SOME RESULTS IN ANALYSIS

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Abstract. This is a list of selected mathematical results placed here to have them handy.

Contents

1. Introduction

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2. Inequalities

2.1. **Inequalities.**

2.1.1. *Abel.* If $u_1 \geq u_2 \geq u_3 \geq \cdots \geq u_p \geq 0$, then

$$\left|\sum_{n=1}^{p} a_n u_n\right| \leq u_1 \max_{1\leq k\leq p} \left|\sum_{n=1}^{k} a_n\right| . \tag{2.1}$$

2.1.2. *Bessel.*

$$|\mathbf{u}|^2 \geq \sum_{k=1}^{n} |\mathbf{u}\cdot\mathbf{x}_k|^2 , \tag{2.2}$$

where $\{\mathbf{x}_k\}_{k=1,\ldots,n}$ is orthogonal normalized, i.e., $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$ (Kronecker delta) for $i, j = 1, 2, \ldots, n$.

2.1.3. *Cauchy.* (Hölder for $p = q = 2$.)

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 . \tag{2.3}$$

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \left|\sum_{i=1}^{n} a_i\right|^2 \left|\sum_{i=1}^{n} b_i\right|^2 . \tag{2.4}$$

2.1.4. *Chebyshev.*

$$P\left((x-\mu)^2 \geq k^2\right) \leq \frac{\langle (x-\mu)^2\rangle}{k^2} = \frac{\sigma^2}{k^2} , \tag{2.5}$$

where $P(S)$ = probability of event S, and $\langle f\rangle$ = mean value of f, $\mu = \langle x\rangle$ = mean value of x, σ^2 = variance of x.

2.1.5. *Hadamard.*

$$\left\|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\right\|^2 \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2\right) \tag{2.6}$$

2.1.6. *Hausdorff-Young.* If $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$q^{1/2q}\, \|\widehat{f}\|_q \leq p^{1/2p}\, \|f\|_p . \tag{2.7}$$

2.1.7. *Hölder.* If $p > 1$ and $1/p + 1/q = 1$ then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q \,. \tag{2.8}$$

Particular cases:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/q} , \tag{2.9}$$

$$\left|\int_\Omega \overline{f} g \, d\mu\right| \leq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^q \, d\mu\right)^{1/q} . \tag{2.10}$$

2.1.8. *Jensen.* If $\mu(\Omega) = 1$, $f : \Omega \to (a, b)$ in $L^1(\mu)$, φ convex on (a, b), then

$$\varphi\left(\int_\Omega f \, d\mu\right) \leq \int_\Omega (\varphi \circ f) \, d\mu \,. \tag{2.11}$$

If φ is convex on (a, b), $x_1, x_2, \ldots, x_n \in (a, b)$, $\lambda_i \geq 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i \varphi(x_i) \,. \tag{2.12}$$

If $a_i > 0$ $(i = 1, 2, \ldots, n)$, $s > t > 0$, then

$$\left(\sum_{i=1}^{n} a_i^s\right)^{1/s} \leq \left(\sum_{i=1}^{n} a_i^t\right)^{1/t} . \tag{2.13}$$

2.1.9. *Markov.* If $a > 0$ and the random variable x takes only non negative values, then

$$P(x \geq a) \leq \frac{\langle x \rangle}{a} , \tag{2.14}$$

where $\langle x \rangle$ = mean value of x.

2.1.10. *Minkowski.* If $p \geq 1$ then

$$\|\mathbf{u} + \mathbf{v}\|_p \leq \|\mathbf{u}\|_p + \|\mathbf{v}\|_p \,, \tag{2.15}$$

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p} , \tag{2.16}$$

$$\left(\int_\Omega |f + g|^p \, d\mu\right)^{1/p} \leq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} + \left(\int_\Omega |g|^p \, d\mu\right)^{1/p} . \tag{2.17}$$

2.1.11. *Newton.* Let p_r be the average value of the $\binom{n}{r}$ terms comprising the rth elementary symmetric function b_r of a set of numbers $a_1, \dots, a_n$ $(0 \le r \le n)$, i.e.,

$$\prod_{i=1}^{n}(x - a_i) = \sum_{k=0}^{n}(-1)^k \, b_k(a_1, \dots, a_n) \, x^{n-k} \,, \quad p_r = b_r / \binom{n}{r} \,.$$

Then:

$$p_{r-1} \, p_{r+1} \le p_r^2 \,. \tag{2.18}$$

2.1.12. *Schwarz.* (Hölder with $p = q = 2$)

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \,, \tag{2.19}$$

$$\left| \sum_{i=1}^{n} a_i b_i \right|^2 \le \left(\sum_{i=1}^{n} |a_i|^2 \right) \left(\sum_{i=1}^{n} |b_i|^2 \right) \,, \tag{2.20}$$

$$\left| \int_\Omega \overline{f} g \, d\mu \right|^2 \le \left(\int_\Omega |f|^2 \, d\mu \right) \left(\int_\Omega |g|^2 \, d\mu \right) \,. \tag{2.21}$$

2.1.13. *Young's Theorem.* Assume $p, q, r \in [1, \infty]$ and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \,.$$

If $f \in L^p$ and $g \in L^q$, then their convolution $f * g$ exists and belongs to L^r. Moreover:

$$\|f * g\|_r \le \|f\|_p \|g\|_q \,.$$

2.1.14. *Young.* If f is continuous, strictly increasing for $x \ge 0$, $f(0) = 0$, $g = f^{-1}$, then for $a, b \ge 0$:

$$ab \le \int_0^a f(x) \, dx + \int_0^b g(y) \, dy \,. \tag{2.22}$$

The equality holds for $b = f(a)$.

3. Some Identities

3.1. **Infinite Sums.**

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z - n} \,. \tag{3.1}$$

$$\left(\frac{\pi}{\sin \pi z} \right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2} \tag{3.2}$$

$$\left(\frac{\pi}{\cos \pi z} \right)^2 = \sum_{n=0}^{\infty} \left\{ \frac{1}{\left(z - n - \frac{1}{2}\right)^2} + \frac{1}{\left(z + n + \frac{1}{2}\right)^2} \right\} \tag{3.3}$$

$$\text{(3.4)}\qquad \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\frac{m\sin mx}{m^2-\alpha^2}=\frac{\sin\{\alpha(\pi-x)\}}{\sin\alpha\pi}\,.$$

$$\text{(3.5)}\qquad \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\frac{\cos mx}{m^2-\alpha^2}=-\frac{\cos\{\alpha(\pi-x)\}}{\alpha\sin\alpha\pi}\,.$$

$$\text{(3.6)}\qquad \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\cos\{(2m+1)x\}}{(2m+1)^2-\alpha^2}=\frac{\cos\{\frac{\alpha}{2}(\pi-x)\}}{2\alpha\sin\frac{\alpha\pi}{2}}-\frac{\cos\{\alpha(\pi-x)\}}{\alpha\sin\alpha\pi}\,.$$

3.2. **Infinite Products.**

$$\text{(3.7)}\qquad \frac{\sin\pi z}{\pi z}=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)=\lim_{N\to\infty}\prod_{\substack{n=-N\\ n\neq 0}}^{N}\left(1-\frac{z}{n}\right)\,.$$

$$\text{(3.8)}\qquad \cos\pi z=\prod_{n=0}^{\infty}\left(1-\frac{z^2}{(n+\frac{1}{2})^2}\right)=\lim_{N\to\infty}\prod_{n=-N}^{N}\left(1-\frac{z}{n+\frac{1}{2}}\right)\,.$$

$$\text{(3.9)}\qquad \begin{aligned}\frac{\sin\{\pi(z-\alpha)\}}{-\sin(\pi\alpha)}&=\left(1-\frac{z}{\alpha}\right)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n+\alpha)^2}\right)\\ &=\lim_{N\to\infty}\prod_{n=-N}^{N}\left(1-\frac{z}{n+\alpha}\right)\,.\end{aligned}$$

$$\text{(3.10)}\qquad \begin{aligned}\frac{\cos\{\pi(z-\alpha)\}}{\cos(\pi\alpha)}&=\prod_{n=0}^{\infty}\left(1-\frac{z^2}{(n+\frac{1}{2}+\alpha)^2}\right)\\ &=\lim_{N\to\infty}\prod_{n=-N}^{N}\left(1-\frac{z}{n+\frac{1}{2}+\alpha}\right)\,.\end{aligned}$$

4. Functions, Measure, Integration

4.1. **Lebesgue Integral.**

Theorem 4.1 (Lebesgue's Monotone Convergence [LMCT])**.**

Let $f_n : X \to [0,\infty]$ be a nondecreasing sequence of measurable functions. Then their limit is measurable and

$$\int_X \lim_{n\to\infty} f_n\,d\mu=\lim_{n\to\infty}\int_X f_n\,d\mu\,.$$

Corollary 4.2. *If $f_n : X \to [0, \infty]$ is measurable for every n,*

$$\int_X \sum_{n=0}^{\infty} f_n \, d\mu = \sum_{n=0}^{\infty} \int_X f_n \, d\mu \, .$$

Theorem 4.3 (Fatou's Lemma)**.** *If $f_n : X \to [0, \infty]$ is measurable for every n,*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu \, .$$

Theorem 4.4 (Lebesgue's Dominated Convergence [LDCT])**.**

Let $f_n : X \to \mathbb{C}$ be measurable, $f(x) = \lim_{n\to\infty} f_n(x)$, and $|f_n(x)| \le g(x)$ for some function $g \in L^1(X)$. Then $f \in L^1(X)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 \, ,$$

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu \, .$$

Theorem 4.5. *Let X, Y be σ-finite measure spaces.*[1]

(1) *(Tonelli) If $f : X \times Y \to [0, \infty]$ is measurable then for a.e. $y \in Y$ the function $x \mapsto f(x, y)$ is measurable and*

$$\int_Y \left\{ \int_X f(x, y) \, dx \right\} \, dy = \int_{X\times Y} f(x, y) \, dxdy \, .$$

(2) *(Fubini) If f is in $L^1(X \times Y)$ then for a.e. $y \in Y$ the function $x \mapsto f(x, y)$ is in $L^1(X)$, the function $y \mapsto \int_X f(x, y) \, dx$ is in $L^1(Y)$ and*

$$\int_Y \left\{ \int_X f(x, y) \, dx \right\} \, dy = \int_{X\times Y} f(x, y) \, dxdy \, .$$

(The integrals are respect to the corresponding measures in X, Y or the product space $X \times Y$.)

4.1.1. *L^p and l^p spaces.*

- $L^p(X)$ = set of functions $f : X \to \mathbb{C}$ such that $\int_X |f|^p \, d\mu < \infty$, where $\int$ is the Lebesgue integral respect to the measure μ in the measure space X.
- l^p = set of complex sequences $\{c_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{\infty} |c_n|^p < \infty$.

4.2. **Various Definitions and Properties of Functions.**

[1] A measure space is σ-finite if it is a countable union of sets of finite measure.

4.2.1. *Variation.* The variation of a function $f : [a, b] \to \mathbb{R}^n$ over a compact interval $I = [a, b]$ is

$$V_I(f) = V_a^b(f) = \sup\left\{\sum_{i=0}^{n} |f(x_i) - f(x_{i+1})| : a = x_0 \leq x_1 \leq \cdots \leq x_n = b\right\} \tag{4.1}$$

It is of *bounded variation* if $V_a^b(f) < \infty$. Notation: $BV(I)$ = set of functions of bounded variation in I.

A function $f : [a, b] \to \mathbb{R}$ has bounded variation iff it is the difference of two nondecreasing functions.

Over non compact (including unbounded) intervals the definition can be generalized like this:

$$V_a^b(f) = \lim_{\substack{a' \to a^+ \\ b' \to b^-}} V_f(a', b')$$

A function is *locally of bounded variation* if it is of bounded variation over every compact interval in its domain.

4.2.2. *Absolutely Continuous.* A function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for every $\epsilon > 0$ there is a δ such that for any non-overlapping collection of subintervals $[x_i, y_i] \subseteq [a, b]$ such that $\sum_{i=1}^{m}(y_i - x_i) < \delta$, then $\sum_{i=1}^{m} |f(y_i) - f(x_i)| < \varepsilon$.

Notation: $C(I)$ = set of continuous functions in I. $AC(I)$ = set of absolutely continuous functions in I.

Any absolutely continuous function is continuous and has bounded variation.

A function is absolutely continuous iff it is an indefinite integral of a Lebesgue-integrable function.

4.2.3. *Singular Function.* A *singular function* is a function whose derivative is zero a.e.

