1. (Inequalities) Find the minimum value of the function

\[ f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \]

for \( x, y, z > 0 \).

- Answer: The answer is 3.

In fact, by the Arithmetic Mean-Geometric Mean inequality

\[ \frac{1}{3} \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \geq \sqrt[3]{\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = 1, \]

hence

\[ f(x, y, z) \geq 3, \]

and the equality is reached for \( x = y = z \).

2. (Algebra) Express \( F = \frac{\sqrt[3]{2}}{1 + \sqrt[3]{2} + \sqrt[3]{4}} \) with a rational denominator.

- Answer: Multiplying numerator and denominator by \( 1 - \sqrt[3]{2} \) we get

\[ F = \frac{\sqrt[3]{2}(1 - \sqrt[3]{2})}{(1 + \sqrt[3]{2} + \sqrt[3]{4})(1 - \sqrt[3]{2})} = \frac{\sqrt[3]{2}(1 - \sqrt[3]{2})}{1 - 2} = \sqrt[3]{2}(\sqrt[3]{2} - 1). \]

3. (Polynomials) Prove that \((2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}\) is rational.

- Answer: Let \( \alpha = (2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3} \). By raising to the third power, expanding and simplifying we get that \( \alpha \) verifies the following polynomial equation:

\[ \alpha^3 + 3\alpha - 4 = 0. \]

We have \( x^3 + 3x - 4 = (x - 1)(x^2 + x + 4) \). The second factor has no real roots, hence \( x^3 + 3x - 4 \) has only one real root equal to 1, i.e., \( \alpha = 1 \).

4. (Number Theory) Find all prime numbers of the form \( n^n + 1 \) which are less than \( 10^{19} \).
For $n = 1, 2$ we get prime numbers 2 and 5. An odd $n$ yields an even $n^2 + 1 > 2$, so $n$ must be even. Since $x + 1$ divides $x^{2k+1} + 1$, we have that $n$ must be a power of 2, $n = 2^u$. Since $n^u = 2^u$, by the same reasoning $u$ cannot have an odd factor, so $u = 2^t$, $n = 2^{2^t}$, and

$$n^u = \left(2^{2^t}\right)^{2^t}.$$ 

For $t = 0, 1, 2$ we get $n^u + 1 = 5, 257, 16^{16} + 1 = 2^64 + 1 > 16 \cdot 10^{18} + 1 > 10^{19}$, so the only solutions are 2, 5, and 257.

5. (Induction) Prove that $3^{n+1}$ divides $2^{3^n} + 1$ for all integers $n \geq 0$.

- Answer: The statement is true for $n = 0$, because 3 indeed divides $2 + 1 = 3$. Next we prove that if the statement is true for some $n \geq 0$, the it is true for $n + 1$ too.

In fact, assume that $3^{n+1}$ divides $2^{3^n} + 1$. Then for $n + 1$ we have

$$2^{3^n+1} + 1 = (2^{3^n})^3 + 1 = (2^{3^n} + 1) \left[(2^{3^n})^2 - 2^{2^n} + 1\right].$$

By inductive hypothesis the first factor is divisible by $3^{n+1}$, and the second factor is divisible by 3 because $2^{3^n} \equiv -1 \pmod{3}$, hence the whole expression is divisible by $3^{n+2}$.

6. (Recurrences) Determine the maximum number of regions in the plane that are determined by $n$ “vee”s. A “vee” is two rays which meet at a point. The angle between them is any positive number.

- Answer: Let $x_n$ be the number of regions in the plane determined by $n$ “vee”s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n + 1)$th “vee” intersects each of the other “vee”s at 4 points, so it is divided into $4n + 1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n + 1$. So the answer is

$$x_n = 2 + (4 + 1) + (4 \cdot 2 + 1) + \cdots + (4 \cdot (n - 1) + 1) = 2n^2 - n + 1.$$ 

7. (Telescoping) Find the infinite product $\prod_{n=0}^{\infty} \left(1 + \frac{1}{3^{2n}}\right)$.

- Answer: We can write

$$1 + \frac{1}{3^{2n}} = \frac{1 - \frac{1}{3^{2n+1}}}{1 - \frac{1}{3^{2n}}}.$$
and use telescoping:

\[
\prod_{n=0}^{N} \left( 1 + \frac{1}{3^{2^n}} \right) = \prod_{n=0}^{N} \frac{1 - \frac{1}{3^{2^{n+1}}}}{1 - \frac{1}{3^{2^n}}} = \\
\frac{1 - \frac{1}{3^2}}{1 - \frac{1}{3}} \cdot \frac{1 - \frac{1}{3^4}}{1 - \frac{1}{3^2}} \cdot \frac{1 - \frac{1}{3^8}}{1 - \frac{1}{3^4}} \cdots \frac{1 - \frac{1}{3^{2N+1}}}{1 - \frac{1}{3^{2N}}} \rightarrow \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.
\]

Remark: A slightly different way to get the same result is by multiplying to the left by \(1 - \frac{1}{3}\) and progressively simplify the product:

\[
(1 - \frac{1}{3}) \prod_{n=0}^{N} \left( 1 + \frac{1}{3^{2^n}} \right) = (1 - \frac{1}{3^2}) \prod_{n=1}^{N} \left( 1 + \frac{1}{3^{2^n}} \right) = \\
\left( 1 - \frac{1}{3^4} \right) \prod_{n=2}^{N} \left( 1 + \frac{1}{3^{2^n}} \right) = \cdots = 1 - \frac{1}{3^{N+1}} \rightarrow \frac{3}{2}.
\]

8. (Pigeonhole Principle) (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

- Answer: A set of 10 elements has \(2^{10} - 1 = 1023\) non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than \(10 \cdot 99 = 990\). There are more subsets than possible sums, so two different subsets \(S_1\) and \(S_2\) must have the same sum. If \(S_1 \cap S_2 = \emptyset\) then we are done. Otherwise remove the common elements and we get two non-intersecting subsets with the same sum.

