1. Show that if $a^2 + b^2 = c^2$, then $3|ab$.

   - Answer: For any integer $n$ we have that $n^2$ only can be 0 or 1 mod 3. So if $3$ does not divide $a$ or $b$ they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.

2. Show that $2^{70} + 3^{70}$ is divisible by 13.

   - Answer: According to Fermat’s Little Theorem, if $p$ is prime and it does not divide $a$, then $a^{p-1} \equiv 1 \pmod{p}$. Hence $2^{72} = (2^6)^{12} \equiv 1 \pmod{13}$, and $3^{72} = (3^6)^{12} \equiv 1 \pmod{13}$. Hence

$2^{70} + 3^{70} \equiv 2^{-2} \cdot 2^{72} + 3^{-2} \cdot 3^{72} \equiv 2^{-2} + 3^{-2} \equiv 6^{-2} \cdot (9 + 4) \equiv 6^{-2} \cdot 13 \equiv 0 \pmod{13}$.

3. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of an integer.

   - Answer: Pick 1999 different prime numbers $p_1, p_2, \ldots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:

$$
\begin{align*}
  x & \equiv 0 \pmod{p_1^3} \\
  x & \equiv -1 \pmod{p_2^3} \\
  x & \equiv -2 \pmod{p_3^3} \\
  \vdots \\
  x & \equiv -1998 \pmod{p_{1999}^3} 
\end{align*}
$$

According to the Chinese Remainder Theorem, that system of congruences has a solution $x$ (modulo $M = p_1^3 \cdots p_{1999}^3$). For $k = 1, \ldots, 1999$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of $p_k^3$.

4. Prove that there are no primes in the following infinite sequence of numbers:

$$1001, 1001001, 1001001001, 1001001001001, \ldots$$
- **Answer:** Each of the given numbers can be written

\[ 1 + 1000 + 1000^2 + \cdots + 1000^n = p_n(10^3) \]

where \( p_n(x) = 1 + x + x^2 + \cdots + x^n, \ n = 1, 2, 3, \ldots \) We have \( (x - 1)p_n(x) = x^{n+1} - 1. \)

If we set \( x = 10^3, \) we get:

\[ 999 \cdot p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1). \]

If \( p_n(10^3) \) were prime it should divide one of the factors on the RHS. It cannot divide \( 10^{n+1} - 1, \) because this factor is less than \( p_n(10^3), \) so \( p_n(10^3) \) must divide the other factor. Hence \( 10^{n+1} - 1 \) must divide 999, but this is impossible for \( n > 2. \) In only remains to check the cases \( n = 1 \) and \( n = 2. \) But 1001 = 7 \( \cdot \) 11 \( \cdot \) 13, and 1001001 = 3 \( \cdot \) 333667, so they are not prime either.

**5.** Let \( a_n = 10 + n^2 \) for \( n \geq 1. \) For each \( n, \) let \( d_n \) denote the \( \gcd \) of \( a_n \) and \( a_{n+1}. \) Find the maximum value of \( d_n \) as \( n \) ranges through the positive integers.

- **Answer:** The answer is 41. In fact, we have:

\[ \gcd(a_n, a_{n+1}) = \gcd(a_n, a_{n+1} - a_n) = \gcd(n^2 + 10, 2n + 1) = \cdots \]

(since \( 2n + 1 \) is odd we can multiply the other argument by 4 without altering the \( \gcd \))

\[ \cdots = \gcd(4n^2 + 40, 2n + 1) = \gcd((2n + 1)(2n - 1) + 41, 2n + 1) = \gcd(41, 2n + 1) \leq 41. \]

The maximum value is attained e.g. at \( n = 20. \)

**6.** Suppose that the positive integers \( x, \ y \) satisfy \( 2x^2 + x = 3y^2 + y. \) Show that \( x - y, 2x + 2y + 1, 3x + 3y + 1 \) are all perfect squares.

- **Answer:** The given condition implies:

\[ (x - y)(2x + 2y + 1) = y^2. \]

Since the right hand side is a square, to prove that the two factors on the left hand side are also squares it suffices to prove that they are relatively prime. In fact, if \( p \) if a prime number dividing \( x - y \) then it divides \( y^2 \) and consequently it divides \( y. \) So \( p \) also divides \( x, \) and \( x + y. \) But then it cannot divide \( 2x + 2y + 1. \)

An analogous reasoning works using the following relation, also implied the given condition:

\[ (x - y)(3x + 3y + 1) = x^2. \]