4.2.4. *Jump Function.* An *elementary nondecreasing (resp. nonincreasing) jump function* is a function of the form

$$h(x) = \begin{cases} A & \text{if } x < x_0 \\ B & \text{if } x = x_0 \\ C & \text{if } x > x_0 \end{cases}$$

where $A \leq B \leq C$ (resp. $A \geq B \geq C$). We also assume $A \neq C$. A *jump function* is a function than can be expressed as an absolutely convergent series of elementary jump functions:

$$f(x) = \sum_{n=0}^{\infty} h_n(x)\,, \tag{4.2}$$

A *jump function* is the difference of two nondecreasing jump functions, hence it is locally of bounded variation.

A *step function* is a sum of finitely many elementary jump functions.

4.2.5. *Regulated Function.* A *regulated function* is a function that has all of its one-sided limits at all points of its domain. A function is regulated iff it is the uniform limit of step functions. All functions of bounded variation are regulated, but the converse is not true.

4.2.6. *Lebesgue Decomposition Theorem.* Every function of bounded variation $f : [a, b] \to \mathbb{R}$ can be expressed in the form:

$$f = f_a + f_s + f_j \,, \tag{4.3}$$

where f_a is *absolutely continuous*, f_s is *continuous singular*, and f_j is a *jump* function. These functions are unique to within additive constants.

If f is nondecreasing (resp. nonincreasing), f_a, f_s and f_j can be chosen to be nondecreasing (resp. nonincreasing). In this case the function is absolutely continuous iff

$$\int_a^b f'(x)\,dx = f(a) - f(b)\,.$$

4.2.7. *Absolutely integrable.* A function $f : \mathbb{R} \to \mathbb{C}$ is *absolutely integrable*, or in $L^1(\mathbb{R})$, if f is integrable and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$.

4.2.8. *Locally integrable.* A function f is *locally integrable*, or in L^1_{loc}, if at every point of its domain there is a neighborhood where f is integrable and its integral is finite.

4.2.9. *Convex.* A function $f : (a, b) \to \mathbb{R}$ is *convex* if for every $x, y \in (a, b)$, $0 \leq \lambda \leq 1$:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)\,.$$

4.3. **Henstock-Kurzweil Integral.**

4.3.1. *Partitions.* Given a nondegenerate closed interval $I = [a, b]$, a *partition* or *division* of I a finite collection $\mathcal{P} = \{I_i \mid i = 1, \dots, n\}$ of intervals $I_i = [x_{i-1}, x_i]$ with $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$. A *tagged partition* of I is a collection $\dot{\mathcal{P}} = \{(I_i, t_i) \mid i = 1, \dots, n\}$, where $\mathcal{P} = \{I_i \mid i = 1, \dots, n\}$ is a partition of I and $t_i \in I_i$ for each i.

4.3.2. *Riemann Sums.* Given a function $f : I = [a, b] \to \mathbb{R}$ and a tagged partition $\dot{\mathcal{P}} = \{(I_i, t_i) \mid i = 1, \dots, n\}$ of I, the *Riemann sum* of f corresponding to $\dot{\mathcal{P}}$ is

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)\, l(I_i)\,,$$

where $l(I_i) = b - a$ is the length of I_i.

4.3.3. *Riemann Integral.* The *Riemann integral* of a function $f : I = [a, b] \to \mathbb{R}$, is the number

$$\int_a^b f(x)\,dx = \lim_{|\dot{\mathcal{P}}|\to 0} S(f;\dot{\mathcal{P}})\,, \tag{4.4}$$

where $|\dot{\mathcal{P}}| = \max_{I_i \text{ in } \dot{\mathcal{P}}} l(I_i)$. In other words, for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\dot{\mathcal{P}} = \{(I_i, t_i) \mid i = 1, \dots, n\}$ is a tagged partition of I verifying $l(I_i) \leq \delta$ for $i = 1, \dots, n$, then

$$\left| \int_a^b f(x)\,dx - S(f;\dot{\mathcal{P}}) \right| \leq \varepsilon\,.$$

4.3.4. *Gauges.* A *gauge* on $I = [a, b]$ is a function $\delta : I \to (0, \infty)$. A tagged partition $\dot{\mathcal{P}} = \{(I_i, t_i) \mid i = 1, \dots, n\}$ of I is *δ-fine* if $I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for all $I = 1, \dots, n$.

4.3.5. *Henstock-Kurzweil Integral.* (See [7, 2]). The *Henstock-Kurzweil integral* (or *gauge integral*, or *generalized Riemann integral*) of a function $f : I = [a, b] \to \mathbb{R}$, is the number $\int_a^b f(x)\,dx$ verifying that for every $\varepsilon > 0$ there is gauge $\delta : I \to (0, \infty)$ such that if $\dot{\mathcal{P}}$ is a δ-fine tagged partition of I, then

$$\left| \int_a^b f(x)\,dx - S(f;\dot{\mathcal{P}}) \right| \leq \varepsilon\,.$$

Note: The Riemann integral coincides with the Henstock-Kurzweil integral restricted to constant gauges.

Both the Riemann and the Henstock-Kurzweil integrals can easily be generalized to functions $f : I = [a, b] \to \mathbb{R}^n$

The Henstock-Kurzweil integral can be generalized to functions defined on unbounded intervals by allowing a and b to be infinity, and omitting in the Riemann sums the terms corresponding to unbounded subintervals. Alternatively define $f(\infty) = f(-\infty) = 0$ and adopt the conventions $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$, $(\pm\infty) + x = x + (\pm\infty) = \pm\infty$, $(\pm\infty) \cdot x = x \cdot (\pm\infty) = \pm\infty$ or $\mp\infty$ depending on whether $x > 0$ or $x < 0$.

4.3.6. *Properties of the Integral.* Here $f, g : \mathbb{R} \to \mathbb{R}$ (although some properties are obviously generalizable to vector-valued functions). The integral is assumed to be Henstock-Kurzweil unless stated otherwise.

(1) *Linearity.*

$$\int_I (f + g) = \int_I f + \int_I g$$

$$\int_I cf = c \int_I f$$

(2) *Interval additivity.*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

(3) *Squeeze Theorem.* f is integrable over I if and only if for every $\varepsilon > 0$ there are two integrable functions ψ_ε, φ_ε integrable over I, such that $\psi_\varepsilon(x) \le f(x) \le \varphi_\varepsilon(x)$ for every $x \in I$ and

$$\int_I (\psi_\varepsilon - \varphi_\varepsilon) \le \varepsilon \,.$$

(4) *Change of Variable.* Let $\tau : [c,d] \to [a,b]$ be continuous and monotone. If $\int_a^b f\,d\alpha$ exists then $\int_a^b f \circ \tau \, d(\alpha \circ \tau)$ exists and

$$\int_a^b f \circ \tau \, d(\alpha \circ \tau) = \int_{\tau(c)}^{\tau(d)} f \, d\alpha \,. \tag{4.5}$$

Theorem 4.6 (Fundamental Theorem of Calculus)**.** *Given $f : [a,b] \to \mathbb{R}$, if f is integrable over I and*

$$F(x) = \int_a^x f(u)\,du \,,$$

then F is continuous on I and $F'(x) = f(x)$ for a.e. x in $[a,b]$.

If f has a right (left) hand limit at $x \in [a,b)$, then

$$f(x\pm) = \lim_{h \to 0^\pm} \frac{F(x+h) - F(x)}{h} \,.$$

Theorem 4.7. *A function f is Lebesgue-integrable iff f and $|f|$ are Henstock-Kurzweil-integrable.*

4.3.7. *Stieltjes Integral.* A Stieltjes integral of a function f respect to another function φ, represented

$$\int_a^b f(x)\,d\varphi(x) \,, \tag{4.6}$$

is defined as in the previous paragraphs replacing Riemann sums with Riemann-Stieltjes sums:

$$\Sigma(f, \varphi; \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)\,[\varphi(b) - \varphi(a)] \,.$$

It this way it is possible to define the *Riemann-Stieltjes* integral, and the *Henstock-Kurzweil-Stieltjes* (or *generalized Riemann-Stieltjes*) integral.

In the following theorem we use the notation

$$\Delta_-(f;c) = f(c) - f(c-)$$
$$\Delta_+(f;c) = f(c+) - f(c)$$
$$\Delta(f;c) = f(c+) - f(c-)$$

Theorem 4.8 (Integration by Parts for the Henstock-Kurzweil-Stieltjes Integral)**.** *Suppose that one of f, φ is a function of bounded variation and the other one is a regulated function over $[a,b]$. Then both $\int_a^b f(x)\,d\varphi(x)$ and $\int_a^b \varphi(x)\,df(x)$ exist and*

$$\int_a^b f(x)\,d\varphi(x) = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi(x)\,df(x) + S\,, \tag{4.7}$$

where

$$\begin{aligned} S &= \sum_n \{\Delta_-(f;c_n)\ \Delta_-(\varphi;c_n) - \Delta_+(f;c_n)\ \Delta_+(\varphi;c_n)\} \\ &= \sum_n \{f(c_n)\ \Delta(\varphi;c_n) + \varphi(c_n)\ \Delta(f;c_n) - \Delta(f\varphi;c_n)\}\,. \end{aligned} \tag{4.8}$$

Here the c_n's are the points at which f and φ are simultaneously discontinuous from the left or from the right (see [7, chap. 7]*.)*

4.4. **Summability.**

4.4.1. *Cesàro Summability.* A series $\sum_{n=1}^{\infty} a_n$ is summable $(C,1)$ to A if

$$A = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S_n = \lim_{N\to\infty} \sum_{n=1}^{N}\left(1 - \frac{n-1}{N}\right) a_n\,,$$

where $S_n = \sum_{i=1}^{n} a_i$.

A series $\sum_{n=1}^{\infty} a_n$ is summable (C,k) to A if

$$A = \lim_{n\to\infty} \frac{T_n^{(k)}}{\binom{n+k-1}{k}}\,,$$

where $T_n^{(k)}$ is defined recursively in the following way:

$$\begin{aligned} T_n^{(0)} &= S_n = \sum_{i=1}^{n} a_i \\ T_n^{(j)} &= \sum_{i=1}^{n} T_i^{(j-1)} \qquad (j>0) \end{aligned}$$

Equivalently:

$$A = \lim_{N\to\infty} \frac{1}{\binom{N+k-1}{k}} \sum_{n=1}^{N} \binom{N+k-n}{k} a_n\,.$$

4.4.2. *Hölder Summability.* A series $\sum_{n=1}^{\infty} a_n$ is summable $(H,1)$ to A if

$$A = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S_n = \lim_{N\to\infty} \sum_{n=1}^{N}\left(1-\frac{n-1}{N}\right) a_n \,,$$

where $S_n = \sum_{i=1}^{n} a_i$.

A series $\sum_{n=1}^{\infty} a_n$ is summable (H,k) to A if

$$A = \lim_{N\to\infty} U_n^{(k)} \,,$$

where $U_n^{(k)}$ is defined recursively in the following way:

$$\begin{aligned} U_n^{(0)} &= S_n = \sum_{i=1}^{n} a_i \\ U_n^{(j)} &= \frac{1}{n}\sum_{i=1}^{n} U_i^{(j-1)} \quad (j>0) \end{aligned}$$

Equivalently:

$$A = \lim_{N\to\infty} \sum_{n=1}^{N} c_n a_n \,.$$

where

$$\begin{aligned} c_n &= \sum_{n\le i_0\le i_1\le\cdots\le i_{k-1}\le N} \frac{1}{i_1 i_2 \ldots i_{k-1} N} \\ &= \sum_{n\le i_1\le\cdots\le i_{k-1}\le i_k = N} \frac{i_1-n+1}{i_1 i_2 \ldots i_k} \end{aligned}$$

5. Fourier Series

5.1. **Definition and properties.** Note: There are two approaches to Fourier series, depending on whether we are interested in studying 1-periodic functions, defined on

$$\mathbb{T}_1 = \mathbb{R}/\mathbb{Z}\,, \tag{5.1}$$

or 2π-periodic functions, defined on

$$\mathbb{T}_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}\,. \tag{5.2}$$

Both $\mathbb{T}_1$ and $\mathbb{T}_{2\pi}$ are endowed with a uniform measure such that their total measure is 1, so if f is an integrable function defined in either $\mathbb{T}_1$ or $\mathbb{T}_{2\pi}$ we have

$$\int_{\mathbb{T}_1} f(x)\,dx = \int_0^1 f(x)\,dx\,, \tag{5.3}$$

$$\int_{\mathbb{T}_{2\pi}} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\,. \tag{5.4}$$

The second approach is perhaps the most common, but here we will use the first one because it provides simpler and more symmetric formulas, so some of the usual functions and classical kernels in Fourier analysis will be replaced with the ones obtained by the variable change $x \to 2\pi x$.