9. (Generating Functions) Find the infinite sum \(\sum_{n=1}^{\infty} \frac{n}{2^n}\).

- Answer: That sum is the value of the following function at \(x = 1\):

\[
f(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{2^n} \quad (|x| < 2)
\]

which in turn is the derivative of the following power series:

\[
g(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \frac{1}{1 - \frac{x}{2}}.
\]
We have \( g'(x) = \frac{2}{(2-x)^2} \), hence the desired sum is \( f(1) = g'(1) = 2 \).

10. (Symmetries) (Putnam 1980) Evaluate \( \int_{0}^{\pi/2} \frac{dx}{1 + (\tan x)\sqrt{2}} \).

- Answer: Writing \( \alpha = \sqrt{2} \), the integrand \( f(x) = \frac{1}{1 + \tan^\alpha x} \) verifies the following symmetry:

\[
\begin{align*}
  f(x) + f(\frac{\pi}{2} - x) &= \frac{1}{1 + \tan x} + \frac{1}{1 + \cot x} \\
  &= \frac{1}{1 + \tan x} + \frac{\tan x}{1 + \tan x} \\
  &= 1.
\end{align*}
\]

On the other hand, making the substitution \( u = \frac{\pi}{2} - x \):

\[
\int_{0}^{\pi/2} f(\frac{\pi}{2} - x) \, dx = - \int_{\pi/2}^{0} f(u) \, du = \int_{0}^{\pi/2} f(x) \, dx = I,
\]

where \( I \) is the desired integral. So:

\[
2I = \int_{0}^{\pi/2} \{ f(x) + f(\pi/2 - x) \} \, dx = \int_{0}^{\pi/2} 1 \, dx = \frac{\pi}{2}.
\]

Hence \( I = \frac{\pi}{4} \).

11. (Combinatorics) A parking lot for compact cars has 12 adjacent spaces, and 8 are occupied. A large sport-utility vehicle arrives, needing 2 adjacent open spaces. What is the probability that it will be able to park?

- Answer: The number of ways 12 parking spaces can be occupied by 8 cars is \( \binom{12}{8} = 495 \). The number of ways 4 empty spaces can be distributed into 12 spaces so that no two of them are contiguous is \( \binom{12-3}{9} = \binom{9}{4} = 126 \) (the parking space to the right of each of the three leftmost empty spaces cannot be empty). Hence the probability that the sport-utility vehicle cannot park is \( \frac{9}{4} / \binom{12}{8} = 126/495 = 14/55 \). Hence the probability that it can park is \( 1 - 14/41/55 = \).

12. (Complex Numbers) Find a close-form expression for \( \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} \).
- Answer: Write $\sin t = (e^{it} - e^{-it})/2i$ and consider the polynomial

$$p(x) = \prod_{k=1}^{n-1} (x - e^{2\pi ik/n}).$$

We have:

$$P = \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \prod_{k=1}^{n-1} \frac{e^{\pi ik/n} - e^{-\pi ik/n}}{2i} = \frac{e^{-\pi i(n-1)/2}}{(2i)^{n-1}} \prod_{k=1}^{n-1} (e^{2\pi ik/n} - 1) = \frac{p(1)}{2^{n-1}}.$$

On the other hand the roots of $p(x)$ are all $n$th roots of 1 except 1, so $(x-1)p(x) = x^n - 1$, and

$$p(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Consequently $p(1) = n$, and Consequently $p(1) = n$, and $P = \frac{n}{2^{n-1}}$.

13. (Calculus) Compute $\lim_{n \to \infty} \left( \prod_{k=1}^{n} \left( 1 + \frac{k}{n} \right) \right)^{1/n}$.

- Answer: Let $P$ be the limit. Then

$$\ln(P) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} \ln \left( 1 + \frac{k}{n} \right).$$

That sum is a Riemann sum for the following integral:

$$\int_{0}^{1} \ln(1 + x) \, dx = [(1 + x)\ln(1 + x)]_{0}^{1} = 2 \ln 2 - 1.$$

Hence $P = e^{2\ln 2 - 1} = 4/e$.

14. (Games) Consider the following two-player game. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny which is already on the table. The table starts out with no pennies. The last player who makes a legal move wins. Does the first player have a winning strategy?

- Answer: The first player does have a winning strategy: place the first penny exactly on the center of the table, and then after the second player places a penny, place the next penny in a symmetric position respect to the center of the table. After each of the first player’s move the configuration of pennies on the table will have radial symmetry, so if the second player can still place a penny somewhere on the table, the radially symmetric position respect to the center of the table will still not be occupied and the first player will also be able to place a penny there.
15. (Other) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( f \circ f \) has a fixed point, i.e., there is some real number \( x_0 \) such that \( f(f(x_0)) = x_0 \). Prove that \( f \) also has a fixed point.

- **Answer:** By contradiction. If the equality \( f(x) = x \) never holds then \( f(x) > x \) for every \( x \), or \( f(x) < x \) for every \( x \). Then \( f(f(x)) > f(x) > x \) for every \( x \), or \( f(f(x)) < f(x) < x \) for every \( x \), contradicting the hypothesis that \( f \circ f \) has a fixed point.