Main Inequalities

1. Arithmetic Mean, Geometric Mean, Harmonic Mean Inequalities. Let $a_1, \ldots, a_n$ be positive numbers. Then the following inequalities hold:

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \leq \sqrt[n]{a_1 \cdot a_2 \cdots \cdot a_n} \leq \frac{a_1 + \cdots + a_n}{n}$$

| Harmonic Mean | Geometric Mean | Arithmetic Mean |

In all cases equality holds if and only if $a_1 = \cdots = a_n$.

2. Power Means Inequality. The AM-GM, GM-HM and AM-HM inequalities are particular cases of a more general kind of inequality called Power Means Inequality. Let $r$ be a non-zero real number. We define the $r$-mean or $r$th power mean of positive numbers $a_1, \ldots, a_n$ as follows:

$$M^r(a_1, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r\right)^{1/r}.$$  

The ordinary arithmetic mean is $M^1$, $M^2$ is the quadratic mean, $M^{-1}$ is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

$$M^0(a_1, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$  

Then, for any real numbers $r$, $s$ such that $r < s$, the following inequality holds:

$$M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n).$$  

Equality holds if and only if $a_1 = \cdots = a_n$.

2.1. Power Means Sub/Superadditivity. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers.

(a) If $r > 1$, then the $r$-mean is subadditive, i.e.:

$$M^r(a_1 + b_1, \ldots, a_n + b_n) \leq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n).$$

(b) If $r < 1$, then the $r$-mean is superadditive, i.e.:

$$M^r(a_1 + b_1, \ldots, a_n + b_n) \geq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n).$$

Equality holds if and only if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

3. Cauchy-Schwarz.

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right).$$
4. Hölder. If \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) then
\[
\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q}.
\]
(For \( p = q = 2 \) we get Cauchy-Schwarz.)

5. Minkowski. If \( p > 1 \) then
\[
\left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/p}.
\]
Equality holds iff \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) are proportional.

6. Norm Monotonicity. If \( a_i > 0 \) \((i = 1, \ldots, n)\), \( s > t > 0 \), then
\[
\left( \sum_{i=1}^{n} a_i^s \right)^{1/s} \leq \left( \sum_{i=1}^{n} a_i^t \right)^{1/t}.
\]

7. Chebyshev. Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be sequences of real numbers which are monotonic in the same direction (we have \( a_1 \leq a_2 \leq \cdots \leq a_n \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \), or we could reverse all inequalities.) Then
\[
\frac{1}{n} \sum_{i=1}^{n} a_i b_i \geq \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} b_i \right).
\]

8. Rearrangement Inequality. For every choice of real numbers \( x_1 \leq \cdots \leq x_n \) and \( y_1 \leq \cdots \leq y_n \), and any permutation \( x_{\sigma(1)}, \ldots, x_{\sigma(n)} \) of \( x_1, \ldots, x_n \), we have
\[
x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n.
\]
If the numbers are different, e.g., \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \), then the lower bound is attained only for the permutation which reverses the order, i.e. \( \sigma(i) = n - i + 1 \), and the upper bound is attained only for the identity, i.e. \( \sigma(i) = i \), for \( i = 1, \ldots, n \).

9. Schur. If \( x, y, z \) are positive real numbers and \( k \) is a real number such that \( k \geq 1 \), then
\[
x^k(x - y)(x - z) + y^k(y - x)(y - z) + z^k(z - x)(z - y) \geq 0.
\]
For \( k = 1 \) the inequality becomes
\[
x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x).
\]
10. **Weighted Power Means Inequality.** Let \( w_1, \ldots, w_n \) be positive real numbers such that \( w_1 + \cdots + w_n = 1 \). Let \( r \) be a non-zero real number. We define the \( r \)th weighted power mean of non-negative numbers \( a_1, \ldots, a_n \) as follows:

\[
M^r_w(a_1, \ldots, a_n) = \left( \sum_{i=1}^{n} w_i a_i^r \right)^{1/r}.
\]

As \( r \to 0 \) the \( r \)th weighted power mean tends to:

\[
M^0_w(a_1, \ldots, a_n) = \left( \prod_{i=1}^{n} a_i^{w_i} \right).
\]

which we call 0th weighted power mean. If \( w_i = 1/n \) we get the ordinary \( r \)th power means. Then for any real numbers \( r, s \) such that \( r < s \), the following inequality holds:

\[
M^r_w(a_1, \ldots, a_n) \leq M^s_w(a_1, \ldots, a_n).
\]

11. **Convexity.** A function \( f: (a, b) \to \mathbb{R} \) is said to be **convex** if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for every \( x, y \in (a, b), 0 \leq \lambda \leq 1 \). Graphically, the condition is that for \( x < t < y \) the point \((t, f(t))\) should lie below or on the line connecting the points \((x, f(x))\) and \((y, f(y))\).

![Convex function](image)