Given $f : \mathbb{T}_1 \to \mathbb{C}$, we define the nth *Fourier coefficient* of f:

$$\widehat{f}(n) = \int_{\mathbb{T}_1} f(x)\,e^{-2\pi inx}\,dx\,. \tag{5.5}$$

The *Fourier series* of f is[2]

$$s[f](x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\,e^{2\pi inx} = \lim_{N\to\infty} s_N[f](x)\,, \tag{5.6}$$

where

$$s_N[f](x) = \sum_{n=-N}^{N} \widehat{f}(n)\,e^{2\pi inx} \tag{5.7}$$

The Cesáro means of the Fourier series are

$$\sigma_N[f](x) = \frac{1}{N+1}\sum_{k=0}^{N} s_k[f](x) = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)\widehat{f}(n)\,e^{2\pi inx}\,. \tag{5.8}$$

The concept of Fourier series can be extended to distribution functions in the following way. Let $F : \mathbb{R} \to \mathbb{C}$ be a function such that $F(x+1) - F(x) = F(1) - F(0)$ for every $x \in \mathbb{R}$. The *Fourier-Stieltjes series* of F, or *Fourier series* of dF is defined

$$s[dF](x) = \sum_{n=-\infty}^{\infty} \widehat{dF}(n)\,e^{2\pi inx}\,, \tag{5.9}$$

where

$$\widehat{dF}(n) = \int_0^1 e^{-2\pi inx}\,dF(x) \tag{5.10}$$

[2]When both limits in a sum are infinite we assume that the index approaches infinity in a symmetric way:

$$\sum_{k=-\infty}^{\infty} f(k) = \lim_{T\to\infty}\sum_{k=-T}^{T} f(k)\,.$$

is the nth *Fourier-Stieltjes coefficient* of F or *Fourier coefficient* of dF—the integral being of Stieltjes type.

For instance the periodic Dirac delta corresponds to the distribution function $F(x) = \lfloor x \rfloor =$ integer part of x. Its Fourier coefficients are so:

$$\widehat{dF}(n) = 1 \quad \text{for every } n\,,$$

and its Fourier series is

$$s[dF](x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x}\,.$$

5.1.1. *Properties.*

(1) *Linearity.*
$$\widehat{af+bg} = a\widehat{f} + b\widehat{g}\,.$$
(2) *Translation.* If $(T_y f)(x) = f(x+y)$ (translated of f by y), then:
$$\widehat{T_y f}(n) = \widehat{f}(n)\, e^{2\pi i y}\,.$$
(3) *Modulation.* If $m \in \mathbb{Z}$ and $(E_m f)(x) = e^{2\pi i m x} f(x)$, then
$$\widehat{E_m f}(n) = \widehat{f}(n-m)\,.$$
(4) *Complex Conjugation.*
$$\widehat{\overline{f}}(n) = \overline{\widehat{f}(-n)}\,.$$
(5) *Differentiation.* If f is differentiable over $\mathbb{T}_1$ then:
$$\widehat{f'}(n) = 2\pi i n \widehat{f}(n)\,.$$

5.1.2. *Convolution.* The convolution of two functions $f, g : \mathbb{T}_1 \to \mathbb{C}$ is

$$(f * g)(x) = \int_{\mathbb{T}_1} f(u) g(x-u)\, du\,. \tag{5.11}$$

If $f, g \in L^1(\mathbb{T}_1)$ then $\widehat{f*g} \in L^1(\mathbb{T}_1)$ and

$$\widehat{f*g} = \widehat{f}\,\widehat{g}\,. \tag{5.12}$$

Note:

$$s_N[f] = f * D_N\,,$$

and

$$\sigma_N[f] = \frac{1}{N+1} \sum_{k=0}^{N} s_k[f] = f * F_N\,,$$

where D_N and F_N are respectively the Dirichlet and Fejér kernels (see bellow).

5.2. **Some Theorems.**

5.2.1. *Inversion Theorems.*

Proposition 5.1. *If c_n is nonincreasing and $c_n \to 0$ then the series*

$$\sum_{n=-\infty}^{\infty} c_n \, e^{2\pi i n x}$$

converges uniformly on compact subsets of $\mathbb{T}_1 \setminus \{0\}$.

Proposition 5.2. *If $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ then the series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{2\pi i n x}$$

converges uniformly on $\mathbb{T}_1$ (so f is continuous) and $c_n = \widehat{f}(n)$.

Theorem 5.3 (Fourier Inversion Theorem)**.** *Assume $f \in L^1(\mathbb{T}_1)$ and $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$. Then $s[f]$ is continuous and $f(x) = s[f](x)$ a.e. If f is continuous then $f(x) = s[f](x)$ for every x.*

Theorem 5.4. *If $f : \mathbb{T}_1 \to \mathbb{C}$ is bounded and verifies the Dirichlet conditions:*

(1) *f is continuous except possibly for a finite number of jump discontinuities,*
(2) *f has a finite number of maxima and minima,*

then (in Cesáro sense)

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \, e^{2\pi i n x} \,,$$

provided that $f(x) = \frac{1}{2}\{f(x+) + f(x-)\}$.

Note that the summation converges in Cesàro sense, i.e.:

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) \, e^{2\pi i n x} \,.$$

Theorem 5.5. *If $f \in L^1(\mathbb{T}_1)$, $f(x) = \frac{1}{2}\{f(x+) + f(x-)\}$, and for some constant C*

$$|f(y) - f(x\pm)| \le \pm C(y - x)$$

for all y in a neighborhood of x, then

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n) \, e^{2\pi i n x} \,.$$

Corollary 5.6. *If f satisfies a Lipschitz condition on $\mathbb{T}_1$ then its Fourier series converges uniformly to f.*

Theorem 5.7 (Dirichlet-Jordan Test)**.** *Suppose $f \in BV(\mathbb{T}_1)$. Then*

(1) *$s[f](x) = \frac{1}{2}\{f(x+) + f(x-)\}$ for every $x \in \mathbb{T}_1$.*

(2) *If f is continuous in a closed interval I, $s[f]$ converges uniformly in I.*

Proof. See [11, thm. 8.1]. □

Theorem 5.8 (Cesáro Sumability of Fourier Series)**.** *The Cesàro means of the Fourier series of f converge to f in the following ways.*

(1) $f \in C(\mathbb{T}) \Rightarrow \sigma_N[f] \to f$ *uniformly.*
(2) $f \in L^1(\mathbb{T})$ *and* f *continuous at* $x_0 \Rightarrow \sigma_N[f](x_0) \to f(x_0)$.
(3) $f \in L^p(\mathbb{T}) \Rightarrow \sigma_N[f] \to f$ *in* $L^p(\mathbb{T})$ *for* $1 \leq p < \infty$.

5.2.2. *Other Theorems.*

Theorem 5.9. *The mapping $f \leftrightarrow \widehat{g}$ is a Hilbert space isomorphism between $L^2(\mathbb{T}_1)$ and $l^2(\mathbb{Z})$. In particular:*

- *(Riesz-Fischer Theorem) If* $\displaystyle\sum_{n=0}^{\infty} |c_n|^2 < \infty$ *then there is an* $f \in L^2(\mathbb{T}_1)$ *such that*

$$c_n = \int_{\mathbb{T}_1} f(x)\, e^{-2\pi i n x}\, dx\,.$$

- *(Parseval Theorem) If* $f, g \in L^2(\mathbb{T}_1)$ *then*

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} = \int_{\mathbb{T}_1} f(x)\overline{g(x)}\, dx\,.$$

Also, if $f \in L^2(\mathbb{T}_1)$ *then* $\displaystyle\lim_{N\to\infty} \|f - s_N[f]\|_2 = 0$.

5.3. **Some Kernels.**

5.3.1. *Dirichlet Kernel.*

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin\{2\pi(N+\frac{1}{2})x\}}{\sin \pi x}\,.$$

Note:

$$D_N(x) = N \lim_{T\to\infty} \sum_{n=-T}^{T} D\{N(x+n)\}\,,$$

where

$$D(x) = \int_{-1}^{1} e^{2\pi i x t}\, dt = \frac{\sin 2\pi x}{\pi x}$$

is the non-periodic Dirichlet Kernel.

5.3.2. *Fejér Kernel.*

$$\begin{aligned} F_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) \\ &= \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right) e^{2\pi i n x} \\ &= \frac{1}{N+1}\left(\frac{\sin\{\pi(N+1)x\}}{\sin \pi x}\right)^2 . \end{aligned}$$

Note:

$$F_N(x) = (N+1)\sum_{n=-\infty}^{\infty} K\{(N+1)(x+n)\},$$

where

$$K(x) = \int_{-1}^{1} (1-|t|)\, e^{2\pi i x t}\, dt = \left(\frac{\sin \pi x}{\pi x}\right)^2$$

is the non-periodic Fejér Kernel.

5.3.3. *Poisson Kernel.* For $0 < r < 1$, the 2π-periodic *Poisson kernel* $P_r : \mathbb{T}_{2\pi} \to \mathbb{C}$ is

$$\begin{aligned} P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{in\theta} = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \\ &= \frac{1-r^2}{1-2r\cos\theta + r^2} = \Re\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}}\right\} . \end{aligned} \tag{5.13}$$

The 1-periodic version of the Poisson kernel is

$$P_r^{(\text{1-per})}(\alpha) = P_r(2\pi\alpha) = \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{2\pi i n \alpha} . \tag{5.14}$$

In the upper half plane $\mathcal{H}^{(+)} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ the Poisson kernel is defined

$$P^{(+)}(x+yi) = \Re\left\{\frac{i}{\pi z}\right\} = P_y^{(+)}(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} , \tag{5.15}$$

where $z = x + yi$. Also:

$$P_y^{(+)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|t|}\, e^{ixt}\, dt = \int_{-\infty}^{\infty} e^{-2\pi y|t|}\, e^{2\pi i x t}\, dt . \tag{5.16}$$

Another relation with the periodic Poisson kernel is

$$\begin{aligned} \frac{1}{2\pi} P_r(\theta) &= \sum_{n=-\infty}^{\infty} P^{(+)}\left(\theta + 2\pi n + i \log \tfrac{1}{r}\right) \\ &= \sum_{\substack{\text{all branches} \\ \text{of log}}} P^{(+)}\left(-i\log\{re^{i\theta}\}\right) \end{aligned} \tag{5.17}$$

or

$$P_r^{(\text{1-per})}(\alpha) = \sum_{n=-\infty}^{\infty} P^{(+)}\left(\alpha + n + \tfrac{i}{2\pi}\log\tfrac{1}{r}\right) \tag{5.18}$$

$$= \sum_{\substack{\text{all branches}\\ \text{of log}}} P^{(+)}\left(-\frac{i}{2\pi}\log\left\{re^{2\pi i\alpha}\right\}\right),$$

where $\alpha = \theta/2\pi$.

