t > 1$, hence $t^3\,h'(t)$ is decreasing. □

Lemma 8.7. *Let $h(t)$ be the function defined by (4.4). Then for every $t > 0$*

$$-\frac{14}{t^4} \;\leq\; h^{(4)}(t) \;\leq\; \frac{1}{t^4}, \tag{8.35}$$

where $h^{(4)}(t)$ is the fourth derivative of $h(t)$. As a consequence

$$|h^{(4)}(t)| \;\leq\; \frac{14}{t^4} \tag{8.36}$$

for every $t > 0$.

Proof. Let f_a be the function:

$$f_a(t) = h^{(4)}(t) - \frac{a}{t^4}. \tag{8.37}$$

After some computations we get

$$f_a(t) \;=\; \frac{240\,p(t)}{(t^2-1)^6}\left\{\log t + \frac{g_a(t)}{t^4\,p(t)}\right\}, \tag{8.38}$$

where

$$p(t) \;=\; (t^2+1)\,(t^4+14t^2+1), \tag{8.39}$$

and g_a is a certain polynomial of 12th degree whose coefficients depend on a. The factor outside the curly brackets is positive for $t \in (0,\infty)\setminus\{1\}$ (note that

$f_a(t)$ has a removable discontinuity at $t = 1$), and the expression inside has the following derivative:

$$\frac{d}{dt}\left\{\log t \;+\; \frac{g_a(t)}{t^4\,p(t)}\right\} \;=\; \frac{(t^2-1)^5\,q_a(t)}{120\,t^5\,p(t)^2}, \tag{8.40}$$

where

$$\begin{aligned} q_a(t) \;=\; & -a\,t^8 \;+\; (120-35a)\,t^6 \;+\; (-1800-105a)\,t^4 \\ & \;+\; (-600-49a)\,t^2 \;+\; (-24-2a). \end{aligned} \tag{8.41}$$

By counting the zeros of $q_a(t)$ in $(0,\infty)$ (e.g., by the Sturm method) we find that $q_1(t)$ and $q_{-14}(t)$ have no zeros in that interval. Also we have that $q_a(1) = -192-2304a$, hence $q_1(1) = -2496$ and $q_{-14}(1) = 384$, which implies $q_1(t) < 0 < q_{-14}(t)$ for $t > 0$. Thus, for $a = 1$ the expression inside the curly brackets is increasing in $(0,1)$, vanishes at 0, and is decreasing in $(1,\infty)$, hence it is negative in $(0,\infty)\setminus\{1\}$. For $a = -14$ it is the other way around. Since the factor outside the curly brackets is positive in $(0,\infty)\setminus\{1\}$, and $f_a(1) = -12-a$, we get that for every $t > 0$:

$$f_1(t) \;<\; 0 \;<\; f_{-14}(t). \tag{8.42}$$

From here the announced result follows. $\square$

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Vita

Miguel Angel Lerma was born in Madrid, Spain, on March 31, 1954, the son of Miguel Lerma and Pilar Usero. After completing his High School studies in Madrid, Spain, in 1971, he entered the Universidad Complutense in Madrid. He received degree of Licenciado (B. S.) in Physics from the Universidad Complutense in 1977, and in Mathematics from the same university in 1978. The next year he got a Lecturer position for one year at the department of Mathematics in the Universidad Complutense, and a tenured position as a Math Teacher in High School. In 1989 he started graduate studies in the department of Computer Science of the Universidad Politécnica in Madrid. The next year he got a position as an untenured Assistant Professor in the Department of Computer Science of the Universidad Politécnica, where he remained until 1993. In 1991 he received a Doctor (Ph. D.) degree in Computer Science. In September 1993, he entered the Graduate School of the University of Texas at Austin.

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