5.3.4. *Conjugate Poisson Kernel.*

For $0 < r < 1$, the *conjugate Poisson kernel* $Q_r : \mathbb{T}_{2\pi} \to \mathbb{C}$ is

$$Q_r(\theta) = \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}\,(n)\, r^{|n|}\, e^{in\theta} = 2\sum_{n=1}^{\infty} r^n \sin n\theta \tag{5.19}$$

$$= \frac{2r\sin\theta}{1-2r\cos\theta+r^2} = \Im\left\{\frac{1+re^{i\theta}}{1-re^{i\theta}}\right\}.$$

6. Integral Transforms

6.1. **Fourier Transform.** Given a function $f : \mathbb{R} \to \mathbb{C}$, the Fourier transform of f is defined in the following way:

$$\mathcal{F}[f(x)](t) = \widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x t}\, dx\,. \tag{6.1}$$

The inverse Fourier transform of $F : \mathbb{R} \to \mathbb{C}$ is

$$\mathcal{F}^{-1}[F(t)](x) = \int_{-\infty}^{\infty} F(t)\, e^{2\pi i x t}\, dt\,. \tag{6.2}$$

Theorem 6.1 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1(\mathbb{R})$, then its Fourier transform $\widehat{f}(t)$ is defined everywhere, uniformly continuous, and $\lim_{t\to\pm\infty} \widehat{f}(t) = 0$.*

Theorem 6.2 (Inversion Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a locally integrable function, of bounded variation in a neighborhood of a point x_0, and such that*

$$f(x_0) = \frac{1}{2}\lim_{h\to 0}\left\{f(x_0+h) + f(x_0-h)\right\}.$$

If f satisfies either one of the following conditions:

1. $f \in L^1(\mathbb{R})$,
2. $x \mapsto f(x)/(1+|x|)$ *is in* $L^1(\mathbb{R})$, *and the Fourier integral of* f *at* x_0 *converges uniformly on every finite interval of* t,

then

$$f(x_0) = \lim_{T\to\infty}\int_{-T}^{T} \widehat{f}(t)\, e^{2\pi i x_0 t}\, dt\,. \tag{6.3}$$

6.1.1. *Properties of the Fourier Transform.*

(1) *Linearity.*

$$\mathcal{F}[af(x) + bg(x)](t) = a\widehat{f}(t) + b\widehat{g}(t)\,.$$

(2) *Scaling.* if $a \neq 0$,

$$\mathcal{F}[f(ax)](t) = \frac{1}{|a|}\widehat{f}\left(\frac{s}{a}\right)\,.$$

(3) *Shifting.*

$$\mathcal{F}[f(x-b)](t) = \widehat{f}(t)\,e^{-2\pi ibt}\,.$$

(4) *Complex conjugation.*

$$\mathcal{F}[\overline{f(x)}](t) = \overline{\widehat{f}(-t)}\,.$$

(5) *Modulation.*

$$\mathcal{F}[f(x)\,e^{2\pi iax}](t) = \widehat{f}(t-a)\,.$$

(6) *Differentiation.* If $f^{(k)} \in L^1(\mathbb{R})$ for $0 \leq k \leq n$ and $\lim_{|t|\to\infty} f^{(k)}(t) = 0$ for $0 \leq k \leq n-1$, then

$$\mathcal{F}[f^{(n)}(x)](t) = (2\pi it)^n\,\widehat{f}(t)\,.$$

(7) *Product by powers.* If $x \mapsto x^k f(x)$ is in $L^1(\mathbb{R})$, then

$$\mathcal{F}[x^k f(x)](t) = \frac{(-1)^k}{(2\pi i)^n}\,\widehat{f}^{(k)}(t)\,.$$

(8) *Convolution.* If $f, g \in L^1(\mathbb{R})$ then $\widehat{f * g} \in L^1(\mathbb{R})$, and

$$\mathcal{F}[fg] = \widehat{f} * \widehat{g}\,, \qquad \mathcal{F}[f * g] = \widehat{f}\,\widehat{g}\,.$$

Here

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\,.$$

6.1.2. *Parseval Theorem.* If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2\,dt \tag{6.4}$$

6.1.3. *Extension to $L^2(\mathbb{R})$.*

Theorem 6.3 (Plancheret Theorem)**.** *(See [9].) Given $f \in L^2(\mathbb{R})$ there is an $\widehat{f} \in L^2(\mathbb{R})$ such that*

(1) *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\widehat{f}$ is the usual Fourier transform of f.*
(2) *For every $f \in L^2(\mathbb{R})$, $\|\widehat{f}\|_2 = \|f\|_2$.*
(3) *The mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.*
(4) *If*

$$(\widehat{f}\,)_T(t) = \int_{-T}^{T} f(x)\,e^{-2\pi ixt}\,dx\,, \quad f_T(x) = \int_{-T}^{T} \widehat{f}(t)\,e^{2\pi ixt}\,dt\,,$$

then

$$\lim_{T\to\infty} \|\widehat{f} - (\widehat{f}\,)_T\|_2 = 0 \quad \textit{and} \quad \lim_{T\to\infty} \|f - f_T\|_2 = 0\,.$$

Theorem 6.4 (Paley-Wiener Theorem)**.** *Suppose $f \in L^2(\mathbb{R})$. Then f is the Fourier transform of a function vanishing outside $[-\sigma, \sigma]$ if and only if f is the restriction to $\mathbb{R}$ of an entire function of exponential type*[3] $2\pi\sigma$ *(see* [10, III.4, p. 108]*).*

Theorem 6.5 (Shannon's Sampling Theorem)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is a function band-limited to $[-\sigma, \sigma]$, i.e.,*

$$f(x) = \int_{-\sigma}^{\sigma} F(t)\, e^{2\pi i x t}\, dt$$

with $F \in L^2([-\sigma, \sigma])$, then it can be reconstructed from its values at the points $x_k = k/2\sigma$, $k \in \mathbb{Z}$, via the formula

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} f(x_k)\, \frac{D(\sigma(x - x_k))}{2} \\ &= \frac{\sin(2\pi\sigma x)}{2\pi\sigma} \sum_{k=-\infty}^{\infty} (-1)^k \frac{f(x_k)}{(x - x_k)}\,, \end{aligned} \tag{6.5}$$

where $D(x) = \sin 2\pi x / \pi x$ is the non-periodic Dirichlet kernel (see bellow), with the series absolutely convergent and uniformly convergent on compact sets.

Partial Proof. Since $\widehat{f}(t) = F(t)$ is supported in $[-\sigma, \sigma]$, for $|t| < \sigma$ we have:

$$F(t) = \sum_{n=-\infty}^{\infty} F(t + 2\sigma n)\,.$$

So, using the Poisson Summation formula:

$$\begin{aligned} F(t) &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} F(t + 2\sigma u)\, e^{2\pi i u k}\, du \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{2\sigma} \int_{-\infty}^{\infty} F(s)\, e^{2\pi i (s-t)k/2\sigma}\, ds \qquad (s = t + 2\sigma u) \\ &= \sum_{k=-\infty}^{\infty} \frac{e^{-2\pi i t k/2\sigma}}{2\sigma} \underbrace{\int_{-\infty}^{\infty} F(s)\, e^{2\pi i s k/2\sigma}\, ds}_{f(k/2\sigma)} \\ &= \sum_{k=-\infty}^{\infty} f(\tfrac{k}{2\sigma})\, \frac{e^{-2\pi i t k/2\sigma}}{2\sigma}\,, \end{aligned}$$

hence:

$$\begin{aligned} f(x) &= \int_{-\sigma}^{\sigma} F(t)\, e^{2\pi i x t}\, dt \\ &= \sum_{k=-\infty}^{\infty} f(\tfrac{k}{2\sigma}) \int_{-\sigma}^{\sigma} \frac{e^{2\pi i t (x - k/2\sigma)}}{2\sigma}\, dt \\ &= \sum_{k=-\infty}^{\infty} f(\tfrac{k}{2\sigma})\, \frac{D(\sigma(x - \frac{k}{2\sigma}))}{2}\,. \end{aligned}$$

[3] Section 7.1.2.

$\square$

6.1.4. *Some Fourier Transforms.*

(1) Non-periodic Dirichlet Kernel.

$$D(x) = \frac{\sin 2\pi x}{\pi x} \quad \Rightarrow \quad \widehat{D}(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ \frac{1}{2} & \text{if } |t| = 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

(2) Non-periodic Fejér Kernel.

$$K(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \quad \Rightarrow \quad \widehat{K}(t) = (1 - |t|)\,, \quad (|t| < 1)\,.$$

(3) Poisson Kernel in the Upper Half-Plane ($y > 0$).

$$P_y^{(+)}(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2} \quad \Rightarrow \quad \widehat{P_y^{(+)}}(t) = e^{-2\pi y|t|}\,.$$

(4) Conjugate Poisson Kernel in the Upper Half-Plane ($y > 0$).

$$Q_y^{(+)}(x) = \frac{1}{\pi}\frac{x}{x^2 + y^2} \quad \Rightarrow \quad \widehat{Q_y^{(+)}}(t) = -i\,\mathrm{sgn}\,(t)\, e^{-2\pi y|t|}\,.$$

(5) Gaussian.

$$G_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \Rightarrow \quad \widehat{G_{\mu,\sigma}}(t) = G_{0,\frac{2\pi}{\sigma}}(t)\, e^{-2\pi i\mu t}\,.$$

6.1.5. *More Fourier Transforms.*

(1) $\mathcal{F}[\mathrm{sgn}(x)](t) = \dfrac{1}{\pi i t}\,.$

(2) $\mathcal{F}[\log|x|](t) = -\dfrac{1}{2|t|}\,.$[4]

(3) $f_x(y) = \log\sqrt{x^2 + y^2} \quad \Rightarrow \quad \widehat{f}_x(t) = -\dfrac{1}{2|t|}e^{-2\pi x|t|} \quad (x > 0)\,.$

(4) $f_x(y) = \arctan\dfrac{y}{x} \quad \Rightarrow \quad \widehat{f}_x(t) = -\dfrac{i}{2t}e^{-2\pi x|t|} \quad (x > 0)\,.$

(5) $f_x(y) = \log(x + iy) = \log\sqrt{x^2 + y^2} + i\arctan\dfrac{y}{x} \quad \Rightarrow$

$$\widehat{f}_x(t) = \left(-\frac{1}{2|t|} + \frac{1}{2t}\right) e^{-2\pi x|t|} \quad (x > 0)\,.$$

6.2. **Laplace Transform.** The Laplace transform of a function $f(t)$ is defined:

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}\,dt\,. \tag{6.6}$$

[4] $\mathcal{F}[\log|x|\, e^{-\lambda|x|}](t) \to -\frac{1}{2|t|}$ as $\lambda \to 0^+$.

Theorem 6.6 (Existence of the Laplace transform)**.** *If f is a locally integrable function on $[0,\infty)$ and of (real) exponential type γ, then the Laplace integral of f, $\int_0^\infty f(t)\,e^{-st}\,dt$, converges for $\Re(s) > \gamma$, and converges uniformly for $\Re(s) \geq \gamma_1 > \gamma$.*

Note: A real function f is said to be of (real) exponential type if

$$|f(x)| \leq M e^{\gamma x}, \qquad \text{for all } x \geq x_0$$

for some constants M, $x_0 > 0$ and real γ.

6.2.1. *Properties of the Laplace Transform.*

Here we denote $F(s) = \mathcal{L}[f(t)](s)$, $G(s) = \mathcal{L}[g(t)](s)$.

(1) *Linearity.*

$$\mathcal{L}[af(t) + bg(t)](s) = aF(s) + bG(s)\,.$$

(2) *Dilation.* if $a > 0$,

$$\mathcal{L}[f(at)](s) = \frac{1}{a}F\left(\frac{s}{a}\right)\,.$$

(3) *Translation.*

$$\mathcal{L}[f(t-a)H(t-a)](s) = e^{-as}F(s)\,,$$

where H is the Heaviside function.

(4) *Multiplication by exponential functions.*

$$\mathcal{L}[e^{at}f(t)](s) = F(s-a)\,.$$

(5) *Differentiation.* If f is an n differentiable function, $f^{(k)}$ $(k = 0, 1, \ldots, n-1)$ is of (real) exponential type, $\lim_{t\to 0^+} f^{(k)}(t) = f^{(k)}(0^+)$ exists, and $f^{(n)}$ is locally integrable on $[0,\infty)$, then its Laplace transform exists and

$$\mathcal{L}[f^{(n)}(t)](s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\,.$$

(6) *Integration.*

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right](s) = \frac{F(s)}{s}\,,$$

if the transform exists.

(7) *Multiplication by powers of t.* Let f be a locally integrable function whose Laplace integral converges absolutely and uniformly for $\Re(s) > \sigma$. Then $F(s)$ is analytic in $\Re(s) > \sigma$, and for $n = 0, 1, 2, \ldots$, $\Re(s) > \sigma$:

$$\mathcal{L}[t^n f(t)](s) = \left(-\frac{d}{ds}\right)^n F(s)\,,$$

$$\mathcal{L}\left[\left(t\frac{d}{dt}\right)^n f(t)\right](s) = \left(-\frac{d}{ds}s\right)^n F(s)\,,$$

where $\left(-\frac{d}{ds}s\right)F(s) = -\frac{d}{ds}(sF(s)) = -F(s) - sF'(s)$.

(8) *Division by t.* If f is a locally integrable function of (real) exponential type, then

$$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(u)\,du\,.$$

if the transform exists. Also:

$$\mathcal{L}\left[\int_0^t \frac{f(x)}{x}\,dx\right](s) = \frac{1}{s}\int_s^\infty F(u)\,du\,.$$

(9) *Periodic functions.* If f is a locally integrable function that is periodic with period T, then:

$$\mathcal{L}[f(t)](s) = \frac{1}{1-e^{-Ts}}\int_0^T f(t)\,e^{-st}\,dt\,.$$

(10) *Convolution.* If $f, g : [0,\infty) \to \mathbb{R}$ are locally integrable on $[0,\infty)$ and their Laplace integrals converge absolutely in some half-plane $\Re(x) > \alpha$, then their convolution $f * g$ is locally integrable on $[0,\infty)$, and is continuous if either f or g is continuous. Additionally it has a Laplace transform give by:

$$\mathcal{L}[(f*g)(t)](s) = F(s)\,G(s)\,.$$

Here

$$(f*g)(x) = \int_0^\infty f(u)\,g(x-u)\,du\,.$$

6.2.2. *Inversion Formula.*

Theorem 6.7. *Let t_0 a real number and $f : \mathbb{R} \to \mathbb{C}$ a function such that*

(1) *$f(t) = 0$ for every $t < 0$,*
(2) *$f(t_0) = \frac{1}{2}\{f(t_0-) + f(t_0+)\}$,*
(3) *is locally integrable,*
(4) *is of bounded variation in a neighborhood of t_0,*
(5) *the Laplace integral of f, $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$, converges absolutely on the line $\Re(s) = c$.*

Then

$$f(t_0) = \lim_{T\to\infty} \frac{1}{2\pi i}\,\mathrm{PV}\int_{c-iT}^{c+iT} F(s)\,e^{st_0}\,ds\,, \tag{6.7}$$

In particular if f is differentiable on $(0,\infty)$ then the inversion formula holds for every $t_0 > 0$.

7. Complex Analysis

7.1. **Some Basic Theorems.**

7.1.1. *Some Definitions and Notations.*

A *domain* or *region* in the complex plane is a non-empty subset of $\mathbb{C}$ that is open and connected.

A *path* in the complex plane is a continuous function $\gamma : [a, b] \to \mathbb{C}$. Its image or trajectory is represented $|\gamma|$. A *cycle* σ is a finite sequence of closed, piecewise smooths paths. Its winding number about $z \in \mathbb{C} \setminus |\sigma|$ is

$$n(\sigma, z) = \frac{1}{2\pi i} \int_\sigma \frac{d\xi}{\xi - z}\,.$$

A cycle in and open set Ω is said to be *homologous to zero* if $n(\sigma, z) = 0$ for every $z \in \mathbb{C} \setminus \Omega$.

Given a function f analytic in the punctured disk $\Delta^*(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ and with an isolated singularity at z_0, it can be represented in $\Delta^*(z_0, r)$ by a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\,.$$

The coefficient a_{-1} is called *residue* of f at z_0, and represented $\mathrm{Res}(z_0, f)$.

An analytic function defined on the whole $\mathbb{C}$ is called *entire*.

7.1.2. *Exponential Type.* An entire function $F : \mathbb{C} \to \mathbb{C}$ is said to be of *exponential type* τ if

$$\limsup_{|z|\to\infty} \frac{\log |F(z)|}{|z|} = \tau \;<\; \infty$$

7.1.3. *Semicontinuity.* A function f with values in the real or extended real line is *lower semicontinuous* if $\{x \mid f(x) > x_0\}$ is open for every real x_0. A function f with values in the real or extended real line is *upper semicontinuous* if $\{x \mid f(x) < x_0\}$ is open for every real x_0.

7.1.4. *Some Theorems.*

Theorem 7.1 (Liouville's Theorem)**.** *Every Bounded entire function is constant.*

Theorem 7.2 (Maximum Principle)**.** *If f is analytic in a domain D and $|f|$ attains its maximum in D, then f is constant.*

Corollary 7.3. *If f is analytic in a bounded domain D and continuous in $\overline{D}$ then $|f|$ attains its maximum at some point on the boundary of D.*

Theorem 7.4. *If f_n are analytic in the domain D and $f_n \to f$ uniformly on compact subsets of D then f is analytic in D and $f_n' \to f'$ uniformly on compact subsets of D.*

Theorem 7.5 (Cauchy's Integral Formula)**.** *Suppose that f is analytic in an open set Ω and that σ is a cycle in Ω which is homologous to zero in this set. Then*

$$n(\sigma, z) f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(\xi)\, d\xi}{\xi - z} \tag{7.1}$$

for every $z \in \Omega \setminus |\sigma|$.

Theorem 7.6 (Residue Theorem)**.** *Suppose that f is analytic modulo isolated singularities in an open set Ω, that E is the singular set of f in Ω, and that σ is a cycle in $\Omega \setminus E$ which is homologous to zero in Ω. Then*

$$\int_\sigma f(z)\,dz = 2\pi i \sum_{z\in E} n(\sigma, z)\,\mathrm{Res}(z, f)\,. \tag{7.2}$$

Theorem 7.7 (Paley-Wiener Theorems)**.**

(1) *Suppose f is analytic in the upper half-plane and*

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^2\,dx = C < \infty\,.$$

Then there exists an $F \in L^2(0,\infty)$ such that

$$f(z) = \int_0^\infty F(t)\,e^{2\pi i t z}\,dt\,, \quad \Im(z) > 0$$

and

$$\int_0^\infty |F(t)|^2\,dt = C\,.$$

(2) *Suppose that f is an entire function such that*

$$|f(z)| \le C e^{2\pi A|z|}$$

for some positive constants A, C and for all z, and

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty\,.$$

Then there exists an $F \in L^2([-A, A])$ such that

$$f(z) = \int_{-A}^{A} F(t) e^{2\pi i t z}\,dt$$

for all z.

Proof. See [9, 19.2–19.3]. □

7.2. **Harmonic Analysis.** Here we use the notations of section 5 for $\mathbb{T}_1$ and $\mathbb{T}_{2\pi}$. Also we denote the unit disk $U = \Delta(0,1)$.

7.2.1. *Harmonic Functions.* Let D be a complex domain. A function $u : D \to \mathbb{R}$ is *harmonic* if it verifies Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0\,.$$

Given u harmonic in a domain D, a *harmonic conjugate* is function $v : D \to \mathbb{R}$ such that $u + vi$ is analytic in D. Two harmonic conjugates of a given function differ in a constant.

Theorem 7.8. *Let D be a complex domain. Every harmonic function in D has a harmonic conjugate if and only if D is simply connected.*

7.2.2. *Mean Value Property.* A continuous function $u : \Omega \to \mathbb{R}$ has the *Mean Value Property* in the open set Ω if for each $z \in \Omega$ there is a $\rho > 0$ such that and

$$u(z) = \int_{\mathbb{T}_{2\pi}} u(z + re^{i\theta})\, d\theta = \int_{\mathbb{T}_1} u(z + re^{2\pi i\alpha})\, d\alpha\,. \tag{7.3}$$

for every $0 < r < \rho$ (the disk $\Delta(z,\rho) = \{z' \in \mathbb{C} : |z' - z| < \rho\}$ is supposed to be contained in Ω.)

Every harmonic function in an open set Ω has the mean value property with any ρ such that the disk $\Delta(z,\rho)$ is contained in Ω. The converse also is true:

Theorem 7.9. *A continuous real function u defined in an open set Ω is harmonic in Ω if and only if it has the mean value property in Ω.*

7.2.3. *Functions Harmonic in Annuli.*

Theorem 7.10. *Suppose a function u is harmonic in the annulus $D = \{z \in \mathbb{C} \mid a < |z - z_0| < b\}$, where $0 \le a < b \le \infty$. Then there exist constants c and d such that*

$$\int_{\mathbb{T}_{2\pi}} u(z_0 + re^{i\theta})\, d\theta = c \log r + d\,.$$

7.2.4. *Subharmonic Functions.* If a function verifies

$$u(z) \le \int_{\mathbb{T}_{2\pi}} u(z + re^{i\theta})\, d\theta \tag{7.4}$$

in place of (7.3), it is called *subharmonic*. The precise definition is as follows ([9, def. 17.1]):

Definition 7.11. A function $u : \Omega \to [-\infty, \infty)$ defined in an open set $\Omega \subseteq \mathbb{C}$ is said to be *subharmonic* if:

(1) u is upper semicontinuous in Ω.
(2) $u(z) \le \int_{\mathbb{T}_{2\pi}} u(z + re^{i\theta})\, d\theta$ for every closed disk $\overline{\Delta}(z,r) \subset \Omega$.
(3) None of the integrals above is $-\infty$.

A subharmonic function always lies below any harmonic function with which it shares the same boundary values.

A twice continuously differentiable function $w : \Omega \to \mathbb{R}$ in an open set Ω is subharmonic in Ω if and only if $\Delta w \ge 0$ throughout Ω.

Theorem 7.12. *If u is harmonic (subharmonic) in a domain D and attains its maximum in D, then u is constant.*

Corollary 7.13. *If u is harmonic (subharmonic) in a bounded domain D and continuous in $\overline{D}$ then u attains its maximum at some point on the boundary of D.*

7.2.5. *Poisson Integrals.* Here we use the following notations ($z = re^{i\theta}$):

(1) *Convolution.* Given $f, g : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} = \mathbb{T}_1$ or $\mathbb{T} = \mathbb{T}_{2\pi}$, the convolution of f and g is:
$$f * g(x) = \int_{\mathbb{T}} f(t) g(x - t)\, dt\,.$$

(2) *Cauchy Kernel.*
$$C(z) = \frac{1}{1-z}$$
$$C_r(\theta) = \frac{1}{1 - re^{i\theta}}\,.$$

(3) *H Kernel.*
$$H(z) = 2C(z) - C(0) = \frac{1+z}{1-z}$$
$$H_r(\theta) = 2C_r(\theta) - C_0(\theta) = \frac{1 + re^{i\theta}}{1 - re^{i\theta}}\,.$$

(4) *Poisson Kernel.*
$$P(z) = \Re\{H(z)\} = C(z) + \overline{C}(z) - C(0)$$
$$P_r(\theta) = C_r(\theta) + \overline{C}_r(\theta) - C_0(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}\,.$$

(5) *Conjugate Poisson Kernel.*
$$Q(z) = \Im\{H(z)\} = -i\left\{C(z) - \overline{C}(z)\right\}$$
$$Q_r(\theta) = \frac{2r\sin\theta}{1 - 2r\cos\theta + r^2}\,.$$

(6) *Poisson Integral.* If $h : \mathbb{T}_{2\pi} \to \mathbb{R}$ is integrable, its *Poisson integral* is
$$P[h](re^{i\theta}) = \int_{\mathbb{T}_{2\pi}} h(t)\, P_r(\theta - t)\, dt = h * P_r(\theta)\,.$$

The main result is the following:

Theorem 7.14. *Suppose u is harmonic in the unit disk U and continuous on $\overline{U}$. Then u is the Poisson integral of its restriction to ∂U, and is the real part of the holomorphic function*

$$f(z) = \int_{\mathbb{T}_{2\pi}} \frac{e^{it} + z}{e^{it} - z}\, u(e^{it})\, dt\,. \tag{7.5}$$

Assume $u, v : \overline{U} \to \mathbb{R}$, $v(0) = 0$, $f = u + iv$ is holomorphic in U and continuous in $\overline{U}$. Then (we use the notation $f_r(\theta) = f(re^{i\theta})$, etc.):

(1) $f_r = f_1 * C_r$ (Cauchy's Integral Formula.)
(2) $f_r = u_1 * H_r$.
(3) $u_r = u_1 * P_r$ (Poisson's Integral Formula.)
(4) $v_r = u_1 * Q_r$.

7.2.6. *Other results.*

Theorem 7.15 (Harnack's Theorem). *Let $\{u_n\}$ be a sequence of harmonic functions in a domain D.*

(1) *If $u_n \to u$ uniformly on compact subsets of D, then u is harmonic in D.*
(2) *If the sequence is nondecreasing then either $\{u_n\}$ converges uniformly on compact subsets of D or $u_n(z) \to \infty$ for every $z \in D$.*

7.3. H^p **Spaces.** Here we use the notation:

$H(D) = \{f : D \to \mathbb{C} \mid \text{f is analytic in } D\}$.
$h(D) = \{f : D \to \mathbb{R} \mid \text{f is harmonic in } D\}$.
$U = \Delta(0,1)$ (unit disk).

7.3.1. *Integral Means.* For $f : U \to \mathbb{C}$ we define ([3]):

(1) $M_p(r,f) = \left\{ \int_{\mathbb{T}_{2\pi}} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p}, \quad 0 < p < \infty.$

(2) $M_\infty(r,f) = \max_{0 \le \theta < 2\pi} |f(re^{i\theta})|.$

(3) $M_0(r,f) = \exp\left\{ \int_{\mathbb{T}_{2\pi}} \log^+ |f(re^{i\theta})| \, d\theta \right\}.$

(4) $\|f\|_p = \sup_{0 \le r < 1} M_p(r,f), \quad 0 \le p \le \infty.$

(5) $H^p = \{f \in H(U) \, : \, \|f\|_p < \infty\}.$

(6) $h^p = \{f \in h(U) \, : \, \|f\|_p < \infty\}.$

Note: $p \le q \Rightarrow H^q \subseteq H^p$.

For $p = 0$, $N = H^0$ is the *Nevanlinna* class of functions of *bounded characteristic*. If $u \in H(U)$, then $u \in N$ if and only if u is the quotient of two bounded analytic functions.

Proposition 7.16. *A function $f \in H(U)$ is the Poisson integral of a function $\varphi \in L^p(\mathbb{T}_{2\pi})$ iff $f \in H^p$.*

7.3.2. *Poisson-Stieltjes Integrals.* A *Poisson-Stieltjes integral* is a function in U of the form:

$$u(re^{i\theta}) = P[d\varphi](re^{i\theta}) = \int_{\mathbb{T}_{2\pi}} P_r(\theta - t) \, d\varphi(t) \, , \tag{7.6}$$

where $P_r(\theta)$ is the Poisson kernel, $\varphi : \mathbb{T}_{2\pi} \to \mathbb{C}$ is of bounded variation, and the integral is of Stieltjes type.[5]

[5] $\int_{\mathbb{T}_{2\pi}} f(x) \, d\varphi(x) = \frac{1}{2\pi} \int_{x_0}^{x_0+2\pi} f(x) \, d\varphi(x)$ for any real x_0.

Theorem 7.17. *The following three classes of functions in U are identical:*

(1) *Poisson-Stieltjes integrals.*
(2) *Differences of two positive harmonic functions.*
(3) h^1.

Theorem 7.18. *Assume $\varphi : \mathbb{T}_{2\pi} \to \mathbb{C}$ is of bounded variation and the following symmetric derivative exists:*

$$D\varphi(\theta_0) = \lim_{t\to 0} \frac{\varphi(\theta_0 + t) - \varphi(\theta_0 - t)}{2t}\,.$$

Let u be the Poisson-Stieltjes integral $u(z) = P[\varphi](z)$. Then $u(z) \to D\varphi(\theta_0)$ as $z \to e^{i\theta_0}$ along any path not tangent to the unit circle.

7.3.3. *Harmonic Conjugation.* The harmonic conjugate of a function $u \in h(U)$ is another function v such that $u + iv$ is analytic in U. It is "normalized" if $v(0) = 0$. Note: if $f(z) = u(z) + iv(z) = \sum_{n=0}^{\infty} c_n z^n$, $c_n = a_n - ib_n$, then:

$$\begin{aligned} u(z) &= \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)\,, \\ v(z) &= \sum_{n=0}^{\infty} r^n (-b_n \cos n\theta + a_n \sin n\theta)\,. \end{aligned} \tag{7.7}$$

Also:

$$u(z) = \sum_{n=-\infty}^{\infty} c'_n\, r^n e^{ni\theta}\,, \quad v(z) = \sum_{n=-\infty}^{\infty} c''_n\, r^n e^{ni\theta}\,, \tag{7.8}$$

where $c'_0 = \Re(c_0)$, $c''_0 = \Im(c_0)$, and

$$c'_n = \begin{cases} c_n/2 & (n > 0) \\ \bar{c}_n/2 & (n < 0) \end{cases}, \quad c''_n = \begin{cases} c_n/2i & (n > 0) \\ -\bar{c}_n/2i & (n < 0) \end{cases}. \tag{7.9}$$

The relation between the coefficients of u and v for $n \neq 0$ is:

$$c''_n = -i\,\mathrm{sgn}(n) c'_n\,. \tag{7.10}$$

Theorem 7.19. *Let v be the harmonic conjugate of $u \in h(U)$.*

(1) *(M. Riesz) For $1 < p < \infty$, $u \in h^p \Rightarrow v \in h^p$. Furthermore there is a constant A_p depending only on p such that*

$$M_p(r, v) \le A_p M_p(r, u), \quad 0 \le r < 1$$

for all $u \in h^p$.

(2) *(Kolmogorov) If $u \in h^1$ then $v \in h^p$ for every $0 < p < 1$. Furthermore there is a constant B_p depending only on p such that*

$$M_p(r, v) \le B_p M_p(r, u), \quad 0 \le r < 1$$

for all $u \in h^1$.

(3) *If both $u, v \in h^1$ then the function φ such that $u(z) = P[\varphi](z)$ is absolutely continuous.*

7.3.4. *H^p Spaces Over General Domains.* Here D is a simply connected domain with at least two boundary points.

(1) $\|f\|_\infty = \sup_{z \in D} |f(z)|$.

(2) $H^\infty(D) = \{f \in H(D) : \|f\|_\infty < \infty\}$.

(3) $H^p(D) = \{f \in H(D) : |f|^p$ has a harmonic majorant in $D\}$ $(0 < p < \infty)$. In this case the norm can be defined as $\|f\|_p = u(z_0)^{1/p}$, where $z_0 \in D$ is fixed and u is the least harmonic majorant of $|f|^p$.

(4) A function f analytic in D belongs to the class $E^p(D)$ if there exists a sequence of rectifiable Jordan curves C_n in D tending to ∂D (C_n eventually surrounds each compact subdomain of D) such that

$$\int_{C_n} |f(z)|^p \, |dz| \le M < \infty \,.$$

Note: for the unit disk, $E^p(U) = H^p(U)$.

7.3.5. *H^p Spaces On the Upper Half-Plane.* (See [3]).

Here $\mathcal{H}^{(+)} = \{z \in \mathbb{C} : \Im(z) > 0\}$ represents the upper half-plane; $\mathcal{H}^{(-)} = \{z \in \mathbb{C} : \Im(z) < 0\}$ is the lower half-plane.

For the upper half-plane we have that $E^p(\mathcal{H}^{(+)}) \subset H^p(\mathcal{H}^{(+)})$ properly. On the other hand for $0 < p < \infty$ we have $E^p(\mathcal{H}^{(+)}) = \mathfrak{H}^p(\mathcal{H}^{(+)})$,

$$\mathfrak{H}^p(\mathcal{H}^{(+)}) = \{f \in H(\mathcal{H}^{(+)}) \mid \forall y > 0,\ f_y \in L^p(\mathbb{R});\ \sup_{0<y<\infty} \mathfrak{M}_p(y, f) < \infty\}, \tag{7.11}$$

where $f_y(x) = f(x + iy)$ and

$$\mathfrak{M}_p(y, f) = \|f_y\|_p \,.$$

Theorem 7.20. *(Poisson Integral in $\mathfrak{H}^p(\mathcal{H}^{(+)})$.)*

If $f \in \mathfrak{H}^p(\mathcal{H}^{(+)})$, $1 \le p \le \infty$, then

$$f(x + iy) = \int_{-\infty}^{\infty} P^{(+)}((x - t) + iy) f(t) \, dt \,, \tag{7.12}$$

where

$$P^{(+)}(x + iy) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

is the Poisson kernel for the upper half-plane.

Conversely, if $h \in L^p(\mathbb{R})$, $1 \le p \le \infty$, and

$$f(x + iy) = \int_{-\infty}^{\infty} P^{(+)}((x - t) + iy) h(t) \, dt$$

is analytic in $\mathcal{H}^{(+)}$*, then* $f \in \mathfrak{H}^p(\mathcal{H}^{(+)})$*, and its boundary function*

$$f_0(x) = \lim_{y \to 0^+} f(x+iy)$$

verifies $f_0(x) = h(x)$ *a.e.*

Proposition 7.21. *If* $f \in \mathfrak{H}^p$*,* $0 < p < \infty$*, then*

$$\lim_{y \to 0^+} \|f_y\|_p = \|f_0\|_p \,.$$

Theorem 7.22. *(Cauchy Integral in* $\mathfrak{H}^p(\mathcal{H}^{(+)})$*.)*

If $f \in \mathfrak{H}^p(\mathcal{H}^{(+)})$*,* $1 \le p < \infty$*, then*

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt\,, \quad \Im(z) > 0\,; \tag{7.13}$$

and the integral vanishes for $\Im(z) < 0$*.*

Conversely, if $h \in L^p(\mathbb{R})$*,* $1 \le p < \infty$*, and*

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h(t)}{t-z}\, dt \equiv 0\,, \quad \Im(z) < 0\,,$$

then for $\Im(z) > 0$ *the integral defines a function* $f \in \mathfrak{H}^p(\mathcal{H}^{(+)})$*, and its boundary function*

$$f_0(x) = \lim_{y \to 0^+} f(x+iy)$$

verifies $f_0(x) = h(x)$ *a.e.*

Proposition 7.23. [4, ex. 8.10] *Every function* $f \in L^2(\mathbb{R})$ *is uniquely expressible in the form* $f = f_{(+)} + f_{(-)}$*,* $f_{(+)} \in \mathfrak{H}^2(\mathcal{H}^{(+)})$*,* $f_{(-)} \in \mathfrak{H}^2(\mathcal{H}^{(-)})$*, where*

$$\begin{aligned} f_{(+)}(w) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-w}\, dx && \Im(w) > 0 \\ f_{(-)}(w) &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-w}\, dx && \Im(w) < 0\,. \end{aligned}$$

Example: For $f(x) = \dfrac{2x}{x^2+1}$ we have

$$\begin{aligned} f_{(+)}(w) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{2x}{x^2+1}}{x-w}\, dx = \frac{1}{w+i} && \Im(w) > 0 \\ f_{(-)}(w) &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{2x}{x^2+1}}{x-w}\, dx = \frac{1}{w-i} && \Im(w) < 0\,, \end{aligned}$$

hence

$$\frac{2x}{x^2+1} = \frac{1}{x+i} + \frac{1}{x-i}\,.$$

7.3.6. *H^p Spaces On the Right Half-Plane.* (See [4]).

Here $\mathcal{H}^{(\rightarrow)} = \{z \in \mathbb{C} : \Re(z) > 0\}$ represents the right half-plane. Also

$$\mathfrak{H}^p(\mathcal{H}^{(\rightarrow)}) = \{f \in H(\mathcal{H}^{(\rightarrow)}) \mid \forall x > 0,\, f_x \in L^p(\mathbb{R});\ \sup_{0<x<\infty} \|f_x\|_p < \infty\}, \tag{7.14}$$

where $f_y(x) = f(x + iy)$.

The Poisson kernel and its conjugate for the right half-plane are ($z = x + iy$, $x > 0$):

$$P^{(\rightarrow)}(z) = \Re\left\{\frac{1}{\pi z}\right\} = \frac{1}{\pi}\frac{x}{x^2+y^2} \tag{7.15}$$

$$= P_x^{(\rightarrow)}(y) = \int_{-\infty}^{\infty} e^{-2\pi x|t|}\, e^{2\pi i y t}\, dt\,,$$

$$Q^{(\rightarrow)}(z) = \Im\left\{\frac{1}{\pi z}\right\} = \frac{1}{\pi}\frac{-y}{x^2+y^2} \tag{7.16}$$

$$= Q_x^{(\rightarrow)}(y) = \int_{-\infty}^{\infty} i\,\mathrm{sgn}(t)\, e^{-2\pi x|t|}\, e^{2\pi i y t}\, dt\,.$$

The relation between P_r and $P_x^{(\rightarrow)}$ is as follows. Consider the following linear fractional map from the right half-plane to the unit disk:[6]

$$w(z) = \frac{z-1}{z+1} \tag{7.17}$$

Assume $w(iy) = e^{i\theta}$. Then

$$\frac{d\theta}{2\pi} = -\frac{dy}{\pi(1+y^2)}\,,$$

so the normalized Lebesgue measure $d\theta/2\pi$ on the circle corresponds to the Cauchy probability measure $dy/\pi(1+y^2)$ on the imaginary axis.

If $g : \partial U \to \mathbb{C}$ is measurable and $f(iy) = g(e^{i\theta}) = g(w(iy))$ then g is Lebesgue-integrable if and only if f is integrable respect to the measure $dy/\pi(1+y^2)$, and

$$\int_{\mathbb{T}_{2\pi}} g(e^{i\theta})\, d\theta = \int_{-\infty}^{\infty} f(iy)\cdot\frac{1}{\pi(1+y^2)}\, dy\,.$$

On the other hand, writing $w(x+iy) = re^{i\theta}$, $w(iy') = e^{i\theta'}$:

$$P_r(\theta-\theta') = \pi\,(1+{y'}^2)\, P_x^{(\rightarrow)}(y-y')\,,$$

so the Poisson formula for the right half-plane becomes as shown in the following result:

Theorem 7.24. *Let h be a measurable function on the imaginary axis which is integrable respect to the measure $(1+t^2)^{-1}\,dt$. Define f in the right half-plane by*

$$f(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(it)\,\frac{x}{x^2+(y-t)^2}\, dt \tag{7.18}$$

$$= \int_{-\infty}^{\infty} h(it)\, P_x^{(\rightarrow)}(y-t)\, dt\,.$$

[6] A similar analysis works for $P_x^{(+)}$ using $w(z) = \frac{z-i}{z+i}$ instead.

Then f is harmonic and has non-tangential limits which exist and agree with h at almost every point of the imaginary axis.

Theorem 7.25. *Assume $p \geq 1$ and $h \in L^p(\mathbb{R})$. Let f be the harmonic function in the right half-plane defined by*

$$f(x+iy) = \int_{-\infty}^{\infty} h(t)\,\frac{x}{x^2+(y-t)^2}\,dt = h * P_x^{(\to)}(y) = f_x(y)\,.$$

Then:

(1) *$f_x \in L^p(\mathbb{R})$ for every $x > 0$.*
(2) *The L^p-norms $\|f_x\|_p$ are bounded for $x > 0$. In fact, $\|f_x\|_p$ is a decreasing function of x for $x > 0$.*
(3) *$\|f_x - h\|_p \to 0$ as $x \to 0^+$.*
(4) *$f(\xi) \to 0$ uniformly as $\xi \to \infty$ inside any fixed half-plane $\Re(\xi) \geq \delta > 0$.*

7.3.7. *Kernels in the Right Half-Plane.* We have already seen by looking at the effects of the linear fractional map (7.17) that the Poisson kernel for the unit circle becomes $P^{(\to)}(z) = \Re(1/\pi z)$ in the right half-plane. On the other hand the equation

$$g(re^{i\theta}) = \int_0^{2\pi} g(e^{i\theta'})\,\frac{1}{1-re^{i(\theta-\theta')}}\,\frac{d\theta'}{2\pi}$$

is transformed into the equation

$$f(x+iy) = \int_{-\infty}^{\infty} f(iy')\,\frac{1}{\frac{x+iy-1}{x+iy+1}-\frac{iy'-1}{iy'+1}}\,\frac{dy'}{\pi(1+iy')^2}\,,$$

where $f(z) = g(w(z)) = g(\frac{z-1}{z+1})$. Hence the Cauchy kernel for the unit circle becomes

$$C_x^{(\to)}(y,y') = \frac{1}{\frac{x+iy-1}{x+iy+1}-\frac{iy'-1}{iy'+1}}\,\frac{1}{\pi(1+iy')^2} \tag{7.19}$$

in the right half-plane. The H kernel then becomes:

$$\begin{aligned} 2C_x^{(\to)}(y,y') - C_1^{(\to)}(0,y') &= \frac{1}{\pi}\left\{\frac{1}{x+(y-y')i} - \frac{iy'}{1+y'^2}\right\} \\ &= H_x^{(\to)}(y-y') - \frac{iy'}{\pi(1+y'^2)}\,, \end{aligned} \tag{7.20}$$

where

$$\begin{aligned} H^{(\to)}(z) = P^{(\to)}(z) + iQ^{(\to)}(z) &= \frac{1}{\pi z} \\ = H_x^{(\to)}(y) &= 2\int_0^{\infty} e^{-2\pi(x+iy)t}\,dt\,. \end{aligned} \tag{7.21}$$

So the H kernel for the unit circle becomes what we have called the H kernel for the right half-plane corrected with the term $-iy'/\pi(1+y'^2)$.

We know that an analytic function in the unit circle that is real at zero can be recovered from the boudary values of its real part by convolution with the H kernel. The corresponding result for an analytic function $f(z) = u(z) + iv(z)$ in the right half-plane such that $v(1) = 0$ can be expressed with the following equation:

$$f(x+iy) = \int_{-\infty}^{\infty} u(it) \left\{ \frac{1}{\pi\{x+(y-t)i\}} - \frac{it}{\pi(1+t^2)} \right\} dt \tag{7.22}$$

The integral converges as long as $t \mapsto u(it)/(1+t^2)$ is in $L^1(\mathbb{R})$. If $t \mapsto u(it)/(1+|t|)$ is in $L^1(\mathbb{R})$ then the equation

$$\begin{aligned} f(x+iy) &= \int_{-\infty}^{\infty} u(it) \frac{1}{\pi\{x+(y-t)i\}} dt \\ &= \int_{-\infty}^{\infty} u(it)\, H_x(y-t)\, dt \end{aligned} \tag{7.23}$$

yields essentially the same function but normalized so that

$$v(\infty) = \lim_{x\to\infty} \Im\{f(x+iy)\} = 0\,.$$

In such case:

$$v(1) = \int_{-\infty}^{\infty} u(it) \frac{it}{\pi(1+t^2)} dt\,.$$

8. Some Special Functions

8.1. **Bernoulli periodic functions.** The Bernoulli polynomials $\mathrm{B}_n^*(x)$ can be defined in various ways.[7] The following are two of them ([8, ch. 1], [1]):

(1) By a generating function:

$$\frac{t\,e^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \mathrm{B}_n^*(x) \frac{t^n}{n!} \tag{8.1}$$

(2) By the following recursive formulas ($n \geq 1$):

$$\mathrm{B}_0^*(x) = 1\,, \tag{8.2}$$

$$\mathrm{B}_n^{*\prime}(x) = n\,\mathrm{B}_{n-1}^*(x)\,, \tag{8.3}$$

$$\int_0^1 \mathrm{B}_n^*(x)\,dx = 0\,. \tag{8.4}$$

The first Bernoulli polynomials are:

$\mathrm{B}_0^*(x) = 1$

$\mathrm{B}_1^*(x) = x - \frac{1}{2}$

$\mathrm{B}_2^*(x) = x^2 - x + \frac{1}{6}$

$\mathrm{B}_3^*(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

[7] Here we use the notation B_n^* for the Bernoulli polynomials, and reserve the notation B_n for the Bernoulli periodic functions.

The Bernoulli numbers are $B_n = \mathrm{B}_n^*(0)$, and the Bernoulli periodic functions are usually defined $\mathrm{B}_n(x) = \mathrm{B}_n^*(\langle x\rangle)$. However here we normalize B_1 defining $\mathrm{B}_1(k) = 0$ instead of $-1/2$ for k integer, so that B_1 coincides with the normalized sawtooth function:

$$\mathrm{B}_1(x) = \begin{cases} \langle x\rangle - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} \tag{8.5}$$

where $\langle x\rangle = x - \lfloor x\rfloor =$ fractional part of x, $\lfloor x\rfloor =$ integer part of x (greatest integer $\leq x$). Also we will leave $B_0(k)$ undefined for k integer—in fact B_0 should be defined as the distribution $B_0(x) = 1 - \delta_{per}(x)$, where $\delta_{per}(x) = \sum_{k=-\infty}^{\infty} \delta(x-k)$ is the periodic Dirac's delta.

8.1.1. *Properties of the Bernoulli Periodic Functions.* ($n \geq 1$):

1. $\mathrm{B}_1(x) =$ sawtooth function (eq. 8.5).
2. $\mathrm{B}_n'(x) = n\,\mathrm{B}_{n-1}(x)$ for $n > 2$ or $x \notin \mathbb{Z}$.
3. $\displaystyle\int_0^1 \mathrm{B}_n(x)\,dx = 0.$

8.1.2. *Fourier expansions.* The Fourier expansion for the Bernoulli periodic functions is ($n \geq 1$):

$$\mathrm{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^n}\,, \tag{8.6}$$

so:

$$\widehat{\mathrm{B}}_n(k) = \begin{cases} 0 & \text{if } k = 0, \\ -\dfrac{n!}{(2\pi i k)^n} & \text{otherwise.} \end{cases} \tag{8.7}$$

This result also holds in the distributional sense for $n = 0$.

8.2. **Polylogarithms.** The Bernoulli periodic functions appear naturally in expressions involving polylogarithms, together with the so called Clausen functions (see [6]):

$$\mathrm{Cl}_{2n-1}(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2n-1}}, \tag{8.8}$$

$$\mathrm{Cl}_{2n}(\theta) = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2n}}, \tag{8.9}$$

for $n \geq 1$.

To be more precise, the polylogarithms can be defined by the series:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{8.10}$$

for $n \geq 0$, $|z| < 1$, or by the following recursive relations:

$$\operatorname{Li}_0(z) = \frac{z}{1-z}, \tag{8.11}$$

$$\operatorname{Li}_n(z) = \int_0^z \frac{\operatorname{Li}_{n-1}(\xi)}{\xi}\, d\xi \qquad (n \geq 1), \tag{8.12}$$

in $\mathbb{C} \setminus [1, \infty)$. Note that $\operatorname{Li}_1(z) = -\log(1-z)$ is the usual logarithm. $\operatorname{Li}_2(z)$ is the *dilogarithm.*[8]

A generating function is

$$\int_0^\infty \frac{z\, e^{(t+1)u}}{(e^u - z)^2}\, du = \sum_{n=0}^\infty \operatorname{Li}_n(z)\, t^n. \tag{8.13}$$

The Bernoulli periodic functions and the Clausen functions are related to the polylogarithms in the following way:

$$-\frac{2in!}{(2\pi i)^n} \operatorname{Li}_n(e^{2\pi i x}) = \mathrm{A}_n(x) \; + \; i\ \mathrm{B}_n(x), \tag{8.14}$$

for $x \notin \mathbb{Z}$, where

$$\mathrm{A}_n(x) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \frac{2n!}{(2\pi)^n} \operatorname{Cl}_n(2\pi x). \tag{8.15}$$

We will call the $\mathrm{A}_n(x)$ *conjugate Bernoulli periodic functions.* The first ones are $A_0(x) = \cot \pi x$, $A_1(x) = \frac{2}{\pi} \log\left(2\,|\sin \pi x|\right), \ldots$

The series (8.10) converges for $|z| = 1$ if $n \geq 2$.

For $n = 1$ both $\operatorname{Li}_1(e^{2\pi i x})$ and $\operatorname{Cl}_1(2\pi x)$ diverge at $x = 0$, but

$$-\pi i\, \mathrm{B}_1(x) = \operatorname{Li}_1(e^{2\pi i x}) - \operatorname{Cl}_1(2\pi x) = i \sum_{k=1}^\infty \frac{\sin 2\pi k x}{k}, \tag{8.16}$$

and the series becomes zero for $x = 0$, so our definition $\mathrm{B}_1(0) = 0$ allows (8.16) to hold also for $x = 0$.

For $n = 0$, $x \notin \mathbb{Z}$, we easily compute

$$\operatorname{Cl}_0(x) = -i\ \operatorname{Li}_0(e^{2\pi i x}) - \frac{i}{2} = \frac{1}{2} \cot(\pi x). \tag{8.17}$$

Also by definition $\operatorname{Cl}_0(k) = 0$ for $k \in \mathbb{Z}$. Hence,

$$\Im\{\operatorname{Li}_0(e^{2\pi i x})\} = \operatorname{Cl}_0(x) \tag{8.18}$$

for every $x \in \mathbb{R}$.

Finally we observe that for $y > 0$

$$\Re \left\{ \int_{-\frac{1}{2}}^{x} \operatorname{Li}_0(e^{2\pi i(u + yi)})\, du \right\} = -\frac{1}{2\pi} \arg\left\{ e^{2\pi y} - e^{2\pi i x} \right\}, \tag{8.19}$$

[8] For some authors the dilogarithm is $\operatorname{Li}_2(1-z)$.

which tends to $-\frac{1}{2}\,\mathrm{B}_1(x)$ as $y \to 0^+$ for every $x \in \mathbb{R}$, hence

$$\lim_{y\to 0^+} \Re\left\{\mathrm{Li}_0(e^{2\pi i(x+yi)})\right\} = -\frac{1}{2}\,\mathrm{B}_0(x) = -\frac{1}{2} + \frac{1}{2}\,\delta_{per}(x) \tag{8.20}$$

(where δ_{per} is the periodic Dirac's delta) in the distributional sense.

We also note that the Bernoulli periodic functions and their conjugates have harmonic extensions to the upper half plane, given by the formula:

$$-\frac{2in!}{(2\pi i)^n}\,\mathrm{Li}_n(e^{2\pi i z}) = \mathrm{A}_n(z) \;+\; i\,\mathrm{B}_n(z), \tag{8.21}$$

for $\Im(z) > 0$.

9. Summation Formulas

9.1. The Euler-Maclaurin Summation Formula.

Theorem 9.1. *Let $f : [a,b] \to \mathbb{C}$ be q times differentiable, $\int_a^b |f^{(q)}(x)|\,dx < \infty$. Then for $1 \le m \le q$:*

$$\begin{aligned}\sum_{a\le n\le b}{}' f(n) &= \int_a^b f(x)\,dx \\ &+ \sum_{k=1}^{m} \frac{(-1)^k}{k!}\,\Big(\mathrm{B}_k(b)\,f^{(k-1)}(b) - \mathrm{B}_k(a)\,f^{(k-1)}(a)\Big) \\ &+ \frac{(-1)^{m+1}}{m!}\int_a^b \mathrm{B}_m(x)\,f^{(m)}(x)\,dx,\end{aligned} \tag{9.1}$$

where $\sum_{a\le k\le b}{}' f(k)$ for $a < b$ represents a summation modified by taking only half of $f(k)$ when $k = a$ or $k = b$.

Proof. (See [5]) We have

$$\begin{aligned}\sum_{a\le k\le b}{}' f(n) &= \int_a^b f(x)\,d(x - \mathrm{B}_1(x)) \\ &= \int_a^b f(x)\,dx - \int_a^b f(x)\,d\,\mathrm{B}_1(x)\end{aligned} \tag{9.2}$$

Next, integrate by parts successively the last integral on the right hand side of (9.2). □

9.1.1. *Sum of Powers.* As an example of application of the Euler-Maclaurin summation formula, we give the sum of the first m rth powers:

$$S(m,r) = \sum_{n=1}^{m} n^r = 1^r + 2^r + 3^r + \cdots + m^r\,.$$

Here $f(x)=x^r$, so $f^{(k)}(x)=r!x^{(r-k)}/(r-k)!$ for $k=0,1,\ldots,r$, $f^{(k)}(x)=0$ for $k>r$, and

$$\begin{aligned}\sum_{0\le n\le m}{}' n^r = \int_0^m x^r\,dx \\ + \sum_{k=1}^{r+1}\frac{(-1)^k}{k!}\Big(\mathrm{B}_k(m)\,f^{(k-1)}(m)-\mathrm{B}_k(0)\,f^{(k-1)}(0)\Big) \\ = \frac{m^{r+1}}{r+1}+\sum_{k=1}^{r+1}\frac{(-1)^k}{k!}\,B_k(0)\,\frac{r!}{(r-k+1)!}\,m^{r-k+1}-\frac{B_{r+1}(0)}{r+1} \\ = \frac{m^{r+1}}{r+1}+\frac{1}{r+1}\sum_{k=1}^{r+1}(-1)^k\binom{r+1}{k}B_k(0)\,m^{r-k+1}-\frac{B_{r+1}(0)}{r+1}\end{aligned}$$

Hence

$$\begin{aligned}S(m,r) &= \sum_{0\le n\le m}{}' n^r+\frac{m^r}{2} \\ &= \frac{1}{r+1}\left\{\Big(\sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}B_k\,m^{r-k+1}\Big)-B_{r+1}\right\}\end{aligned}$$

where B_k are the Bernoulli numbers $B_0=1$, $B_1=-1/2$, $B_k=B_k(0)$ for $k>1$.

9.2. **The Poisson Summation Formula.** Here f represents a function $f:\mathbb{R}\to\mathbb{C}$.

Theorem 9.2. *If f is absolutely integrable over $\mathbb{R}$ then*

$$f_{per}(x)=\lim_{N\to\infty}\sum_{n=-N}^{N}f(x+n)$$

exists for a.e. x and is periodic: $f_{per}(x+1)=f_{per}(x)$.

Furthermore:

$$\widehat{f}(k)=\int_0^1 f_{per}(x)\,e^{-2\pi ikx}\,dx\,. \tag{9.3}$$

i.e., $\widehat{f}(k)=\widehat{f_{per}}(k)$ (the Fourier transform $\widehat{f}(k)$ of f coincides with the k-th Fourier coefficient of f_{per}.)

Proof. We have (LMCT):

$$\begin{aligned}\int_0^1\lim_{N\to\infty}\sum_{n=-N}^{N}|f(x+n)|\,dx &= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_n^{n+1}|f(x)|\,dx \\ &= \lim_{N\to\infty}\int_{-N}^{N+1}|f(x)|\,dx \\ &= \int_{-\infty}^{\infty}|f(x)|\,dx<\infty\,,\end{aligned}$$

so the series converges absolutely at almost every x. Also, if the series converges at x, then

$$f_{per}(x+1) - f_{per}(x) = \lim_{N\to\infty} \{f(x+N+1) - f(x-N)\} = 0\,,$$

so f_{per} is periodic. Furthermore, since

$$\left|\sum_{n=-N}^{N} f(x+n)\, e^{-2\pi i k x}\right| \le \lim_{N\to\infty} \sum_{n=-N}^{N} |f(x+n)|\,,$$

and the right hand side being integrable in $\mathbb{T}_1$, by the LDCT we have:

$$\begin{aligned}
\widehat{f}_{per}(k) &= \int_0^1 f_{per}(x)\, e^{-2\pi i k x}\, dx \\
&= \int_0^1 \lim_{N\to\infty} \sum_{n=-N}^{N} f(x+n)\, e^{-2\pi i k x}\, dx \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_n^{n+1} f(x)\, e^{-2\pi i k x}\, dx \\
&= \lim_{N\to\infty} \int_{-N}^{N+1} f(x)\, e^{-2\pi i k x}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \\
&= \widehat{f}(k)\,.
\end{aligned}$$

□

Theorem 9.3 (Poisson Summation Formula)**.** *If f is absolutely integrable over $\mathbb{R}$, of bounded variation and normalized in the sense that for every x,*

$$f(x) = \frac{1}{2} \lim_{h\to 0} \{f(x+h) + f(x-h)\}\,,$$

then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \lim_{T\to\infty} \sum_{k=-T}^{T} \widehat{f}(k)\, e^{2\pi i k x}. \tag{9.4}$$

The following is a formulation covering the case in which f is not absolutely integrable (see [11, II.13]).

Theorem 9.4. *Assume that*

(1) *f is integrable over every finite interval;*
(2) $\displaystyle\lim_{n\to\pm\infty} \int_n^{n+1} |f(x)|\, dx = 0\,;$
(3) *the function*

$$f^*(x) = f(x) - \int_n^{n+1} f(u)\, du \quad \textit{for} \quad n \le x < n+1,\ (n \in \mathbb{Z})$$

is absolutely integrable over $\mathbb{R}$.

Then

$$\widehat{f}(k) = \lim_{M\to\infty} \int_{-M}^{M} f(x)\, e^{-2\pi i k x}\, dx\,, \tag{9.5}$$

exists for every $k \in \mathbb{Z} \setminus \{0\}$*, and if* g^* *is the function*

$$g^*(x) = \lim_{N\to\infty} \left\{ \sum_{n=-N}^{N} f(n+x) - \int_{-N}^{N} f(u)\, du \right\}$$

then $g^*(x)$ *exists for a.e.* x*, is periodic:* $g^*(x+1) = g^*(x)$*, and the* k*-th Fourier coefficient of* g^* *is*

$$\widehat{g^*}(k) = \int_0^1 g^*(x)\, e^{-2\pi i k x}\, dx = \begin{cases} 0 & \text{if } k = 0 \\ \widehat{f}(k) & \text{if } k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

10. Miscelanea

10.1. **Various Results.**

10.1.1. Here we want to justify the following expression, which can be considered as a formal application of the Poisson Summation Formula:

Proposition 10.1. *For* $x, y \in \mathbb{R}$*,* $x > 0$*:*

$$\begin{aligned} \lim_{N\to\infty} &\left\{ \sum_{n=-N}^{N} \log\{x + (y+n)i\} - \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \log\{x + (y+u)i\}\, du \right\} \\ &= \lim_{T\to\infty} \sum_{\substack{k=-T \\ k\neq 0}}^{T} \left\{ -\frac{1}{2|k|} + \frac{1}{2k} \right\} e^{-2\pi x|k|}\, e^{2\pi i y k} \\ &= -\sum_{k=1}^{\infty} \frac{e^{-2\pi k(x+iy)}}{k} \\ &= \log\left(1 - e^{-2\pi(x+iy)}\right). \end{aligned} \tag{10.1}$$

Proof. We define

$$S_N(x+iy) = \sum_{n=-N}^{N} \log\{x + (y+n)i\} - \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \log\{x + (y+u)i\}\, du\,.$$

Using the Euler-Maclaurin summation formula:

$$S_N(x+iy) = \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} B_1(u)\, \frac{i}{x + (y+u)i}\, du\,, \tag{10.2}$$

where $B_1(u) = \langle u \rangle - \frac{1}{2}$ for $u \notin \mathbb{Z}$, $\langle u \rangle = u - \lfloor u \rfloor$, and we have used $B_1(-N - \frac{1}{2}) = B_1(N + \frac{1}{2}) = 0$ for N integer. Also, using one more step of the Euler-Maclaurin summation formula, or integrating (10.2) by parts:

$$
\begin{aligned}
S_N(x+iy) &= \frac{1}{2} B_2(\tfrac{1}{2}) \left\{ \frac{i}{x + i(y + N + \frac{1}{2})} - \frac{i}{x + i(y - N - \frac{1}{2})} \right\} \\
&\quad + \frac{1}{2} \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} B_2(u) \, \frac{1}{\{x + i(y+u)\}^2} \, du \,,
\end{aligned} \tag{10.3}
$$

where $B_2(u) = \langle x \rangle^2 - \langle x \rangle + \frac{1}{6}$.

First, (10.3) shows that the sequence converges:

$$
\begin{aligned}
S(x+iy) &= \lim_{N\to\infty} S_N(x+iy) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} B_2(u) \, \frac{1}{\{x + i(y+u)\}^2} \, du \,.
\end{aligned}
$$

From here we also get:

$$
\lim_{x\to\infty} S(x+iy) = 0 \,.
$$

Next changing $t = -u$ and taking the limit as $N \to \infty$ in (10.2), we get formally:

$$
\begin{aligned}
S(x+iy) &= i \int_{-\infty}^{\infty} B_1(-t) \, \frac{1}{x + (y-t)i} \, dt \\
&= i\pi \int_{-\infty}^{\infty} B_1(-t) \, H_x^{(\to)}(y-t) \, dt \,,
\end{aligned}
$$

where $H_x^{(\to)}(y) = P_x^{(\to)}(y) + iQ_x^{(\to)}(y) = 1/\pi(x+iy)$ is the H kernel for the right half-plane. Unfortunately the integral does not converge absolutely, so we rewrite it like this:

$$
\begin{aligned}
\frac{1}{\pi i} S(x+iy) &= \int_{-\infty}^{\infty} B_1(-t) \, \left\{ \frac{1}{\pi\{x + (y-t)i\}} - \frac{it}{\pi(1+t^2)} \right\} \, dt \\
&\quad + \lim_{N\to\infty} \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} B_1(-t) \, \frac{it}{\pi(1+t^2)} \, dt \,.
\end{aligned}
$$

Since $t \mapsto B_1(t)/(1+t^2)$ is in $L^1(\mathbb{R})$, the first integral converges absolutely and equals an analytic function in the right half-plane such that the boundary values of its real part coincide with $B_1(-y)$ almost everywhere and is real at $z = 1$ (sec. 7.3.7). The last term, being the difference of two converging sequences, converges and amounts to adding an imaginary constant to that function.

Next, using the harmonic extension of B_1 to the upper half-plane we have:

$$
B_1(iz) = \Re \left\{ \frac{i}{\pi} L_1(e^{-2\pi z}) \right\} \,, \quad \Re(z) > 0 \,,
$$

where $L_1(z) = -\log(1-z)$ $(z \in \mathbb{C} \setminus [1,\infty))$ is the first polylogarithm. Hence

$$
\frac{1}{\pi i} S(x+iy) = \frac{i}{\pi} L_1(e^{-2\pi z}) + iC \,,
$$

where C is a real constant. Since $L_1(e^{-2\pi(x+iy)}) \to 0$ as $x \to \infty$ the constant must be zero, so:

$$\begin{aligned} S(x+iy) &= -L_1(e^{-2\pi(x+iy)}) \\ &= \log\left(1 - e^{-2\pi(x+iy)}\right) \end{aligned}$$

for $x > 0$.

$\square$

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